This homework is due on Thursday, March 10.

Define $\tilde{\mathbf{x}}$ as the reflection of $\mathbf{x}$ about the plane $\partial \mathbb{R}^2_+$ and $\mathbf{x}^*$ as the inversion of $\mathbf{x}$ about the unit circle $\partial B(0, 1)$. We used $\tilde{\mathbf{x}}$ for both in class. Create additional notation as required.

1. Use the Poisson formula for the unit ball $B(0, 1) \subset \mathbb{R}^2$ given by

   $$u(\mathbf{x}) = \frac{1 - |\mathbf{x}|^2}{2\pi} \int_{\partial B(0,1)} \frac{u(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \, ds_{\mathbf{y}}$$

   for $\mathbf{x} \in B^0(0, 1)$ to prove that

   $$\frac{1 - |\mathbf{x}|}{1 + |\mathbf{x}|} u(0) \leq u(\mathbf{x}) \leq \frac{1 + |\mathbf{x}|}{1 - |\mathbf{x}|} u(0)$$

   whenever u is positive and harmonic in $B^0(0, 1)$.

   Hint: You must use the triangle inequality and the mean-value property.

2. In a previous homework, you found the harmonic function

   $$u(\mathbf{x}) = a \arctan\left(\frac{x_1}{x_2}\right) + b$$

   for $\mathbf{x} = (x_1, x_2)$ depending only on the quotient $\frac{x_1}{x_2}$.

   (a) Find the boundary data $\lim_{x_2 \to 0^+} u(\mathbf{x})$.

   (b) Use your result as boundary data for the Poisson formula

   $$u(\mathbf{x}) = \frac{x_2}{\pi} \int_{\partial \mathbb{R}^2_+} \frac{g(\mathbf{y})}{|\mathbf{y} - \mathbf{x}|^2} \, d\mathbf{y} = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{g(y)}{(y - x_1)^2 + x_2^2} \, dy$$

   to obtain the same harmonic function $u(\mathbf{x})$ in the upper half-plane $\mathbb{R}^2_+$.

3. Find the Green's function for the upper half-disc $\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1, \ x_2 > 0\}$ in terms of the fundamental solution Φ.

   Hint: Start building your corrector by cancelling the boundary values of Φ on the interval $0 < x_1 < 1$, $x_2 = 0$. Add the required term and check the boundary values of the resulting function on the upper half-circle. Cancel these values. Repeat this process, alternating pieces of the boundary, until you've eliminated the boundary values everywhere on the boundary.

4. Find the Green's function for the wedge $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 > x_2, \ x_2 > 0\}$ in terms of the fundamental solution Φ.

5. Find the Green's function for the strip $\{\mathbf{x} \in \mathbb{R}^2 \mid 0 < x_2 < 1\}$ in terms of the fundamental solution Φ.