Name: Solutions

This exam contains 5 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any graphing calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Determine if each of the following statements is true or false and give a short justification for your choice.

(a) The bisection method will always converge to a root provided that the initial interval satisfies $f(a_0) \cdot f(b_0) < 0$.

True. The intermediate value theorem guarantees that at each step there is a root in the chosen subinterval, and the size of the subintervals shrinks to zero.

(b) Given a good enough initial guess, fixed point iteration is guaranteed converge to the fixed point $x^* = 1$ of the function $g(x) = \sqrt{2-x}$.

True. $g'(x) = -\frac{1}{2}(2-x)^{-1/2}$ and so $|g'(1)| = \frac{1}{2} < 1$.

(c) An iterative method that converges quadratically eventually satisfies $\varepsilon_{i+1}^2 \approx M\varepsilon_i$ for some constant $M < \infty$.

False. It satisfies $\varepsilon_{i+1} \approx M\varepsilon_i^2$.

2. (25 points) Provide a short response to each of the following questions about iterative methods.

   (a) An iterative method generates a sequence of scalars $\{x_1, x_2, \ldots, x_n\}$. What is a stopping criterion that ensures x_n has approximately 10 correct decimal places?

   $$|x_n - x_{n-1}| < 0.5 \times 10^{-10}.$$

   (b) You observe that an iterative method is increasing the accuracy of an approximate solution by 3 digits per iteration. What is the order of convergence?

   Linear.

   (c) What are three trade-offs you might consider when choosing one iterative method over another?

   Cost, Convergence, Stability, etc.

3. (25 points) The function $f(x) = x - x^3$ has the roots $r = -1, 0, 1$. Find a function $g(x)$ for which fixed point iteration will converge locally to all three roots.

There are many possible solutions, but the easiest way is to use the Newton-Raphson iteration:

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{x - x^3}{1 - 3x^2}$$

$$= \frac{-2x^3}{1 - 3x^2}.$$

4. (25 points) Determine how many evaluations of the objective function are required in order to ensure that your approximation has absolute error less than 10^{-6} when using the bisection method with an initial interval of width 1. *Hint*: $\log_2(10^{-6}) \approx -19.93$.

iter. #	max. error	total f'n evals.
0	$1/2$	2 $(f(a_0), f(b_0))$
1	$1/4$	3 $(f(c_0))$
2	$1/8$	4 $(f(c_1))$
⋮	⋮	⋮
n	$1/2^{n+1}$	$2+n$

$$\Rightarrow \quad 1/2^{n+1} < 10^{-6}$$

$$-(n+1) < \log_2 10^{-6}$$

$$n > -\log_2 10^{-6} - 1 \approx 18.93$$

$\Rightarrow$ $n = 19$. is the closest integer.

$\Rightarrow$ $2+n = 21$ function evaluations are required.