Name: Solutions

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	25	
3	25	
4	30	
Total:	100	

1. For each of the statements that follow, answer 'true' if the statement is always true and 'false' otherwise. In the case of a true statement, explain or prove your answer. In the case of a false statement, give an example to show that the statement is not always true.

   (a) (5 points) Observing a line on a plot with logarithmic axes for both x and y indicates that y is proportional to an exponential function of x.

   False. Observing a line on a loglog plot indicates that y is proportional to a power of x.

   (b) (5 points) A stationary iterative method will converge if the matrix A for the given linear system $A\mathbf{x} = \mathbf{b}$ has a spectral radius less than one.

   False. It is the spectral radius of the iteration matrix that needs to be less than one.

   (c) (5 points) Using the Chebyshev points will always yield a more accurate interpolating polynomial than using equally spaced points.

   False. Interpolating a constant function yields the same polynomial regardless of points, for instance.

   (d) (5 points) The unique interpolating polynomial for n data points at distinct values of x will always have degree $n - 1$.

   False. The interpolating polynomial may have degree less than n-1 as well.

2. Complete the following steps to evaluate the interpolating polynomial $P(x)$ for the three data points $(0,2)$, $(2,4)$, and $(4,0)$ using Newton's divided differences.

   (a) (10 points) Find the divided differences $f[x_1]$, $f[x_1, x_2]$, and $f[x_1, x_2, x_3]$.

X	Y
0	$\boxed{2} = f[x_1]$
2	4
4	0

$$\frac{4-2}{2-0} = \boxed{1} = f[x_1, x_2]$$

$$\frac{0-4}{4-2} = -2$$

$$\frac{-2-1}{4-0} = \boxed{-\frac{3}{4}} = f[x_1, x_2, x_3]$$

   (b) (10 points) Find the Newton polynomials $p_1(x)$, $p_2(x)$, and $p_3(x)$.

$$p_1(x) = 1$$

$$p_2(x) = x$$

$$p_3(x) = x(x-2)$$

   (c) (5 points) Use your solutions from the previous steps to evaluate $P(1)$.

$$P(1) = f[x_1]p_1(1) + f[x_1,x_2]p_2(1) + f[x_1,x_2,x_3]p_3(1)$$

$$= 2(1) + 1(1) - \frac{3}{4}(1(1-2))$$

$$= \frac{15}{4}.$$

3. Consider the three data points $(1, 2)$, $(2, 4)$, and $(3, 8)$. Let $\mathcal{L}(x)$ be the linear spline function

$$\mathcal{L}(x) = \begin{cases} 2 + a_1\,(x-1) & \text{for } 1 < x < 2, \\ 4 + a_2\,(x-2) & \text{for } 2 < x < 3. \end{cases}$$

(a) (10 points) Determine the coefficients a_1 and a_2 if $\mathcal{L}(x)$ interpolates the data.

$$a_1 = \frac{4-2}{2-1} = 2.$$

$$a_2 = \frac{8-4}{3-2} = 4.$$

(b) (15 points) Suppose that the data are sampled from a function $f(x)$ satisfying

$$|f''(x)| < 2 \quad \text{for} \quad 1 < x < 3.$$

Find an upper bound for the interpolation error at $x = 1.5$.
Hint: On each subinterval, the spline is an interpolating polynomial for the two data points at its ends.

$$|f(x) - \mathcal{L}(x)| = \frac{|(x-1)(x-2)|}{2}|f''(c)|$$

(where $1 < c < 2$)

$$< |(x-1)(x-2)|.$$

$$\Rightarrow |f(1.5) - \mathcal{L}(1.5)| < |(1.5-1)(1.5-2)|$$

$$= 0.25.$$

4. Consider solving the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 7 & 1 & 5 \\ 4 & 9 & 3 \\ 2 & 5 & 8 \end{pmatrix}.$$

(a) (20 points) A general stationary iterative method has the form $\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}$. Find the iteration matrix B for Jacobi's method.

$$B = -D^{-1}(L+U)$$

$$= -\begin{pmatrix} 1/7 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/8 \end{pmatrix} \begin{pmatrix} 0 & 1 & 5 \\ 4 & 0 & 3 \\ 2 & 5 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1/7 & -5/7 \\ -4/9 & 0 & -1/3 \\ -1/4 & -5/8 & 0 \end{pmatrix}.$$

(b) (10 points) Will Jacobi's method converge for the given matrix A? Explain why or why not.

Yes, because A is strictly diagonally dominant.