Name: Solutions

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
4	15	
5	15	
6	40	
Total:	100	

1. (10 points) Determine the values of $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ for the vector $\mathbf{x} = (-3, 2, 0)$.

$$\|x\|_1 = 3 + 2 + 0 = 5$$

$$\|x\|_2 = (3^2 + 2^2 + 0^2)^{1/2} = \sqrt{13}$$

$$\|x\|_\infty = \max(3, 2, 0) = 3.$$

2. (10 points) You have written a program to solve the scalar problem $2x = b$ for x given the right hand side $b = 8$. It produces the approximate solution $\tilde{x} = 2$. What are the relative forward and backward errors?

$$\begin{cases} \tilde{b} = 4 \\ x = 4 \end{cases} \Rightarrow$$

relative forward error $= \dfrac{x - \tilde{x}}{x} = \dfrac{2}{4} = \dfrac{1}{2}$

relative backward error $= \dfrac{b - \tilde{b}}{b} = \dfrac{4}{8} = \dfrac{1}{2}.$

3. (10 points) A 3×3 matrix A has eigenvalues $\lambda = 0.1, 0.2, 0.3$. What can you say about the convergence of Jacobi's method on the linear system $A\mathbf{x} = \mathbf{b}$?

Nothing.

4. (15 points) Let $P(x)$ be the polynomial of degree at most 2 interpolating the points $(-1,3)$, $(0,1)$, $(2,3)$. Use the method of your choice to find $P(1)$.

Using Lagrange interpolation:

$$\ell_1(1) = \frac{(1-0)(1-2)}{(-1-0)(-1-2)} = \frac{-1}{3}$$

$$\ell_2(1) = \frac{(1-(-1))(1-2)}{(0-(-1))(0-2)} = \frac{-2}{-2} = 1$$

$$\ell_3(1) = \frac{(1-(-1))(1-0)}{(2-(-1))(2-0)} = \frac{2}{6} = \frac{1}{3}$$

$$\Rightarrow P(1) = -\frac{1}{3}(3) + 1(1) + \frac{1}{3}(3) = 1.$$

5. (15 points) Compute the ∞-norm[1] condition number of the matrix $A = \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix}$.

$$\|A\|_\infty = 7$$

$$A^{-1} = \begin{pmatrix} -1 & 2 \\ -3 & 5 \end{pmatrix} \Rightarrow \|A^{-1}\|_\infty = 8$$

$$\Rightarrow \operatorname{cond}(A) = \|A\|_\infty \|A^{-1}\|_\infty = 56.$$

[1]The ∞-norm of a matrix is the maximum absolute row sum.

6. Consider the linear system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}.$$

(a) (20 points) Find the solution $\mathbf{x}$ using Gaussian elimination with partial pivoting and back substitution.

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & 2 & 1 & 6 \\ 1 & 2 & 3 & 9 \end{array}\right) \xrightarrow[\;]{\text{swap } r1 + r2} \left(\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 1 & 2 & 1 & 5 \\ 1 & 2 & 3 & 9 \end{array}\right)$$

$$\xrightarrow{\substack{r2 = r2 - \frac{1}{2}r1 \\ r3 = r3 - \frac{1}{2}r1}} \left(\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 0 & 1 & 1/2 & 2 \\ 0 & 1 & 5/2 & 6 \end{array}\right)$$

$$\xrightarrow{r3 = r3 - r2} \left(\begin{array}{ccc|c} 2 & 2 & 1 & 6 \\ 0 & 1 & 1/2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right)$$

$$\Rightarrow \quad x_3 = 4/2 = 2$$

$$x_2 = 2 - \frac{1}{2}(2) = 1$$

$$x_1 = \frac{6 - 2(1) - 1(2)}{2} = 1$$

(b) (10 points) Find a permutation matrix P, a unit lower triangular matrix L, and an upper triangular matrix U based on the previous part such that $PA = LU$.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 2 \end{pmatrix}.$$

(c) (10 points) Explain how you can use the matrices P, L, and U from the previous part to solve the linear system.

① Solve $Lc = Pb$ for c using forward substitution.

② Solve $Ux = c$ for x using back-substitution.