Remember to adequately label all plots and include any MATLAB scripts and functions with your solutions. A clear and complete presentation of your solutions is required for full credit.

1. Sauer, Section 2.1, Exercise 2(c).

2. Sauer, Section 2.1, Exercise 5.

3. Write the following MATLAB functions.

   (a) `function [L,U] = naive_gauss(A)` that returns the unit lower triangular matrix L and upper triangular matrix U such that $A = LU$ obtained by Gaussian elimination without partial pivoting.

   (b) `function x = utri_solve(U,b)` that solves the upper triangular system $U\boldsymbol{x} = \boldsymbol{b}$ using backward substitution.

   (c) `function x = ltri_solve(L,b)` that solves the lower triangular system $L\boldsymbol{x} = \boldsymbol{b}$ using forward substitution.

   Use your functions to solve the system from Problem 1. Remember, to use the LU decomposition to solve $A\boldsymbol{x} = \boldsymbol{b}$, you must first solve $L\boldsymbol{y} = \boldsymbol{b}$ then solve $U\boldsymbol{x} = \boldsymbol{y}$.

4. This problem explores the relationship between condition number and the volume of a random parallelepiped in 4-dimensional space whose sides are unit vectors. Let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$ be vectors which emanate from a given vertex of the parallelepiped and describe its sides. By a well-known result from geometry, the volume of the parallelepiped is then $|\det(A)|$, where A is the matrix having $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$ as its columns. Your program for this problem should do the following.

   (a) Generate 4 random column vectors of size 4-by-1 by viewing them as the columns of a 4-by-4 random matrix `A = rand(4)`.

   (b) Normalize the columns using the command `A = A*diag(1./sqrt(sum(A.*A)))`. Provide some comments to explain what this operation does.

   (c) Compute the volume of the parallelepiped defined by the normalized vectors. Your program must use the LU factorization for computing the determinant (i.e. do not use the command `det(A)`).

   (d) Compute the condition number $\kappa(A)$ with the command `cond(A)`.

   (e) Repeat the process 1000 times, and produce a scatter plot of $|\det(A)|$ versus $1/\kappa(A)$.

   What are the maximum and minimum values found for $|\det(A)|$? What is the condition number corresponding to each of these values? How can you explain the extreme values of the condition number and the corresponding values of the determinant?