Name: Solutions

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	25	
3	25	
4	35	
Total:	100	

1. (15 points) Determine if each of the following statements is true or false and give a short justification for your choice.

   (a) If the row echelon form of A involves free variables, then the system $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.

   False. The system could be inconsistent, in which case there will be no solutions.

   (b) If A and B are $n \times n$ matrices, then $(A - B)^2 = A^2 - 2AB + B^2$.

   False. $(A-B)^2 = A^2 - AB - BA + B^2$ and $AB \neq BA$ in general.

   (c) If A is a 4×3 matrix and $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_3$, then the system $A\mathbf{x} = \mathbf{b}$ must be consistent.

   True. This means that $(1, 0, 1)$ is a solution.

2. (25 points) Given that A and B are nonsingular matrices with

$$A^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 8 & 6 \end{pmatrix} \quad \text{and} \quad B^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 3 & 8 & 8 \end{pmatrix},$$

compute $(AB)^{-1}$ or explain why you cannot.

The best way to do this is to use the fact that $(AB)^{-1} = B^{-1}A^{-1}$!

Then,

$$(AB)^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 3 & 8 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 3 & 8 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 16 & 11 \\ 13 & 35 & 25 \\ 35 & 94 & 67 \end{pmatrix}.$$

3. (25 points) Compute the LU factorization of

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 6 & 8 & 8 \end{pmatrix}.$$

That is, find an upper triangular matrix U and a lower triangular matrix L with ones on the diagonal such that $A = LU$.

$$\begin{pmatrix} \boxed{2} & 2 & 1 \\ 2 & 3 & 2 \\ 6 & 8 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 2 & 5 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} = U.$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

4. (35 points) Find all solutions to the following system of equations.

$$\begin{cases} 2x_1 + 2x_2 + x_3 + 2x_4 = 3 \\ 2x_1 + 3x_2 + 2x_3 + 3x_4 = 6 \\ 6x_1 + 8x_2 + 8x_3 + 8x_4 = 18 \end{cases}$$

$\Rightarrow$ If $x_4 = \alpha$, then $(-1, 2-\alpha, 1, \alpha)$ is a solution.

$$\left(\begin{array}{cccc|c} \boxed{2} & 2 & 1 & 2 & 3 \\ 2 & 3 & 2 & 3 & 6 \\ 6 & 8 & 8 & 8 & 18 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 2 & 2 & 1 & 2 & 3 \\ 0 & \boxed{1} & 1 & 1 & 3 \\ 0 & 2 & 5 & 2 & 9 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 2 & 2 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 0 & 3 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1/2 & 1 & 3/2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & \boxed{1} & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$