This homework is due on Thursday, May 5.

1. Assume $F(0) = 0$, that u is a continuous integral solution of the conservation law

$$\begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

and that u has compact support in $\mathbb{R} \times [0, \infty]$. Prove that

$$\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} g(x)\,dx$$

for all $t > 0$.

2. Find explicitly the entropy solution of the conservation law

$$\begin{cases} u_t + (\frac{u^2}{2})_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1. \end{cases}$$

Draw a few pictures documenting your answer, being sure to illustrate what happens for all times $t > 0$.

3. Find explicitly the entropy solution of the conservation law

$$\begin{cases} u_t + (u^2 + u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

for

$$g(x) = \begin{cases} 1 & \text{if } x < 0 \\ -3 & \text{if } x > 0 \end{cases}$$

4. Find explicitly the entropy solution of the conservation law

$$\begin{cases} u_t + (\frac{u^3}{3})_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

for

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ -2 & \text{if } x > 0 \end{cases}$$

Make sure that your solution satisfies the entropy condition. *Hint*: The solution should involve a rarefaction wave of the form $v(\frac{x}{t})$.