

1. In the following way, solve the problem for Poisson's equation on the ball $B(0, 1) \subset \mathbb{R}^2$ given by

$$\begin{cases} \Delta u = x_2 & \text{in } B^0(0, 1) \\ u = 1 & \text{on } \partial B(0, 1). \end{cases}$$

Look for a solution in polar coordinates $x_1 = r \cos(\theta)$, $x_2 = r \sin(\theta)$ of the form

$$u(r, \theta) = \frac{1}{2} A_0(r) + \sum_{n=1}^{\infty} (A_n(r) \cos(n\theta) + B_n(r) \sin(n\theta)),$$

assuming that u is bounded near the origin.

- (a) Write the function x_2 and the operator Δ in polar coordinates. Deduce the ODEs that A_0 , A_1 , B_1 , \dots satisfy along with the corresponding initial conditions.
- (b) Solve these initial value problems and substitute the solutions into the ansatz for u . Write your answer as a function of x_1 and x_2 .

Hint: You should obtain the ODE $r^2 B_1'' + r B_1' - B_1 = r^3$ when solving for B_1 . This ODE has a particular solution of the form $B_1(r) = c r^3$. Use this fact to obtain the general solution to the ODE.

2. (Evans §2.5, #3) Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \oint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0, r) \\ u = g & \text{on } \partial B(0, r). \end{cases}$$

3. (Evans §2.5, #4) We say $v \in C^2(\bar{U})$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

- (a) Prove for subharmonic v that

$$v(x) \leq \oint_{B(x,r)} v dy \quad \text{for all } B(x, r) \subset U.$$

- (b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.
- (c) Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v = \phi(u)$. Prove v is subharmonic.
- (d) Prove $v = |Du|^2$ is subharmonic, whenever u is harmonic.