This homework is due on Thursday, April 14.

1. Use the Kirchoff formula

$$u(x,t) = \fint_{\partial B(x,t)} \big(th(y) + g(y) + Dg(y)\cdot(y-x)\big)\, dS_y$$

to solve the wave equation in three dimensions

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^3 \times (0,\infty), \quad u = g,\ u_t = h \text{ on } \mathbb{R}^3 \times \{t=0\}$$

when $g = 0$ and h is harmonic.

2. Solve the wave equation in three dimensions when $g = 0$ and

$$h = |x|^2 = x_1^2 + x_2^2 + x_3^2.$$

Hint: It may be easier to go back and solve the Euler-Poisson-Darboux equation directly.

3. Where does a solution of the wave equation in three dimensions have to vanish if its initial data g and h vanish inside the unit sphere? What if they vanish outside the unit sphere, but not inside?

4. Find all three-dimensional plane waves. That is, find all solutions of the wave equation in three dimensions of the form $u(x,t) = f(k\cdot x - t)$, where k is a fixed vector and f is a function of one variable.

5. Take $\Omega = (0,\ell) \subset \mathbb{R}$ and suppose we define the evolution equation

$$u_{tt} = -\frac{\delta L}{\delta u}$$

for some function $L(u_x, u, x)$.

(a) Show that the energy

$$\int_0^\ell \left(\frac{1}{2}(u_t)^2 + L(u_x,u,x)\right)dx$$

is conserved.

(b) Find an appropriate function L to give the Fermi-Pasta-Ulam equation

$$u_{tt} = u_{xx}\,(1 + \varepsilon\, u_x).$$