Lecture 4.3
Estimating homographies from feature correspondences

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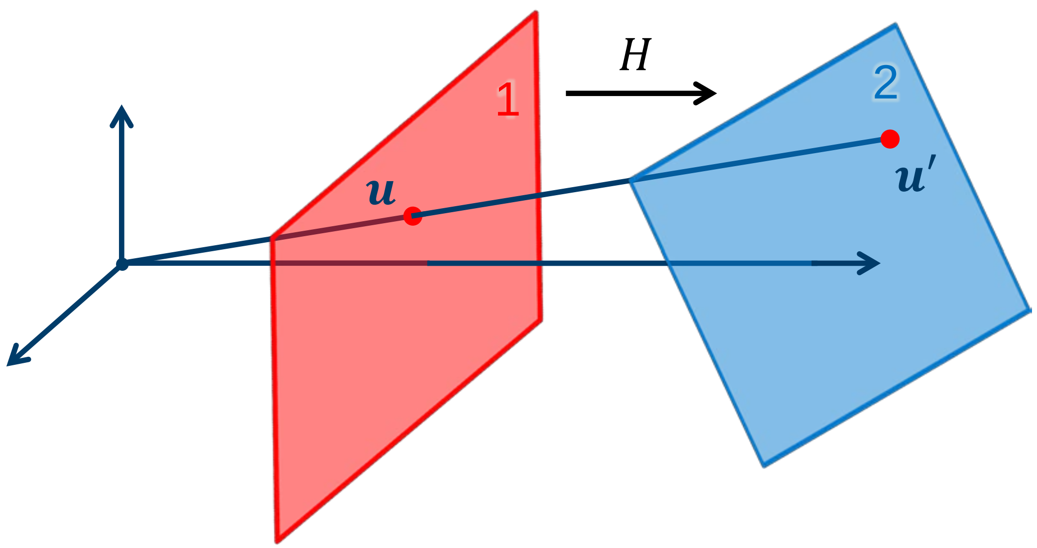
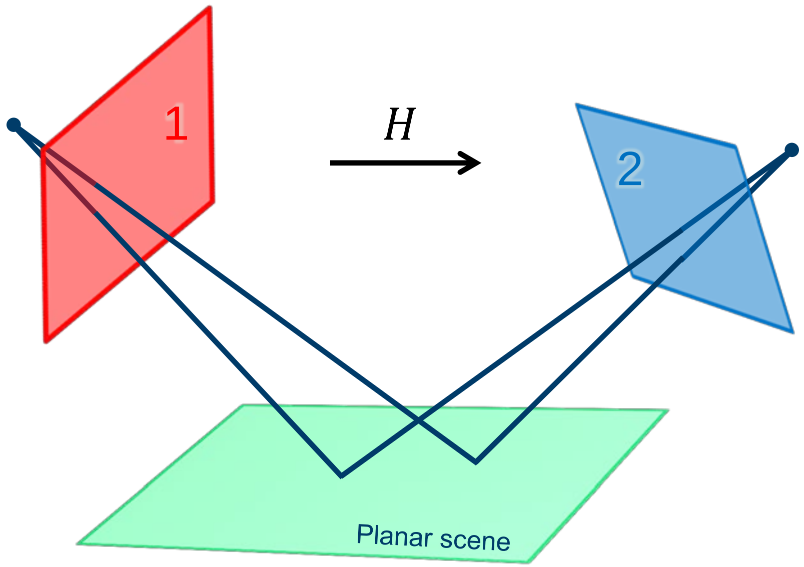
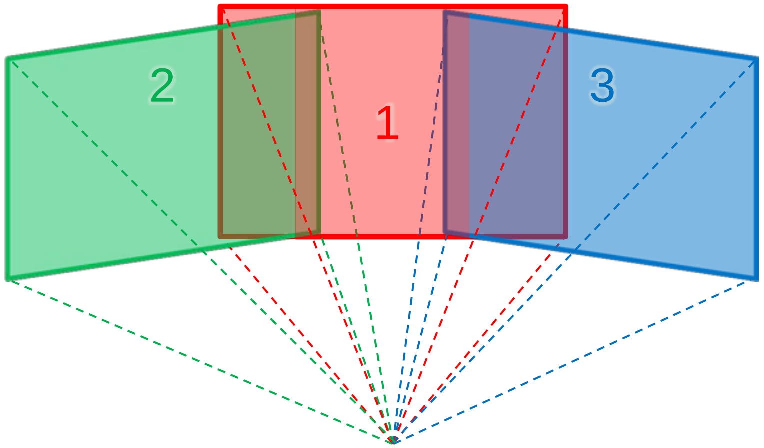
Homographies induced by central projection





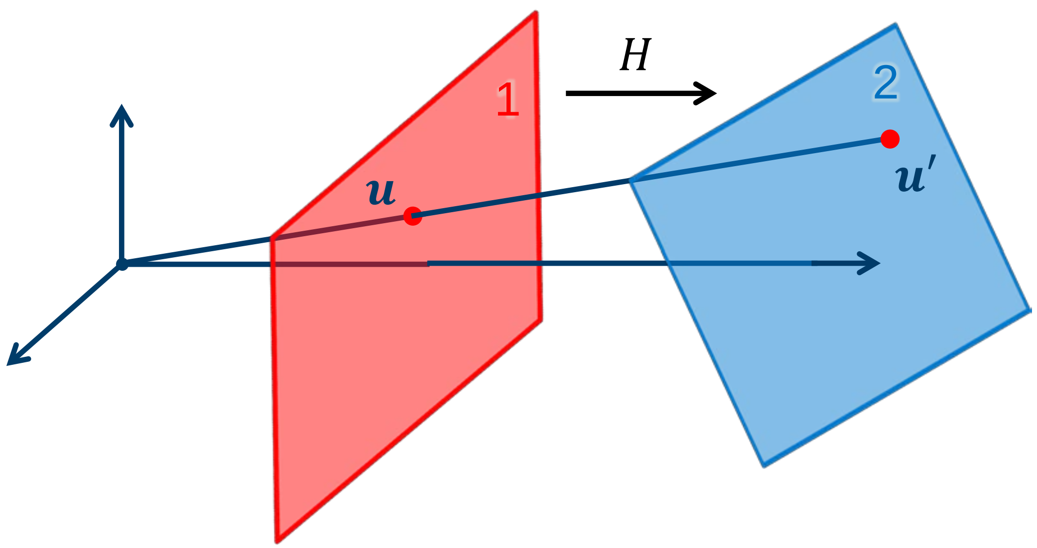


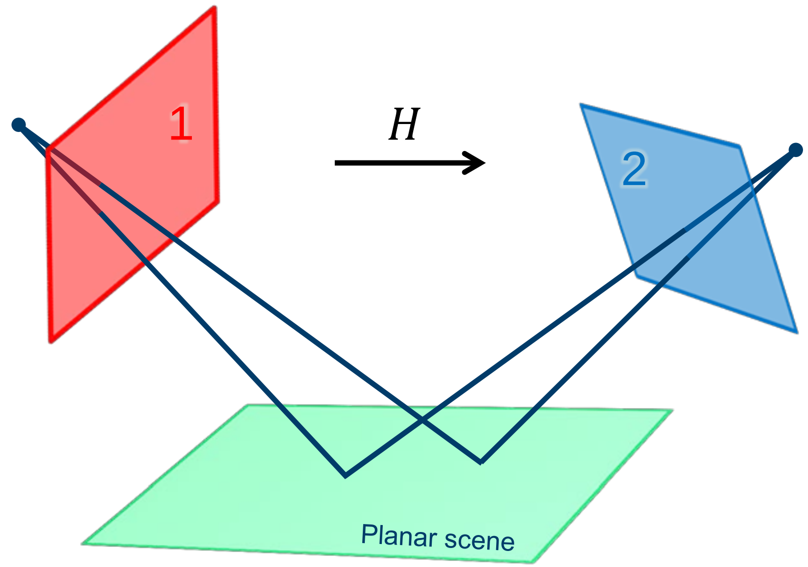
- Homography $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$

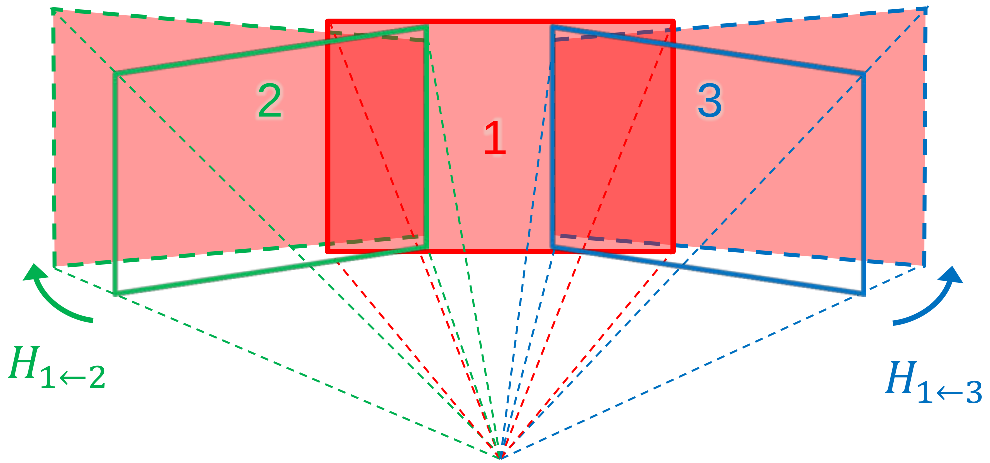
$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

- Point-correspondences can be determined automatically
- Erroneous correspondences are common
- Robust estimation is required to find H

Homographies induced by central projection







- Homography $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$

$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

- Point-correspondences can be determined automatically
- Erroneous correspondences are common
- Robust estimation is required to find H

Estimating the homography between overlapping images

- Establish point correspondences $\boldsymbol{u}_i \leftrightarrow \boldsymbol{u}'_i$
  - Find key points $\{\boldsymbol{u}_i \in \mathrm{Img1}\}$ and $\{\boldsymbol{u}'_i \in \mathrm{Img2}\}$
  - Represent key points by suitable descriptors
  - Determine correspondences $\boldsymbol{u}_i \leftrightarrow \boldsymbol{u}'_i$ by matching descriptors
  - Some wrong correspondences are to be expected

- Estimate the homography H such that $\boldsymbol{u}'_i = H\boldsymbol{u}_i \; \forall i$
  - Robust estimation with RANSAC
  - Improved estimation based on RANSAC inliers

- This homography enables us to compose the images into a larger image
  - Image mosaicing
  - Panorama

Adaptive RANSAC

Objective

To robustly fit a model $\boldsymbol{y} = f(\boldsymbol{x}; \boldsymbol{\alpha})$ to a data set S containing outliers

Algorithm

1. Let $N = \infty$, $S_{IN} = \emptyset$ and $\#iterations = 0$

2. while $N > \#iterations$ repeat 3-5

3. Estimate parameters $\boldsymbol{\alpha}_{tst}$ from a random n-tuple from S

4. Determine inlier set S_{tst}, i.e. data points within a distance t of the model $\boldsymbol{y} = f(\boldsymbol{x}; \boldsymbol{\alpha}_{tst})$

5. If $|S_{tst}| > |S_{IN}|$, set $S_{IN} = S_{tst}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{tst}$, $\omega = \frac{|S_{IN}|}{|S|}$ and $N = \frac{log(1-p)}{log(1-\omega^n)}$ with $p = 0.99$

   Increase $\#iterations$ by 1

Estimating the homography

- Estimating the homography in a RANSAC scheme requires
  1. A basic homography estimation method for n point-correspondences
  2. A way to determine the inlier set of point-correspondences for a given homography

Estimating the homography

- Estimating the homography in a RANSAC scheme requires
  1. **A basic homography estimation method for n point-correspondences**
  2. A way to determine the inlier set of point-correspondences for a given homography

- The homography has 8 degrees of freedom, but it is custom to treat all 9 entries of the matrix as unknowns instead of setting one of the entries to 1 which excludes all potential solutions where this entry is 0

- Let us solve the equation $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$ for the entries of the homography matrix

$$H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$$

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$$

Basic homography estimation

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \Leftrightarrow \begin{Bmatrix} uh_1 + vh_2 + h_3 = u' \\ uh_4 + vh_5 + h_6 = v' \\ uh_7 + vh_8 + h_9 = 1 \end{Bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & -u & -v & -1 & v'u & v'v & v' \\ u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ -v'u & -v'v & -v' & u'u & u'v & u' & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow A\mathbf{h} = \mathbf{0}$$

Basic homography estimation

$$\begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} \Leftrightarrow \begin{Bmatrix} uh_1 + vh_2 + h_3 = u' \\ uh_4 + vh_5 + h_6 = v' \\ uh_7 + vh_8 + h_9 = 1 \end{Bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 0 & 0 & -u & -v & -1 & v'u & v'v & v' \\ u & v & 1 & 0 & 0 & 0 & -u'u & -u'v & -u' \\ -v'u & -v'v & -v' & u'u & u'v & u' & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow A\mathbf{h} = \mathbf{0}$$

Observe that the third row in A is a linear combination of the first and second row

$$row_3 = -u' \cdot row_1 - v' \cdot row_2$$

Hence every correspondence $\boldsymbol{u}_i \leftrightarrow \boldsymbol{u}'_i$ contribute with 2 equations in the 9 unknown entries

Basic homography estimation

- Since H (and thus $\boldsymbol{h}$) is homogeneous, we only need the matrix A to have rank 8 in order to determine $\boldsymbol{h}$ up to scale

- It is sufficient with 4 point correspondences where no 3 points are collinear

- We can calculate the non-trivial solution to the equation $A\boldsymbol{h} = \boldsymbol{0}$ by SVD
$$\operatorname{svd}(A) = USV^T$$

- The solution is given by the right singular vector without a singular value which is the last column of V, i.e. $\boldsymbol{h} = \boldsymbol{v}_9$

$$\begin{bmatrix} 0 & 0 & 0 & -u_1 & -v_1 & -1 & v_1'u_1 & v_1'v_1 & v_1' \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1'u_1 & -u_1'v_1 & -u_1' \\ 0 & 0 & 0 & -u_2 & -v_2 & -1 & v_2'u_2 & v_2'v_2 & v_2' \\ u_2 & v_2 & 1 & 0 & 0 & 0 & -u_2'u_2 & -u_2'v_2 & -u_2' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$$A\mathbf{h} = \mathbf{0}$$

Basic homography estimation

- Estimating the homography in a RANSAC scheme requires
  1. **A basic homography estimation method for n point-correspondences**
  2. A way to determine which of the point correspondences that are inliers for a given homography

Direct Linear Transform

$$A = \begin{bmatrix} 0 & 0 & 0 & -u_1 & -v_1 & -1 & v_1' u_1 & v_1' v_1 & v_1' \\ u_1 & v_1 & 1 & 0 & 0 & 0 & -u_1' u_1 & -u_1' v_1 & -u_1' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

1. Build the matrix A from at least 4 point-correspondences $(u_i, v_i) \leftrightarrow (u'_i, v'_i)$
2. Obtain the SVD of A: $A = USV^T$
3. If S is diagonal with positive values in descending order along the main diagonal, then $\boldsymbol{h}$ equals the last column of V
4. Reconstruct H from $\boldsymbol{h}$

Basic homography estimation

- The basic DLT algorithm is never used with more than 4 point-correspondences
- This is because the algorithm performs better when all the terms of A has a similar scale
  - Note that some of the terms will always be of scale 1
- To achieve this, it is common to extend the algorithm with a normalization and a denormalization step

Normalized Direct Linear Transform

1. Normalize the set of points $\boldsymbol{u}_i = [u_i, v_i]^T$ by computing a similarity transform T that translates the centroid to the origin and scales such that the average distance from the origin is $\sqrt{2}$
2. In the same way normalize the set of points $\boldsymbol{u'}_i = [u'_i, v'_i]^T$ by computing a similarity transform T'
3. Apply the basic DLT algorithm on the normalized points to obtain a homography $\hat{H}$
4. Denormalize to get the homography: $H = T'^{-1}\hat{H}T$

Basic homography estimation

- Estimating the homography in a RANSAC scheme requires
  1. A basic homography estimation method for n point-correspondences
  2. **A way to determine the inlier set of point-correspondences for a given homography**

- For a point correspondence $(u_i, v_i) \leftrightarrow (u'_i, v'_i)$ and homography H, we can choose from several errors
  - Algebraic error: $\varepsilon_i = ||A_i \boldsymbol{h}||$ where

$$A_i = \begin{bmatrix} 0 & 0 & 0 & -u_i & -v_i & -1 & v_i' u_i & v_i' v_i & v_i' \\ u_i & v_i & 1 & 0 & 0 & 0 & -u_i' u_i & -u_i' v_i & -u_i' \end{bmatrix}$$

  - Geometric errors:
    1. $\varepsilon_i = d(H\boldsymbol{u}_i, \boldsymbol{u}'_i) + d(\boldsymbol{u}_i, H^{-1}\boldsymbol{u}'_i)$ **(Reprojection error)**
    2. $\varepsilon_i = d(\boldsymbol{u}_i, H^{-1}\boldsymbol{u}'_i)$
    3. $\varepsilon_i = d(H\boldsymbol{u}_i, \boldsymbol{u}'_i)$

Notation	
Euclidean distance	$d(\cdot,\cdot)$
Inhomogenous $H\tilde{\boldsymbol{u}}_i$	$H\boldsymbol{u}_i$
Inhomogeneous $H^{-1}\tilde{\boldsymbol{u}}'_i$	$H^{-1}\boldsymbol{u}'_i$

Robust homography estimation

RANSAC estimation of homography
For a set of point-correspondences $S = \{\boldsymbol{u_i} \mapsto \boldsymbol{u'}_i\}$, perform N iterations, where N is determined adaptively

1. Estimate H_{tst} from 4 random correspondences $\boldsymbol{u_i} \mapsto \boldsymbol{u'}_i$ using the basic DLT algorithm
2. Determine the set of inlier-correspondences $S_{tst} = \{\boldsymbol{u_i} \mapsto \boldsymbol{u'}_i \; such \; that \; \epsilon_i < t\}$
   Here one can choose $\epsilon_i = d(H\boldsymbol{u}_i, \boldsymbol{u'}_i) + d(\boldsymbol{u}_i, H^{-1}\boldsymbol{u'}_i)$ and $t = \sqrt{5.99}\sigma$ where σ is the expected uncertainty in key-point positions
3. If $|S_{tst}| > |S_{IN}|$ update N, homography and inlier set: $H = H_{tst}$, $S_{IN} = S_{tst}$

- Finally we would typically re-estimate H from all correspondences in S_{IN}
  - Normalized DLT
  - Minimize $\epsilon = \sum \epsilon_i$ in an iterative optimization method like Levenberg Marquardt

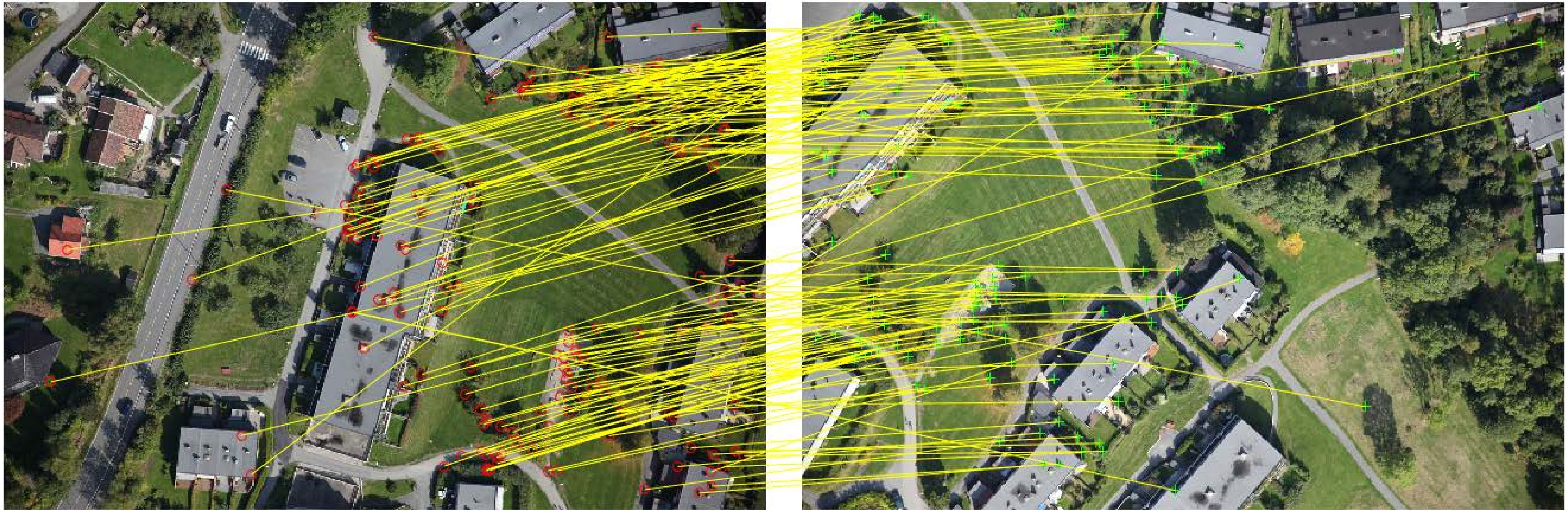
Image mosaicing

- Let us compose these two images into a larger image

Image mosaicing

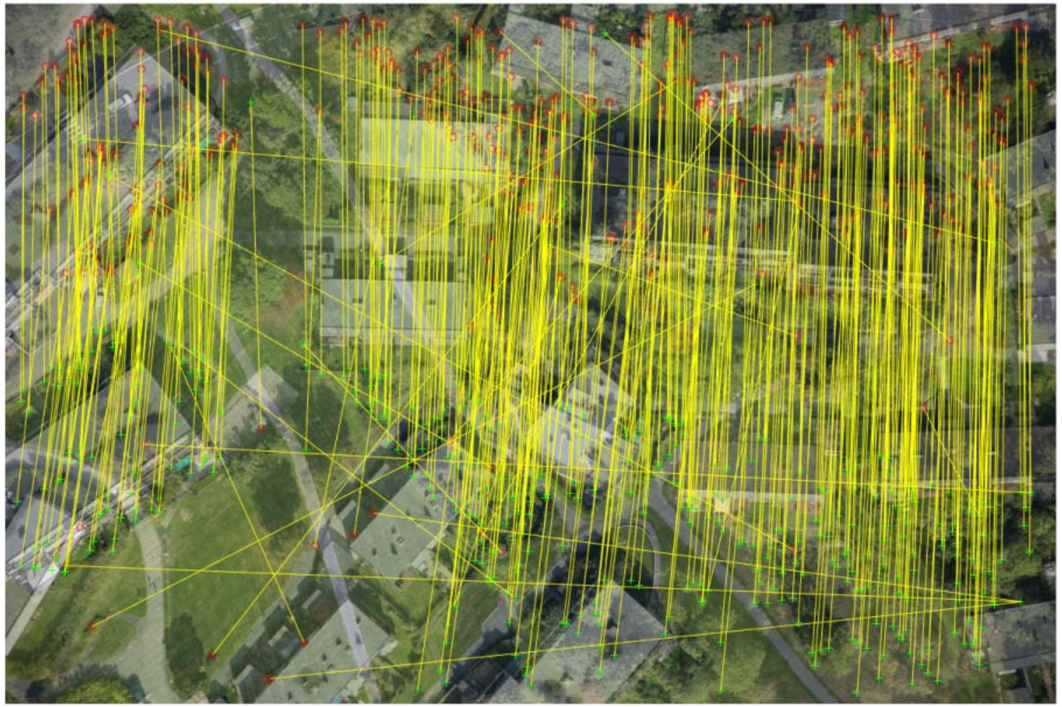
- Find key points and represent by descriptors

Image mosaicing



- Establish point-correspondences by matching descriptors
- Several wrong correspondences

Image mosaicing



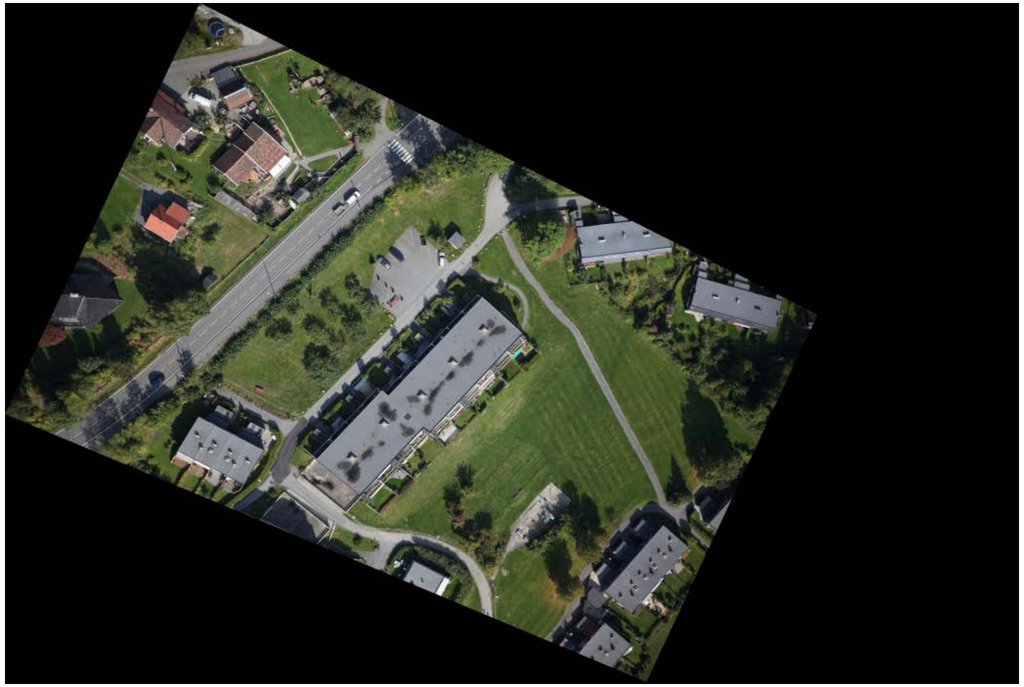
- Establish point-correspondences by matching descriptors
- Several wrong correspondences

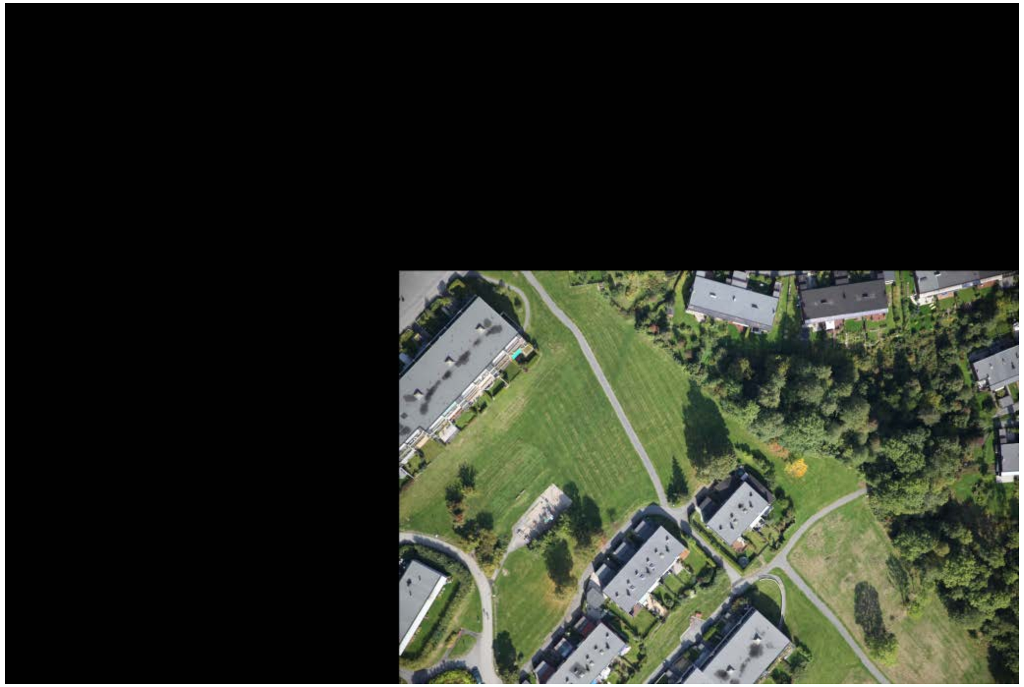
Image mosaicing

$\xrightarrow{H}$

- Estimate homography $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$
  - OpenCV
```
#include "opencv2/calib3d.hpp"
cv::findHomography(srcPoints, dstPoints, CV_RANSAC);
```
  - Matlab
```
tform = estimateGeometricTransform(srcPoints,dstPoints,'projective');
```

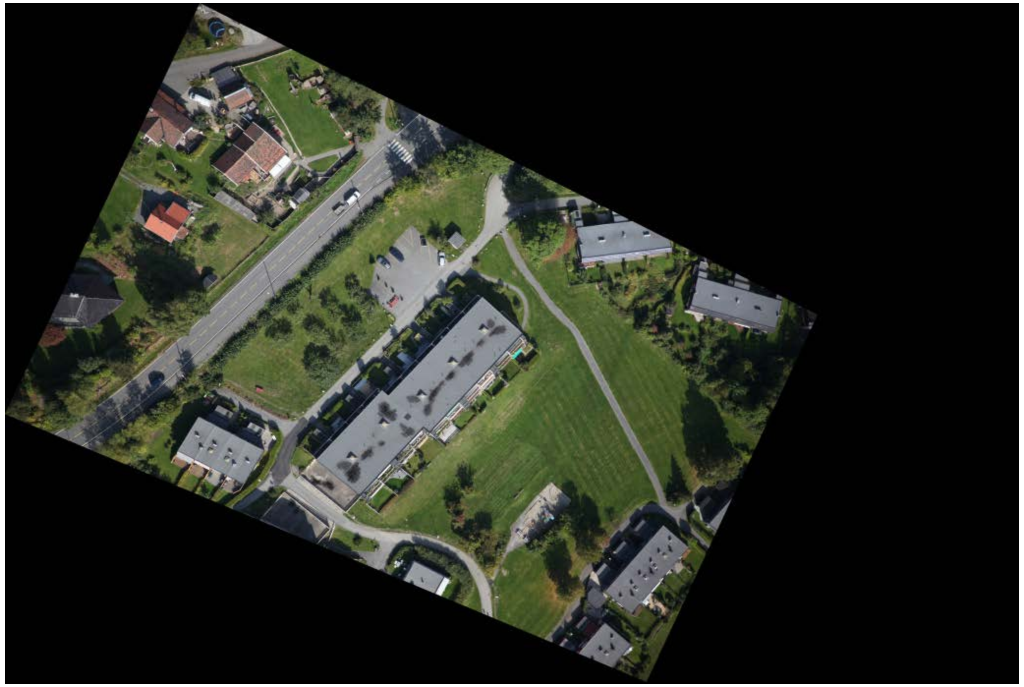
Image mosaicing

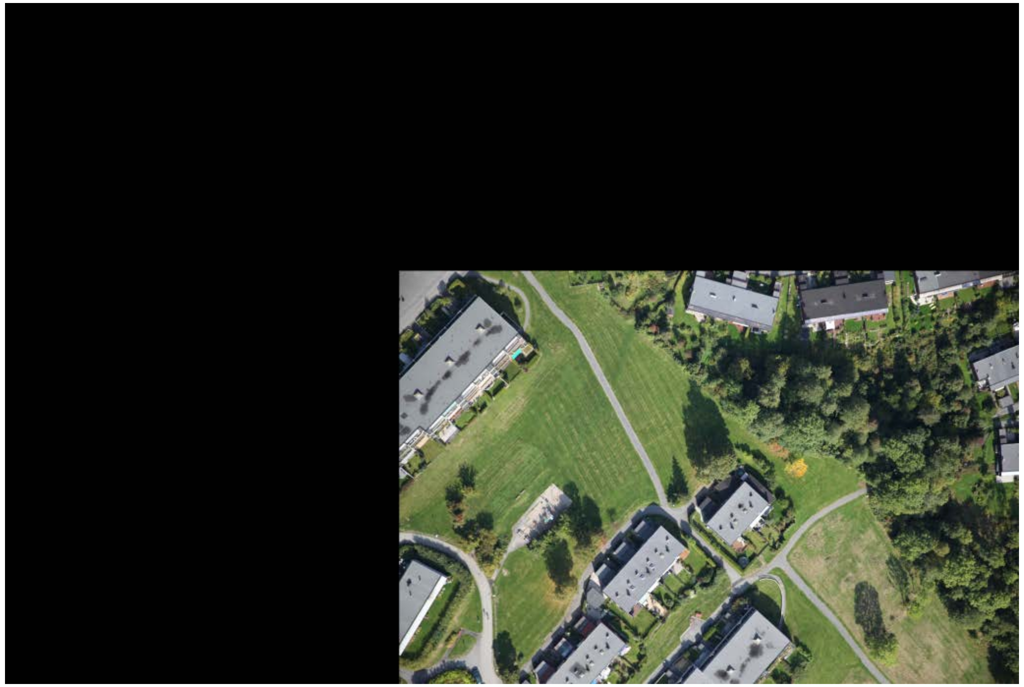




- Represent the images in common coordinates (Note the additional translation!)
  - OpenCV
```
#include "opencv2/calib3d.hpp"
cv::warpPerspective(img1, img2, H, output_size);
```
  - Matlab
```
img2 = imwarp(img1,tform);
```

Image mosaicing





- Now we can compose the images

Overwriting

Blending with a ramp

Blending with a ramp + histogram equalization

SVD

Singular Value Decomposition

The singular value decomposition of a real $m \times n$ matrix A is a factorization $A = USV^T$

Here U is a orthogonal $m \times m$ matrix, V is a orthogonal $n \times n$ matrix and S is a real positive diagonal $m \times n$ matrix

The diagonal entries of $S = \operatorname{diag}\left(s_1, \dots, s_{\min(m,n)}\right)$ are known as the singular values of A and the columns of $U = [\boldsymbol{u}_1, \dots, \boldsymbol{u}_m]$ and $V = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$are known as the left and right singular vectors of A respectively

The nullspace of A is the span of the right singular vectors $\boldsymbol{v}_i$ that corresponds to a zero singular value s_i (or does not have a corresponding singular value)

How to use

- Matlab
```
[U,S,V] = svd(A);
```
Right singular vectors are columns in `V`

- OpenCV
```
cv::SVD::compute(A, S, U, Vtranspose,
cv::SVD::FULL_UV);
```
Right singular vectors are rows in `Vtranspose`

- Eigen
```
Eigen::JacobiSVD<Eigen::MatrixXd> svd(A,
Eigen::ComputeFullU | Eigen::ComputeFullV);
```
Right singular vectors are columns in `svd.matrixV()`

SVD

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Applications of SVD

Solving homogeneous linear equations like

$$A\boldsymbol{h} = \boldsymbol{0}$$

Method

For theoretical problems, $\mathbf{h} \in \mathrm{null}(A)$ so $\boldsymbol{h}$ is a linear combination of the right singular vectors $\boldsymbol{v}_i$ that correspond to a zero singular value s_i

$$\boldsymbol{h} = \sum k_i \boldsymbol{v}_i\,; \quad k_i \in \mathbb{R}, s_i = 0 \text{ (or missing)}$$

For practical problems, the presence of noise force us to expand the solution by including those right singular vectors that correspond to small singular values $s_i \approx 0$

$$\boldsymbol{h} = \sum k_i \boldsymbol{v}_i\,; \quad k_i \in \mathbb{R}, s_i \approx 0 \text{ (or missing)}$$

SVD

Example

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From `[U,S,V] = svd(A);` we get

$$U = \begin{bmatrix} -0.3863 & -0.9224 \\ -0.9224 & 0.3863 \end{bmatrix} \quad S = \begin{bmatrix} 9.5080 & 0 & 0 \\ 0 & 0.7729 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.4287 & 0.8060 & 0.4082 \\ -0.5663 & 0.1124 & -0.8165 \\ -0.7039 & -0.5812 & 0.4082 \end{bmatrix}$$

From this we see that A has:

- 2 left singular vectors

$$\mathbf{u}_1 = \begin{bmatrix} -0.3863 \\ -0.9224 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.9224 \\ 0.3863 \end{bmatrix}$$

- 2 nonzero singular values

$$s_1 = 9.5080 \quad s_2 = 0.7729$$

- 3 right singular vectors

$$\mathbf{v}_1 = \begin{bmatrix} -0.4287 \\ -0.5663 \\ -0.7039 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8060 \\ 0.1124 \\ -0.5812 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix}$$

Since $\boldsymbol{v}_3$ does not have a corresponding singular value, $\boldsymbol{x} = \boldsymbol{v}_3$ is a non-trivial solution to $A\boldsymbol{x} = 0$ and $\boldsymbol{x} = k \cdot \boldsymbol{v}_3;\ k \in \mathbb{R}\backslash\{0\}$ is the family of all non-trivial solutions

SVD

Example

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1.0792 & 2.0656 & 3.0849 \\ 4.0959 & 5.0036 & 6.0934 \\ 1.0679 & 2.0743 & 3.0655 \\ 4.0758 & 5.0392 & 6.0171 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This time singular value decomposition give us the following singular values and right singular vectors:

$$s_1 = 13.6295 \quad s_2 = 1.0849 \quad s_3 = 0.0506$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.4336 \\ -0.5635 \\ -0.7032 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.8103 \\ -0.0975 \\ 0.5778 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.3942 \\ -0.8203 \\ 0.4143 \end{bmatrix}$$

This time all right singular vectors correspond to a non-zero singular value, so the equation does not have any non-trivial solutions!

If this equation came from a practical problem, instead of looking for solutions to $A\boldsymbol{x} = 0$, we might be looking for the $\boldsymbol{x}$ that minimize $\|A\boldsymbol{x}\|$

Since $s_1 \not\approx 0, s_2 \not\approx 0, s_3 \approx 0$, we would conclude that $\boldsymbol{x} = \boldsymbol{v}_3$ solves the equation in a least-squares sense

Check:

$$A\mathbf{v}_3 = \begin{bmatrix} 0.0091 \\ 0.4288 \\ -0.0105 \\ -0.0341 \end{bmatrix}$$

Summary

- Homography $H\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{u}}'$

$$H = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix}$$

- Automatic point-correspondences

- Wrong correspondences are common

- RANSAC estimation
  - Basic DLT (Direct Linear Transform) on 4 random correspondences
  - Inliers determined from the reprojection error $\epsilon_i = d(H\boldsymbol{u}_i, \boldsymbol{u}'_i) + d(\boldsymbol{u}_i, H^{-1}\boldsymbol{u}'_i)$

- Improve estimate by normalized DLT on inliers or iterative methods for an even better estimate

- Additional reading
  - Szeliski: 6.1.1 – 6.1.3