Lecture 1.3
Basic projective geometry

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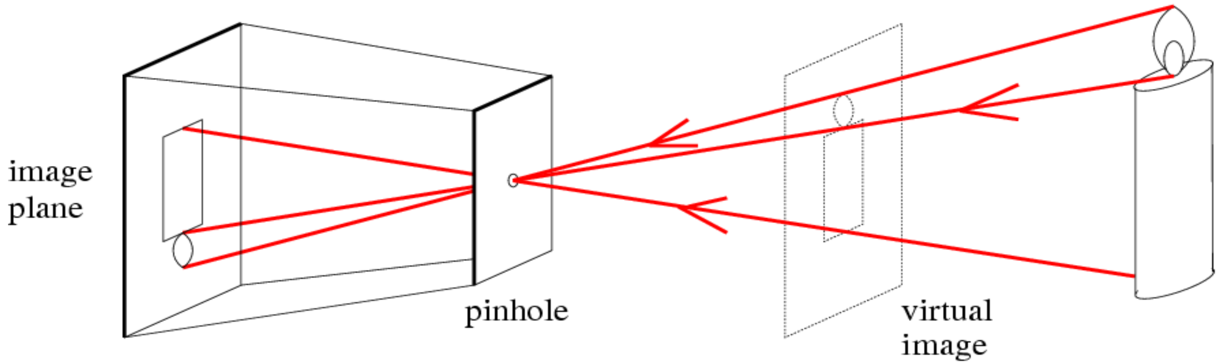
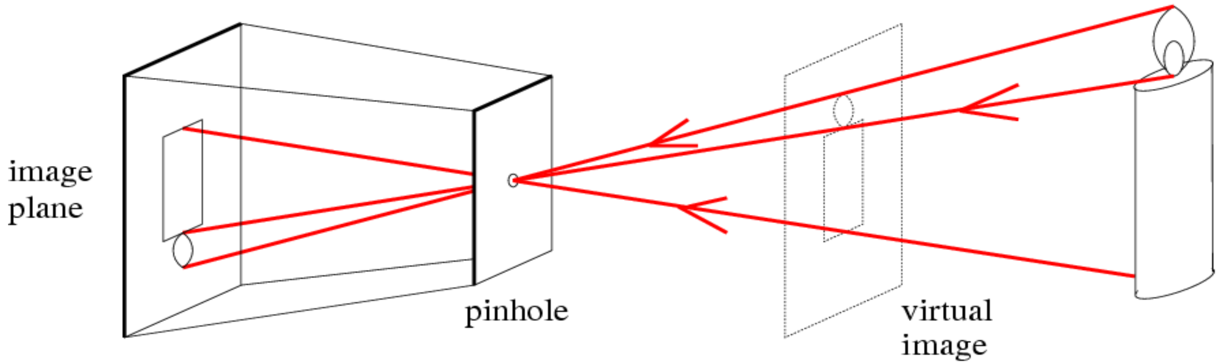
Motivation



- For the pinhole camera, the correspondence between observed 3D points in the world and 2D points in the captured image is given by straight lines through a common point (pinhole)
- This correspondence can be described by a mathematical model known as "*the perspective camera model*" or "*the pinhole camera model*"
- This model can be used to describe the imaging geometry of many modern cameras, hence it plays a central part in computer vision

Motivation

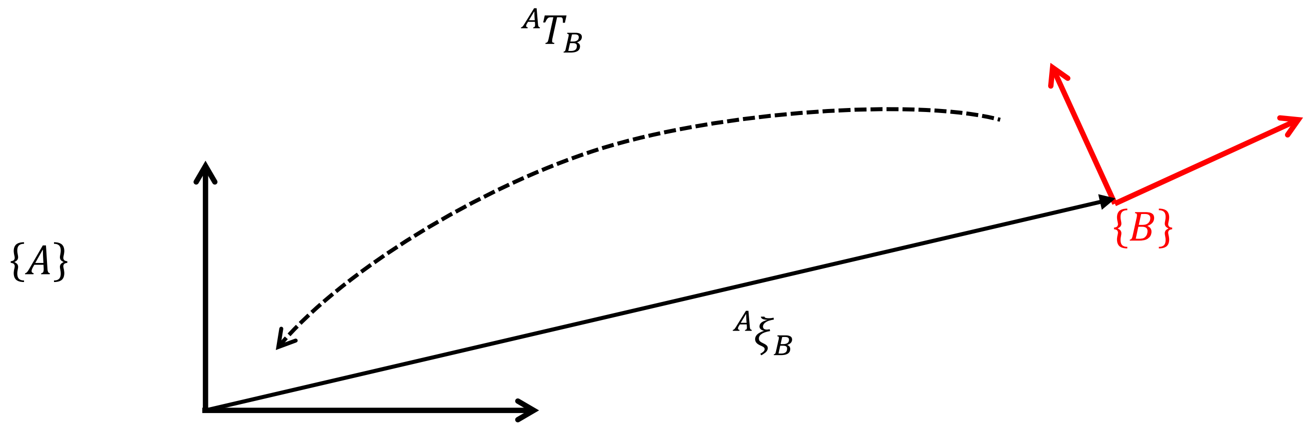


- Before we can study the perspective camera model in detail, we need to expand our mathematical toolbox

- We need to be able to mathematically describe the position and orientation of the camera relative to the world coordinate frame

- Also we need to get familiar with some basic elements of projective geometry, since this will make it MUCH easier to describe and work with the perspective camera model

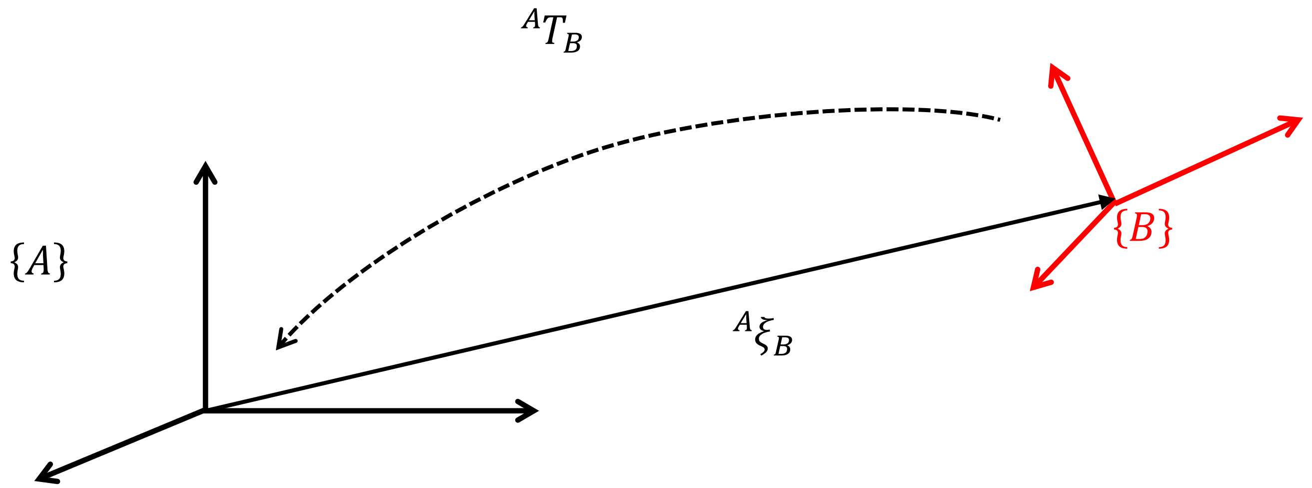
Introduction



- We have seen that the pose of a coordinate frame $\{B\}$ relative to a coordinate frame $\{A\}$, denoted ${}^{A}\xi_{B}$, can be represented as a homogeneous transformation ${}^{A}T_{B}$ in 2D

$$
{}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & {}^{A}t_{Bx} \\ r_{21} & r_{22} & {}^{A}t_{Bx} \\ 0 & 0 & 1 \end{bmatrix} \in SE(2)
$$

Introduction



- We have seen that the pose of a coordinate frame $\{B\}$ relative to a coordinate frame $\{A\}$, denoted ${}^{A}\xi_{B}$, can be represented as a homogeneous transformation ${}^{A}T_{B}$ in 2D and 3D

$$
{}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{A}t_{Bx} \\ r_{21} & r_{22} & r_{23} & {}^{A}t_{By} \\ r_{31} & r_{32} & r_{33} & {}^{A}t_{Bz} \\ 0 & 0 & 0 & 1 \end{bmatrix} \in SE(3)
$$

Introduction

- And we have seen how they can transform points from one reference frame to another if we represent points in homogeneous coordinates

$$\boldsymbol{p} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mapsto \quad \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad\qquad \boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mapsto \quad \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- The main reason for representing pose as homogeneous transformations, was the nice algebraic properties that came with the representation

Introduction

- Euclidean geometry
  - ${}^{A}\xi_{B} \longmapsto ({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$
  - Complicated algebra

$$
\begin{aligned}
{}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} \quad &\mapsto \quad {}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \\
{}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} \quad &\mapsto \quad \left({}^{A}R_{C}, {}^{A}\boldsymbol{t}_{C}\right) = \left({}^{A}R_{B}\,{}^{B}R_{C}, {}^{A}R_{B}\,{}^{B}\boldsymbol{t}_{C} + {}^{A}\boldsymbol{t}_{B}\right) \\
\ominus {}^{A}\xi_{B} \quad &\mapsto \quad \left({}^{A}R_{C}{}^{T}, -{}^{A}R_{C}{}^{T}\,{}^{A}\boldsymbol{t}_{C}\right)
\end{aligned}
$$

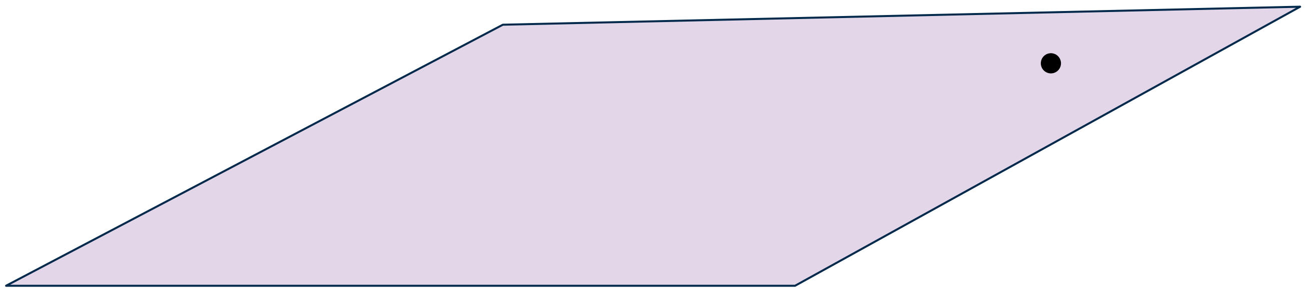
- Projective geometry
  - ${}^{A}\xi_{B} \longmapsto {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix}$
  - Simple algebra

$$
\begin{aligned}
{}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} \quad &\mapsto \quad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}} \\
{}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} \quad &\mapsto \quad {}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C} \\
\ominus {}^{A}\xi_{B} \quad &\mapsto \quad {}^{A}T_{B}{}^{-1}
\end{aligned}
$$

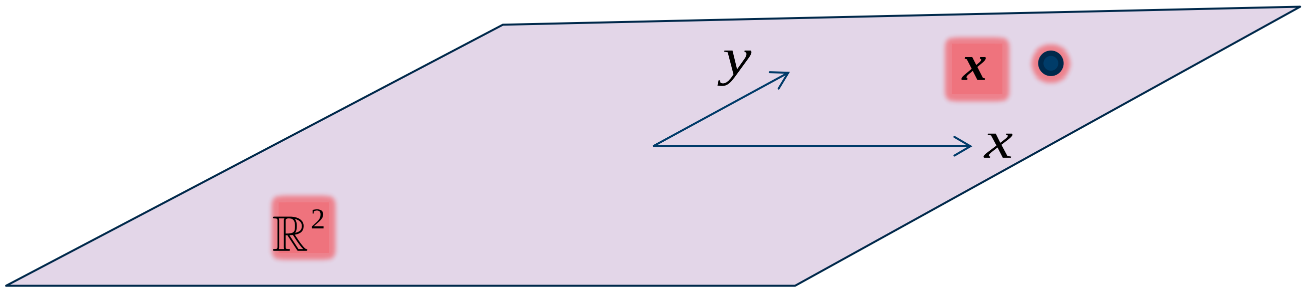
- In the following we will take a closer look at some basic elements of projective geometry that we will encounter when we study the geometrical aspects of imaging
  - Homogeneous coordinates, homogeneous transformations

The projective plane
Points

How to describe points in the plane?



The projective plane
Points



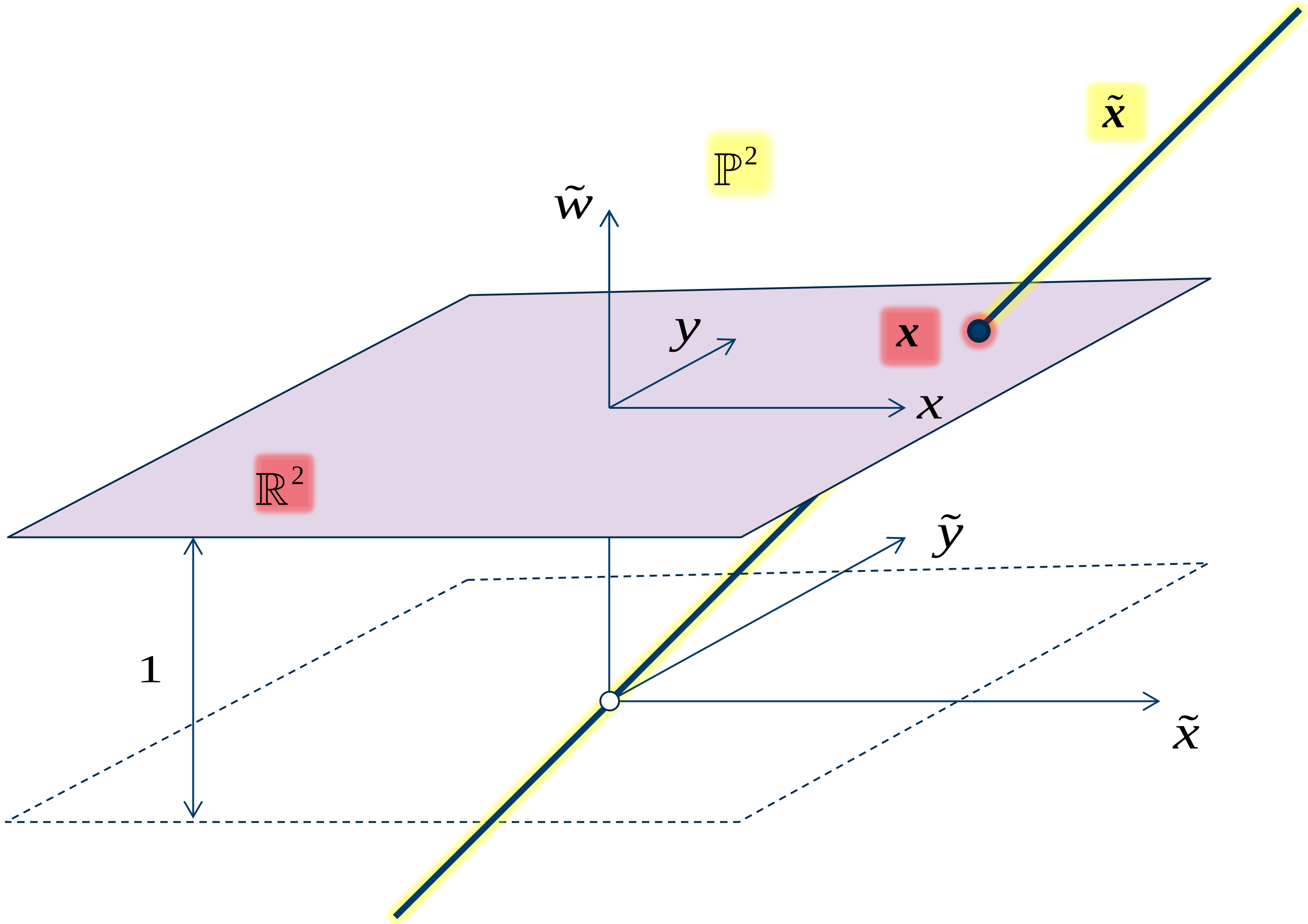
How to describe points in the plane?

Euclidean plane $\mathbb{R}^2$

- Choose a 2D coordinate frame
- Each point corresponds to a unique pair of Cartesian coordinates

$$\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \longmapsto \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The projective plane
Points



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- Choose a 2D coordinate frame
- Each point corresponds to a unique pair of Cartesian coordinates

$$\boldsymbol{x} = (x, y) \in \mathbb{R}^2 \longmapsto \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Projective plane $\mathbb{P}^2$

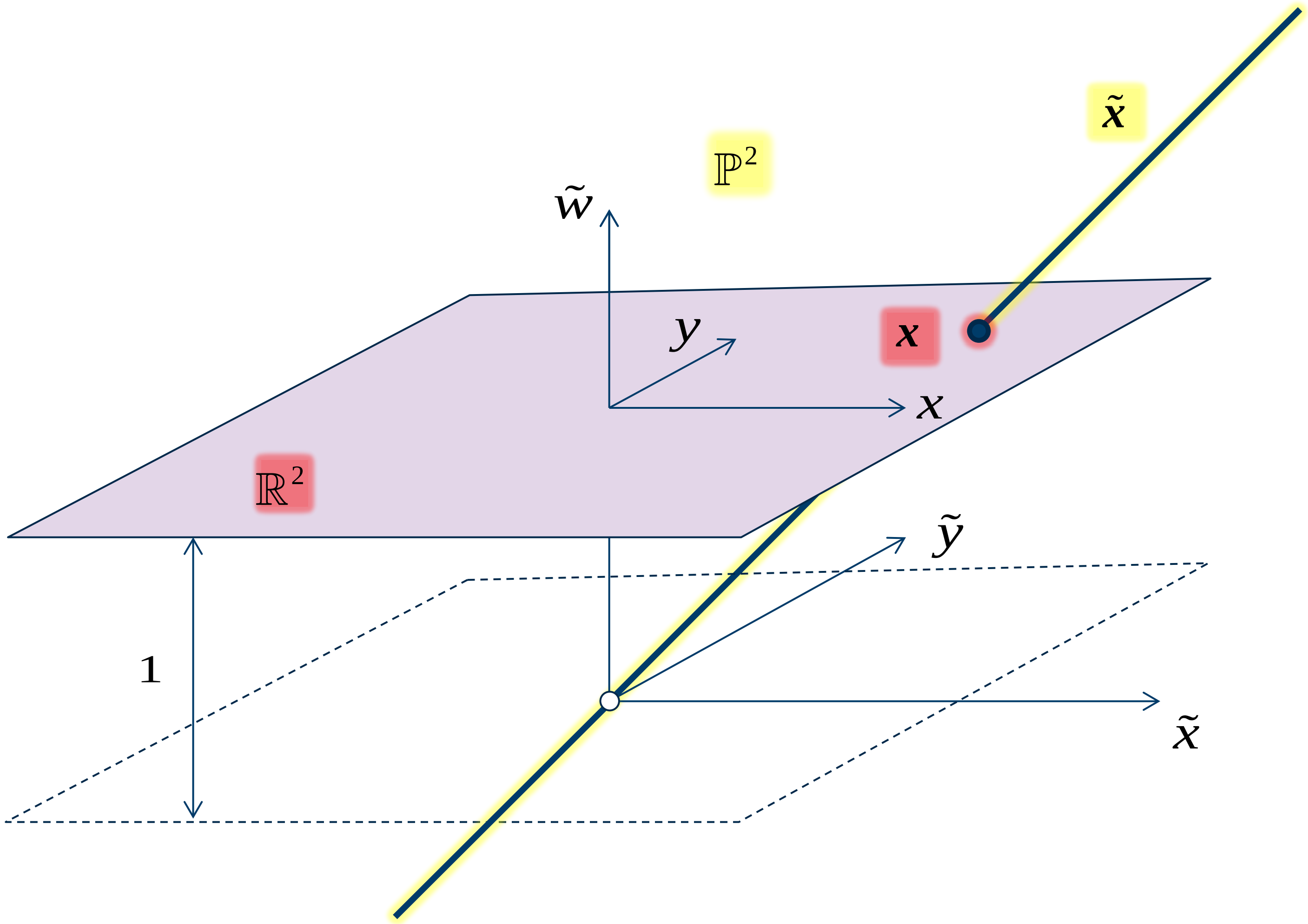
- Expand coordinate frame to 3D
- Each point corresponds to a triple of homogeneous coordinates

$$\tilde{\boldsymbol{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) \in \mathbb{R}^2 \longmapsto \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

s.t.

$$(\tilde{x}, \tilde{y}, \tilde{w}) = \lambda(\tilde{x}, \tilde{y}, \tilde{w}) \;\; \forall \lambda \in \mathbb{R} \backslash \{0\}$$

The projective plane
Points



Observations

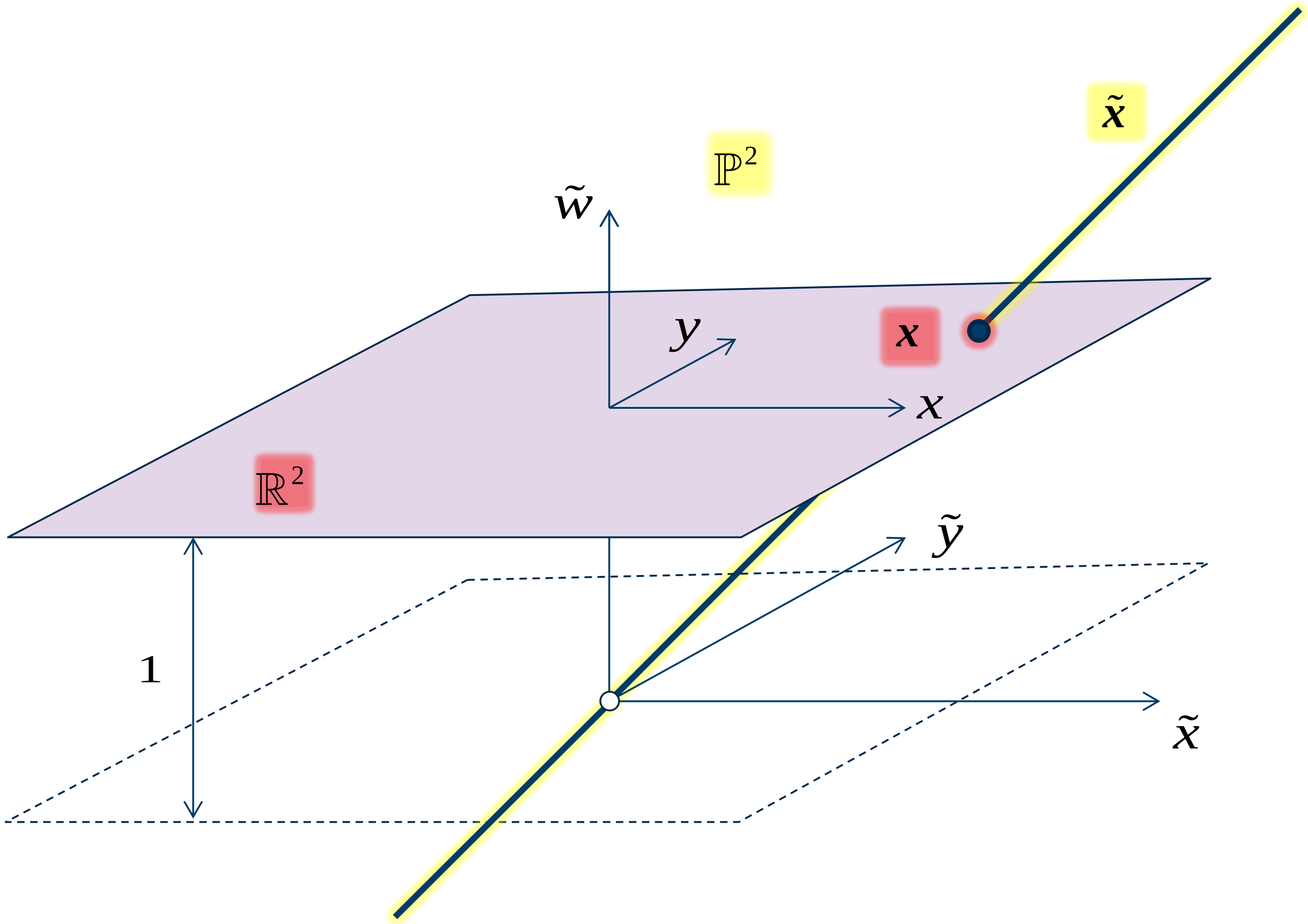
1. Any point $\boldsymbol{x} = (x, y)$ in the Euclidean plane has a corresponding homogeneous point $\tilde{\boldsymbol{x}} = (x, y, 1)$ in the projective plane

2. Homogeneous points of the form $(\tilde{x}, \tilde{y}, 0)$ does not have counterparts in the Euclidean plane

   They correspond to points at infinity and are called *ideal points*

The projective plane

Points



Observations

3. When we work with geometrical problems in the plane, we can swap between the Euclidean representation and the projective representation

$$\mathbb{R}^2 \ni \boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \mapsto \quad \tilde{\boldsymbol{x}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

$$\mathbb{P}^2 \ni \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix} \quad \mapsto \quad \boldsymbol{x} = \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \end{bmatrix} \mathbb{R}^2$$

Example

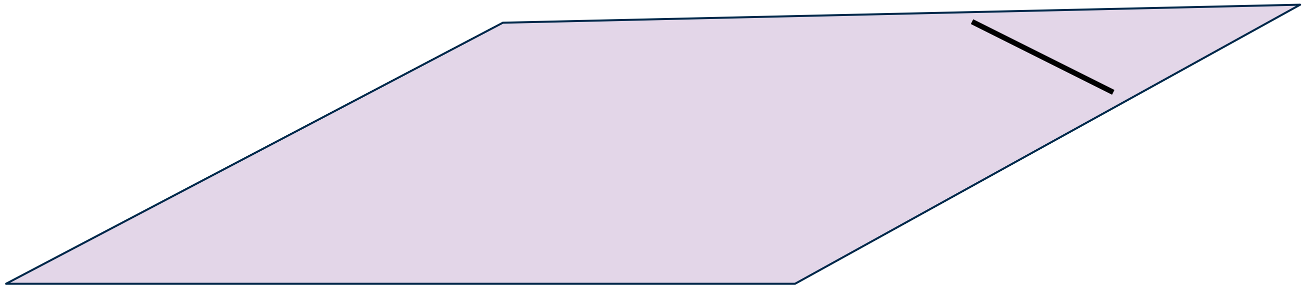
1. These homogeneous vectors are different numerical representations of the same point in the plane

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -30 \\ -20 \\ -10 \end{bmatrix} \in \mathbb{P}^2$$

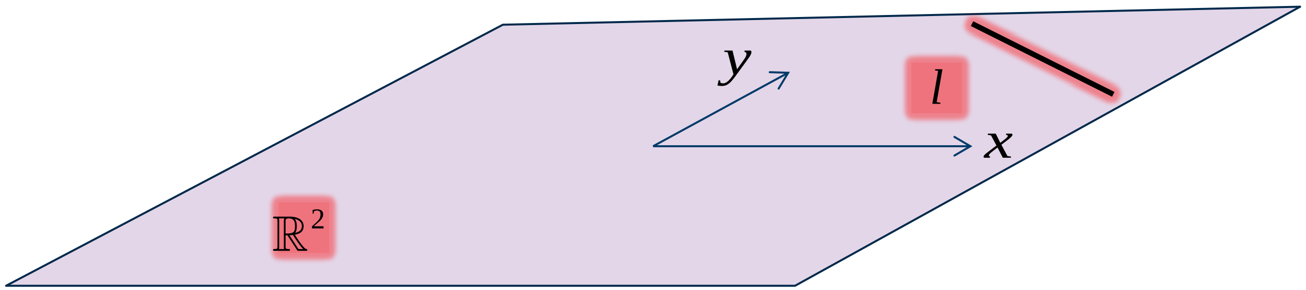
2. The homogeneous point $(1,2,3) \in \mathbb{P}^2$ represents the same point as $\left(\frac{1}{3}, \frac{2}{3}\right) \in \mathbb{R}^2$

The projective plane
Lines

How to describe lines in the plane?



The projective plane
Lines

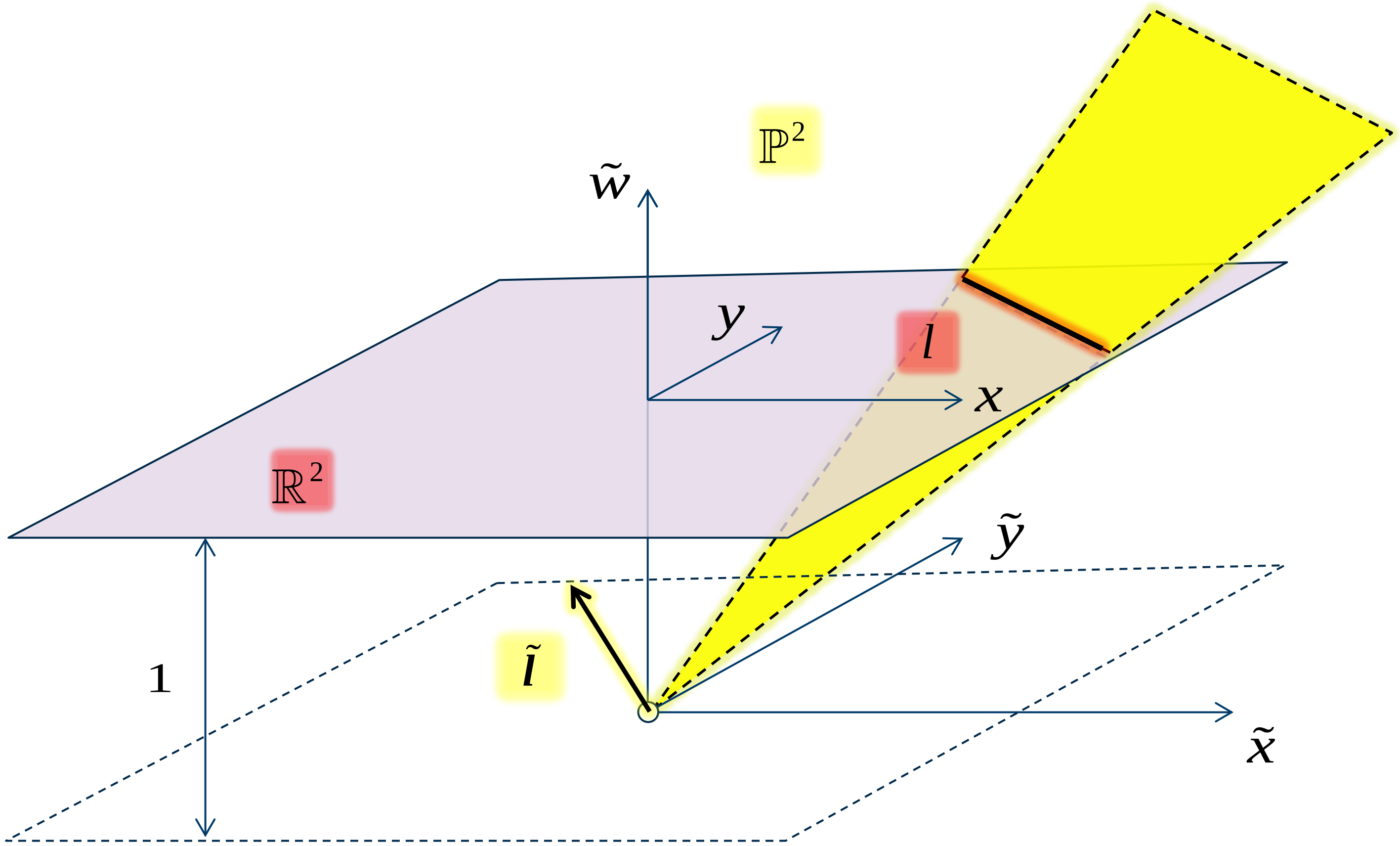


How to describe lines in the plane?

Euclidean plane $\mathbb{R}^2$

- 3 parameters $a, b, c \in \mathbb{R}$
  $$l = \{(x, y) \mid ax + by + c = 0\}$$

The projective plane
Lines



How to describe lines in the plane?

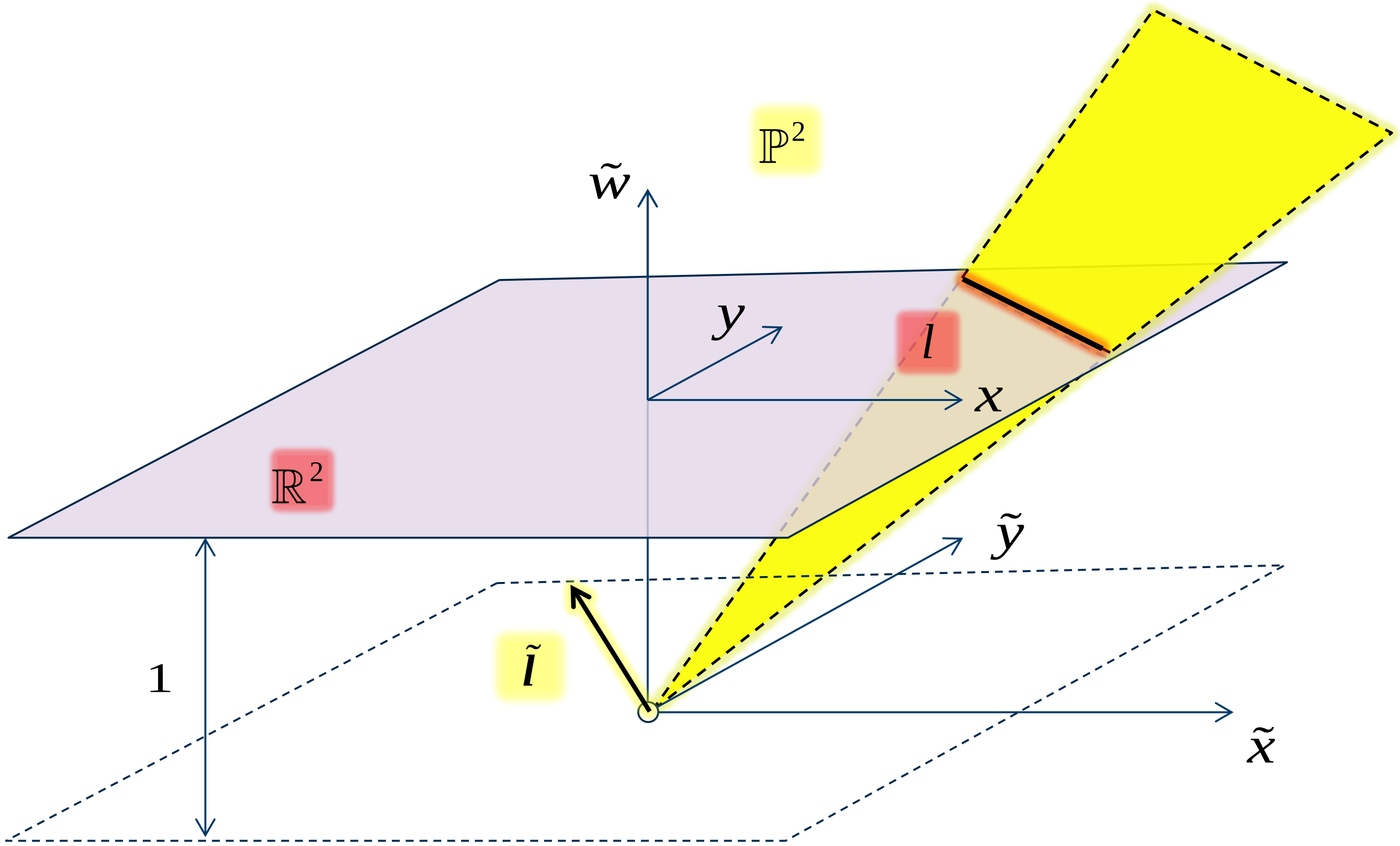
Euclidean plane $\mathbb{R}^2$

- Triple $(a, b, c) \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$
  $$l = \{(x, y) \mid ax + by + c = 0\}$$

Projective plane $\mathbb{P}^2$

- Homogeneous vector $\tilde{\boldsymbol{l}} = [a, b, c]^T$
  $$l = \left\{\tilde{\boldsymbol{x}} \in \mathbb{P}^2 \mid \tilde{\boldsymbol{l}}^T \tilde{\boldsymbol{x}} = 0\right\}$$

The projective plane
Lines

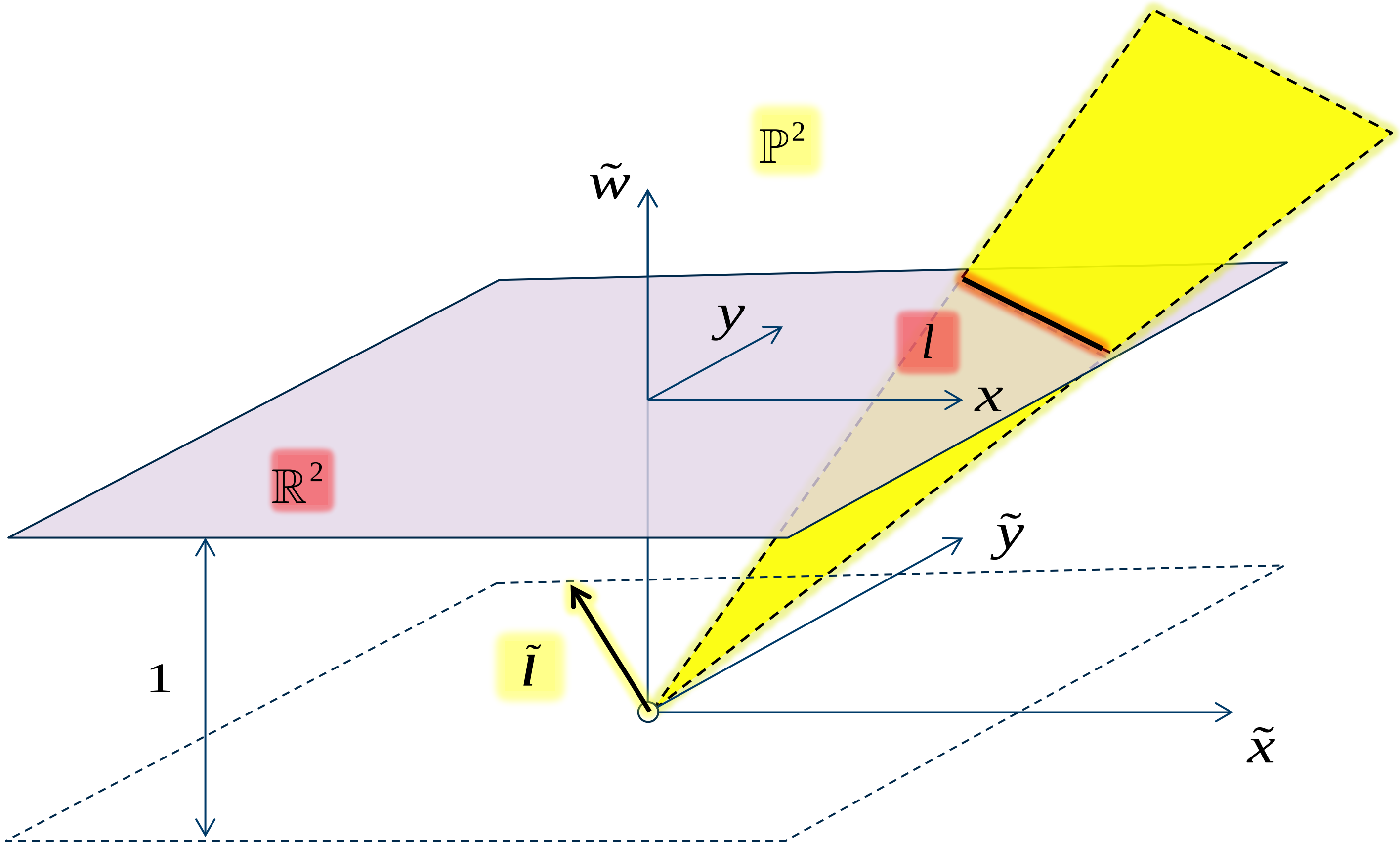


Observations

1. Points and lines in the projective plane have the same representation, we say that points and lines are dual objects in $\mathbb{P}^2$
2. All lines in the Euclidean plane have a corresponding line in the projective plane
3. The line $\tilde{\boldsymbol{l}} = [0,0,1]^T$ in the projective plane does not have an Euclidean counterpart

   This line consists entirely of ideal points, and is know as *the line at infinity*

The projective plane
Lines



Properties of lines in the projective plane

1. In the projective plane, all lines intersect, parallel lines intersect at infinity

   Two lines $\tilde{\boldsymbol{l}}_1$ and $\tilde{\boldsymbol{l}}_2$ intersect in the point
   $$\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{l}}_1 \times \tilde{\boldsymbol{l}}_2$$

2. The line passing through points $\tilde{\boldsymbol{x}}_1$ and $\tilde{\boldsymbol{x}}_2$ is given by
   $$\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2$$

Example

Determine the line passing through the two points $(2, 4)$ and $(5, 13)$

Homogeneous representation of points

$$\tilde{\boldsymbol{x}}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{P}^2 \qquad \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

Homogeneous representation of line

$$\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2 = [\tilde{\boldsymbol{x}}_1]_\times \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Equation of the line

$$-3x + y + 2 = 0 \quad \Leftrightarrow \quad y = 3x - 2$$

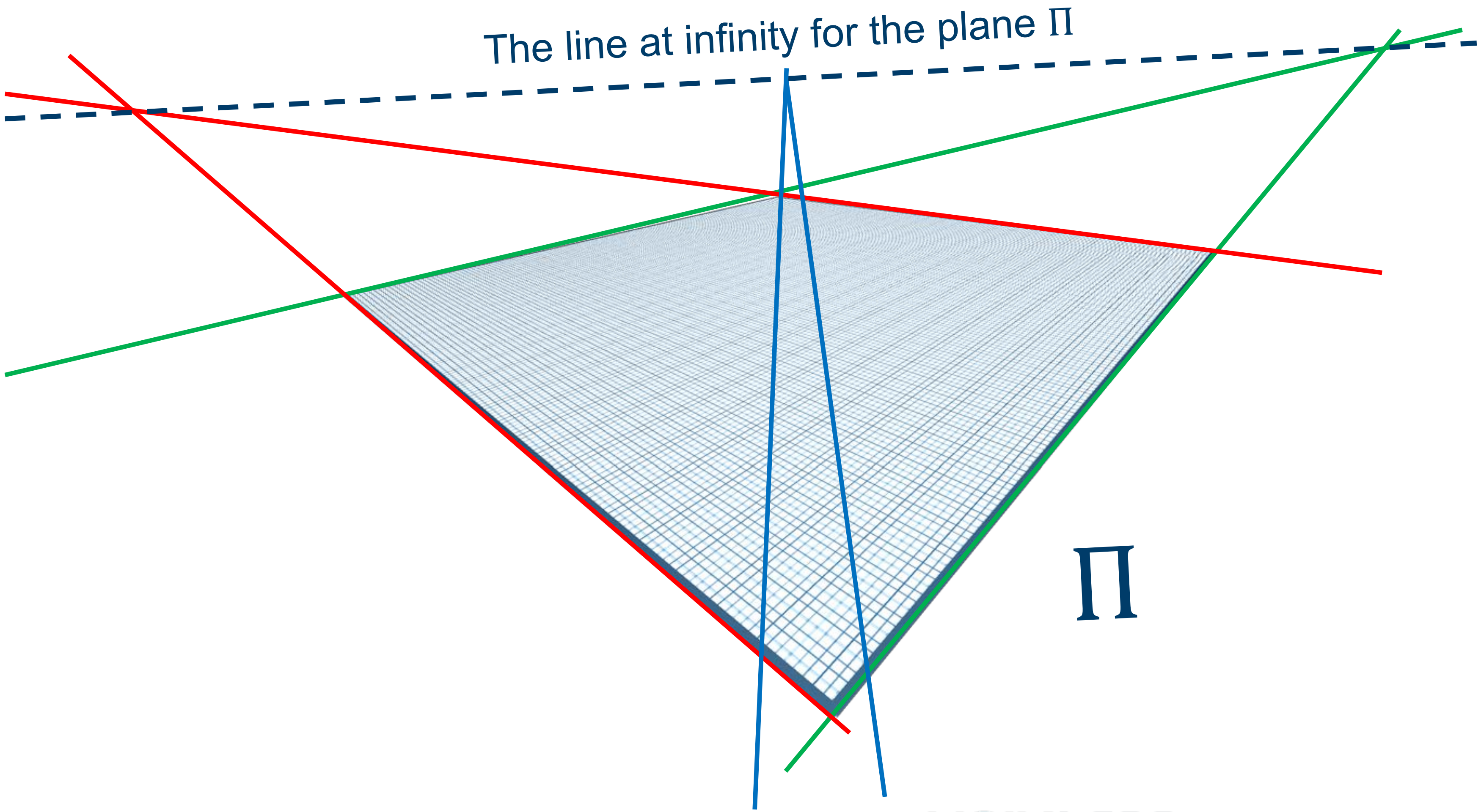
Matrix representation of the cross product

$$\boldsymbol{u} \times \boldsymbol{v} \longmapsto [\boldsymbol{u}]_\times \boldsymbol{v}$$

where

$$[\boldsymbol{u}]_\times \overset{def}{=} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Example



A point at infinity

https://en.wikipedia.org/wiki/Projective_plane

The projective plane
Transformations

- Some important transformations – like the action of a pose ξ on points in the plane – happen to be linear in the projective plane and non-linear in the Euclidean plane

- The most general invertible transformations of the projective plane are known as homographies
  - or projective transformations / linear projective transformations / projectivities / collineations

Definition

A homography of $\mathbb{P}^2$ is a linear transformation on homogeneous 3-vectors represented by a homogeneous, non-singular 3×3 matrix H

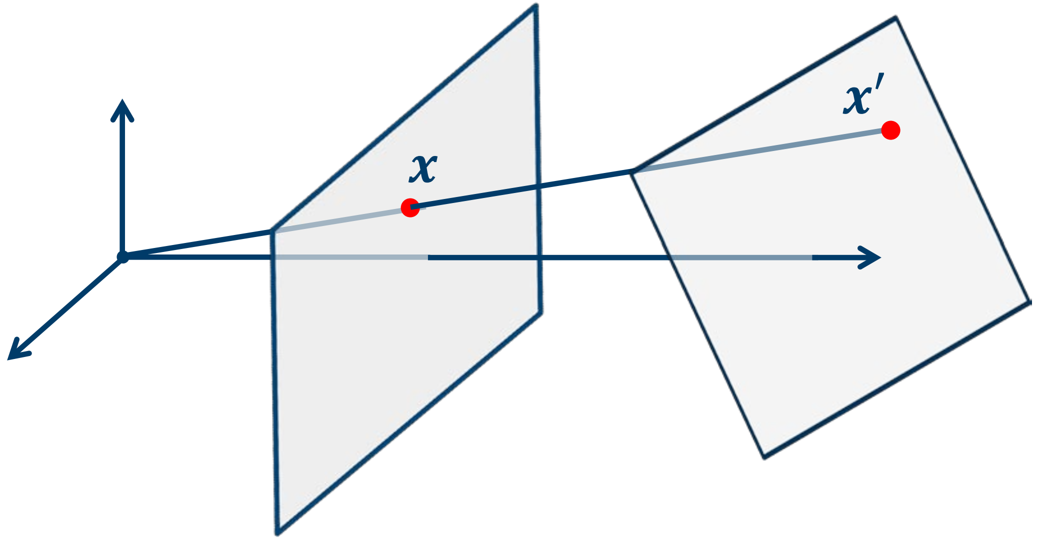
$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \\ \tilde{w}' \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{w} \end{bmatrix}$$

So H is unique up to scale, i.e. $H = \lambda H \;\forall\; \lambda \in \mathbb{R} \backslash \{0\}$

The projective plane
Transformations

- One characteristic of homographies is that they preserve lines, in fact any invertible transformation of $\mathbb{P}^2$ that preserves lines is a homography

- Examples
  - Central projection from one plane to another is a homography
    Hence if we take an image with a perspective camera of a flat surface from an angle, we can remove the perspective distortion with a homography



Perspective distortion

Without distortion

Images from http://www.robots.ox.ac.uk/~vgg/hzbook.html

The projective plane
Transformations

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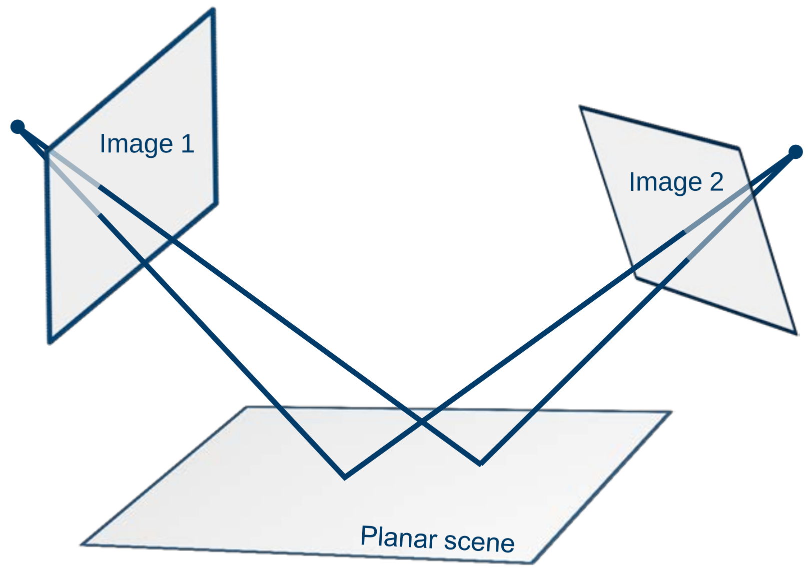
- Examples
  - Central projection from one plane to another is a homography
  - Two images, captured by perspective cameras, of the same planar scene is related by a homography



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Transformations

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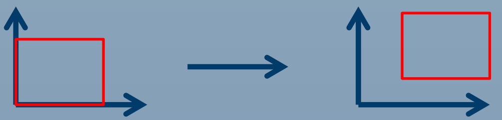
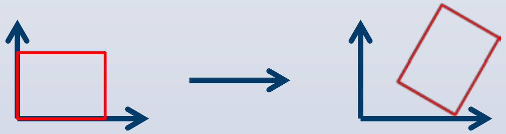
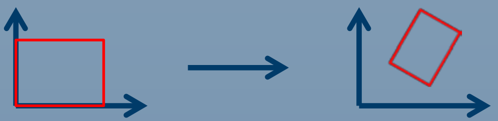
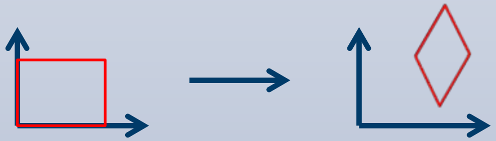
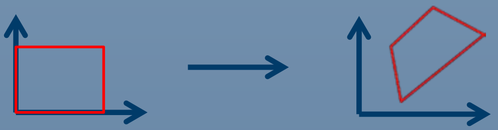
- One can show that the product of two homographies also must be a homography
  We say that the homographies constitute a group – the projective linear group $PL(3)$

- Within this group there are several more specialized subgroups

Transformations of the projective plane

Transformation of $\mathbb{P}^2$	Matrix	#DoF	Preserves	Visualization
Translation	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	2	Orientation + all below	
Euclidean	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	3	Lengths + all below	
Similarity	$\begin{bmatrix} sR & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	4	Angles + all below	
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$	6	Parallelism, line at infinity + all below	
Homography /projective	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$	8	Straight lines	

The projective space

- The relationship between the Euclidean space $\mathbb{R}^3$ and the projective space $\mathbb{P}^3$ is much like the relationship between $\mathbb{R}^2$ and $\mathbb{P}^2$

- In the projective space
  - We represent points in homogeneous coordinates

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} \lambda\tilde{x} \\ \lambda\tilde{y} \\ \lambda\tilde{z} \\ \lambda\tilde{w} \end{bmatrix} \forall\lambda \in \mathbb{R}\backslash\{0\}$$

  - Points at infinity have last homogeneous coordinate equal to zero

  - Planes and points are dual objects

$$\tilde{\Pi} = \{\tilde{\boldsymbol{x}} \in \mathbb{P}^3 \mid \tilde{\boldsymbol{\pi}}^T\tilde{\boldsymbol{x}} = 0\}$$

  - The plane at infinity are made up of all points at infinity

$$\mathbb{R}^3 \ni \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \tilde{\boldsymbol{x}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \in \mathbb{P}^3$$

$$\mathbb{P}^3 \ni \tilde{\boldsymbol{x}} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} \mapsto \boldsymbol{x} = \begin{bmatrix} \tilde{x}/\tilde{w} \\ \tilde{y}/\tilde{w} \\ \tilde{z}/\tilde{w} \end{bmatrix} \in \mathbb{R}^3$$

Transformations of the projective space

Transformation of $\mathbb{P}^3$	Matrix	#DoF	Preserves
Translation	$\begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	3	Orientation + all below
Euclidean	$\begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	6	Volumes, volume ratios, lengths + all below
Similarity	$\begin{bmatrix} sR & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	7	Angles + all below
Affine	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$	12	Parallelism of planes, The plane at infinity + all below
Homography /projective	$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$	15	Intersection and tangency of surfaces in contact, straight lines

Summary

- The projective plane $\mathbb{P}^2$
  - Homogeneous coordinates
  - Line at infinity
  - Points & lines are dual
- The projective space $\mathbb{P}^3$
  - Homogeneous coordinates
  - Plane at infinity
  - Points & planes are dual
- Linear transformations of $\mathbb{P}^2$ and $\mathbb{P}^3$
  - Represented by homogeneous matrices
  - Homographies $\supset$ Affine $\supset$ Similarities $\supset$ Euclidean $\supset$ Translations

- Additional reading
  - Szeliski: 2.1.2, 2.1.3,

Summary

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  - Homographies ⊃ Affine ⊃ Similarities ⊃ Euclidean ⊃ Translations

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MATLAB WARNING

When we work with linear transformations, we represent them as matrices that act on points by right multiplication

$$\begin{aligned} T: \quad \mathbb{R}^n &\rightarrow \quad \mathbb{R}^n \\ \boldsymbol{x} &\mapsto \quad \boldsymbol{y} = M_R \boldsymbol{x} \end{aligned}$$

Matlab seem to prefer left multiplication instead

$$\begin{aligned} T: \quad \mathbb{R}^n &\rightarrow \quad \mathbb{R}^n \\ \boldsymbol{x}^T &\mapsto \quad \boldsymbol{y}^T = \boldsymbol{x}^T M_L \end{aligned}$$

So if you use built in matlab functions when you work with transformations, be careful!!!