This homework is due on Thursday, March 3.

1. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain. We will prove uniqueness of solutions to the problem

$$\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \alpha u = h \text{ on } \partial\Omega$$

for $\alpha > 0$.

   (a) Determine the problem solved by $w = u - v$ if u and v solve the problem above.

   (b) At the risk of giving away part of the previous answer, you should have

$$\int_\Omega w \Delta w \, dx = 0.$$

   Integrate by parts and use the boundary condition on w to show that $w \equiv 0$ in Ω.

2. Let Ω be the open annulus with inner radius 1 and outer radius 2. Consider the problem of finding u with

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + u = h(\theta) \text{ on } r = 1, \quad \frac{\partial u}{\partial n} + u = 0 \text{ on } r = 2.$$

Find a solution using separation of variables. By the previous problem, this solution is unique.

3. Again, let Ω be the open annulus with inner radius 1 and outer radius 2. Consider the problem of finding u with

$$\Delta u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + 2u = 0 \text{ on } r = 1, \quad \frac{\partial u}{\partial n} - u = 0 \text{ on } r = 2.$$

It should be clear that $u \equiv 0$ is a solution. Find another solution using separation of variables. Thus, uniqueness may fail if $\alpha \leq 0$ anywhere on $\partial\Omega$.

4. Let Ω be the unit interval $(0, 1)$. Recall that the Green's function in Ω is given by

$$G(x, y) = \frac{1}{2}\big(|y - x| - x - y\big) + xy.$$

Use this to solve the problem

$$\Delta u = 1 \text{ in } \Omega, \quad u(0) = 3, \quad u(1) = 2.$$