This homework is due on Thursday, March 10.

Define $\tilde{\mathbf{x}}$ as the reflection of $\mathbf{x}$ about the plane $\partial\mathbb{R}^2_+$ and $\mathbf{x}^*$ as the inversion of $\mathbf{x}$ about the unit circle $\partial B(0,1)$. We used $\tilde{\mathbf{x}}$ for both in class. Create additional notation as required.

1. Use the Poisson formula for the unit ball $B(0,1) \subset \mathbb{R}^2$ given by

$$u(\mathbf{x}) = \frac{1-|\mathbf{x}|^2}{2\pi} \int_{\partial B(0,1)} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\, ds_{\mathbf{y}}$$

for $\mathbf{x} \in B^0(0,1)$ to prove that

$$\frac{1-|\mathbf{x}|}{1+|\mathbf{x}|}\, u(0) \le u(\mathbf{x}) \le \frac{1+|\mathbf{x}|}{1-|\mathbf{x}|}\, u(0)$$

whenever u is positive and harmonic in $B^0(0,1)$.

Hint: You must use the triangle inequality and the mean-value property.

2. In a previous homework, you found the harmonic function

$$u(\mathbf{x}) = a \arctan\left(\frac{x_1}{x_2}\right) + b$$

for $\mathbf{x} = (x_1, x_2)$ depending only on the quotient $\frac{x_1}{x_2}$.

(a) Find the boundary data $\lim_{x_2 \to 0^+} u(\mathbf{x})$.

(b) Use your result as boundary data for the Poisson formula

$$u(\mathbf{x}) = \frac{x_2}{\pi} \int_{\partial\mathbb{R}^2_+} \frac{g(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^2}\, d\mathbf{y} = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{g(y)}{(y-x_1)^2 + x_2^2}\, dy$$

to obtain the same harmonic function $u(\mathbf{x})$ in the upper half-plane $\mathbb{R}^2_+$.

3. Find the harmonic function $u(\mathbf{x})$ in the upper half-plane $\mathbb{R}^2_+$ with

$$u(x,0) = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$$

4. Find the Green's function for the upper half-disc $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1,\ x_2 > 0\}$ in terms of the fundamental solution Φ.

Hint: Start building your corrector by cancelling the boundary values of Φ on the interval $0 < x_1 < 1$, $x_2 = 0$. Add the required term and check the boundary values of the resulting function on the upper half-circle. Cancel these values. Repeat this process, alternating pieces of the boundary, until you've eliminated the boundary values everywhere on the boundary.

5. Find the Green's function for the wedge $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 > x_2,\ x_2 > 0\}$ in terms of the fundamental solution Φ.