Remember to adequately label all plots and include any requested code listings with your solutions. *Only include those scripts and functions which are requested.* Show your work for problems that you do by hand. A clear and complete presentation of your solutions is required for full credit.

1. *Initial Value Problems.* This homework is intended to demonstrate that there is a difference between good and not so good numerical methods for solving initial value problems.

   Consider the second order IVP for the function z(t) given by

   $$z'' + 0.05\, z' + z^3 = 7.5\cos(t), \quad z(0) = 0, \quad z'(0) = 1.$$

   Your assignment is to compute $z(70)$ as accurately as your computer allows.

   (a) By introducing the new variables $u_1 = z$ and $u_2 = z'$, rewrite the IVP as a system $\boldsymbol{u}' = f(t, \boldsymbol{u}(t))$, where $\boldsymbol{u} = (u_1, u_2)$.

   (b) Try the following two methods, with a variety of step sizes h:

   - Euler's Method:
   $$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + h\, f(t_n, \boldsymbol{u}_n).$$
   - Fourth Order Runge-Kutta:
   $$\begin{aligned} \boldsymbol{k}_1 &= \boldsymbol{f}(t_n, \boldsymbol{u}_n), \\ \boldsymbol{k}_2 &= \boldsymbol{f}(t_n + \tfrac{h}{2}, \boldsymbol{u}_n + \tfrac{h}{2}\,\boldsymbol{k}_1), \\ \boldsymbol{k}_3 &= \boldsymbol{f}(t_n + \tfrac{h}{2}, \boldsymbol{u}_n + \tfrac{h}{2}\,\boldsymbol{k}_2), \\ \boldsymbol{k}_4 &= \boldsymbol{f}(t_n + h, \boldsymbol{u}_n + h\,\boldsymbol{k}_3), \\ \boldsymbol{u}_{n+1} &= \boldsymbol{u}_n + \tfrac{h}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4). \end{aligned}$$

   Use the values

   $$u_1^{\text{ex}}(70) = 1.582857756103056, \quad u_2^{\text{ex}}(70) = -2.835763853877514$$

   as "exact" and plot the mean-square error $\varepsilon = \sqrt{(u_1 - u_1^{\text{ex}})^2 + (u_2 - u_2^{\text{ex}})^2}$ at $t = 70$ as a function of h.

   (c) If h is small enough, the error is expected to follow a power law $\varepsilon \approx ch^p$. Determine c and p from your plot and estimate the number of time steps required to reach $\varepsilon < 10^{-5}$ with Euler's method. Is your computer fast enough? Discuss your results.

   Note: Unless you take h small enough, neither of these two methods will be stable. To be on the safe side (inside the stability region), take $h < 8.75 \times 10^{-3}$ for Euler and $h < 0.35$ for Runge-Kutta.