HW25 1016

1. 給定AR(p)序列,求(a)以lag operators表示(令L為lag operator,滿足$Ly_t = y_{t-1}$)

Lag operator 定義: $Ly_t = y_{t-1}$, $L^2 y_t = y_{t-2}$, $L^k y_t = y_{t-k}$

⇒把模型改寫成含L的形式,原式

(I) $y_t = 1.3y_{t-1} - 0.5y_{t-2} + \varepsilon_t$

⇒ $y_t = 1.3Ly_t - 0.5L^2 y_t + \varepsilon_t$

⇒ $y_t - 1.3Ly_t + 0.5L^2 y_t = \varepsilon_t$, $(1-1.3L+0.5L^2)y_t = \varepsilon_t$

⇒ $\Phi(L) = 1-1.3L+0.5L^2$, 因此AR(2)模型可寫成$\Phi(L)y_t = \varepsilon_t$

(II) $y_t = 1.2y_{t-1} - 0.4y_{t-2} + \varepsilon_t$

$y_t = 1.2Ly_t - 0.4L^2 y_t + \varepsilon_t$

⇒ $y_t - 1.2Ly_t + 0.4L^2 y_t = \varepsilon_t$

⇒ $(1-1.2L+0.4L^2)y_t = \varepsilon_t$

⇒ $\Phi(L) = 1-1.2L+0.4L^2$, 因此AR(2)模型可寫成$\Phi(L)y_t = \varepsilon_t$

(III) $y_t = 1.2y_{t-1} + 1.2y_{t-2} + \varepsilon_t$

$y_t(1-1.2L-1.2L^2) = \varepsilon_t$

(VI) $y_t = 0.7y_{t-1} + 0.25y_{t-2} - 0.175y_{t-3} + \varepsilon_t$

$y_t(1-0.7L-0.25L^2+0.175L^3) = \varepsilon_t$

(IV) $y_t = -1.2y_{t-1} + \varepsilon_t$

$y_t(1+1.2L) = \varepsilon_t$

(b) 判斷是否為平穩序列 Stationary

↳ AR(p)過程平穩 ⇔ 其特徵方程的根(in terms of L)皆滿足$|z|>1$.

(I) $1-1.2L+0.2L^2=0$, $1-1.2z+0.2z^2=0$, $z=1, 0.2$ <1, → 不平穩

(II) $1-1.2L+0.4L^2=0$, $z=0.6\pm0.2i$, $|z|<0.63<1$ → 不平穩

(III) $1-1.2L-1.2L^2=0$，$z=1.8, -0.6$ → 不平穩

(IV) $1+1.2L=0$，$L=-\frac{1}{1.2}=-0.833$，$|L|<1$ → 不平穩

(VI) $1-0.7L-0.15L^2+0.175L^3=0$，$z=1.15, 1.2, -2$，全部$|z|>1$ → 平穩

4. $y_t=y_{t-1}+\varepsilon_t$，$\varepsilon_t \overset{iid}{\sim} (0,\sigma^2)$ pf y_t為非定態

平穩序列(弱定態)條件：1. $E(y_t)$為常數

2. $Var(y_t)$為常數

3. $Cov(y_t, y_{t-k})$只與k有关

$y_t=y_{t-1}+\varepsilon_t$，代入$y_{t-1}=y_{t-2}+\varepsilon_{t-1}$得 $y_t=y_{t-2}+\varepsilon_{t-1}+\varepsilon_t$

繼續展開，$y_t=y_0+\sum_{i=1}^{t}\varepsilon_i$

$E(y_t)=E(y_0)+\sum_{i=1}^{t}E(y_1)=E(y_0)$ → 期望值為常數

$Var(y_t)=Var(\sum_{i=1}^{t}\varepsilon_i)=\sum_{i=1}^{t}Var(\varepsilon_i)=t\sigma^2$ → 隨t↑，Var↑ (線性)

$Cov(y_t, y_s)=E[(y_t-E(y_t))(y_s-E(y_s))]$

$\because E(y_t)=E(y_s)=E(y_0) \Rightarrow Cov(y_t, y_s)=E[(y_t-y_0)(y_s-y_0)]$

代入展開式 $y_t-y_0=\sum_{i=1}^{t}\varepsilon_i$，$y_s-y_0=\sum_{j=1}^{s}\varepsilon_j$

$\therefore Cov(y_t, y_s)=E[(\sum_{i=1}^{t}\varepsilon_i)(\sum_{j=1}^{s}\varepsilon_j)]$ $\because \varepsilon_i$之間iid，$E(\varepsilon_i\varepsilon_j)=\begin{cases}\sigma^2, & i=j\\ 0, & i\neq j\end{cases}$

$\therefore Cov(y_t, y_s)=\sum_{i=1}^{t}\sum_{j=1}^{s}E(\varepsilon_i\varepsilon_j)=\sum_{i=1}^{\min(t,s)}\sigma^2$

若 $t>s$，$Cov(y_t, y_s)=s\sigma^2=Var(y_s)$

平穩序列應滿足 $Cov(y_t, y_s)=\gamma(t-s)$，即只和時間差 $t-s$ 有关

but這裡卻是和s有关，這代表往後 observe 更久(s更大)，Cov↑，

data 波動幅度↑，∴此过程為 Non-stationary

5. 給定 y_t 為定態 AR(2) 序列 $y_t = c+\phi_1 y_{t-1}+\phi_2 y_{t-2}+\varepsilon_t$，
其中 $\varepsilon \overset{iid}{\sim} (0,\sigma^2)$，求 (a) $E(y_t)$ (b) $Var(y_t)$ (c) $\gamma(j)$ (d) $\rho(j)$

(a) $E(y_t)$ 令 $E(y_t)=\mu$，$\mu=c+\phi_1\mu+\phi_2\mu \Rightarrow \mu(1-\phi_1-\phi_2)=c$

$$\Rightarrow E(y_t)=\mu=\frac{c}{1-\phi_1-\phi_2}$$

(b) $Var(y_t)$ 令 $y_t-\mu=x_t$，則 $x_t=\phi_1 x_{t-1}+\phi_2 x_{t-2}+\varepsilon_t$

利用 Yule-Walker 方程式 $\begin{cases}\gamma(0)=\phi_1\gamma(1)+\phi_2\gamma(2)+\sigma^2 & —① \\ \gamma(1)=\phi_1\gamma(0)+\phi_2\gamma(1) & —②\end{cases}$

② $\gamma(1)(1-\phi_2)=\phi_1\gamma(0) \Rightarrow \gamma(1)=\frac{\phi_1}{1-\phi_2}\gamma(0)$ 代入①，$\gamma(0)=\phi_1\frac{\phi_1}{1-\phi_2}\gamma(0)+\phi_2\gamma(2)+\sigma^2$

$\gamma(2)=\phi_1\gamma(1)+\phi\gamma(0) \Rightarrow \gamma(0)\left[1-\frac{\phi_1^2(1+\phi_2)}{1-\phi_2}-\phi_2^2\right]=\sigma^2$

$$Var(y_t)=\gamma(0)=\frac{(1-\phi_2)\sigma^2}{(1+\phi_2)((1-\phi_2)^2-\phi_1^2)}$$

(c) $\gamma(j)$ $\gamma(j)=\phi_1\gamma(j-1)+\phi_2\gamma(j-2)$，$j\geq 2$

$\gamma(0)$，$\gamma(1)=\frac{\phi_1}{1-\phi_2}\gamma(0)$

(d) $\rho(j)$ $\rho(j)=\frac{\gamma(j)}{\gamma(0)}$，$\rho(j)=\phi_1\rho(j-1)+\phi_2\rho(j-2)$，$\rho(0)=1$

6. $y_t=1.3y_{t-1}-0.5y_{t-2}+\varepsilon_t$，$\varepsilon_t \overset{iid}{\sim} (0,\sigma^2)$

(a) 以 lag operator 表示式

$y_t=1.3Ly_t-0.5L^2y_t+\varepsilon_t \Rightarrow (1-1.3L+0.5L^2)y_t=\varepsilon_t$

$\Phi(L)y_t=\varepsilon_t$，$\Phi(L)=1-1.3L+0.5L^2$

(b) 判斷是否為平穩過程

$1-1.3z+0.5z^2=0$, $z=0.65\pm0.278i$; $|z|=\sqrt{0.65^2+0.278^2}=0.709<1$

$\Rightarrow$ AR(2) model 不平穩

(c) $E(y_t)$

$\because$ 沒有常數項, $\therefore E(y_t)=1.3E(y_{t-1})-0.5E(y_{t-2})$

令 $E(y_t)=\mu$, $\mu=1.3\mu-0.5\mu$, $(1-1.3+0.5)\mu=0$. $\mu=0$, $E(y_t)=0$ #

(d) $\gamma(j)$

$$\begin{cases}\gamma(0)=1.3\gamma(1)-0.5\gamma(2)+\sigma^2\\ \gamma(1)=1.3\gamma(0)-0.5\gamma(1)\end{cases}$$

$\rightarrow \gamma(1)(1+0.5)=1.3\gamma(0) \rightarrow \gamma(1)=0.867\gamma(0)$

$\Rightarrow \gamma(j)=1.3\gamma(j-1)-0.5\gamma(j-2)$, $j\geq2$

(e) Yule-Walker

$$\begin{cases}\gamma(0)=1.3\gamma(1)-0.5\gamma(2)+\sigma^2\\ \gamma(1)=1.3\gamma(0)-0.5\gamma(1)\\ \gamma(2)=1.3\gamma(1)-0.5\gamma(0)\end{cases}$$

(f) $Var(y_t)=\gamma(0)$

$\gamma(1)=\frac{1.3}{1.5}\gamma(0)$

$\gamma(2)=1.3\gamma(1)-0.5\gamma(0)=1.3(0.867\gamma(0))-0.5\gamma(0)=0.6267\gamma(0)$

$$\gamma(0) = 1.3(0.867\gamma(0)) - 0.5(0.6267\gamma(0)) + \sigma^2$$

$$\gamma(0)[1 - (1.12711 - 0.31335)] = \sigma^2$$

$$\gamma(0) = \frac{\sigma^2}{0.18625} = 5.37\sigma^2$$

$$, \quad Var(y_t) = 5.37\sigma^2$$