This homework is due on Thursday, April 7.

1. Let $\Omega = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_2 \le 1\}$ be the infinite strip of unit width. Solve the boundary-value problem

$$\Delta u = 0 \text{ in } \Omega, \quad u(x_1, 0) = g(x_1), \quad u(x_1, 1) = h(x_1)$$

as a Fourier integral involving $\hat{g}(y)$ and $\hat{h}(y)$.

2. Consider the problem

$$u_t + u_{xxx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \quad u = f \text{ on } \mathbb{R} \times \{t = 0\}.$$

This is a linearized version of the Korteweg-de Vries equation, which is a mathematical model for waves on shallow water surfaces. Find a function $K(x, t)$ in terms of the Airy function

$$\mathrm{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(xz - z^3/3)}\, dz$$

such that a solution to this problem is given by $u(x, t) = K * f$.

3. Let u solve the problem

$$u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty), \quad u = g,\ u_t = h \text{ on } \mathbb{R} \times \{t = 0\}$$

where g and h are smooth and have compact support[1]. Define the kinetic energy and potential energy as

$$k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2\, dx \quad \text{and} \quad p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2\, dx$$

respectively.

   (a) Prove that $e(t) = k(t) + p(t)$ is constant in t. You can follow what we did in class for a bounded domain, except you need to use the d'Alambert solution to justify that the boundary terms disappear.

   (b) Prove that $k(t) = p(t)$ for all large enough times t. This is called the equipartition of energy.

[1] There exists an M so that $g(x) = h(x) = 0$ for all $|x| > M$.