This homework is due on Tuesday, February 9.

1. Show that $u(x,y)$ defined for $y \geq 0$ by

$$u(x,y) = \begin{cases} -\frac{2}{3}\left(y + \sqrt{3x + y^2}\right) & \text{for } 4x + y^2 > 0 \\ 0 & \text{for } 4x + y^2 < 0 \end{cases}$$

is a weak solution of

$$\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0$$

for the choices $R(u) = u$, $S(u) = \frac{1}{2}u^2$.

2. Consider the Cauchy problem for $u_x^2 + u_y^2 = 1$ subject to the boundary condition $u = 0$ on the circle of radius 1 in the xy-plane.

   (a) You may parameterize the Cauchy data $\Gamma : (f, g, h, \phi, \psi)$ with

   $$f(s) = \cos(s), \quad g(s) = \sin(s), \quad h(s) = 0.$$

   Determine admissible[1] functions $\phi(s)$, $\psi(s)$. You should get two sets of solutions.

   (b) Find the solution $u(x,y)$ for each set of Cauchy data using the characteristic equations. Be careful to make sure your solution satisfies the boundary condition.

3. Recall that

$$F(x,y,z,p,q) = a(x,y,z)\,p + b(x,y,z)\,q - c(x,y,z)$$

for a quasilinear first-order equation. Determine admissible functions ϕ, ψ in terms of f, g, h for the Cauchy data $\Gamma : (f, g, h, \phi, \psi)$. You should be able to express each function as a quotient. What condition guarantees that the denominator remains nonzero?

[1]Recall that Cauchy data are admissible if $F(f, g, h, \phi, \psi) = 0$ and $h' = \phi f' + \psi g'$.