Design of Engineering Experiments – The Blocking Principle

- Text Reference, Chapter 4
- **Blocking** and **nuisance factors** 干擾
- The randomized complete block design or the **RCBD** 實驗順序本身也可能是 nuisance factors
- Extension of the ANOVA to the RCBD
- Other blocking scenarios…Latin square designs, balanced incomplete block design

The Blocking Principle

- **Blocking** is a technique for dealing with **nuisance factors**
- A **nuisance** factor is a factor that probably has some effect on the response, but it's of no interest to the experimenter…however, the variability it transmits to the response needs to be minimized
- Typical nuisance factors include batches of raw material, operators, pieces of test equipment, time (shifts, days, etc.), different experimental units
- **Many** industrial experiments involve blocking (or should)
- Failure to block is a common flaw in designing an experiment (consequences?)

The Blocking Principle

- If the nuisance variable is **known** and **controllable**, we use **blocking**
- If the nuisance factor is **known** and **uncontrollable**, ANCOVA sometimes we can use the **analysis of covariance** (see Chapter 15) to remove the effect of the nuisance factor from the analysis
- If the nuisance factor is **unknown** and **uncontrollable** (a **"lurking" variable**), we hope that **randomization** balances out its impact across the experiment
- Sometimes several sources of variability are **combined** in a block, so the block becomes an aggregate variable

The Hardness Testing Example

- Recall in Chapter 2

2個針是否一樣

Paired t-test

IQC

Incoming

TABLE 2.6

Data for the Hardness Testing Experiment

Specimen	Tip 1	Tip 2
1	7	6
2	3	3
3	3	5
4	4	3
5	8	8
6	3	2
7	2	4
8	9	9
9	5	4
10	4	5

Block

Let

$$d_j = y_{1j} - y_{2j} \qquad j = 1, 2, \ldots, 10$$

We test for the hypothesis

$$H_0: \mu_d = 0$$

$$H_1: \mu_d \neq 0$$

In fact, the statistical model behind is

$$y_{ij} = \mu_i + \beta_j + \epsilon_{ij} \begin{cases} i = 1, 2 \\ j = 1, 2, \ldots, 10 \end{cases}$$

The Hardness Testing Example

- Text reference, pg 139, 140 tip1 ~ tip4
- We wish to determine whether 4 different tips produce different (mean) hardness reading on a Rockwell hardness tester
- **Gauge & measurement systems** capability studies are frequent areas for applying DOX
- Assignment of the tips to an **experimental unit**; that is, a test coupon
- The 4 test coupons are a source of **nuisance variability**
- The need for blocking

CRD

```
coupon_no <- rep(1:4, each = 4)
tip_no <- rep(sprintf("tip%d", 1:4), 4)
set.seed(1)
crd <- sample(1:16, 16)
tip_run <- tip_no[crd]
data.frame(rbind(coupon_no, tip_run))
```

coupon_no	1	1	1	1	2	2	2	2
tip_run	tip1	tip4	tip3	tip1	tip2	tip2	tip4	tip3
coupon_no	3	3	3	3	4	4	4	4
tip_run	tip1	tip1	tip3	tip2	tip2	tip3	tip4	tip4

The Hardness Testing Example

- To conduct this experiment as a RCBD, assign all 4 tips to each coupon
- Each coupon is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the tips
- Variability **between** blocks can be large, variability **within** a block should be relatively small
- In general, a **block** is a specific level of the nuisance factor
- A complete replicate of the basic experiment is conducted in each block

4 tips : treatment
4 coupon : block

- A block represents a **restriction on randomization**
- All runs **within** a block are **randomized**

The Hardness Testing Example

- Suppose that we use $b = 4$ blocks:

■ **TABLE 4.1**
Randomized Complete Block Design for the Hardness Testing Experiment

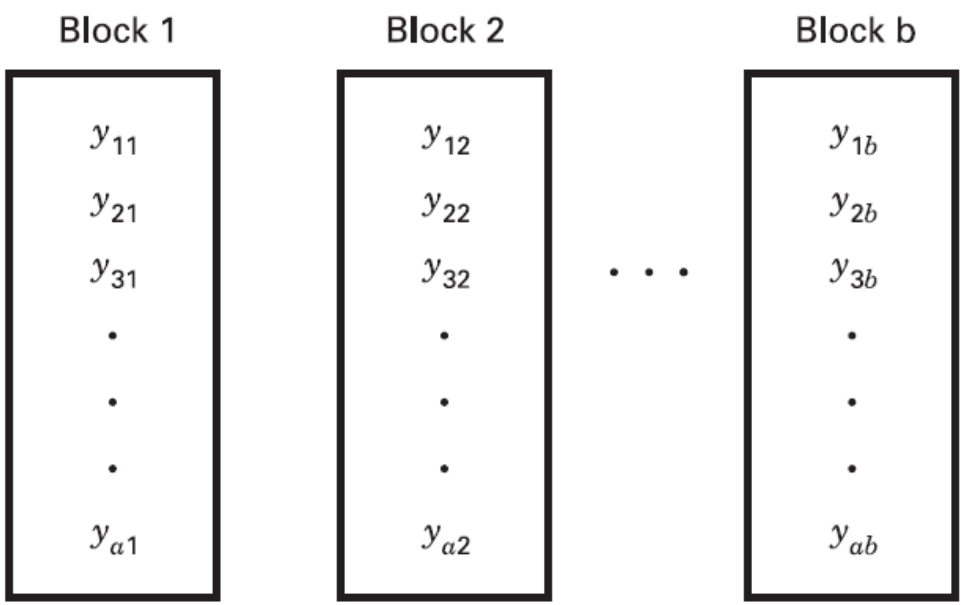
Test Coupon (Block)			
1	**2**	**3**	**4**
Tip 3	Tip 3	Tip 2	Tip 1
Tip 1	Tip 4	Tip 1	Tip 4
Tip 4	Tip 2	Tip 3	Tip 2
Tip 2	Tip 1	Tip 4	Tip 3

- Notice the **two-way structure** of the experiment
- Once again, we are interested in testing the equality of treatment means, but now we have to remove the variability associated with the nuisance factor (the blocks)

coupons

The Randomized Complete Block Design

- Units of test equipment or machinery are often different in their operating characteristics.
- Batches of raw material, people, and time, …



Extension of the ANOVA to the RCBD

- Suppose that there are *a* treatments (factor levels) and *b* blocks
- A **statistical model** (effects model) for the RCBD is

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

ϵ_{ij} is the usual NID $(0, \sigma^2)$ random error term

- The relevant (fixed effects) hypotheses are $\sum_{j=1}^{b} \beta_j = 0$

μ_i = treatment mean

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_a \text{ where } \mu_i = (1/b)\sum_{j=1}^{b}(\mu + \tau_i + \beta_j) \overset{?}{=} \mu + \tau_i$$

Notations

- a treatments
- b blocks
- $N = ab$ is the total number of observations

Sum of ith the treatment level

$$y_{i.} = \sum_{j=1}^{b} y_{ij} \qquad i = 1, 2, \ldots, a$$

$\bar{y}_i = \frac{1}{b} y_{i.}$

jth

$$y_{.j} = \sum_{i=1}^{a} y_{ij} \qquad j = 1, 2, \ldots, b$$

$\bar{y}_j = \frac{1}{a} y_{.j}$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij} = \sum_{i=1}^{a} y_{i.} = \sum_{j=1}^{b} y_{.j}$$

$$\bar{y}_{i.} = y_{i.}/b \quad \bar{y}_{.j} = y_{.j}/a \quad \bar{y}_{..} = y_{..}/N$$

Extension of the ANOVA to the RCBD

ANOVA partitioning of total variability:

SST=

$$\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{..})^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}[(\bar{y}_{i.}-\bar{y}_{..})+(\bar{y}_{.j}-\bar{y}_{..})$$

$$+(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})]^2$$

sum of square total

$$=\boxed{b\sum_{i=1}^{a}(\bar{y}_{i.}-\bar{y}_{..})^2} \; ① +\boxed{a\sum_{j=1}^{b}(\bar{y}_{.j}-\bar{y}_{..})^2} \; ②$$

treatment effect ① 냥 $\bar{y}_{i.}-\bar{y}_{..}$

② 냥 $\bar{y}_{.j}-\bar{y}_{..}$ block effect

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}(y_{ij}-\bar{y}_{i.}-\bar{y}_{.j}+\bar{y}_{..})^2$$

random error

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

① ②

Extension of the ANOVA to the RCBD

The degrees of freedom for the sums of squares in

$$SS_T = SS_{Treatments} + SS_{Blocks} + SS_E$$

are as follows:

$$ab - 1 = a - 1 + b - 1 + (a-1)(b-1)$$

- Ratios of sums of squares (SS) to their degrees of freedom (df) result in mean squares (MS)

$E(MS_{Tr}) = \sigma^2 + A \sum_{i=1}^{a} \tau_A^2$

- Ratio of the mean square for treatments ($MS_{Treatments}$) to the error mean square (MSE) is an F statistic that can be used to test the hypothesis of equal treatment means

ANOVA Display for the RCBD

TABLE 4.2

Analysis of Variance for a Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	
Treatments	$SS_{\text{Treatments}}$	$a - 1$	$\dfrac{SS_{\text{Treatments}}}{a - 1}$	$\dfrac{MS_{\text{Treatments}}}{MS_E}$	P_{Tr}
Blocks	SS_{Blocks}	$b - 1$	$\dfrac{SS_{\text{Blocks}}}{b - 1}$	$\dfrac{MS_{Block}}{MS_E}$	P_B
Error	SS_E	$(a - 1)(b - 1)$	$\dfrac{SS_E}{(a - 1)(b - 1)}$		
Total	SS_T	$N - 1$			

ANOVA Display for the RCBD

Manual computing:

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$SS_{\text{Treatments}} = \frac{1}{b} \sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}$$

$$SS_{\text{Blocks}} = \frac{1}{a} \sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}$$

and the error sum of squares is obtained by subtraction as

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

Vascular Graft Example (pg. 145)

EXAMPLE 4.1

A medical device manufacturer produces vascular grafts (artificial veins). These grafts are produced by extruding billets of polytetrafluoroethylene (PTFE) resin combined with a lubricant into tubes. Frequently, some of the tubes in a production run contain small, hard protrusions on the external surface. These defects are known as “flicks.” The defect is cause for rejection of the unit.

The product developer responsible for the vascular grafts suspects that the extrusion pressure (X) affects the occurrence of flicks and therefore intends to conduct an experiment to investigate this hypothesis. However, the resin is manufactured by an external supplier and is delivered to the medical device manufacturer in batches. The engineer also suspects that there may be significant batch-to-batch variation, because while the material should be consistent with respect to parameters such as molecular weight, mean particle size, retention, and peak height ratio, it probably isn’t due to manufacturing variation at the resin supplier and natural variation in the material. Therefore, the product developer decides to investigate the effect of four different levels of extrusion pressure on flicks using a randomized complete block design considering batches of resin as blocks. The RCBD is shown in Table 4.3. Note that there are four levels of extrusion pressure (treatments) and six batches of resin (blocks). Remember that the order in which the extrusion pressures are tested within each block is random. The response variable is yield, or the percentage of tubes in the production run that did not contain any flicks.

X

Treatment is Extrusion pressure

Y

Response is yield 良率

品質的 data

Block is Resin Material

X_B

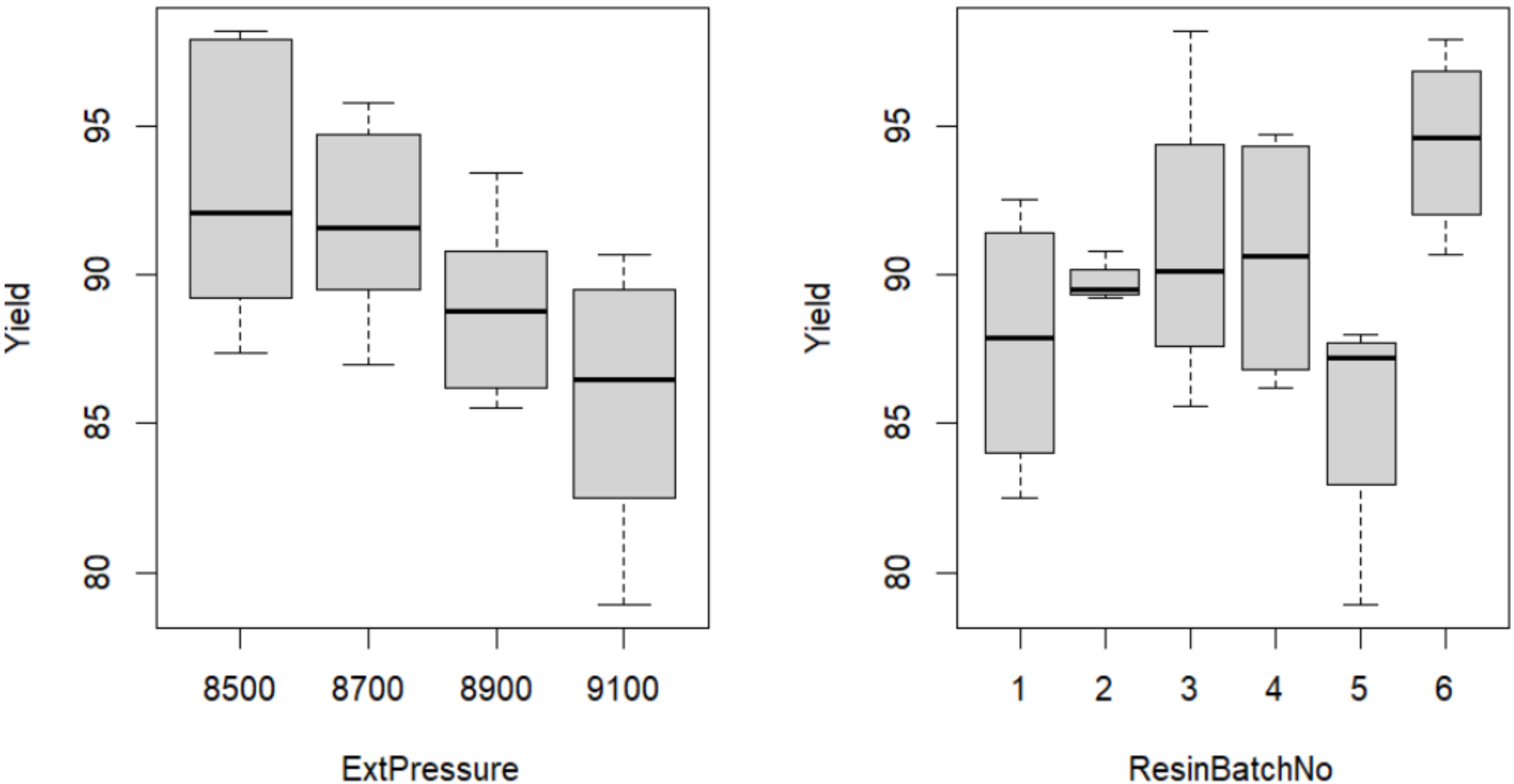
Vascular Graft Example (pg. 145)

- To conduct this experiment as a RCBD, assign all 4 pressures to each of the 6 batches of resin
- Each batch of resin is called a "**block**"; that is, it's a more homogenous experimental unit on which to test the extrusion pressures

■ **TABLE 4.3**
Randomized Complete Block Design for the Vascular Graft Experiment

Extrusion Pressure (PSI)	Batch of Resin (Block) 1	2	3	4	5	6	Treatment Total
8500	90.3	89.2	98.2	93.9	87.4	97.9	556.9
8700	92.5	89.5	90.6	94.7	87.0	95.8	550.1
8900	85.5	90.8	89.6	86.2	88.0	93.4	533.5
9100	82.5	89.5	85.6	87.4	78.9	90.7	514.6
Block Totals	350.8	359.0	364.0	362.2	341.3	377.8	$y_{..} = 2155.1$

Vascular Graft Example (pg. 145)



To perform the analysis of variance, we need the following sums of squares:

$$SS_T = \sum_{i=1}^{4}\sum_{j=1}^{6} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= 193{,}999.31 - \frac{(2155.1)^2}{24} = 480.31$$

$$SS_{\text{Treatments}} = \frac{1}{b}\sum_{i=1}^{4} y_{i.}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{6}[(556.9)^2 + (550.1)^2 + (533.5)^2 + (514.6)^2] - \frac{(2155.1)^2}{24} = 178.17$$

$$SS_{\text{Blocks}} = \frac{1}{a}\sum_{j=1}^{6} y_{.j}^2 - \frac{y_{..}^2}{N}$$

$$= \frac{1}{4}[(350.8)^2 + (359.0)^2 + \cdots + (377.8)^2] - \frac{(2155.1)^2}{24} = 192.25$$

$$SS_E = SS_T - SS_{\text{Treatments}} - SS_{\text{Blocks}}$$

$$= 480.31 - 178.17 - 192.25 = 109.89$$

The ANOVA is shown in Table 4.4. Using $\alpha = 0.05$, the critical value of F is $F_{0.05,3,15} = 3.29$. Because $8.11 > 3.29$, we conclude that extrusion pressure affects the mean yield. The P-value for the test is also quite small. Also, the resin batches (blocks) seem to differ significantly, because the mean square for blocks is large relative to error.

■ TABLE 4.4
Analysis of Variance for the Vascular Graft Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Treatments (extrusion pressure)	178.17	3	59.39	8.11	0.0019
Blocks (batches)	192.25	5	38.45		
Error	109.89	15	7.33		
Total	480.31	23			

If CRD Only

■ **TABLE 4.5**

Incorrect Analysis of the Vascular Graft Experiment as a Completely Randomized Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Extrusion pressure	178.17	3	59.39	3.95	$\boxed{0.0235}$
Error	302.14	20	$\boxed{15.11}$		
Total	480.31	23			

RCBD

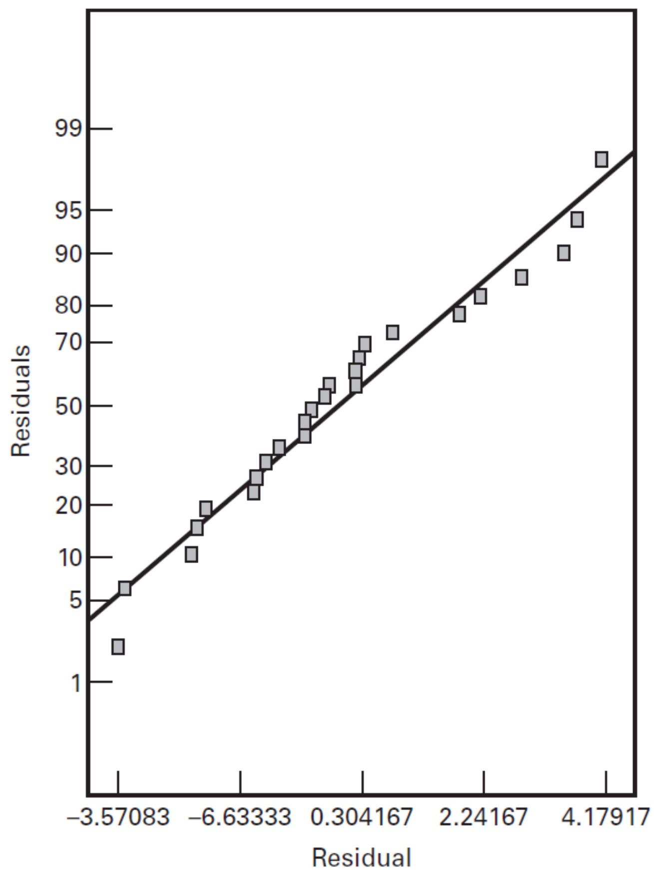
多考慮干擾因子 → 結果更 beautiful

■ **TABLE 4.4**

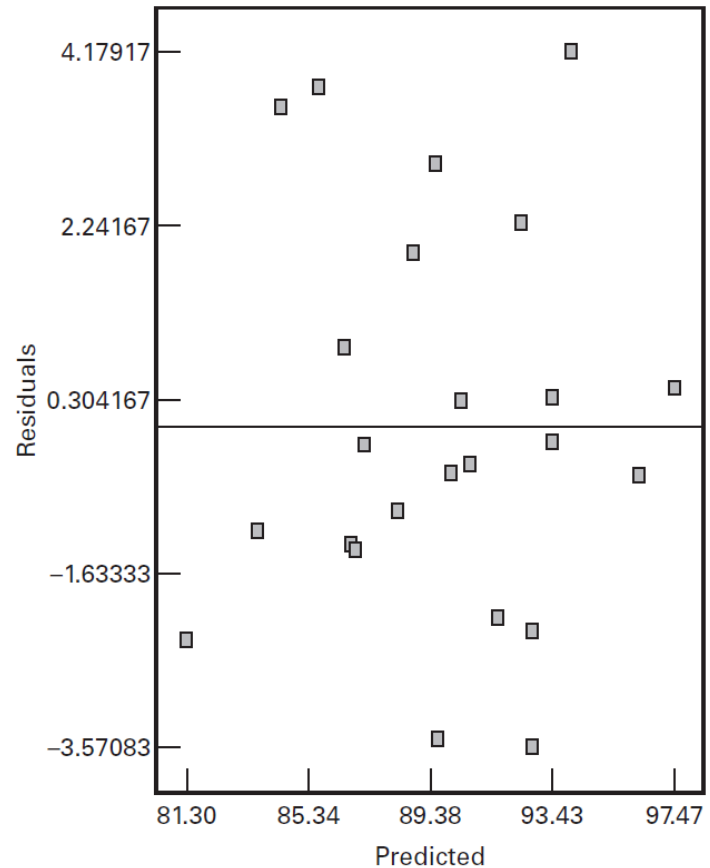
Analysis of Variance for the Vascular Graft Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Treatments (extrusion pressure)	178.17	3	59.39	8.11	$\boxed{0.0019}$
Blocks (batches)	192.25	5	38.45		
Error	109.89	15	$\boxed{7.33}$		
Total	480.31	23			

Residual Analysis for the Vascular Graft Example

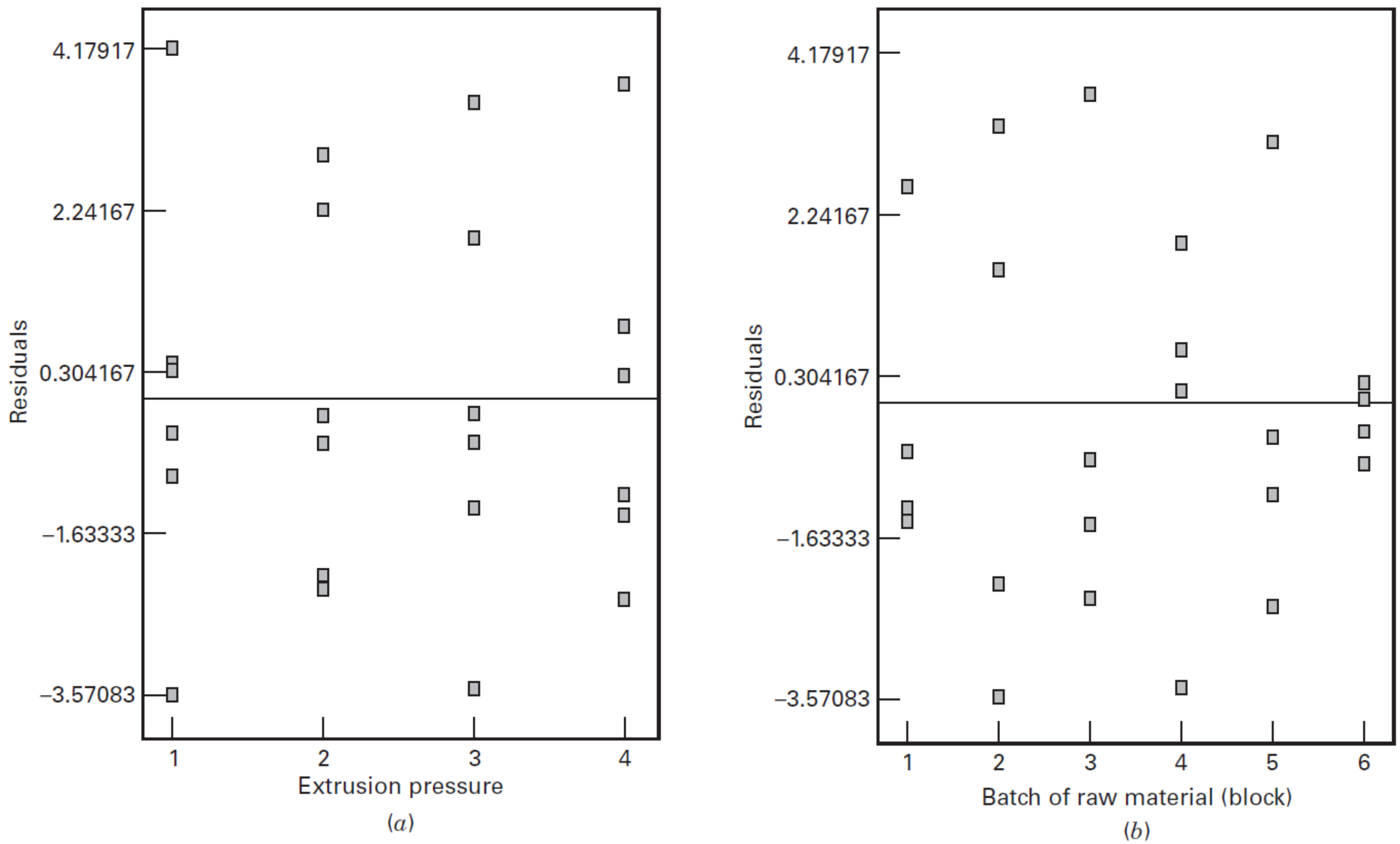


■ **FIGURE 4.4** **Normal probability plot of residuals for Example 4.1**



■ **FIGURE 4.5** **Plot of residuals versus $\hat{y}_{ij}$ for Example 4.1**

Residual Analysis for the Vascular Graft Example



■ **FIGURE 4.6** **Plot of residuals by extrusion pressure (treatment) and by batches of resin (block) for Example 4.1**

Residual Analysis for the Vascular Graft Example

- Basic residual plots indicate that **normality**, **constant variance** assumptions are satisfied
- No obvious problems with **randomization**
- **No patterns in the residuals vs. block**
- Can also plot residuals versus the pressure (residuals by factor)
- These plots provide more information about the constant variance assumption, possible outliers

Multiple Comparisons for the Vascular Graft Example – Which Pressure is Different?

Tukey's multiple comparisons

```
  Tukey multiple comparisons of means
    95% family-wise confidence level

Fit: aov(formula = Yield ~ factor(ExtPressure) + factor(ResinBatchNo), data = df1)

$`factor(ExtPressure)`
               diff        lwr       upr     p adj
8700-8500 -1.133333  -5.637161  3.370495 0.8854831
8900-8500 -3.900000  -8.403828  0.603828 0.1013084
9100-8500 -7.050000 -11.553828 -2.546172 0.0020883
8900-8700 -2.766667  -7.270495  1.737161 0.3245644
9100-8700 -5.916667 -10.420495 -1.412839 0.0086667
9100-8900 -3.150000  -7.653828  1.353828 0.2257674
```

Fisher's LSD method

```
> out <- LSD.test(fit, "ExtPressure", p.adj = "bonferroni")
> out$group
        Yield groups
8500 92.81667      a
8700 91.68333      a
8900 88.91667     ab
9100 85.76667      b
```

a= {8500, 8700, 8900}

b= {8900, 9100}

The Latin Square Design

- Text reference, Section 4.2, pg. 158
- These designs are used to simultaneously control (or eliminate) **two sources of nuisance variability**
- A significant assumption is that the three factors (treatments, nuisance factors) **do not interact**
- If this assumption is violated, the Latin square design will not produce valid results
- Latin squares are not used as much as the RCBD in industrial experimentation

The Rocket Propellant Problem – A Latin Square Design

- An experimenter is studying the effects of **five different formulations** of a rocket propellant used in aircrew escape systems on the observed burning rate.
- Each formulation is mixed from a batch of raw material that is only large enough for five formulations to be tested.
- The formulations are prepared by several operators, and there may be substantial differences in the skills and experience of the operators.

(5 levels)

Treatment is Formulation of RP **Response is** Burning Rate

Blocks are raw material, operators

5 份 5 people

Treatment : 5 lvs

Block 25 lv < block 1 : 5 lvs, block 2 : 5 lvs

The Rocket Propellant Problem – A Latin Square Design

■ **TABLE 4.9**
Latin Square Design for the Rocket Propellant Problem

Batches of Raw Material	Operators 1	2	3	4	5
1	$A = 24$	$B = 20$	$C = 19$	$D = 24$	$E = 24$
2	$B = 17$	$C = 24$	$D = 30$	$E = 27$	$A = 36$
3	$C = 18$	$D = 38$	$E = 26$	$A = 27$	$B = 21$
4	$D = 26$	$E = 31$	$A = 26$	$B = 23$	$C = 22$
5	$E = 22$	$A = 30$	$B = 20$	$C = 29$	$D = 31$

- This is a 5×5 Latin square design
- Page 159 shows some other Latin squares
- Table 4-13 (page 162) contains properties of Latin squares
- Statistical analysis?

Latin Square Designs

- The Latin square design is used to eliminate two nuisance sources of variability; that is, it systematically allows blocking in two directions.
- In general, a Latin square for p factors, or a $p \times p$ Latin square, is a square containing p rows and p columns.
- Each of the resulting p^2 cells contains one of the p letters that corresponds to the treatments, and **each letter occurs once and only once in each row and column**.

■ **TABLE 4.13**

Standard Latin Squares and Number of Latin Squares of Various Sizes[a]

Size	3×3	4×4	5×5	6×6	7×7	$p \times p$
Examples of standard squares	*A B C* *B C A* *C A B*	*A B C D* *B C D A* *C D A B* *D A B C*	*A B C D E* *B A E C D* *C D A E B* *D E B A C* *E C D B A*	*A B C D E F* *B C F A D E* *C F B E A D* *D E A B F C* *E A D F C B* *F D E C B A*	*A B C D E F G* *B C D E F G A* *C D E F G A B* *D E F G A B C* *E F G A B C D* *F G A B C D E* *G A B C D E F*	*ABC . . . P* *BCD . . . A* *CDE . . . B* ⋮ *PAB . . .* $(P-1)$
Number of standard squares	1	4	56	9408	16,942,080	—
Total number of Latin squares	12	576	161,280	818,851,200	61,479,419,904,000	$p!(p-1)! \times$ (number of standard squares)

[a]Some of the information in this table is found in Fisher and Yates (1953). Little is known about the properties of Latin squares larger than 7×7.

The Rocket Propellant Problem – A Latin Square Design

- The data of a Latin square design

i	RawMaterialBatch	Operator	Formulation	BurningRate
1	1	1	A	24
2	1	2	B	20
3	1	3	C	19
4	1	4	D	24
5	1	5	E	24
6	2	1	B	17
7	2	2	C	24
8	2	3	D	30
9	2	4	E	27
10	2	5	A	36
11	3	1	C	18
12	3	2	D	38
13	3	3	E	26
14	3	4	A	27
15	3	5	B	21

i	RawMaterialBatch	Operator	Formulation	BurningRate
16	4	1	D	26
17	4	2	E	31
18	4	3	A	26
19	4	4	B	23
20	4	5	C	22
21	5	1	E	22
22	5	2	A	30
23	5	3	B	20
24	5	4	C	29
25	5	5	D	31

Statistical Analysis of the Latin Square Design

- The statistical (effects) model is

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, p & \text{row effect} \\ j = 1, 2, \ldots, p & \text{treatment effect} \\ k = 1, 2, \ldots, p & \text{column effect} \end{cases}$$

- The statistical analysis (ANOVA) is much like the analysis for the RCBD.

$$SS_T = SS_{\text{Rows}} + SS_{\text{Columns}} + SS_{\text{Treatments}} + SS_E$$

with dof. $p^2 - 1 = p - 1 + p - 1 + p - 1 + (p - 2)(p - 1)$

Statistical Analysis of the Latin Square Design

- The ANOVA table for the rocket propellant example

TABLE 4.10
Analysis of Variance for the Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments	$SS_{\text{Treatments}} = \frac{1}{p}\sum_{j=1}^{p} y_{.j.}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Treatments}}}{p-1}$	$F_0 = \frac{MS_{\text{Treatments}}}{MS_E}$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i..}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Rows}}}{p-1}$	distributed as $F_{p-1,(p-2)(p-1)}$ under the null hypothesis.
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{k=1}^{p} y_{..k}^2 - \frac{y_{...}^2}{N}$	$p-1$	$\frac{SS_{\text{Columns}}}{p-1}$	
Error	SS_E (by subtraction)	$(p-2)(p-1)$	$\frac{SS_E}{(p-2)(p-1)}$	
Total	$SS_T = \sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{N}$	p^2-1		

The Rocket Propellant Problem – A Latin Square Design

■ **TABLE 4.12**
Analysis of Variance for the Rocket Propellant Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
Formulations	330.00	4	82.50	7.73	0.0025
Batches of raw material	68.00	4	17.00		
Operators	150.00	4	37.50		
Error	128.00	12	10.67		
Total	676.00	24			

As in any design problem, the experimenter should investigate the adequacy of the model by inspecting and plotting the residuals. For a Latin square, the residuals are given by

$$\begin{aligned} e_{ijk} &= y_{ijk} - \hat{y}_{ijk} \\ &= y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...} \end{aligned} \longrightarrow \text{model adequacy}$$

Graeco-Latin Square Designs

- A $p \times p$ Latin square, and superimpose on it a second $p \times p$ Latin square in which the treatments are denoted by Greek letters.
- Each Greek letter appears once and only once with each Latin letter.
- Controls systematically **three** sources of extraneous variability.
- Investigates four factors (rows, columns, Latin letters, and Greek letters), each at p levels in only p^2 runs.

TABLE 4.18
4×4 Graeco-Latin Square Design

	Column			
Row	**1**	**2**	**3**	**4**
1	$A\alpha$	$B\beta$	$C\gamma$	$D\delta$
2	$B\delta$	$A\gamma$	$D\beta$	$C\alpha$
3	$C\beta$	$D\alpha$	$A\delta$	$B\gamma$
4	$D\gamma$	$C\delta$	$B\alpha$	$A\beta$

Graeco-Latin Square Designs

- The statistical model for the Graeco-Latin square design

$$y_{ijkl} = \mu + \theta_i + \tau_j + \omega_k + \Psi_l + \epsilon_{ijkl} \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, p \end{cases}$$

■ **TABLE 4.19**
Analysis of Variance for a Graeco-Latin Square Design

Source of Variation	Sum of Squares	Degrees of Freedom
Latin letter treatments	$SS_L = \frac{1}{p}\sum_{j=1}^{p} y_{.j..}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Greek letter treatments	$SS_G = \frac{1}{p}\sum_{k=1}^{p} y_{..k.}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Rows	$SS_{\text{Rows}} = \frac{1}{p}\sum_{i=1}^{p} y_{i...}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Columns	$SS_{\text{Columns}} = \frac{1}{p}\sum_{l=1}^{p} y_{...l}^2 - \frac{y_{....}^2}{N}$	$p - 1$
Error	SS_E (by subtraction)	$(p - 3)(p - 1)$
Total	$SS_T = \sum_i \sum_j \sum_k \sum_l y_{ijkl}^2 - \frac{y_{....}^2}{N}$	$p^2 - 1$

Incomplete Block Designs

- Text reference, Section 4.4, pg. 168
- In certain experiments using randomized block designs, we may not be able to run all the treatment combinations in each block.
- When all treatment comparisons are equally important, the treatment combinations used in each block should be selected in a balanced manner. → **Balanced Incomplete Block Design (BIBD)**
- Any pair of treatments occur together the same number of times as any other pair.

Catalysts Experiment Example

As an example, suppose that a chemical engineer thinks that the time of reaction for a chemical process is a function of the type of catalyst employed. Four catalysts are currently being investigated. The experimental procedure consists of selecting a batch of raw material, loading the pilot plant, applying each catalyst in a separate run of the pilot plant, and observing the reaction time. Because variations in the batches of raw material may affect the performance of the catalysts, the engineer decides to use batches of raw material as blocks. However, each batch is only large enough to permit three catalysts to be run. Therefore, a randomized incomplete block design must be used. The balanced incomplete block design for this experiment, along with the observations recorded, is shown in Table 4.22. The order in which the catalysts are run in each block is randomized.

(4 level)

Treatment is Type of catalyst **Response is** Reaction time

Block is raw material

Other Requirements: Each batch of raw material

Catalysts Experiment Example

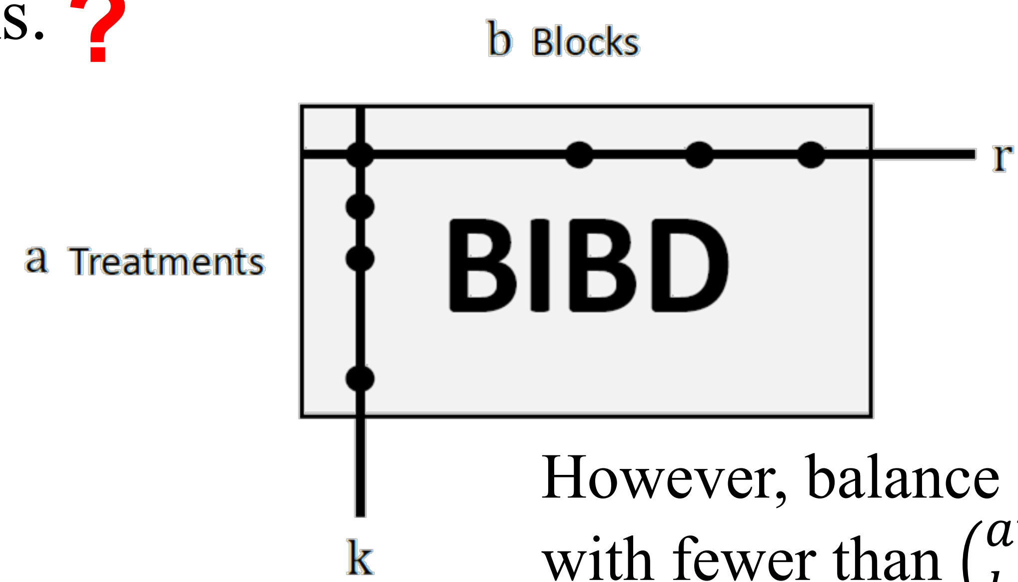
TABLE 4.22

Balanced Incomplete Block Design for Catalyst Experiment

Treatment (Catalyst)	Block (Batch of Raw Material) 1	2	3	4	$y_{i.}$
1	73	74	—	71	218
2	—	75	67	72	214
3	73	75	68	—	216
4	75	—	72	75	222
$y_{.j}$	221	224	207	218	$870 = y_{..}$

Balanced Incomplete Block Designs

- Suppose there are a treatments
- Each block can hold exactly $k < a$ treatments
- A BIBD may be constructed by taking $b = \binom{a}{k}$ blocks. ?



However, balance can be obtained with fewer than $\binom{a}{k}$ blocks.

Balanced Incomplete Block Designs

- There are a treatments and b blocks.
- Each block contasins k treatments.
- Each treatments occurs r times (r blocks).

The BIBD satisfies

- There are $N = ar = bk$ total observations
- The number of times each pair of treatments appears in the same block

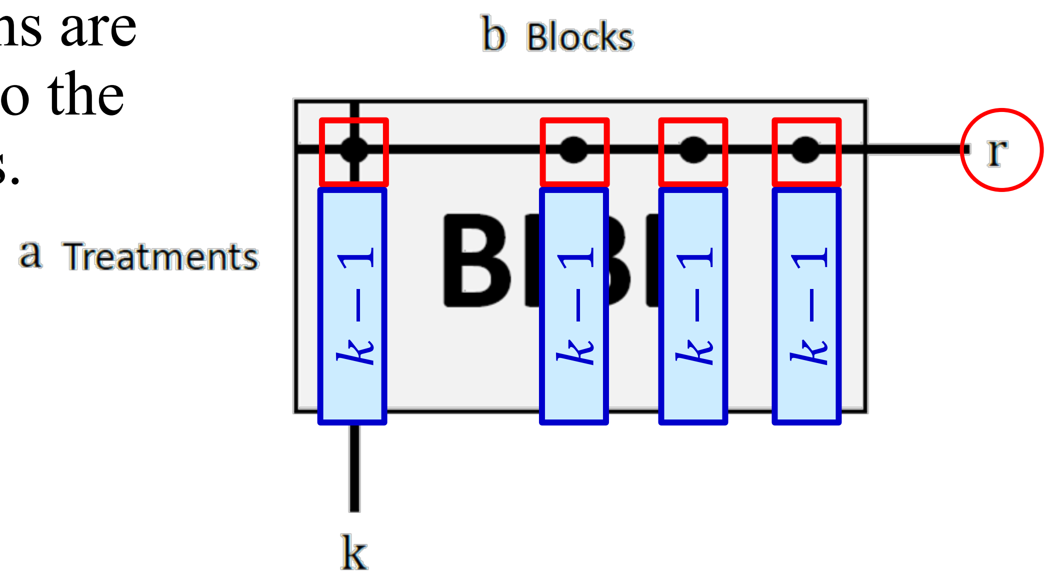
$$\lambda = \frac{r(k-1)}{a-1}$$ must be an integer.

Balanced Incomplete Block Designs

To derive the relationship for λ, take Treatment 1 as the example.

- Treatment 1 appears in r blocks.
- In each of those blocks, there are $k-1$ other treatments.
- Overall there are $r(k-1)$ observations in the blocks containing Treatment 1.
- These $r(k-1)$ observations are distributed evenly λ times to the remaining $a-1$ treatments.
- Therefore,

$$\lambda(a-1) = r(k-1)$$



Balanced Incomplete Block Designs

- Total variability may be partitioned into

$$SS_T = SS_{\text{Treatments(adjusted)}} + SS_{\text{Blocks}} + SS_E$$

- The sum of squares for treatments is **adjusted** to separate the treatment and the block effects
- Thus, differences between unadjusted treatment totals $y_{1\cdot}, y_{2\cdot}, \ldots, y_{a\cdot}$ are also affected by differences between blocks.

Balanced Incomplete Block Designs

- The block sum of squares $$SS_{\text{Blocks}} = \frac{1}{k}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}$$

- The adjusted treatment sum of squares

$$SS_{\text{Treatments(adjusted)}} = \frac{k\sum_{i=1}^{a} Q_i^2}{\lambda a}$$

Q_i is the adjusted total for the *i*th treatment.

$Q_i = y_{i\cdot} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{\cdot j}$, $i = 1, 2, \ldots, a$

with $n_{ij} = 1$ if treatment *i* appears in block *j* and $n_{ij} = 0$ otherwise.

- The error sum of squares is

$$SS_E = SS_T - SS_{\text{Treatments(adjusted)}} - SS_{\text{Blocks}}$$

Balanced Incomplete Block Designs

The ANOVA table of the BIBD

TABLE 4.23
Analysis of Variance for the Balanced Incomplete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Treatments (adjusted)	$\dfrac{k \sum Q_i^2}{\lambda a}$	$a-1$	$\dfrac{SS_{\text{Treatments(adjusted)}}}{a-1}$	$F_0 = \dfrac{MS_{\text{Treatments(adjusted)}}}{MS_E}$
Blocks	$\dfrac{1}{k}\sum y_{.j}^2 - \dfrac{y_{..}^2}{N}$	$b-1$	$\dfrac{SS_{\text{Blocks}}}{b-1}$	
Error	SS_E (by subtraction)	$N-a-b+1$	$\dfrac{SS_E}{N-a-b+1}$	
Total	$\sum\sum y_{ij}^2 - \dfrac{y_{..}^2}{N}$	$N-1$		

Catalysts Experiment Example

■ **TABLE 4.24**
Analysis of Variance for Example 4.5

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Treatments (adjusted for blocks)	22.75	3	7.58	11.66	0.0107
Blocks	55.00	3	—		
Error	3.25	5	0.65		
Total	81.00	11			

<u>Find the answer by yourself:</u>

What if we want to test the significance of the block effect?

$Y \sim B + X$

B → block

X → treatment

習慣 block 在前

Chap 4 !!!

問題解讀

block

treatment treatment

event model !

什麼實驗設計舉例

良心場牌 Name ?

講師 !