

- The partial differential equation  $u_{tt} = c^2 \Delta u - q(x)u$  arises in the study of wave propagation in a nonhomogeneous elastic medium, where  $q(x)$  is non-negative and represents the coefficient of elasticity at  $x$ .
  - Define an appropriate notion of energy for solutions, and justify your choice by showing that the energy is constant under reasonable assumptions about boundary conditions.
  - Use the energy result to show that solutions are uniquely determined by their initial condition.
- Consider a flexible beam with clamped ends at  $x = 0$  and  $x = 1$ . Small amplitude waves in this beam satisfy the problem

$$\begin{cases} u_{tt} + \gamma^2 u_{xxxx} = 0 & \text{in } (0, 1) \times (0, \infty) \\ u(x, t) = u_x(x, t) = 0 & \text{on } \{x = 0, 1\} \times (0, \infty), \end{cases}$$

where  $\gamma^2$  is a constant depending on the shape of the beam and the elastic material in the beam. Show that the following energy function is conserved:

$$e(t) = \frac{1}{2} \int_0^1 (u_t^2 + \gamma^2 u_{xx}^2) dx.$$

- (Evans §2.5, #24) Let  $u \in C^2(\mathbb{R} \times [0, \infty))$  solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose  $g, h$  have compact support. The *kinetic energy* is  $k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(x, t) dx$  and the *potential energy* is  $p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(x, t) dx$ . Prove

- $k(t) + p(t)$  is constant in  $t$ ,
  - $k(t) = p(t)$  for all large enough times  $t$
- Let  $u$  solve

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

where  $g, h$  are smooth functions and have compact support. Show there exists a constant  $C$  such that

$$|u(x, t)| \leq C/t \quad (x \in \mathbb{R}^3, t > 0).$$