Name: Solutions

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	25	
3	25	
4	35	
Total:	100	

1. (15 points) For each of the statements that follow, answer 'true' if the statement is always true and 'false' otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.

   (a) $\det(A + B) = \det(A) + \det(B)$ for any $n \times n$ matrices A and B.

   False. This can be seen with $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

   (b) If S is a subspace of a vector space V, then S is a vector space.

   True. A subspace is a vector space by definition.

   (c) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ span a vector space V, then they are linearly independent.

   False. Repeating a vector doesn't change the span, but it makes them linearly dependent.

2. Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}.$$

(a) (10 points) Find $\det(A)$.

$$\det(A) = 1 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 1(4) - 2(5) + 1(2)$$

$$= -4$$

(b) (15 points) Use Cramer's rule to find the value of x_3.

$$\det(A_3) = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 6 \\ 2 & 9 \end{vmatrix} - 2 \begin{vmatrix} 2 & 6 \\ 1 & 9 \end{vmatrix} + 5 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 1(6) - 2(12) + 5(2)$$

$$= -8$$

$$\Rightarrow x_3 = \det(A_3) / \det(A) = -8/-4 = 2.$$

3. (25 points) Determine whether or not the set $S = \{\mathbf{x} \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 0\}$ forms a subspace of $\mathbb{R}^2$. Justify your response.

① If $\bar{x} \in S$, then $2x_1 + 3x_2 = 0$ and

$$2(\alpha x_1) + 3(\alpha x_2) = \alpha(2x_1 + 3x_2) = 0,$$

so $\alpha \bar{x} \in S$.

② If $\bar{x}, \bar{y} \in S$, then also $2y_1 + 2y_2 = 0$ and

$$2(x_1 + y_1) + 3(x_2 + y_2) = (2x_1 + 3x_2) + (2y_1 + 3y_2) = 0 + 0 = 0,$$

so $\bar{x} + \bar{y} \in S$

③ $\bar{0} \in S$ since $2(0) + 3(0) = 0$.

$\Rightarrow$ S is a subspace of $\mathbb{R}^2$.

4. Consider the following set of vectors in $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

(a) (20 points) Determine if this set is a spanning set for $\mathbb{R}^3$. Justify your response.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

A system involving this matrix will always be consistent, so its columns are a spanning set for $\mathbb{R}^3$.

(b) (15 points) Determine if this set is linearly independent. Justify your response.

This set is not linearly indep. since there are free variables in its RREF.