1. (Evans §2.5, #15) Given $g : [0, \infty) \to \mathbb{R}$. with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} \, e^{\frac{-x^2}{4(t-s)}} \, g(s) \, ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t = 0\} \\ u = g & \text{on } \{x = 0\} \times [0, \infty) \end{cases}$$

Hint: Let $v(x,t) = u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.

2. (Evans §2.5, #16) Give a direct proof (i.e. one that does not rely on the mean-value property) that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

Hint: Define $u_\varepsilon = u - \epsilon t$ for $\varepsilon > 0$, and show that u_ε cannot attain its maximum over $\bar{U}_T$ at a point in U_T.

3. (Evans §2.5, #17) We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \le 0 \quad \text{in } U_T.$$

   (a) Prove for a subsolution v that

$$v(x,t) \le \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \, \frac{|x-y|^2}{(t-s)^2} \, dyds$$

   for all $E(x,t;r) \subset U_T$. *Note*: This was first proven in 1973!

   (b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

   (c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v = \phi(u)$. Prove v is a subsolution.

   (d) Prove $v = |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.