Name: Solutions

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	25	
3	30	
4	30	
Total:	100	

1. (15 points) For each of the statements that follow, answer 'true' if the statement is always true and 'false' otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.

   (a) If A is an $m \times n$ matrix, then A and A^T have the same rank.

   True. If A has r linearly independent rows, then A^T has r linearly indep. columns. Both of these are equal to the rank.

   (b) If U is the reduced row echelon form of A, then A and U have the same column space.

   False. Row operations potentially change the column space of a matrix (but not the row space).

   (c) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If $L(\mathbf{x}_1) = L(\mathbf{x}_2)$, then the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ must be equal.

   False. Consider the linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(\bar{x}) = \bar{0}$.

   $L(\bar{e}_1) = L(\bar{e}_2) = \bar{0}$, but $\bar{e}_1 \neq \bar{e}_2$.

2. Let A be a 6×5 matrix with linearly independent column vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. The remaining columns of A satisfy

$$\mathbf{a}_4 = \mathbf{a}_1 + 3\mathbf{a}_2 + \mathbf{a}_3 \quad \text{and} \quad \mathbf{a}_5 = 2\mathbf{a}_1 - \mathbf{a}_3.$$

(a) (10 points) What is the rank of A? What is the nullity of A?

The rank of A is 3.

The nullity of A is 2.

(b) (15 points) Determine the reduced row echelon form of A.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3. Let $E = \{\mathbf{u}_1, \mathbf{u}_2\}$ and $F = \{\mathbf{v}_1, \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

and

$$\mathbf{v}_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

(a) (15 points) Determine the transition matrix corresponding to a change of basis from F to E.

$$S = (\bar{u}_1\ \bar{u}_2)^{-1}(\bar{v}_1\ \bar{v}_2)$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}^{-1} \begin{pmatrix} 5 & 4 \\ 2 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & 10 \\ -13 & -3 \end{pmatrix}.$$

(b) (15 points) Use this transition matrix to find the coordinates of $\mathbf{z} = 2\mathbf{v}_1 + 3\mathbf{v}_2$ with respect to E.

The coordinate vector of $\bar{z}$ w.r.t F is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so the coord. vector of $\bar{z}$ w.r.t. E is $\begin{pmatrix} 31 & 10 \\ -13 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 92 \\ -35 \end{pmatrix}$.

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{pmatrix}.$$

(a) (15 points) Show that L is a linear transformation.

$$L(\alpha\bar{x} + \beta\bar{y}) = \begin{pmatrix} (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) \\ (\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) \\ 3(\alpha x_1 + \beta y_1) + 2(\alpha x_2 + \beta y_2) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(x_1 + x_2) + \beta(y_1 + y_2) \\ \alpha(x_1 - x_2) + \beta(y_1 - y_2) \\ \alpha(3x_1 + 2x_2) + \beta(3y_1 + 2y_2) \end{pmatrix} = \alpha \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_1 + 2x_2 \end{pmatrix}$$

$$+ \beta \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \\ 3y_1 + 2y_2 \end{pmatrix} = \alpha L(\bar{x}) + \beta L(\bar{y}).$$

(b) (15 points) Find a matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$ in $\mathbb{R}^2$.

$$L(\bar{e}_1) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \quad L(\bar{e}_2) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 3 & 2 \end{pmatrix}.$$