Name: Solutions

This exam contains 5 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work for each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	15	
4	15	
5	25	
6	25	
Total:	100	

1. (10 points) Are solutions to Poisson's equation in $\mathbb{R}^n$ unique? Explain why or why not.

No. You can add any harmonic function and still have a solution.

2. (10 points) Let $U \subset \mathbb{R}^n$ be open and bounded. Fix a final time $T > 0$ and suppose that u solves the heat equation in the parabolic cylinder U_T. If the maximum of u in $\bar{U}_T$ is attained in U at the final time T, is there anything more you can say about the solution?

Since the maximum of u in $\bar{U}_T$ occurs in U_T, the strong maximum principle requires u to be constant.

3. (15 points) Let u be harmonic in $\mathbb{R}^2$ with $\int_{B(0,1)} u(x)\,dx = 1$. Find $\int_{\partial B(0,1)} u(x)\,dx$.

By the mean-value property,

$$\fint_{B(0,1)} u(x)\,dx = \fint_{\partial B(0,1)} u(x)\,dx.$$

$$\Rightarrow \frac{1}{\pi}\int_{B(0,1)} u(x)\,dx = \frac{1}{2\pi}\int_{\partial B(0,1)} u(x)\,dx$$

$$\Rightarrow \int_{\partial B(0,1)} u(x)\,dx = 2.$$

4. (15 points) Show that the equation $u_{\xi\eta} = 0$ becomes $u_{tt} - u_{xx} = 0$ under the change of variable $x = \xi - \eta$ and $t = \xi + \eta$.

$$u_\xi = u_t t_\xi + u_x x_\xi$$

$$= u_t + u_x$$

$$u_{\xi\eta} = (u_t + u_x)_\eta$$

$$= u_{tt} t_\eta + u_{tx} x_\eta + u_{xt} t_\eta + u_{xx} x_\eta$$

$$= u_{tt} - \cancel{u_{tx}} + \cancel{u_{tx}} - u_{xx}$$

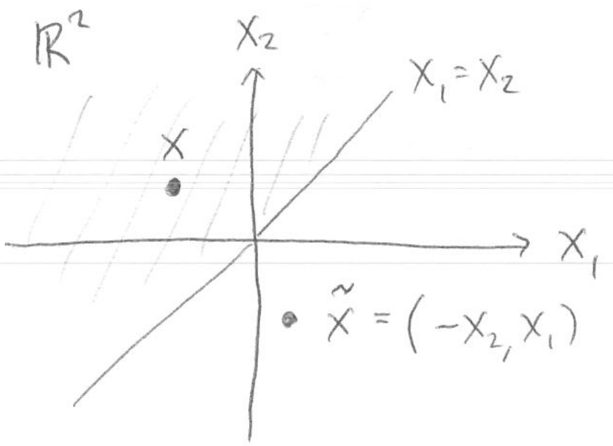
$$= u_{tt} - u_{xx} = 0.$$

5. Let $U \subset \mathbb{R}^2$ be open (but not necessarily bounded) and let Φ be the fundamental solution of Laplace's equation in $\mathbb{R}^2$.

   (a) (5 points) Fix $x \in U$. What problem does the corrector function ϕ^x solve?

$$\begin{cases} \Delta \phi^x = 0 & \text{in } U \\ \phi^x = \Phi(x-y) & \text{on } \partial U \end{cases}$$

   (b) (20 points) Find ϕ^x for the half-space $U = \{x \in \mathbb{R}^2 \mid x_1 < x_2\}$. Be sure to define any notation that you use.



Let $\tilde{x} = (-x_2, x_1)$ be the reflection of x across the line $x_1 = x_2$.

Then $\phi^x(y) = \Phi(\tilde{x} - y)$.

6. Let $U \subset \mathbb{R}^n$ be open and bounded. Consider the initial value problem

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } U \times (0, \infty) \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

where f and g are given functions and $c > 0$.

(a) (5 points) Suppose u and v are two solutions. What problem does $w = u - v$ solve?

$$\begin{cases} w_t - \Delta w + cw = 0 & \text{in } U \times (0,\infty) \\ w = 0 & \text{on } U \times \{t=0\}. \end{cases}$$

(b) (20 points) Differentiate the energy

$$e(t) = \int_U w^2(x,t)\, dx$$

to show that $w \equiv 0$, and so the original problem has a unique solution.

$$\frac{de}{dt}(t) = 2\int_U w w_t \, dx$$

$$= 2\int_U w(\Delta w - cw)\, dx$$

$$= -2\int_U (|Dw|^2 + cw^2)\, dx + \cancel{\int_{\partial U} w \frac{\partial w}{\partial \nu} dS}^{\,0}$$

$$\leq 0.$$

But $e(0) = 0$ and $e'(t) \leq 0$, so $e(t) = 0$ for all $t \geq 0$. Thus $w \equiv 0$ for all $t \geq 0$.

Note: Formally, you also need a zero B.C. for w.