

1. Let  $U = \{x \in \mathbb{R}^n \mid |x| > 1\}$  be the exterior of the unit ball in  $\mathbb{R}^n$  for  $n \geq 2$ . Show that there are infinitely many solutions to the problem

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = 1 & \text{on } \partial U. \end{cases}$$

*Hint:* Find an infinite family of solutions! You will need to handle the cases  $n = 2$  and  $n \geq 3$  separately.

2. Let  $U$  be the wedge  $\{x \in \mathbb{R}^2 \mid x_1 > x_2, x_2 > 0\} \subset \mathbb{R}^2$ . Find the Green's function  $G(x, y)$  in terms of the fundamental solution  $\Phi$ .

*Hint:* Start building your corrector by cancelling the boundary values of  $\Phi$  on the half-line  $x_1 > 0, x_2 = 0$ . Add the required term to your corrector and check the boundary values of the resulting function on the half-line  $x_1 > 0, x_2 = x_1$ . Cancel these values. Repeat this process, alternating sections of the boundary, until you've eliminated the boundary values everywhere.

3. Suppose  $f \in C_c^2(\mathbb{R})$ . Following the proof of Theorem 1 in Evans, prove that

$$u(x) = \int_{-\infty}^{\infty} \Phi(x - y) f(y) dy = \int_{-\infty}^{\infty} \Phi(y) f(x - y) dy$$

solves  $-u_{xx} = f$  in  $\mathbb{R}$ , where  $\Phi(x) = -\frac{1}{2}|x|$  is the fundamental solution.

4. Let  $U$  be the unit interval  $(0, 1) \subset \mathbb{R}$ . Recall that the Green's function is given by

$$G(x, y) = \frac{1}{2}(x + y - |y - x|) - xy.$$

Use this to find an explicit solution the problem

$$\begin{cases} -u_{xx} = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

5. Again, let  $U$  be the unit interval  $(0, 1) \subset \mathbb{R}$ . Consider the problem

$$\begin{cases} -u_{xx} + u = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

- Find the fundamental solution  $\Phi(x)$  in  $\mathbb{R}$ .
- Find the Green's function  $G(x, y)$  in  $U$ .
- Use the Green's function to find an explicit solution.