1. Let $U = \{x \mid |x| > 1\}$ be the exterior of the unit ball in $\mathbb{R}^n$ for $n \geq 2$. Show that there are infinitely many solutions to the problem

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = 1 & \text{on } \partial U. \end{cases}$$

Hint: Find an infinite family of solutions! You will need to handle the cases $n = 2$ and $n \geq 3$ separately.

2. Let U be the wedge $\{x \in \mathbb{R}^2 \mid x_1 > x_2,\ x_2 > 0\} \subset \mathbb{R}^2$. Find the Green's function $G(x, y)$ in terms of the fundamental solution Φ.

Hint: Start building your corrector by cancelling the boundary values of Φ on the half-line $x_1 > 0,\ x_2 = 0$. Add the required term to your corrector and check the boundary values of the resulting function on the half-line $x_1 > 0,\ x_2 = x_1$. Cancel these values. Repeat this process, alternating sections of the boundary, until you've eliminated the boundary values everywhere.

3. Let U be the unit interval $(0, 1) \subset \mathbb{R}$. Recall that the Green's function on is given by

$$G(x, y) = \frac{1}{2}\big(|y - x| - x - y\big) + xy.$$

Use this to find an explicit solution the problem

$$\begin{cases} -u_{xx} = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

4. Again, let U be the unit interval $(0, 1) \subset \mathbb{R}$. Consider the problem

$$\begin{cases} -u_{xx} + u = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

   (a) Find the Green's function $G(x, y)$.

   (b) Use the Green's function to find an explicit solution.

5. (Evans, §2.5, #12) Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

   (a) Show $u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

   (b) Use (a) to show $v(x, t) = x \cdot Du(x, t) + 2tu_t(x, t)$ solves the heat equation as well. *Hint*: Differentiate u_λ with respect to λ.