1. In the following way, solve the problem for Poisson's equation on the ball $B(0,1) \subset \mathbb{R}^2$ given by

$$\begin{cases} \Delta u = y & \text{in } B^0(0,1) \\ u = 1 & \text{on } \partial B(0,1). \end{cases}$$

Look for a solution in polar coordinates of the form

$$u(r,\theta) = \frac{1}{2} A_0(r) + \sum_{n=1}^{\infty} \big(A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta) \big),$$

assuming that u is bounded near the origin.

(a) Write the function y and the operator Δ in polar coordinates. Deduce the ODEs that A_0, A_1, B_1, $\ldots$ satisfy along with the corresponding initial conditions.

(b) Solve these initial value problems and substitute the solutions into the ansatz for u. Write your answer as a function of x and y.

Hint: You should obtain the ODE $r^2 B_1'' + rB_1' - B_1 = r^3$ when solving for B_1. This ODE has a particular solution of the form $B_1(r) = c\,r^3$. Use this fact to obtain the general solution to the ODE.

2. (Evans §2.5, #3) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g\,dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f\,dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ u = g & \text{on } \partial B(0,r). \end{cases}$$

3. (Evans §2.5, #4) We say $v \in C^2(\bar{U})$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v\,dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove v is subharmonic.

(d) Prove $v = |Du|^2$ is subharmonic, whenever u is harmonic.