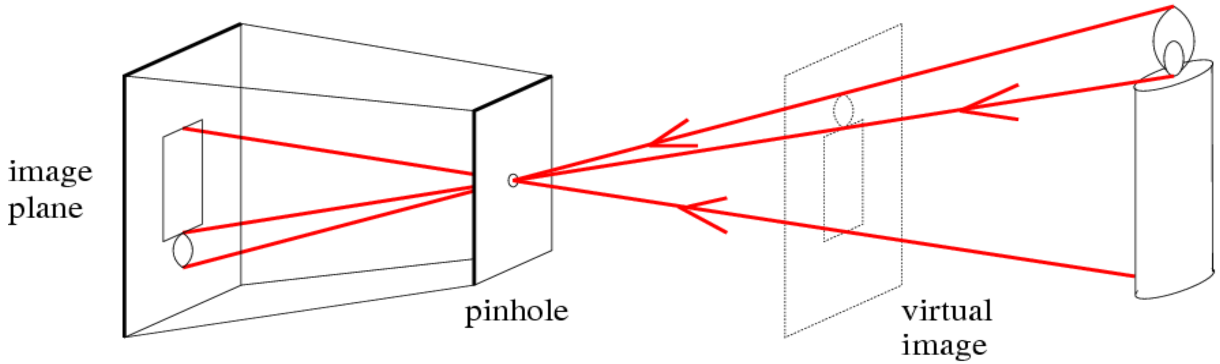
Lecture 1.2
Pose in 2D and 3D

Thomas Opsahl

Motivation

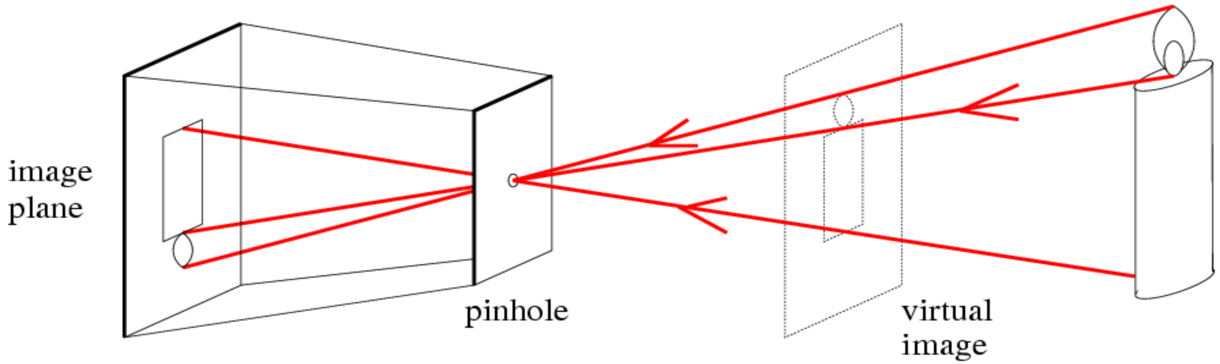


- For the pinhole camera, the correspondence between observed 3D points in the world and 2D points in the captured image is given by straight lines through a common point (pinhole)

- This correspondence can be described by a mathematical model known as “*the perspective camera model*” or “*the pinhole camera model*”

- This model can be used to describe the imaging geometry of many modern cameras, hence it plays a central part in computer vision

Motivation



- Before we can study the perspective camera model in detail, we need to expand our mathematical toolbox
- We need to be able to mathematically describe the position and orientation of the camera relative to the world coordinate frame
- Also we need to get familiar with some basic elements of projective geometry, since this will make it MUCH easier to describe and work with the perspective camera model

Introduction

- In computer vision we are often interested in the geometrical aspects of imaging
  - Points in the world ↔ pixels in an image
  - Pixels in image 1 ↔ pixels in image 2

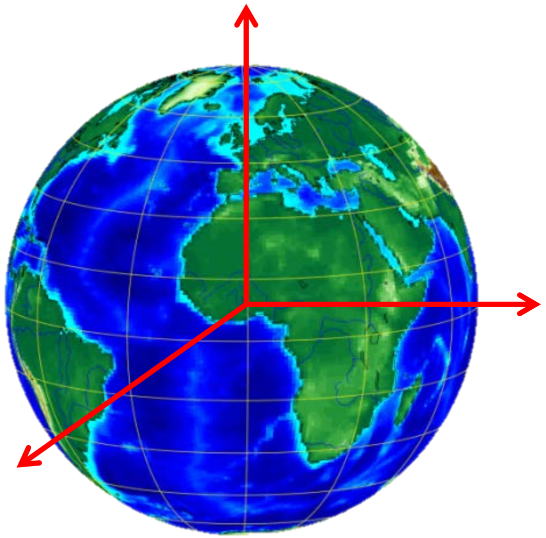
- In order to express and study geometrical problems related to imaging, we first need to know how to describe the position and orientation of objects

- Position and orientation is together known as *pose*

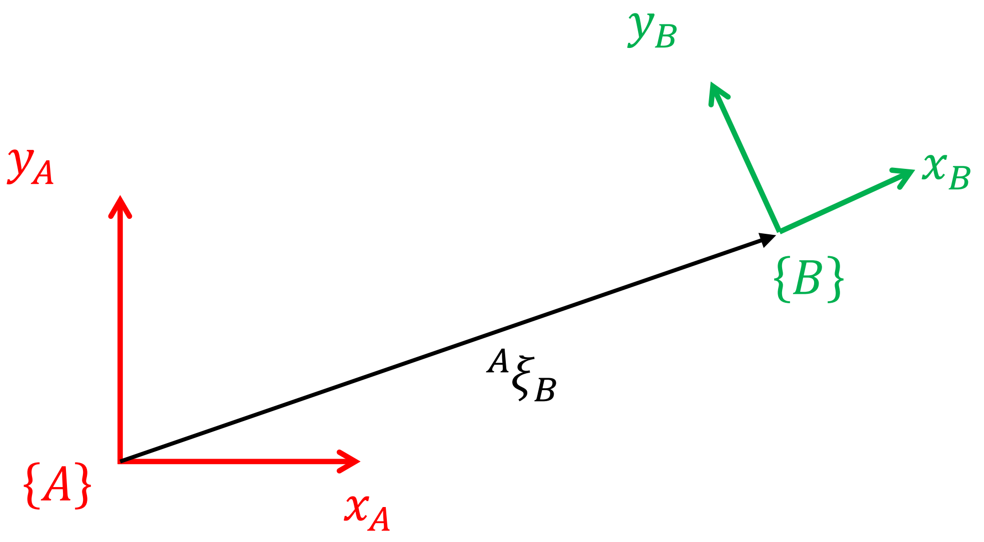
Introduction

- By representing all relevant objects by coordinate frames, it is possible to numerically represent the pose of one object relative to another

- In the following we will look at pose
  - General properties
  - Representation in 2D
  - Representation in 3D



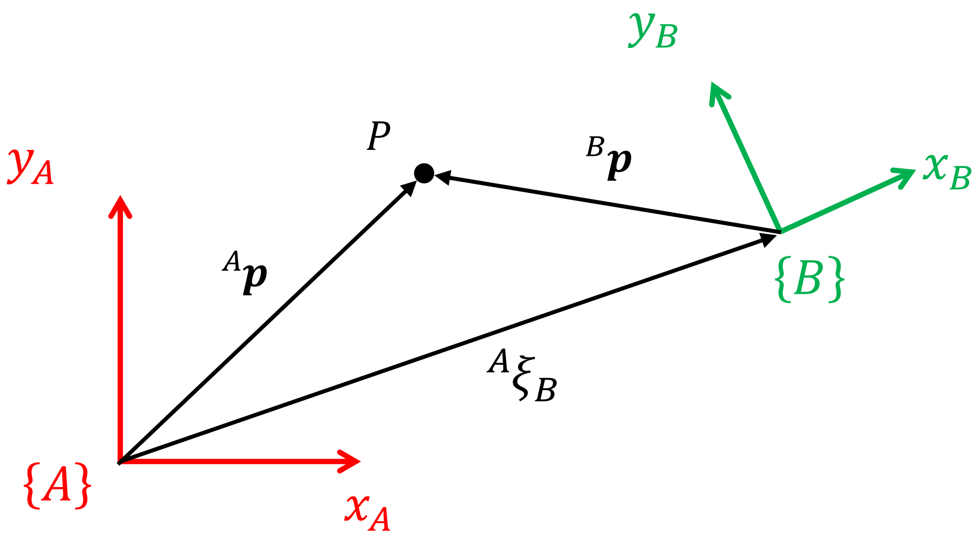
General properties



- Let us denote by ${}^{A}\xi_{B}$ the pose of frame $\{B\}$ relative to frame $\{A\}$

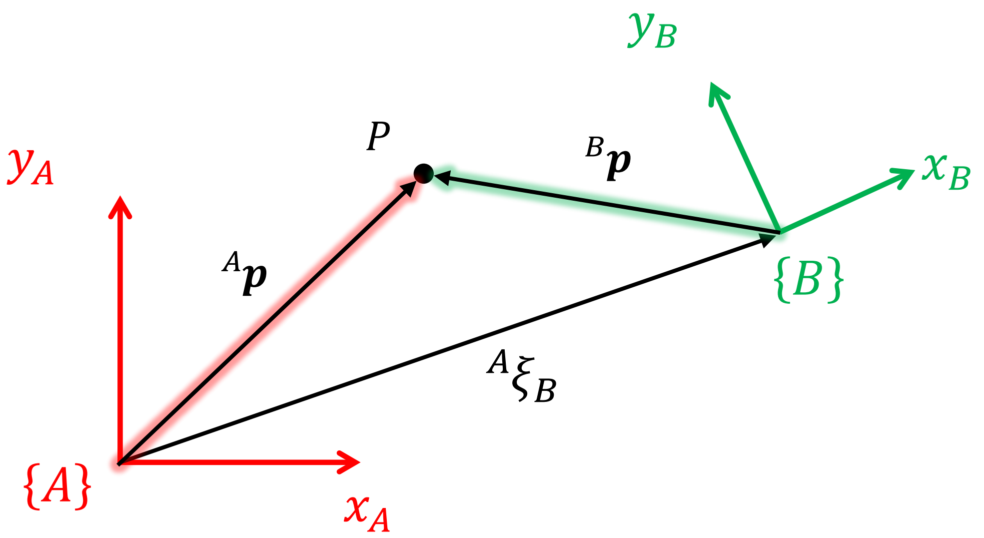
- We can think of ${}^{A}\xi_{B}$ as the translation and rotation required in order to make $\{A\}$ coincide with $\{B\}$

General properties



- A point P can be described with respect to either frame
- These descriptions are related by the pose
- Formally we write this as $^A\boldsymbol{p} = {}^A\xi_B \cdot {}^B\boldsymbol{p}$

General properties



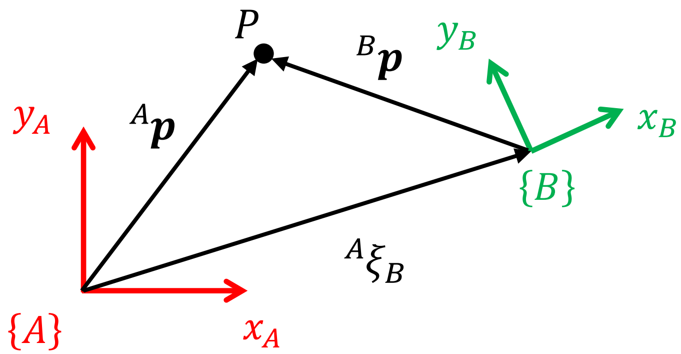
Note that $^A\boldsymbol{p}$ and $^B\boldsymbol{p}$ are different vectors!!!

- A point P can be described with respect to either frame
- These descriptions are related by the pose
- Formally we write this as $^A\boldsymbol{p} = {}^A\xi_B \cdot {}^B\boldsymbol{p}$

General properties

- Action on points

$$^{A}\boldsymbol{p} = {^{A}\xi_{B}} \cdot {^{B}\boldsymbol{p}}$$



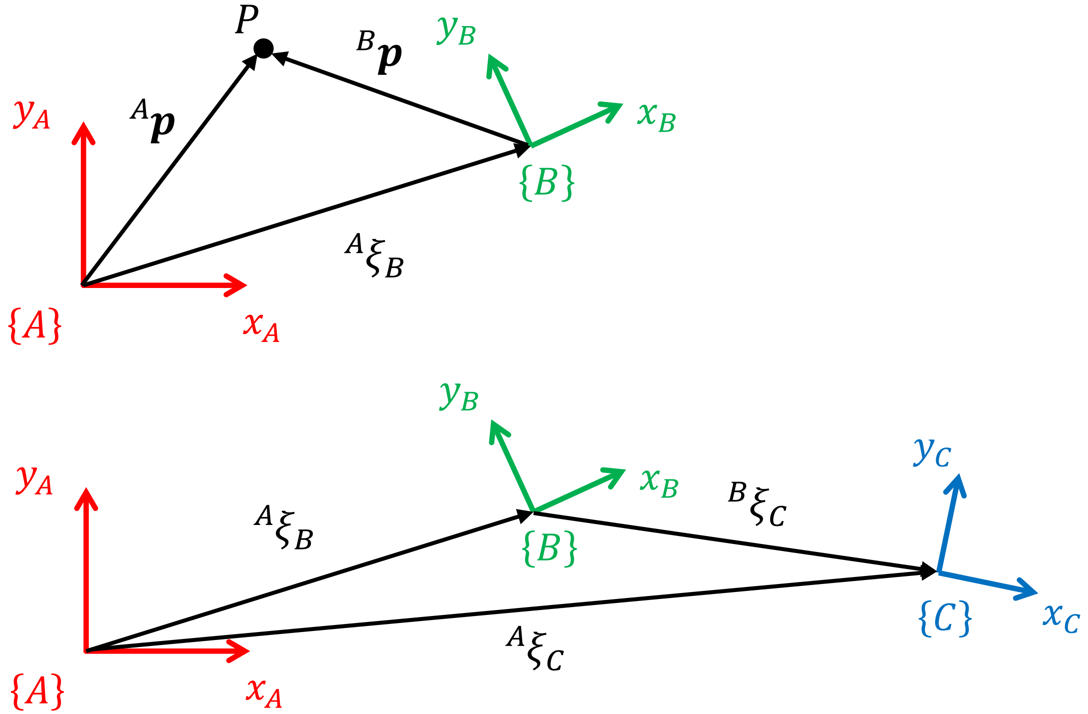
General properties

- Action on points

$$^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p}$$

- Composition

$$^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C}$$



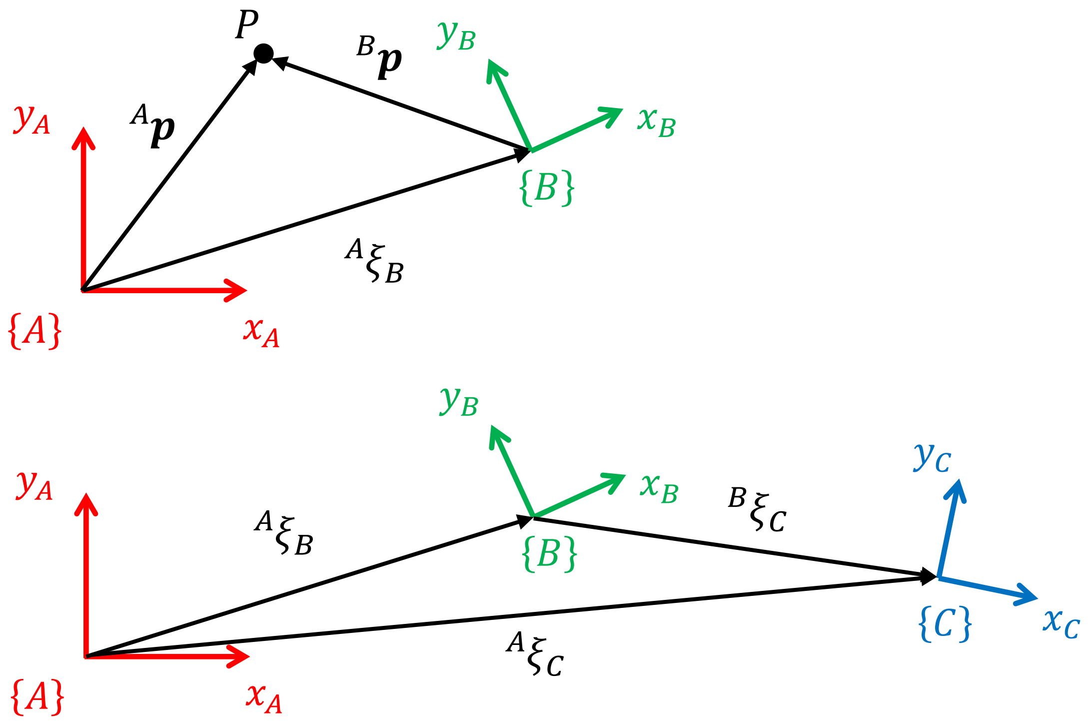
General properties

- Action on points

$${}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p}$$

- Composition

$${}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C}$$



What about ${}^{A}\xi_{B} \oplus {}^{B}\xi_{A}$?

General properties

- Action on points

$$^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p}$$

- Composition

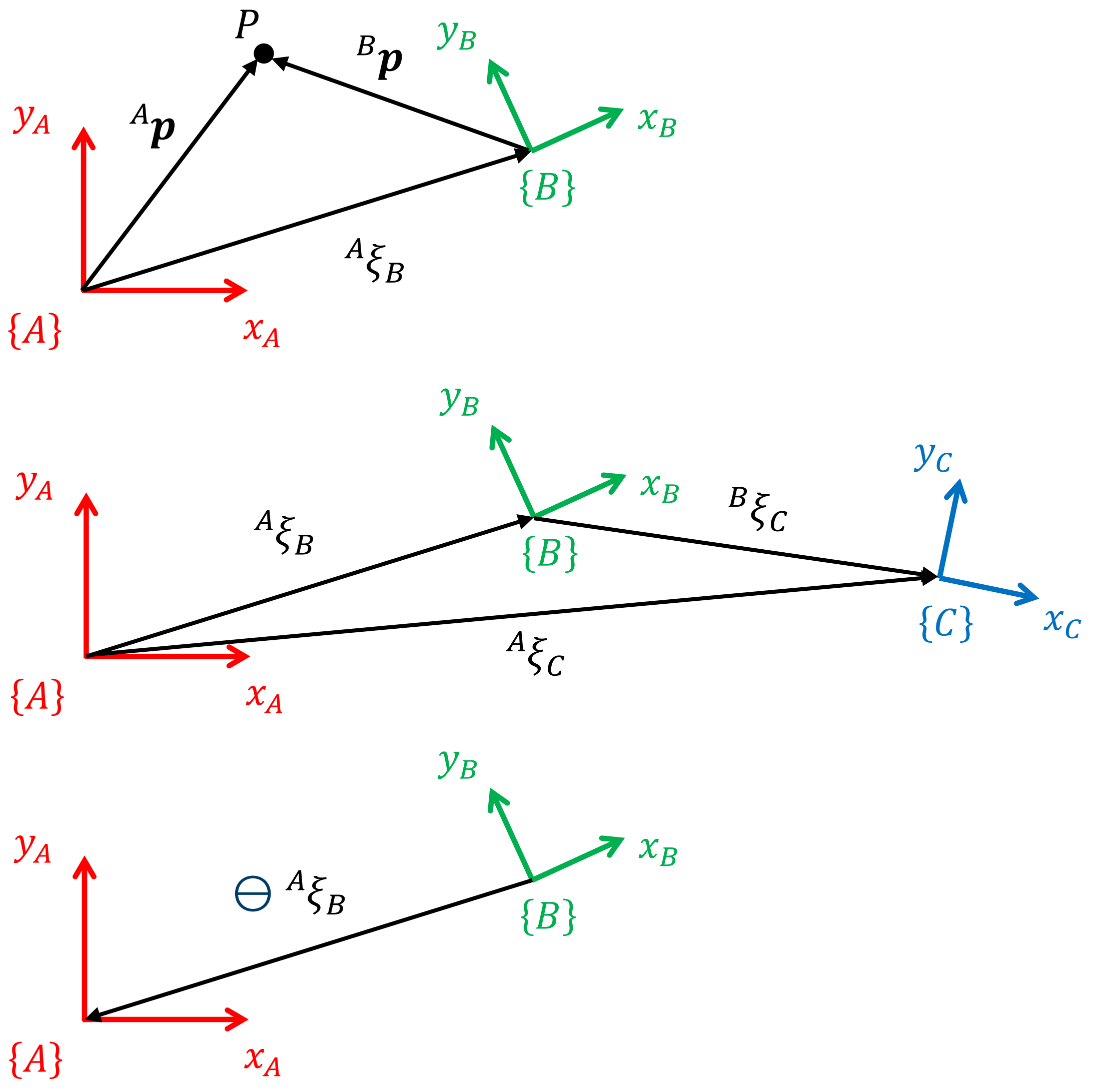
$$^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C}$$

- Inverse

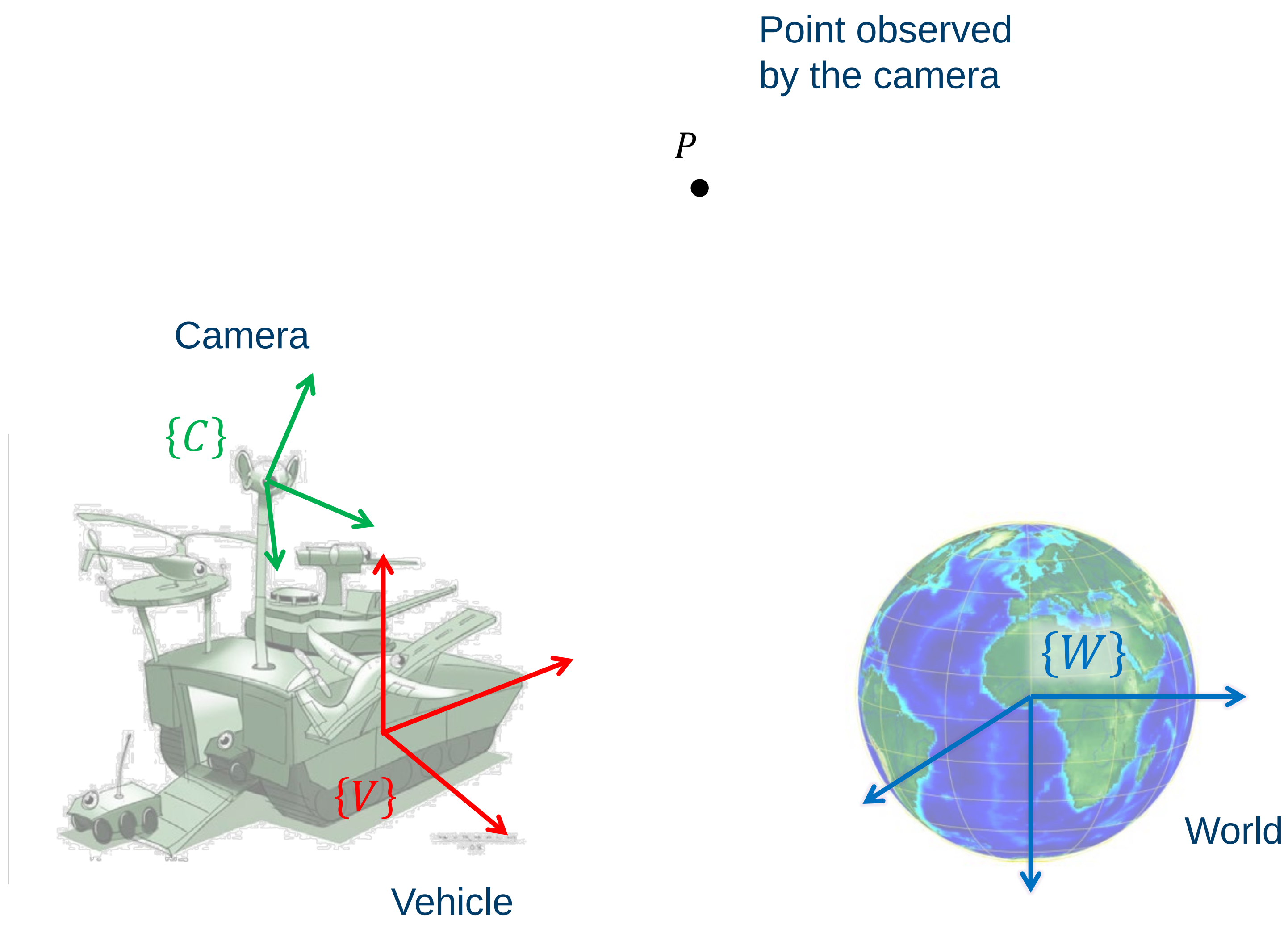
$$\ominus\, {}^{A}\xi_{B} = {}^{B}\xi_{A}$$

- Neutral element

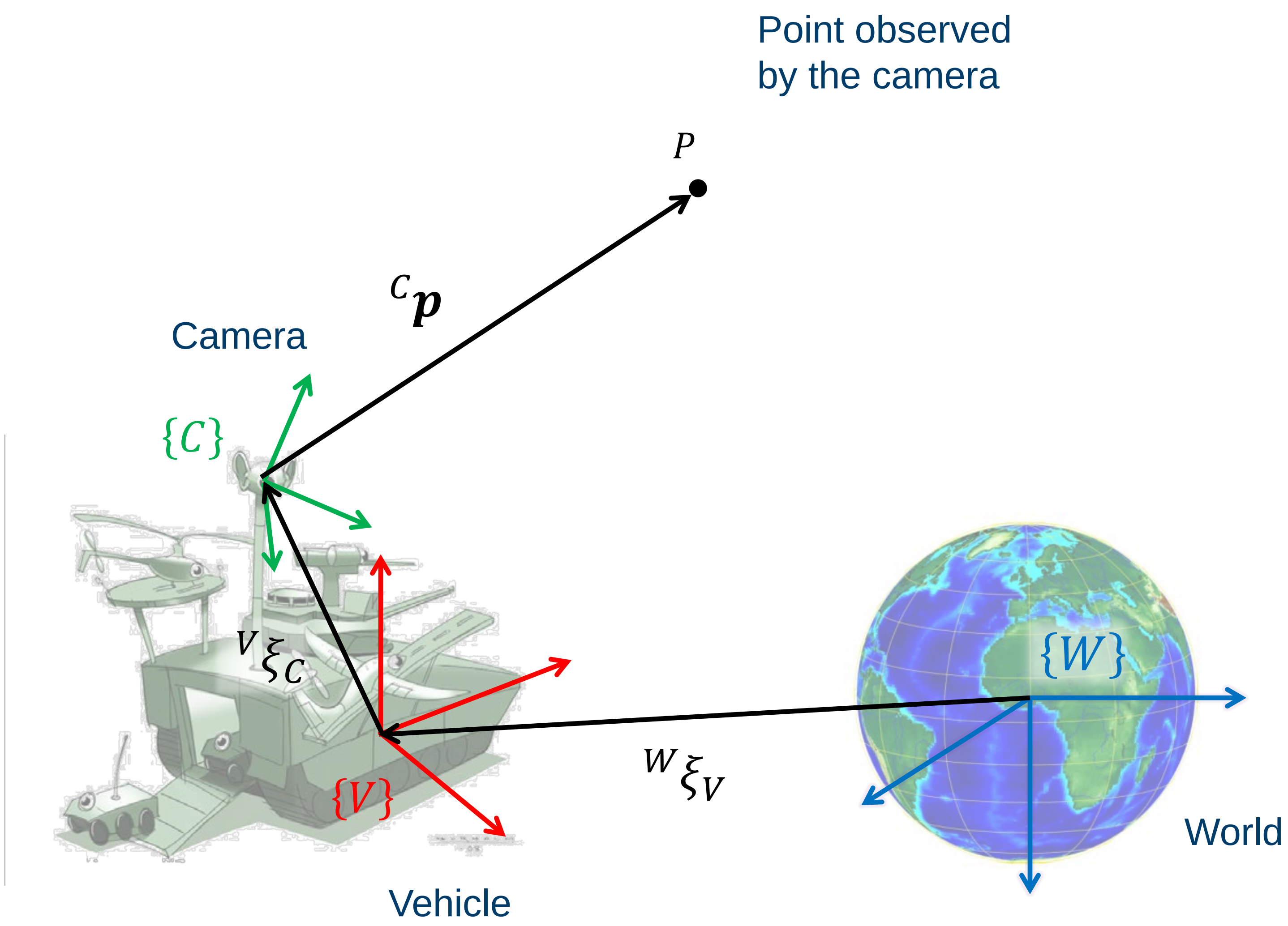
$$0 = {}^{A}\xi_{A}$$
$$0 \oplus \xi = \xi$$
$$\xi \ominus 0 = \xi$$
$$\xi \ominus \xi = 0$$
$$\ominus\, \xi \oplus \xi = 0$$



Example

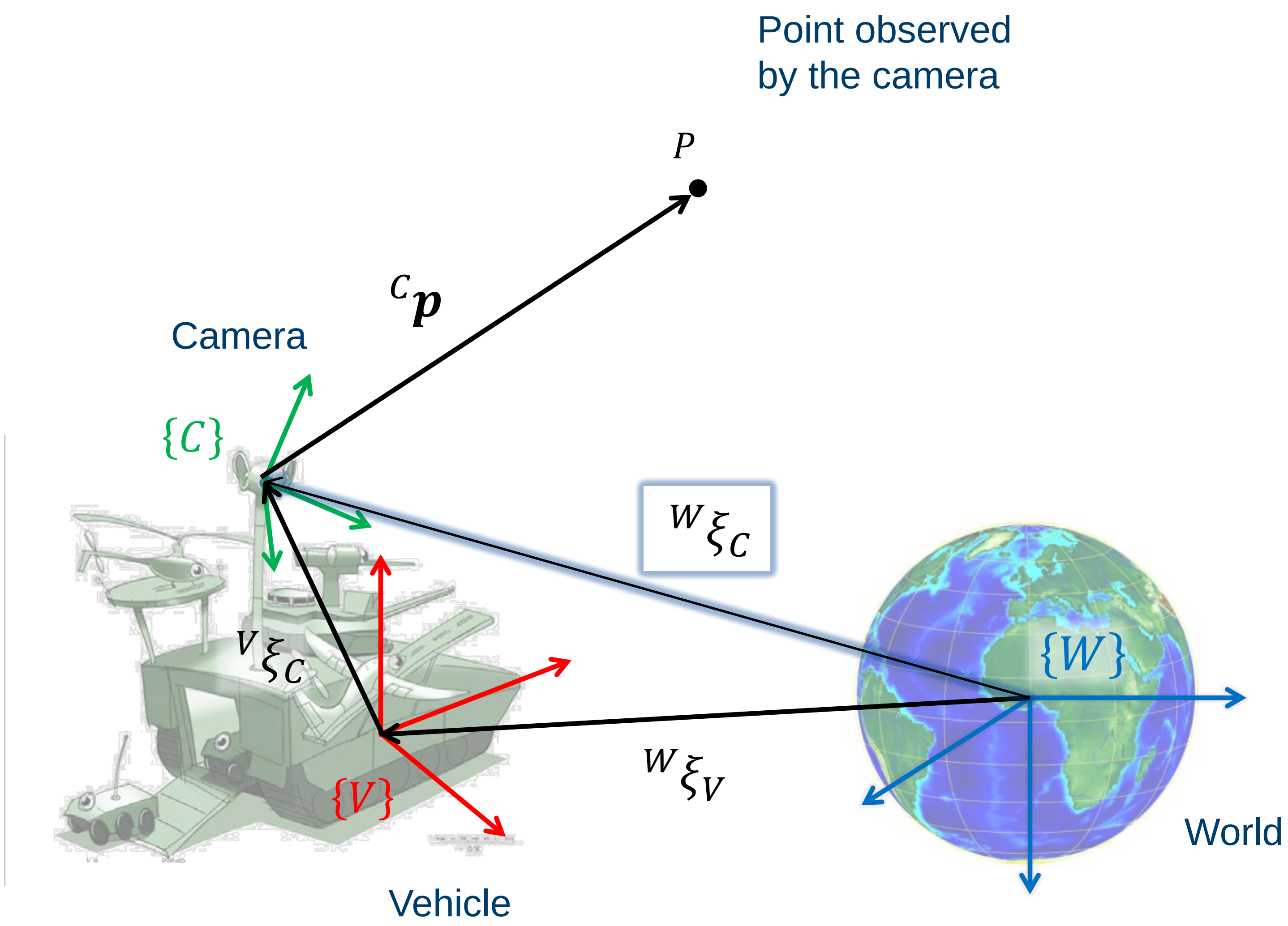


Example



Point observed
by the camera
P
${}^{C}\boldsymbol{p}$
Camera
$\{C\}$
${}^{V}\xi_{C}$
$\{V\}$
${}^{W}\xi_{V}$
$\{W\}$
World
Vehicle

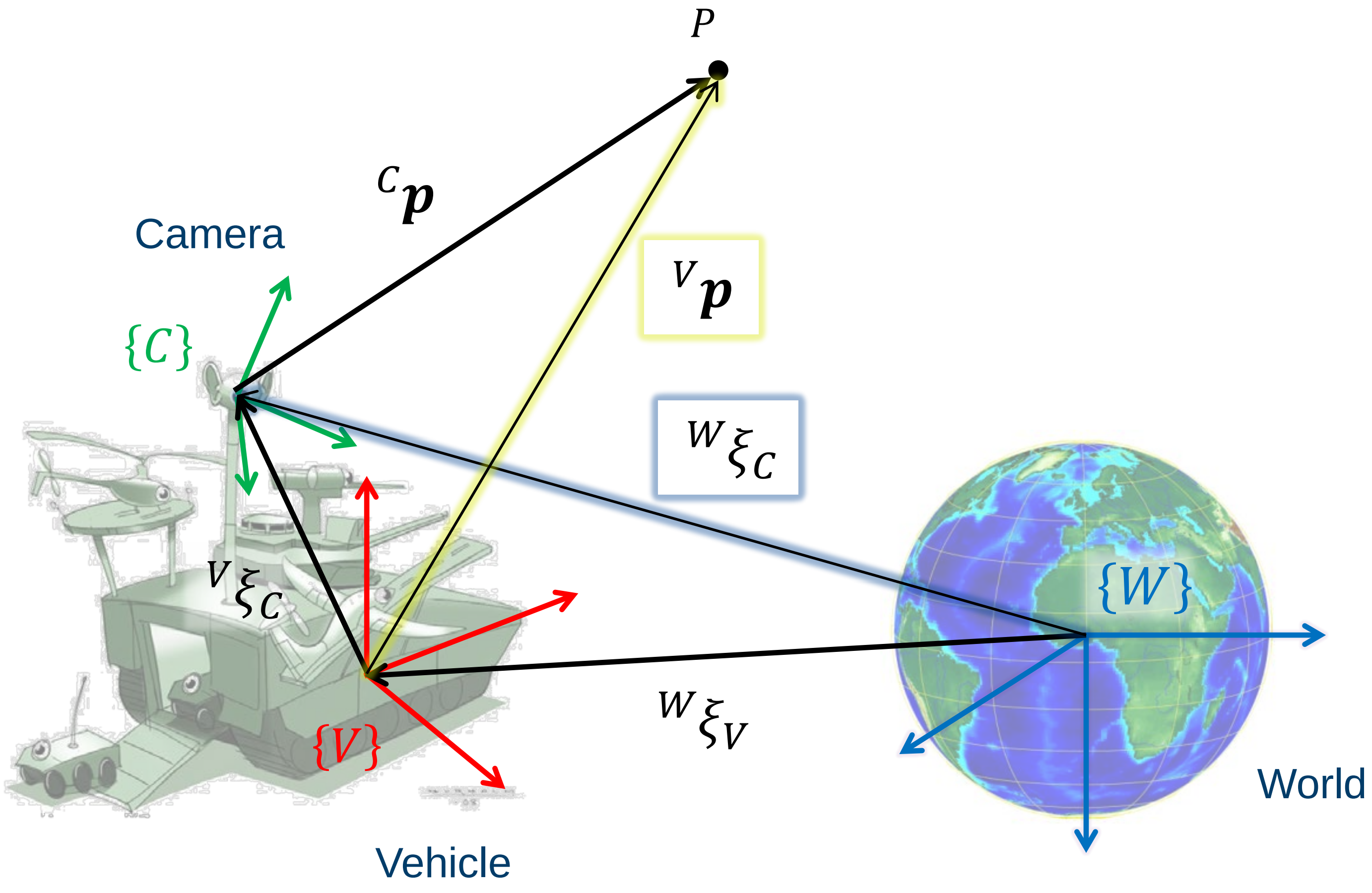
Example



$${}^{W}\xi_C = {}^{W}\xi_V \oplus {}^{V}\xi_C$$

Example

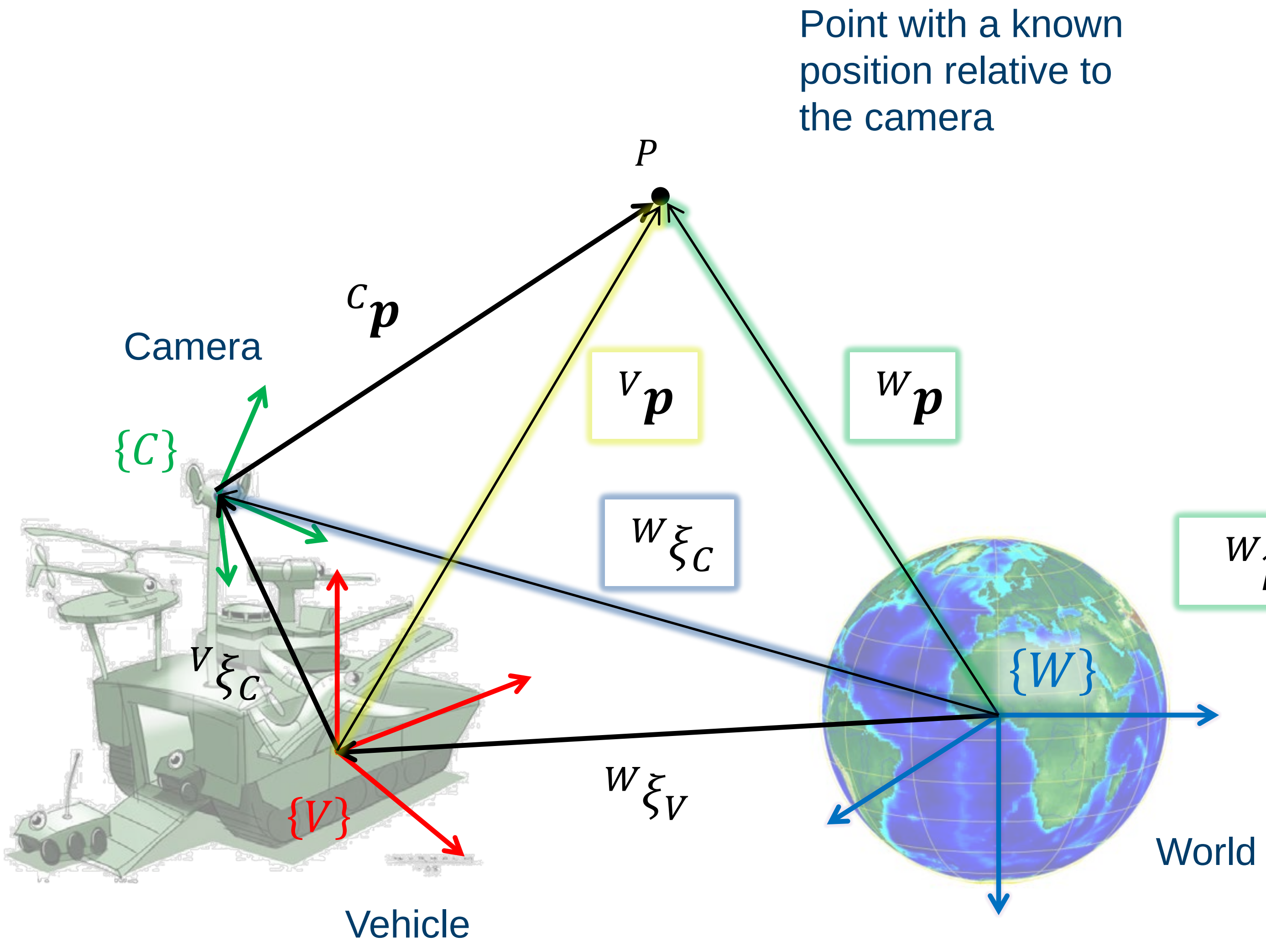
Point with a known position relative to the camera



$$\boxed{{}^{W}\xi_C = {}^{W}\xi_V \oplus {}^{V}\xi_C}$$

$$\boxed{{}^{V}\boldsymbol{p} = {}^{V}\xi_C \cdot {}^{C}\boldsymbol{p}}$$

Example

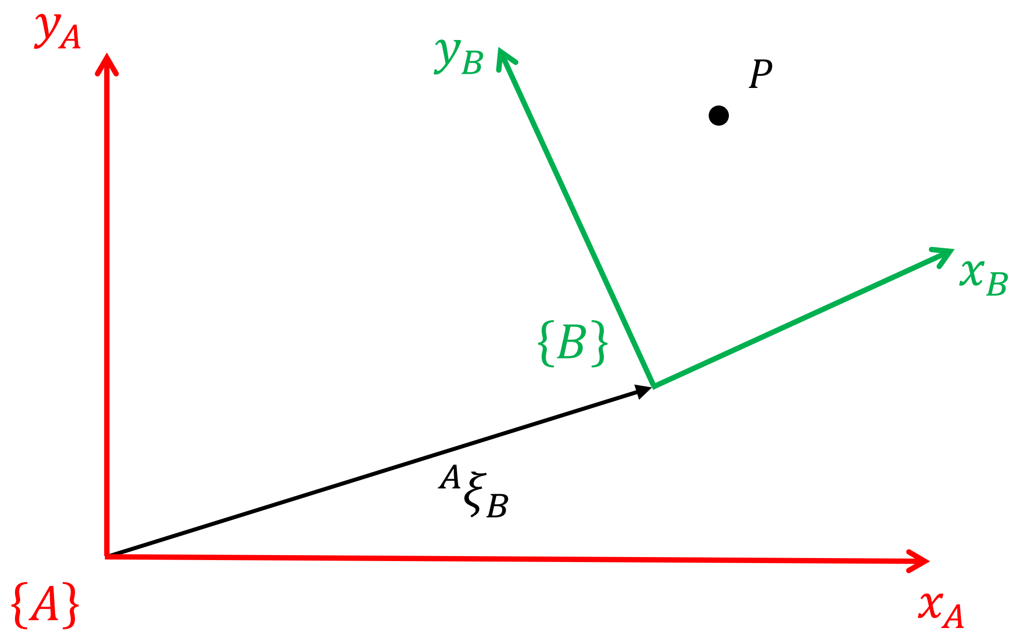


$$\boxed{{}^{W}\xi_{C} = {}^{W}\xi_{V} \oplus {}^{V}\xi_{C}}$$

$$\boxed{{}^{V}\boldsymbol{p} = {}^{V}\xi_{C} \cdot {}^{C}\boldsymbol{p}}$$

$$\boxed{{}^{W}\boldsymbol{p} = \left({}^{W}\xi_{V} \oplus {}^{V}\xi_{C}\right) \cdot {}^{C}\boldsymbol{p}}$$

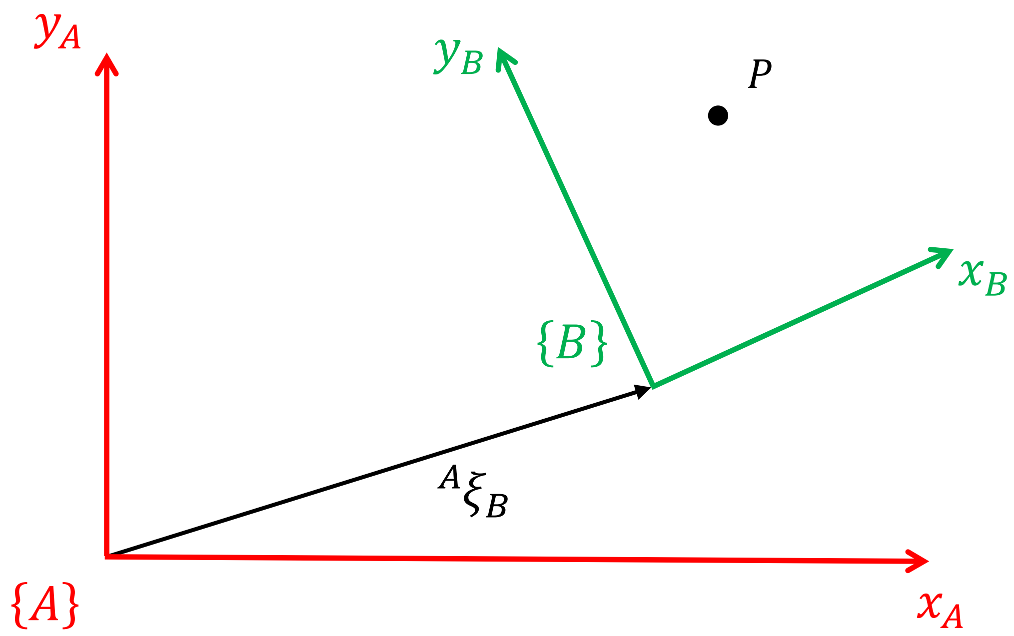
Investigating pose in 2D



- Given two 2D frames, $\{A\}$ and $\{B\}$, how can we represent the pose ${}^{A}\xi_{B}$?

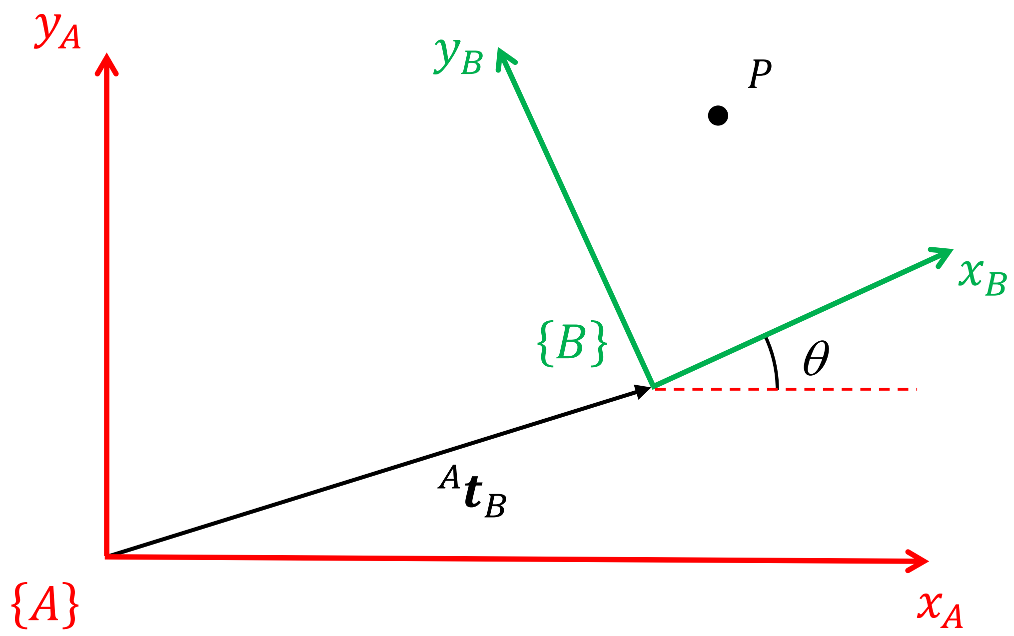
- We need a numerical representation in order to compute ${}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p}$

Investigating pose in 2D



- Recall that we can think of $^A\xi_B$ as the translation and rotation required in order to make $\{A\}$ coincide with $\{B\}$

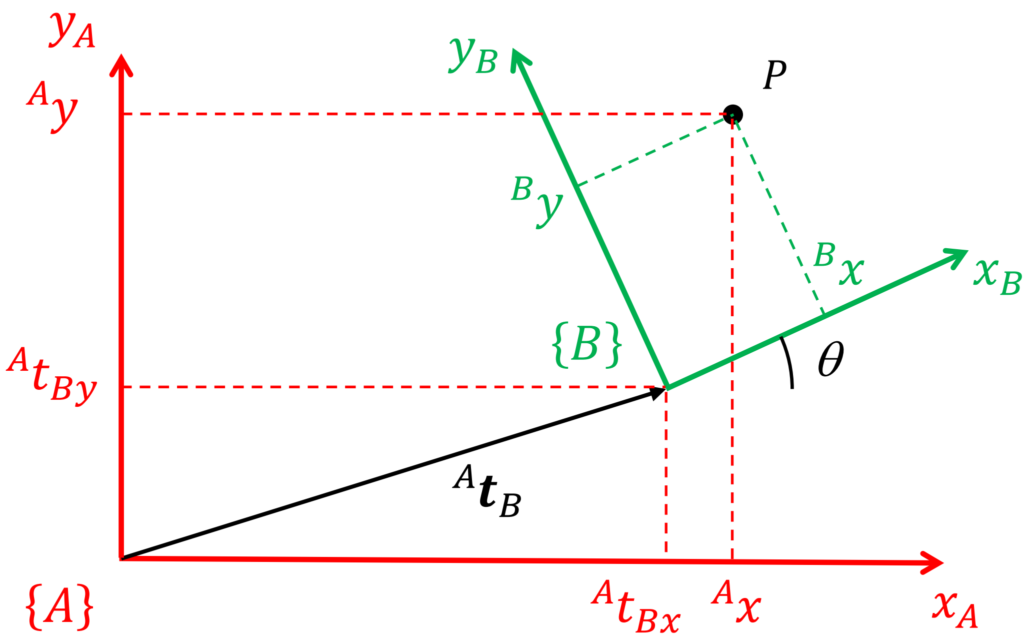
Investigating pose in 2D



- Recall that we can think of ${}^{A}\xi_B$ as the translation and rotation required in order to make $\{A\}$ coincide with $\{B\}$

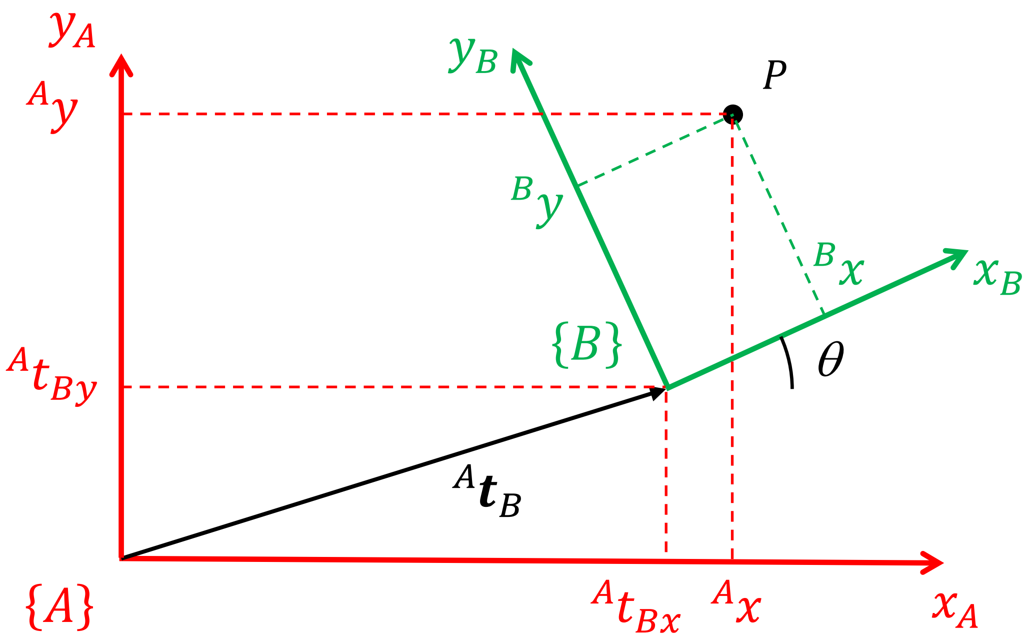
- To coincide with $\{B\}$, $\{A\}$ must undergo a translation ${}^{A}\boldsymbol{t}_B$ and a rotation by an angle θ

Investigating pose in 2D



- Let $^A\boldsymbol{p} = [^Ax, {}^Ay]^{\boldsymbol{T}}$, $^B\boldsymbol{p} = [^Bx, {}^By]^{\boldsymbol{T}}$ and $^A\boldsymbol{t}_B = \left[^At_{Bx}, {}^At_{By}\right]^{\boldsymbol{T}}$

Investigating pose in 2D



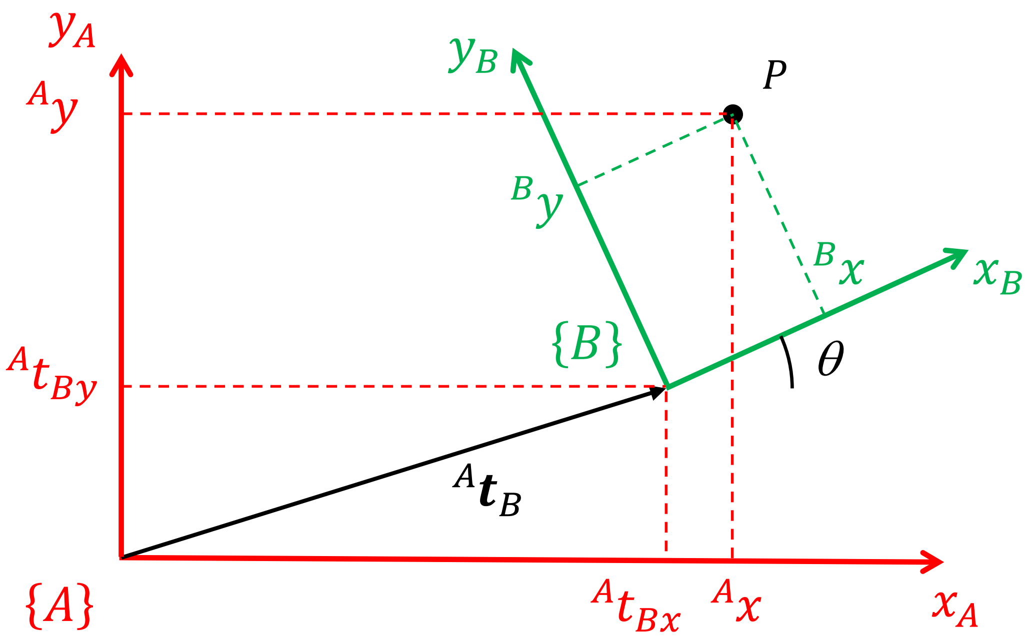
- Let $^A\boldsymbol{p} = [^A x, {}^A y]^{\boldsymbol{T}}$, $^B\boldsymbol{p} = [^B x, {}^B y]^{\boldsymbol{T}}$ and $^A\boldsymbol{t}_B = \left[^A t_{Bx}, {}^A t_{By}\right]^{\boldsymbol{T}}$

- From the figure we can see that

$$^A x = {}^A t_{Bx} + {}^B x \cos\theta - {}^B y \sin\theta$$

$$^A y = {}^A t_{By} + {}^B x \sin\theta + {}^B y \cos\theta$$

Investigating pose in 2D

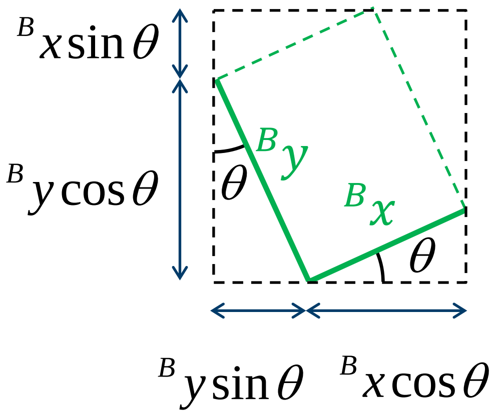


- Let $^A\boldsymbol{p} = [^Ax, {^Ay}]^{\boldsymbol{T}}$, $^B\boldsymbol{p} = [^Bx, {^By}]^{\boldsymbol{T}}$ and $^A\boldsymbol{t}_B = \left[^At_{Bx}, {^At_{By}}\right]^{\boldsymbol{T}}$

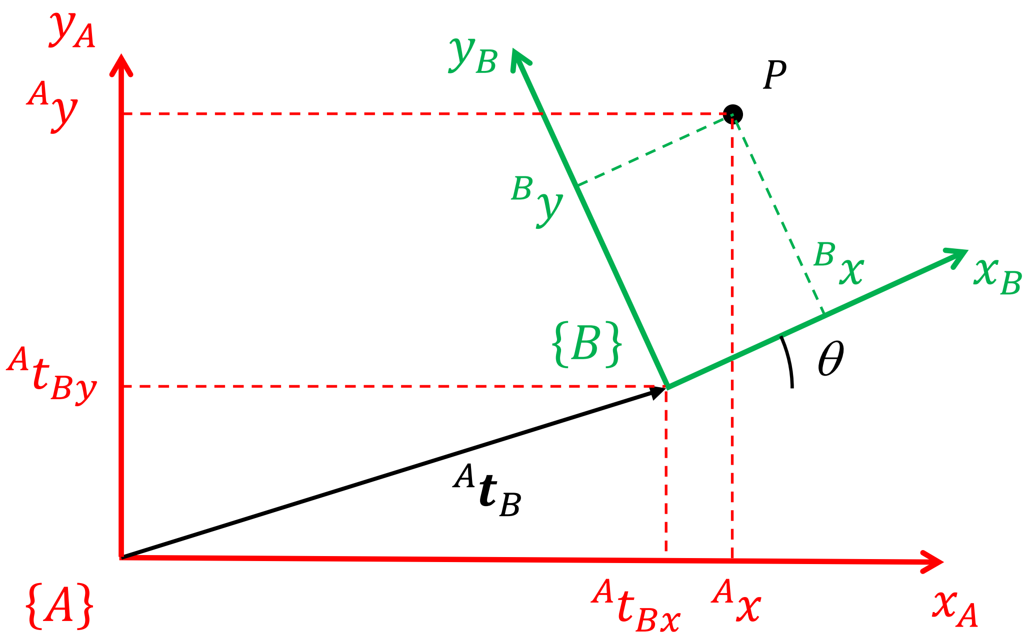
- From the figure we can see that

$$^Ax = {^At_{Bx}} + {^Bx}\cos\theta - {^By}\sin\theta$$

$$^Ay = {^At_{By}} + {^Bx}\sin\theta + {^By}\cos\theta$$



Investigating pose in 2D



- In matrix form

$$\begin{bmatrix} ^A x \\ ^A y \end{bmatrix} = \begin{bmatrix} ^A t_{Bx} \\ ^A t_{By} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} ^B x \\ ^B y \end{bmatrix}$$

$$^A \boldsymbol{p} = {}^A \boldsymbol{t}_B + {}^A R_B \, {}^B \boldsymbol{p}$$

Investigating pose in 2D

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = \begin{bmatrix} {}^{A}t_{Bx} \\ {}^{A}t_{By} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{B} + {}^{A}R_{B}\,{}^{B}\boldsymbol{p}$$

- Can we represent the pose ${}^{A}\xi_{B}$ by the pair $({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$?

Investigating pose in 2D

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = \begin{bmatrix} {}^{A}t_{Bx} \\ {}^{A}t_{By} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{B} + {}^{A}R_{B}\,{}^{B}\boldsymbol{p}$$

- Can we represent the pose ${}^{A}\xi_{B}$ by the pair $({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$?

$$\begin{array}{ccc} {}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} & \mapsto & {}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \\ {}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} & \mapsto & \left({}^{A}R_{C}, {}^{A}\boldsymbol{t}_{C}\right) = \left({}^{A}R_{B}\,{}^{B}R_{C}, {}^{A}R_{B}\,{}^{B}\boldsymbol{t}_{C} + {}^{A}\boldsymbol{t}_{B}\right) \\ \ominus{}^{A}\xi_{B} & \mapsto & \left({}^{A}R_{C}{}^{T}, -{}^{A}R_{C}{}^{T}\,{}^{A}\boldsymbol{t}_{C}\right) \end{array}$$

Investigating pose in 2D

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = \begin{bmatrix} {}^{A}t_{Bx} \\ {}^{A}t_{By} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{t}_{B} + {}^{A}R_{B}\,{}^{B}\boldsymbol{p}$$

- Can we represent the pose ${}^{A}\xi_{B}$ by the pair $({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$?

$$\begin{array}{ccc} {}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} & \mapsto & {}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \\ {}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} & \mapsto & \left({}^{A}R_{C}, {}^{A}\boldsymbol{t}_{C}\right) = \left({}^{A}R_{B}\,{}^{B}R_{C}, {}^{A}R_{B}\,{}^{B}\boldsymbol{t}_{C} + {}^{A}\boldsymbol{t}_{B}\right) \\ \ominus {}^{A}\xi_{B} & \mapsto & \left({}^{A}R_{C}{}^{T}, -{}^{A}R_{C}{}^{T}\,{}^{A}\boldsymbol{t}_{C}\right) \end{array}$$

- Yes, but there is a better option!

Investigating pose in 2D

- Observe the following equivalence

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = {}^{A}R_{B} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix} + {}^{A}\boldsymbol{t}_{B} \quad \Leftrightarrow \quad \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \quad \Leftrightarrow \quad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}}$$

Investigating pose in 2D

- Observe the following equivalence

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = {}^{A}R_{B} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix} + {}^{A}\boldsymbol{t}_{B} \quad \Leftrightarrow \quad \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \quad \Leftrightarrow \quad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}}$$

- Can we represent the pose ${}^{A}\xi_{B}$ by the matrix ${}^{A}T_{B}$?

Investigating pose in 2D

- Observe the following equivalence

$$\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix} = {}^{A}R_{B} \begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix} + {}^{A}\boldsymbol{t}_{B} \qquad \Leftrightarrow \qquad \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}$$

$${}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \qquad \Leftrightarrow \qquad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}}$$

- Can we represent the pose ${}^{A}\xi_{B}$ by the matrix ${}^{A}T_{B}$?

$$\begin{array}{ccc} {}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} & \mapsto & {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}} \\ {}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} & \mapsto & {}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C} \\ \ominus {}^{A}\xi_{B} & \mapsto & {}^{A}T_{B}^{\ -1} \end{array}$$

- Yes, and the algebraic properties are nice!

Investigating pose in 2D

- But…

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix}}_{{}^{A}\mathbf{p}} = {}^{A}R_{B} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}}_{{}^{B}\mathbf{p}} + {}^{A}\boldsymbol{t}_{B} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix}}_{{}^{A}\tilde{\mathbf{p}}} = \underbrace{\begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix}}_{{}^{A}T_{B}} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}}_{{}^{B}\tilde{\mathbf{p}}}$$

- We are describing points in the plane with 3 coordinates despite that they only have 2 degrees of freedom…

- The non linear transformation ${}^{B}\boldsymbol{p} \mapsto {}^{A}\boldsymbol{p}$ then becomes a linear transformation ${}^{B}\tilde{\boldsymbol{p}} \mapsto {}^{A}\tilde{\boldsymbol{p}}$

- What is going on?

Investigating pose in 2D

- We have "discovered" some basic constructions from projective geometry

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix}}_{{}^{A}\mathbf{p}} = {}^{A}R_{B} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}}_{{}^{B}\mathbf{p}} + {}^{A}\mathbf{t}_{B}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix}}_{{}^{A}\tilde{\mathbf{p}}} = \underbrace{\begin{bmatrix} {}^{A}R_{B} & {}^{A}\mathbf{t}_{B} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}}_{{}^{A}T_{B}} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}}_{{}^{B}\tilde{\mathbf{p}}}$$

Investigating pose in 2D

- We have "discovered" some basic constructions from projective geometry

Non-linear transformation of the Euclidean plane $\mathbb{R}^2$

Points in the Euclidean plane are described by Cartesian coordinates

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix}}_{{}^{A}\mathbf{p}} = {}^{A}R_{B} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}}_{{}^{B}\mathbf{p}} + {}^{A}\boldsymbol{t}_{B}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix}}_{{}^{A}\tilde{\mathbf{p}}} = \underbrace{\begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix}}_{{}^{A}T_{B}} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}}_{{}^{B}\tilde{\mathbf{p}}}$$

Investigating pose in 2D

- We have "discovered" some basic constructions from projective geometry

Non-linear transformation of the Euclidean plane $\mathbb{R}^2$

Points in the Euclidean plane are described by Cartesian coordinates

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \end{bmatrix}}_{{}^{A}\mathbf{p}} = {}^{A}R_{B} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \end{bmatrix}}_{{}^{B}\mathbf{p}} + {}^{A}\mathbf{t}_{B}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix}}_{{}^{A}\tilde{\mathbf{p}}} = \underbrace{\begin{bmatrix} {}^{A}R_{B} & {}^{A}\mathbf{t}_{B} \\ \mathbf{0}_{1\times 2} & 1 \end{bmatrix}}_{{}^{A}T_{B}} \underbrace{\begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}}_{{}^{B}\tilde{\mathbf{p}}}$$

Linear transformation of the projective plane $\mathbb{P}^2$

Points in the projective plane are described by homogeneous coordinates

This means that they are only unique up to scale, i.e.
$(x, y, 1) = (\lambda x, \lambda y, \lambda)\ \forall\ \lambda \in \mathbb{R}\backslash\{0\}$

The matrix representing the projective transformation is also homogeneous, i.e.

$${}^{A}T_{B} = \lambda\, {}^{A}T_{B} \forall\ \lambda \in \mathbb{R}\backslash\{0\}$$

Investigating pose in 2D

- Euclidean geometry
  - ${}^{A}\xi_{B} \longmapsto ({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$
  - Complicated algebra

$$
\begin{array}{ccc}
{}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} & \mapsto & {}^{A}\boldsymbol{p} = {}^{A}R_{B}\,{}^{B}\boldsymbol{p} + {}^{A}\boldsymbol{t}_{B} \\
{}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} & \mapsto & \left({}^{A}R_{C}, {}^{A}\boldsymbol{t}_{C}\right) = \left({}^{A}R_{B}\,{}^{B}R_{C}, {}^{A}R_{B}\,{}^{B}\boldsymbol{t}_{C} + {}^{A}\boldsymbol{t}_{B}\right) \\
\ominus {}^{A}\xi_{B} & \mapsto & \left({}^{A}R_{C}{}^{T}, -{}^{A}R_{C}{}^{T}\,{}^{A}\boldsymbol{t}_{C}\right)
\end{array}
$$

- Projective geometry
  - ${}^{A}\xi_{B} \longmapsto {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix}$
  - Simple algebra

$$
\begin{array}{ccc}
{}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} & \mapsto & {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}} \\
{}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} & \mapsto & {}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C} \\
\ominus {}^{A}\xi_{B} & \mapsto & {}^{A}T_{B}{}^{-1}
\end{array}
$$

- Many problems in computer vision are easier to express and solve if we choose to think of points and transformations in terms of projective geometry
  - Algebra and computations become simpler

Representing pose in 2D

- The pose of $\{B\}$ relative to $\{A\}$ can be represented by a homogeneous transformation ${}^{A}T_{B} \in SE(2)$

$$
{}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & {}^{A}t_{Bx} \\ \sin\theta & \cos\theta & {}^{A}t_{By} \\ 0 & 0 & 1 \end{bmatrix}
$$

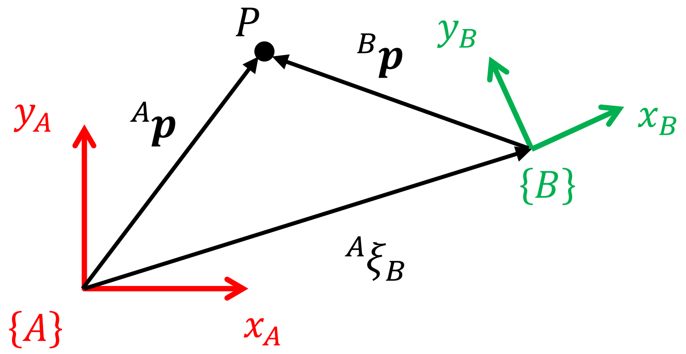
- Properties

$$
\begin{aligned}
{}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} \quad &\mapsto \quad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\, {}^{B}\tilde{\boldsymbol{p}} \\
{}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} \quad &\mapsto \quad {}^{A}T_{C} = {}^{A}T_{B}\, {}^{B}T_{C} \\
\ominus {}^{A}\xi_{B} \quad &\mapsto \quad {}^{A}T_{B}^{\ -1}
\end{aligned}
$$

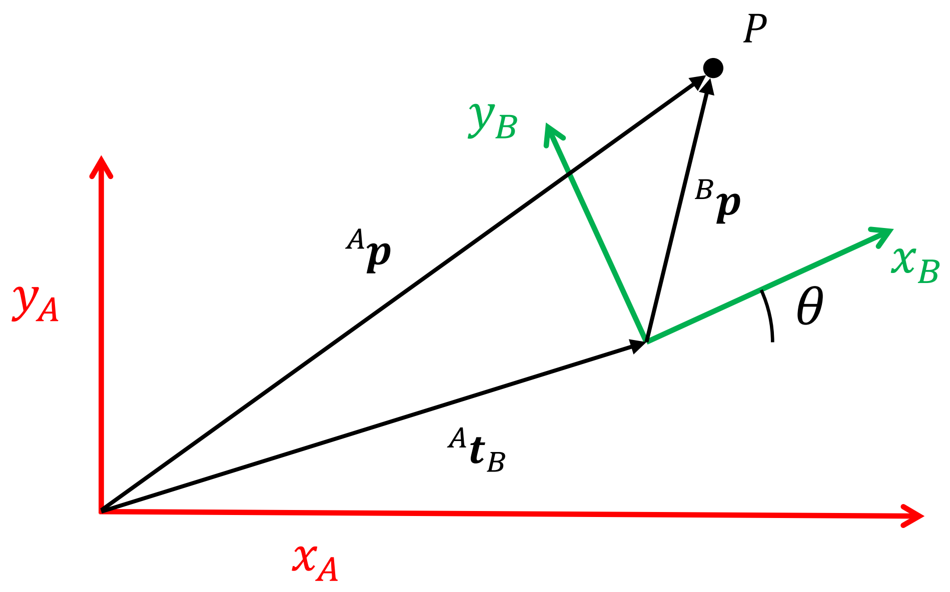
- Points are represented in homogeneous coordinates

$$
{}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\, {}^{B}\tilde{\boldsymbol{p}}
$$

$$
\begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ 1 \end{bmatrix}
$$



Example



- Let $^A\boldsymbol{t}_B = [4,1]^T$, $^B\boldsymbol{p} = [2,3]^T$ and $\theta = 27°$
- Determine the pose of of $\{B\}$ relative to $\{A\}$, i.e. ^AT_B
- Determine the coordinates of P in $\{A\}$, i.e. $^A\boldsymbol{p}$

Example

- From the previous slides we know that

$$ {}^{A}T_{B} = \begin{bmatrix} \cos\theta & -\sin\theta & {}^{A}t_{Bx} \\ \sin\theta & \cos\theta & {}^{A}t_{By} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 27° & -\sin 27° & 4 \\ \sin 27° & \cos 27° & 1 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.8910 & -0.4540 & 4 \\ 0.4540 & 0.8910 & 1 \\ 0 & 0 & 1 \end{bmatrix} $$

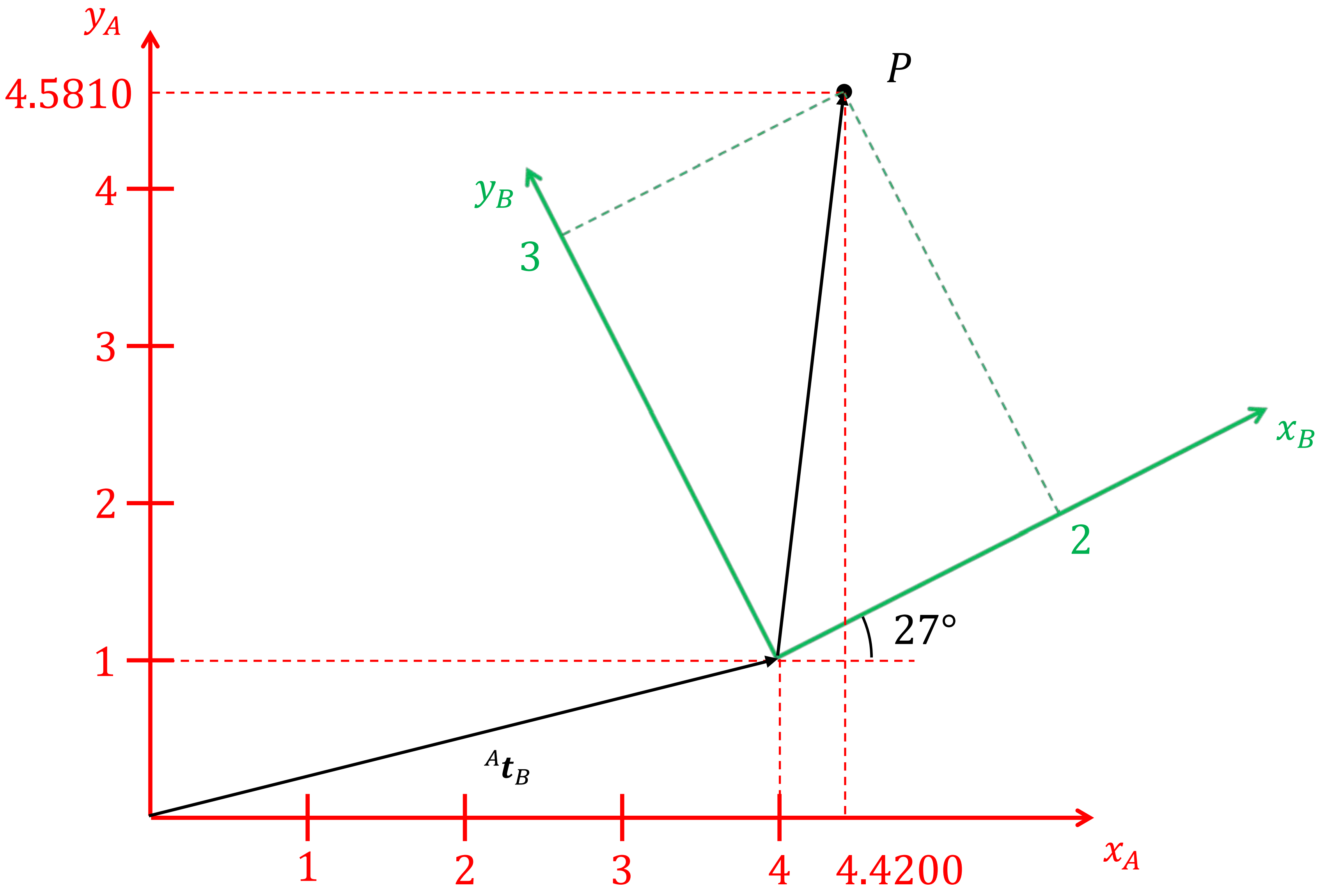
- This allows us to compute ${}^{A}\tilde{\boldsymbol{p}}$

$$ {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\,{}^{B}\tilde{\boldsymbol{p}} $$

$$ \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8910 & -0.4540 & 4 \\ 0.4540 & 0.8910 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4.4200 \\ 4.5810 \\ 1 \end{bmatrix} \quad \Rightarrow \quad {}^{A}\boldsymbol{p} = \begin{bmatrix} 4.4200 & 4.5810 \end{bmatrix}^{T} $$

- This can also be verified by drawing

Example



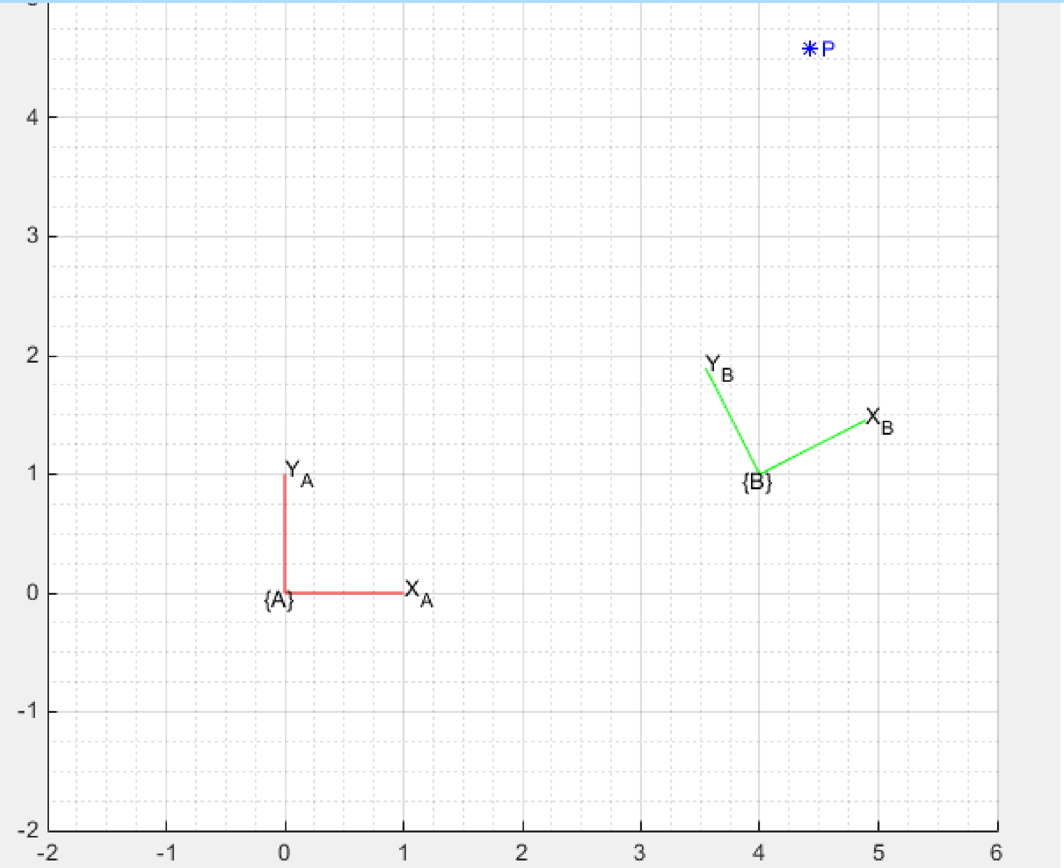
Example

```
%% Example: Visualize {A}, {B} and P in coordinates of {A}
robotics_path = 'G:\MATLAB\BIBLIOTEKER\PeterCork_Robotics\robot-9.10\rvctools';
addpath(genpath(robotics_path));

%Pose of {A} relative to {A}
t_AA = [0;0];
theta_AA = 0;
T_AA = se2(t_AA(1),t_AA(2),theta_AA*pi/180);

%Pose of {B} relative to {A}
t_AB = [4;1];
theta_AB = 27;
T_AB = se2(t_AB(1),t_AB(2),theta_AB*pi/180);

%Point P relative to {B}
P_B = [2;3];

%Transform point P to {A} using homogeneous coordinates
hP_B = e2h(P_B);     % e2h: Changes representation Euclidean --> homogeneous
hP_A = T_AB*hP_B;    % Transformation by multiplication
P_A = h2e(hP_A);     % h2e: Changes representation homogeneous --> Euclidean

%Visualize {A}, {B} and P relative to {A}
figure(1);
clf
axis equal
grid on
grid minor
axis([-2, 6, -2, 6]);
hold on
trplot2(T_AA, 'frame', 'A', 'color', 'r')
trplot2(T_AB, 'frame', 'B', 'color', 'g')
plot_point(P_A, '*b','printf',{' P',P_A},'textcolor','b')
```

You can visualize this example in matlab using the toolboxes created by Peter Cork

- Robotics Toolbox
- Machine Vision Toolbox

www.petercorke.com/Toolboxes.html



Representing pose in 3D

- The pose of $\{B\}$ relative to $\{A\}$ can be represented by a homogeneous transformation ${}^{A}T_{B} \in SE(3)$

$$ {}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & {}^{A}t_{Bx} \\ r_{21} & r_{22} & r_{23} & {}^{A}t_{By} \\ r_{31} & r_{32} & r_{33} & {}^{A}t_{Bz} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Properties

$$ {}^{A}\boldsymbol{p} = {}^{A}\xi_{B} \cdot {}^{B}\boldsymbol{p} \quad \mapsto \quad {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\, {}^{B}\tilde{\boldsymbol{p}} $$

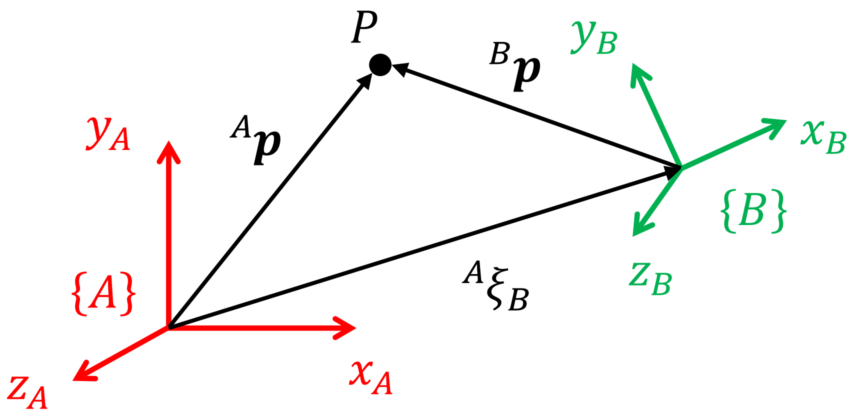
$$ {}^{A}\xi_{C} = {}^{A}\xi_{B} \oplus {}^{B}\xi_{C} \quad \mapsto \quad {}^{A}T_{C} = {}^{A}T_{B}\, {}^{B}T_{C} $$

$$ \ominus {}^{A}\xi_{B} \quad \mapsto \quad {}^{A}T_{B}^{\ -1} $$

- Points are represented in homogeneous coordinates

$$ {}^{A}\tilde{\boldsymbol{p}} = {}^{A}T_{B}\, {}^{B}\tilde{\boldsymbol{p}} $$

$$ \begin{bmatrix} {}^{A}x \\ {}^{A}y \\ {}^{A}z \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}\boldsymbol{t}_{B} \\ \boldsymbol{0}_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}x \\ {}^{B}y \\ {}^{B}z \\ 1 \end{bmatrix} $$

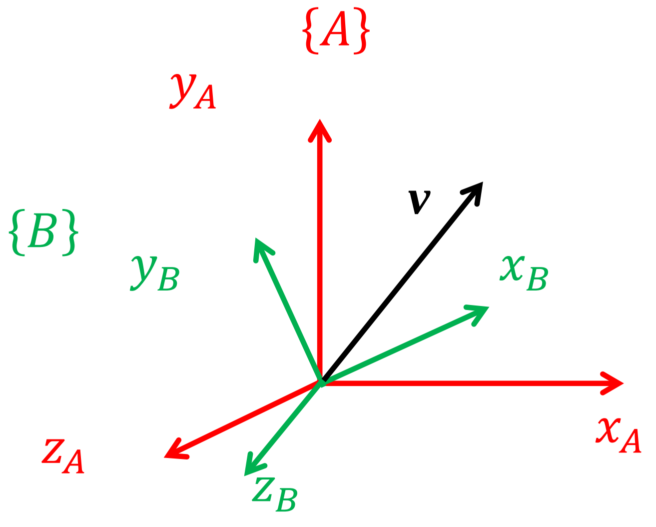


Representing pose in 3D

- The main difference between 3D and 2D is that rotation is far less intuitive in 3D
  Also there are several different representations of rotation in 3D
  - Orthonormal rotation matrix $R \in SO(3)$
  - Euler angles $(\theta_1, \theta_2, \theta_3)$
  - Angle-axis $(\theta, \boldsymbol{e})$ or just $\boldsymbol{\theta} = \theta\boldsymbol{e}$
  - Unit quaternions $q = r + xi + yj + zk$

- Hence there are several ways to represent pose
  - Rotation matrix and translation vector $({}^{A}R_{B}, {}^{A}\boldsymbol{t}_{B})$
  - Homogeneous transformation ${}^{A}T_{B}$
  - Euler angles and translation vector $(\theta_1, \theta_2, \theta_3, {}^{A}\boldsymbol{t}_{B})$
  - Angle-axis and translation vector $(\theta, \boldsymbol{e}, {}^{A}\boldsymbol{t}_{B})$
  - Unit quaternion and translation vector $({}^{A}q_{B}, {}^{A}\boldsymbol{t}_{B})$

Representing rotation in 3D



- Orientation of $\{B\}$ relative to $\{A\}$
  - How $\{B\}$ should rotate to coincide with $\{A\}$

- Orthonormal rotation matrix ${}^AR_B \in SO(3)$

$$ {}^AR_B = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

- Properties

$$ \det\left({}^AR_B\right) = 1 $$

$$ {}^AR_B{}^{-1} = {}^AR_B{}^T $$

$$ {}^AR_C = {}^AR_B\,{}^BR_C $$

$$ {}^A\boldsymbol{v} = {}^AR_B\,{}^B\boldsymbol{v} $$

- Nice algebraic properties
- Computations are simple
- Provides little intuitive understanding

Representing rotation in 3D

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Euler angles $(\theta_1, \theta_2, \theta_3)$ describes rotation in terms of basic rotations about 3 coordinate axis that must be specified
  - 12 compositions: xyx, xzx, yxy, yzy, zxz, zyz, xyz, xzy, yzx, yxz, zxy, zyx
  - E.g. $R_{xyz} = R_x(\theta_1)R_y(\theta_2)R_z(\theta_3)$

- Suffers from singularities
  - $\theta_2 = \pm\frac{\pi}{2}$ for sequences without repetition
  - $\theta_2 \in \{0, \pi\}$ for sequences with repetition

- It is possible to determine the Euler angles that corresponds to a given rotation matrix

$$R_{xyz} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \Rightarrow \begin{aligned} \theta_y &= \operatorname{atan2}\left(r_{13}, \sqrt{{r_{11}}^2 + {r_{12}}^2}\right) & -\frac{\pi}{2} < \theta_y < \frac{\pi}{2} \\ \theta_x &= \operatorname{atan2}\left(-\frac{r_{23}}{\cos\theta_y}, \frac{r_{33}}{\cos\theta_y}\right) & -\pi < \theta_x < \pi \\ \theta_z &= \operatorname{atan2}\left(-\frac{r_{12}}{\cos\theta_y}, \frac{r_{11}}{\cos\theta_y}\right) & -\pi < \theta_z < \pi \end{aligned}$$

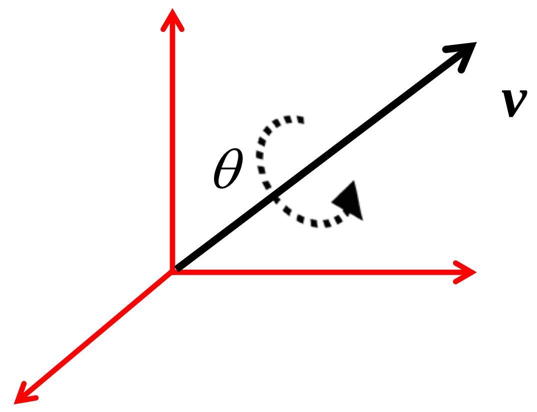
Representing rotation in 3D

- An angle-axis pair $(\theta, \boldsymbol{v})$ describes rotation in terms of an angle and an axis of revolution

- Euler's rotation theorem states that any sequence of rotations in 3 dimensions is equivalent to a single rotation θ about an axis $\boldsymbol{v}$

- The rotation matrix R corresponding to an angle-axis pair $(\theta, \boldsymbol{v})$ can be found by Rodrigues' rotation formula

$$R = I_3 + \sin\theta[\mathbf{v}]_\times + (1-\cos\theta)\left(\mathbf{v}\mathbf{v}^T - I_3\right)$$



Where $[\boldsymbol{v}]_\times$ is the matrix representation of the cross product

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$$

- The angle-axis pair $(\theta, \boldsymbol{v})$ corresponding to a rotation matrix R can be found from the eigenvalues and eigenvectors of R

Representing rotation in 3D

- Unit quaternions are commonly used to work with rotation/orientation in 3D

- A quaternion $q \in \mathbb{H}$ is a number, with 1 real term and 3 imaginary terms

$$q = q_0 + q_1 i + q_2 j + q_3 k = q_0 + \boldsymbol{v}$$

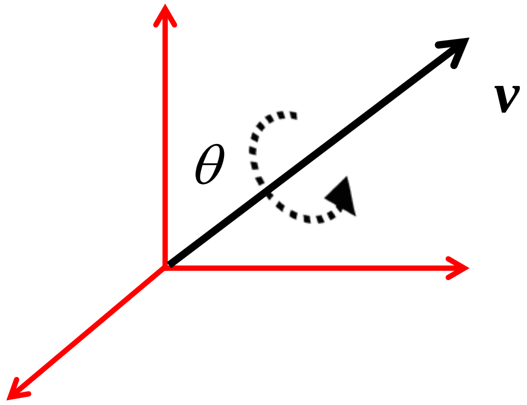
$$i^2 = j^2 = k = ijk = -1$$

- Conjugate quaternion

$$q^* = q_0 - q_1 i - q_2 j - q_3 k = q_0 - \boldsymbol{v}$$

- A unit quaternion satisfy

$$\|q\| = \sqrt{q^* q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$



$$q_0 = \cos\left(\frac{\theta}{2}\right)$$

$$\boldsymbol{v} = \sin\left(\frac{\theta}{2}\right)\boldsymbol{e}$$

- Quaternion multiplication

$$qp = (q_0 + \boldsymbol{v})(p_0 + \boldsymbol{w})$$
$$= \underbrace{q_0 p_0 - \boldsymbol{v} \cdot \boldsymbol{w}}_{\in \mathbb{R}} + \underbrace{q_0 \boldsymbol{w} + p_0 \boldsymbol{v} + \boldsymbol{v} \times \boldsymbol{w}}_{\in \mathbb{R}^3}$$

- Composing two unit quaternions takes 16 multiplications and 12 additions, while composing two rotation matrixes takes 27 multiplications and 18 additions

Representing rotation in 3D

- The unit quaternion corresponding to the rotation matrix R is given by

$$q_0 = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$q_1 = \frac{r_{32} - r_{23}}{4q_0}$$

$$q_2 = \frac{r_{13} - r_{31}}{4q_0}$$

$$q_3 = \frac{r_{21} - r_{12}}{4q_0}$$

- The rotation matrix corresponding to the unit quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is given by

$$R = \begin{bmatrix} 1 - 2q_2^2 - 2q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 1 - 2q_1^2 - 2q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 1 - 2q_1^2 - 2q_2^2 \end{bmatrix}$$

- SLERP – Spherical linear interpolation
  An algorithm for determining a rotation partway between two given rotations

  Useful for visualizing rotations

Summary

- Pose
  - General properties
- Representing pose in 2D
  - Homogeneous transformations
- Representing pose in 3D
  - Homogeneous transformations
  - Other alternatives
- Representing rotation in 3D
  - Rotation matrix
  - Euler angles
  - Angle-axis
  - Unit quaternion

- Additional information
  - Szeliski 2.1.1, 2.1.2, 2.1.4