1. Prove the following properties of the Fourier transform in $\mathbb{R}$.
   (a) If u is purely real, then $\hat{u}(-y) = \overline{\hat{u}(y)}$.
   (b) If u is purely real and even, then $\hat{u}$ is purely real and even.

2. Find the Fourier transform in $\mathbb{R}$ of the function

$$u(x) = \begin{cases} 1 - \frac{|x|}{a} & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$$

3. Use the Fourier transform to find a solution of

$$u'' + xu' + u = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

   Hint: First show that $(x\,u'(x))\hat{}\, = -(\hat{u}(y) + y\,\hat{u}'(y))$.

4. Use the Fourier transform to find an explicit formula for the solution of

$$\begin{cases} u_t - u_{xx} + \alpha u = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

   where $g \in L^2(\mathbb{R})$ and $\alpha \in \mathbb{R}$ is a constant.

5. Consider the initial value problem

$$\begin{cases} u_t = au_x + bu_{xx} + cu_{xxxx} + f & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

   where $a, b, c \in \mathbb{R}$ are constant and $f, g \in L^2(\mathbb{R})$.
   (a) Find all values of a, b, c for which this problem is well-posed in $L^2(\mathbb{R})$. That is, $\|u(\cdot, t)\|_{L^2(\mathbb{R})} < \infty$ for all $t \geq 0$.
   (b) Find an explicit formula for the solution of the problem.