

This homework is due on Tuesday, March 29.

1. Let us think a bit more about finding Green's functions for wedge-shaped domains.
  - (a) Try to use the method from last week's homework to find the Green's function for the wedge  $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 > -x_2, x_2 > 0\}$ . Can you find a satisfactory corrector function? If not, explain why.
  - (b) Under what conditions on the angle at the vertex of the wedge can you use this method to find a Green's function?
2. Prove the following properties of the Fourier transform in  $\mathbb{R}^n$ .
  - (a)  $(u(\mathbf{x} - \mathbf{z}))^\wedge = e^{-i\mathbf{z} \cdot \mathbf{y}} \hat{u}(\mathbf{y})$ .
  - (b)  $(u(a\mathbf{x}))^\wedge = \frac{1}{|a|^n} \hat{u}\left(\frac{\mathbf{y}}{a}\right)$ .
3. Prove the following properties of the Fourier transform in  $\mathbb{R}$ .
  - (a) If  $u$  is purely real, then  $\hat{u}(-y) = \overline{\hat{u}(y)}$ .
  - (b) If  $u$  is purely real and even, then  $\hat{u}$  is purely real and even.
4. Find the Fourier transform in  $\mathbb{R}$  of the function

$$u(x) = \begin{cases} 1 - \frac{|x|}{a} & \text{if } |x| \leq a \\ 0 & \text{if } |x| > a \end{cases}$$

5. Use the Fourier transform to find a solution of

$$u'' + xu' + u = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

*Hint:* First show that  $(xu'(x))^\wedge = -(\hat{u}(y) + y\hat{u}'(y))$ .

6. Use the Fourier transform to find an explicit formula for the solution of

$$\begin{cases} u_t - u_{xx} + \alpha u = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where  $g \in L^2(\mathbb{R})$  and  $\alpha \in \mathbb{R}$  is a constant.