Gaussian Elimination

Write down the algorithms for Gaussian elimination (with and without partial pivoting) and back substitution. Use Gaussian elimination (with and without partial pivoting) and back substitution to solve the linear system

$$\begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & -2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

for x_1, x_2, and x_3. Determine the asymptotic operation counts for Gaussian elimination (with and without partial pivoting) and back substitution applied to an $n \times n$ linear system. Why might you prefer Gaussian elimination with partial pivoting over the alternative?

LU Decomposition

Find decompositions of the form $A = LU$ and $PA = LU$ for the matrix

$$A = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 0 & -2 \\ 2 & 2 & 4 \end{pmatrix}$$

What properties do the matrices P, L, and U possess? Explain how you would solve the linear system $Ax = b$ using each decomposition. Why might it be useful to perform such a decomposition?

Vector and Matrix Norms

What three properties must every norm possess? Define the three vector norms $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$. Define the two matrix norms $\|A\|_1$ and $\|A\|_\infty$. Find $\|A\|_1$ and $\|A\|_\infty$ for the matrix A above.

Errors and Conditioning

What are the two sources of error in any numerical method? Define forward error and backward error. Find the relative forward and backward error associated with solving the linear system $Ax = b$. How is this related to the condition number of the matrix? Write the condition number of a matrix in terms of matrix norms. Find the ∞-norm condition number of the matrix A above.

Jacobi's Method and Stationary Iterative Methods

Write down the step equation for Jacobi's method. What are the conditions for convergence? What are the conditions of convergence for an arbitrary stationary iterative method? What is the rate of convergence? What is the computational cost of Jacobi's method? Why might Jacobi's method be a good choice for a sparse matrix?

Polynomial Interpolation

Suppose that you are given the data

$$(x, y) = (0, 1), (1, 2), (2, 0), (3, 1).$$

Write down a system of equations using the Vandermonde matrix that you could solve for the coefficients of the degree at most 3 polynomial $P(x)$ interpolating these data points. Evaluate $P(0.5)$ using the Lagrange interpolating polynomials. Evaluate $P(0.5)$ using Newton's divided differences. What are the advantages and disadvantages of each approach?