This homework is due on Thursday, April 21.

1. Using integration by parts, find a formula involving partial derivatives for the variational derivative of $L(u_{xx}, u_x, u, x)$ depending additionally on u_{xx}. What boundary terms need to vanish in order to properly define the variational derivative?

2. Find an energy other than $\int u\,dx$ or $\int u^2\,dx$ that is either conserved or dissipated for each of the following equations.
   (a) $u_t = u - u^3 + u_{xx}$
   (b) $u_t = 2u - u^2 - u^3 - 2u_{xx} - u_{xxxx}$
   (c) $u_{tt} = (u_x)^2\,u_{xx}$

3. Consider the Korteweg-de Vries equation in the form
$$u_t = -\frac{\partial}{\partial x}\left(3u^2 + u_{xx}\right).$$
   (a) Substitute $u = w - \varepsilon w_x - \varepsilon^2 w^2$ and show that u solves the KdV equation if w solves
$$w_t + 6\left(w - \varepsilon^2 w^2\right)w_x + w_{xxx} = 0.$$
   (b) Show that $\int w\,dx$ is conserved.

4. A function u is said to be a weak solution to the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ in the open set $\Omega \subset \mathbb{R}^2$ if for every smooth function v with compact support in Ω we have
$$\iint_\Omega u(v_{tt} - v_{xx})\,dx\,dt = 0.$$
   (a) Show that $u \in C^2(\Omega)$ is a weak solution of the one-dimensional wave equation if it is a classical solution.
   (b) Let $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. Show that $u = H(x - t)$ is a weak solution in $\mathbb{R}^2$.

5. Recall that using the Cole-Hopf transformation, we were able to solve the viscous Burgers' equation
$$u_t - au_{xx} + uu_x = 0$$
for $a > 0$ by solving the heat equation
$$w_t - aw_{xx} = 0$$
and applying the map $u = -2a(w_x/w)$. Check that
$$w(x,t) = 1 + b\exp(-kx + ak^2 t)$$
for $b > 0$ is a solution of the heat equation. What solution of Burgers' equation does it correspond to? Describe this solution qualitatively (velocity, amplitude, etc.) in terms of its parameters.