Name: Solutions

This exam contains 6 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total:	100	

1. (15 points) Determine if each of the following statements is true or false and give a short justification for your choice.

   (a) The bisection method will always converge to a root provided that the initial interval satisfies $f(a_0) \cdot f(b_0) < 0$.

   True. At each step there must be at least one subinterval on which f changes signs, and therefore contains a root.

   (b) Given a good enough initial guess, fixed point iteration is guaranteed converge to the fixed point $x^* = 1$ of the function $g(x) = \sqrt{2 - x}$.

   True. $g'(x) = -\frac{1}{2}(2-x)^{-1/2}$

   $\Rightarrow |g'(1)| = |-\frac{1}{2}| = \frac{1}{2} < 1.$

   (c) An iterative method that converges quadratically eventually satisfies $\varepsilon_{i+1}^2 \approx M\varepsilon_i$ for some constant $M < \infty$.

   False. It eventually satisfies $\varepsilon_{i+1} \approx M \varepsilon_i$.

2. (15 points) Provide a short response to each of the following questions.

(a) An iterative method generates a sequence of scalars $\{x_1, x_2, \ldots\}$. What is a stopping criterion that ensures x_n has approximately 6 correct decimal places?

$$|x_n - x_{n-1}| < 0.5 \times 10^{-6}.$$

(b) Consider the following Matlab code. What is a single Matlab statement, not using a `for` loop, which performs the same operation?

```
for k = 2:10
  x(k) = x(k)/x(k-1);
end
```

There is a typo in this question. Should be y(x).

$\Rightarrow$ y(2:10) = x(2:10)./x(1:9).

(c) In terms of $\varepsilon_{\mathbf{mach}}$, what do you know about the relative error of the floating point approximation to any real number using the round-to-nearest rule?

The relative error is at most $\varepsilon_{mach}/2$.

3. (15 points) Determine the exact decimal representation of $(0.1\overline{001})_2$. *Note*: Only the values '001' repeat in the fractional part.

Let $x = 0.1\overline{001}$.

Then $2^4 x = 1001.\overline{001}$

$$- \quad 2x = 1.\overline{001}$$

$$(2^4 - 2)x = (1001)_2 - (1)_2$$

$$x = \frac{9-1}{2^4-2} = \frac{8}{14} = \boxed{\frac{4}{7}}.$$

4. (15 points) Using the round-to-nearest rule, determine the double precision floating point representation of $\frac{27}{8} = (3.375)_{10}$.

$$(3.375)_{10} = (11.011)_2$$

$$\Rightarrow \boxed{1.1011 \times 2^1}$$

5. (20 points) The function $f(x) = x^3 - 3x^2 + 2x$ has the roots $r = 0, 1, 2$. Find a function $g(x)$ for which fixed point iteration will converge locally to all three roots.

The easiest choice is the $g(x)$ from Newton's method:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{x^3 - 3x^2 + 2x}{3x^2 - 6x}.$$

6. (20 points) Determine how many evaluations of the objective function are required to ensure that your approximation of a root has absolute error less than 10^{-8} when using the bisection method with an initial interval of width 1. *Hint*: $\log_2(10^{-8}) \approx -26.58$.

Maximum error after step k is $1/2^{k+1}$.

$$\Rightarrow 1/2^{k+1} < 10^{-8}$$

$$-k-1 < \log_2(10^{-8}) \approx -26.58$$

$$k > 25.58 \Rightarrow 26 \text{ steps.}$$

Number of evaluations after k steps is $k+2$.

$$\Rightarrow \boxed{28 \text{ evaluations}}$$