

This assignment is optional. If completed, it will replace your lowest homework grade. In addition to calculations and proofs, please hand in neatly formatted output (figures and numbers) from your programs. Hand in code unless otherwise noted, but it should be clear from the output if the program works or not.

1. *Initial Value Problems.* This homework is intended to demonstrate that there is a difference between good and not so good numerical methods for solving initial value problems.

Consider the second order IVP for the function $z(t)$ given by

$$z'' + 0.05 z' + z^3 = 7.5 \cos(t), \quad z(0) = 0, \quad z'(0) = 1.$$

Your assignment is to compute $z(70)$ as accurately as your computer allows.

- (a) By introducing the new variables $u_1 = z$ and $u_2 = z'$, rewrite the IVP as a system $\mathbf{u}' = \mathbf{f}(t, \mathbf{u}(t))$, where $\mathbf{u} = (u_1, u_2)$.
- (b) Try the following two methods, with a variety of step sizes h :
 - Euler's Method:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + h \mathbf{f}(t_n, \mathbf{u}_n).$$

- Fourth Order Runge-Kutta:

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(t_n, \mathbf{u}_n), \\ \mathbf{k}_2 &= \mathbf{f}(t_n + \frac{h}{2}, \mathbf{u}_n + \frac{h}{2} \mathbf{k}_1), \\ \mathbf{k}_3 &= \mathbf{f}(t_n + \frac{h}{2}, \mathbf{u}_n + \frac{h}{2} \mathbf{k}_2), \\ \mathbf{k}_4 &= \mathbf{f}(t_n + h, \mathbf{u}_n + h \mathbf{k}_3), \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + \frac{h}{6} (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4). \end{aligned}$$

Use the values

$$u_1^{\text{ex}}(70) = 1.582857756103056, \quad u_2^{\text{ex}}(70) = -2.835763853877514$$

as “exact” and plot the mean-square error $\varepsilon = \sqrt{(u_1 - u_1^{\text{ex}})^2 + (u_2 - u_2^{\text{ex}})^2}$ at $t = 70$ as a function of h .

- (c) If h is small enough, the error is expected to follow a power law $\varepsilon \approx ch^p$. Determine c and p from your plot and estimate the number of time steps required to reach $\varepsilon < 10^{-5}$ with Euler's method. Is your computer fast enough? Discuss your results.

Note: Unless you take h small enough, neither of these two methods will be stable. To be on the safe side (inside the stability region), take $h < 8.75 \times 10^{-3}$ for Euler and $h < 0.35$ for Runge-Kutta.