1. Let U be the open unit square $(0,1) \times (0,1)$. Consider the problem of trying to find u with

$$\Delta u = 0 \text{ in } U, \quad \frac{\partial u}{\partial \nu} = h \text{ on } \partial U$$

for $h = 0$ on all sides but the bottom, where $h = h(x)$ with $\int_0^1 h(x)\,dx = 0$. Find a family of solutions to this problem using separation of variables.

2. Let U be the annulus with inner radius 1 and outer radius 2. Consider the problem of trying to find u with

$$\Delta u = 0 \text{ in } U, \quad u = g \text{ on } \partial U$$

for $g = 0$ on the inner radius and $g = g(\theta)$ on the outer radius. Find a solution to this problem using separation of variables.

3. Consider the PDE

$$u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0 \quad \text{in } \mathbb{R}^2.$$

(a) Use separation of variables with $u(x,y) = X(x) + Y(y)$ to find a nontrivial solution.

(b) Show that in the polar coordinates r and θ, where $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the PDE becomes

$$u_r^2 u_{rr} + \frac{2}{r^2} u_r u_\theta u_{r\theta} + \frac{1}{r^4} u_\theta^2 u_{\theta\theta} - \frac{1}{r^3} u_r u_\theta^2 = 0.$$

(c) Looking for a solution of the form $u(r,\theta) = r^k \Theta(\theta)$, show that the ODE satisfied by Θ is

$$(\Theta')^2 \Theta'' + \Theta(\Theta')^2(2k^2 - k) + k^3(k-1)\Theta^3 = 0.$$

(d) Using the ODE in the previous part, find a nontrivial solution $u(x,y)$ of the PDE for each of $k = 0, \frac{1}{2}, 1$.