1. Let $U = \{x \in \mathbb{R}^n \mid |x| > 1\}$ be the exterior of the unit ball in $\mathbb{R}^n$ for $n \geq 2$. Show that there are infinitely many solutions to the problem

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = 1 & \text{on } \partial U. \end{cases}$$

Hint: Find an infinite family of solutions! You will need to handle the cases $n = 2$ and $n \geq 3$ separately.

2. Let U be the wedge $\{x \in \mathbb{R}^2 \mid x_1 > x_2,\ x_2 > 0\} \subset \mathbb{R}^2$. Find the Green's function $G(x, y)$ in terms of the fundamental solution Φ.

Hint: Start building your corrector by cancelling the boundary values of Φ on the half-line $x_1 > 0$, $x_2 = 0$. Add the required term to your corrector and check the boundary values of the resulting function on the half-line $x_1 > 0$, $x_2 = x_1$. Cancel these values. Repeat this process, alternating sections of the boundary, until you've eliminated the boundary values everywhere.

3. Suppose $f \in C_c^2(\mathbb{R})$. Following the proof of Theorem 1 in Evans, prove that

$$u(x) = \int_{-\infty}^{\infty} \Phi(x-y) f(y)\, dy = \int_{\infty}^{\infty} \Phi(y) f(x-y)\, dy$$

solves $-u_{xx} = f$ in $\mathbb{R}$, where $\Phi(x) = -\frac{1}{2}|x|$ is the fundamental solution.

4. Let U be the unit interval $(0, 1) \subset \mathbb{R}$. Recall that the Green's function is given by

$$G(x, y) = \frac{1}{2}\big(x + y - |y - x|\big) - xy.$$

Use this to find an explicit solution the problem

$$\begin{cases} -u_{xx} = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

5. Again, let U be the unit interval $(0, 1) \subset \mathbb{R}$. Consider the problem

$$\begin{cases} -u_{xx} + u = 1 & \text{in } U \\ u(0) = 1 \\ u(1) = 1. \end{cases}$$

   (a) Find the fundamental solution $\Phi(x)$ in $\mathbb{R}$, being careful to select the proper constants of integration.

   (b) Find the Green's function $G(x, y)$ in U.

   (c) Use the Green's function to find an explicit solution.