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1. (a) What is an iid sequence and what is a white noise?
   (b) How is weak stationarity different from strict stationarity?
   (c) What extra condition is required to conclude weak stationarity from strict stationarity?
   (d) What extra condition is required to conclude strict stationarity from weak stationarity?

1. a) iid sequence (独立同分布): is the sequence of independent random variables from the same distribution.

white noise (白噪声): is the sequence of uncorrelated r.v with same mean and variance.

1 b).

弱平稳的两个重要性质

- 均值 $\mu_t = \mathbb{E}[x_t]$ 是一个常数（不依赖于 t）。
- 协方差 $R_t(s) = Cov\{x_t, x_{t+s}\}$ 是仅依赖于时间差 s 的函数。

lag

1. 严格平稳性的定义

时间序列 $\{x_t\}$ 被称为严格平稳（strictly stationary），如果任意时间点的联合分布不随时间平移而变化。具体来说，对于任意 $n \geq 1$ 和任意时间点 $t_1, t_2, \ldots, t_n \in \mathbb{Z}$ 以及 $s \in \mathbb{Z}$：

$(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) \sim (x_{t_1+s}, x_{t_2+s}, \ldots, x_{t_n+s})$.

这意味着无论时间点如何移动，联合分布保持相同。

weak stationary and strictly stationary 区别

1. 严格平稳性是指联合分布不会随着时间点的平移而发生相同的变化。
2. 弱平稳性是指均值、方差和协方差具有平移不变性。
3. 严格平稳性和有限的一阶和二阶矩的存在 ⇒ 弱平稳性。
4. 弱平稳性和高斯性 ⇒ 严格平稳性。
5. 具有有限均值和有限方差的 iid 序列是白噪声过程。
6. 高斯白噪声过程是 iid 序列。

∴ Weak stationarity only focus on first and second moment (mean and covariance), and after shifting one or two time points, the mean variance remain the same. and the covariance only rely on time lag.

Strictly stationarity says the distribution of r.v at any number of time points remains the same when shifted by the same amount when more than one.

1c). If the first and second moments exists and finite.

1d) If the time series is Gaussian

2. Use the definition of weak stationarity to show that $\{x_t\}$ given by

$$x_t = \varepsilon_t + \theta\varepsilon_{t-1},\ t \in \mathbf{Z}$$

is weakly stationary and find its autocovariance and autocorrelation functions when $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$ with $\sigma > 0$. [Hint: Show that the mean of x_t is constant and the covariance of x_t and x_{t+s} is a function of s only.]

$$\begin{aligned} E(x_t) &= E(\varepsilon_t + \theta\varepsilon_{t-1}) \\ &= E(\varepsilon_t) + \theta E(\varepsilon_{t-1}) \qquad \because \{\varepsilon_t\} \sim WN(0, \sigma^2) \\ &= 0 + \theta \cdot 0 = 0 \qquad \text{not dependent on } t. \end{aligned}$$

$$\begin{aligned} \mathrm{COV}\{x_t, x_{t+s}\} &= \mathrm{cov}\{\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t+s} + \theta\varepsilon_{t+s-1}\}. \\ &= \mathrm{Cov}\{\varepsilon_t, \varepsilon_{t+s}\} + \theta \mathrm{cov}\{\varepsilon_t, \varepsilon_{t+s-1}\} + \theta \mathrm{cov}\{\varepsilon_{t-1}, \varepsilon_{t+s}\} + \theta^2\{\varepsilon_{t-1}, \varepsilon_{t+s-1}\}. \\ &= R(s) + \theta R(s-1) + \theta R(1+s) + \theta^2 R(s) \\ &= (1+\theta^2)R(s) + \theta R(s-1) + \theta R(1+s). \qquad (1) \end{aligned}$$

自协方差函数

函数 $R(s) = Cov\{x_t, x_{t+s}\}$ 称为自协方差函数（autocovariance function）。其中 s 是滞后（lag），即两个时间点之间的间隔。

特别地，当 $s = 0$ 时：

$R(0) = Cov\{x_t, x_t\} = Var\{x_t\}$

因此，时间序列的方差是一个常量。

自协方差函数

白噪声的自协方差函数可以用 Kronecker delta 函数 δ_s 表示：

$R(s) = \sigma^2\delta_s, \quad s \in \mathbb{Z}.$

其中 $\delta_s = 1$ 当 $s = 0$，否则 $\delta_s = 0$。

这反映了白噪声的独立性：只有 $s = 0$ 时有协方差，其余滞后时为零。

$$\therefore (1) \Rightarrow (1+\theta^2)\sigma^2\delta_s + \theta\sigma^2\delta_{s-1} + \theta\sigma^2\delta_{1+s}$$

$$\text{where } \begin{cases} \delta_s = 1 & s = 0 \\ \delta_s = 0 & \text{otherwise} \end{cases}$$

$$R(s) = \begin{cases} (1+\theta^2)\sigma^2 & s = 0. \\ \theta\sigma^2 & s = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

在 $s=0$

$(1+\theta^2)\sigma^2\delta_0 + \theta\sigma^2\delta_{-1} + \theta\sigma^2\delta_2$

$= (1+\theta^2)\sigma^2 + 0 + 0 = (1+\theta^2)\sigma^2$

$s = 1$

$(1+\theta^2)\sigma^2\delta_1 + \theta\sigma^2\delta_0 + \theta\sigma^2\delta_2$

$= 0 + \theta\sigma^2 + 0 = \theta\sigma^2.$

自相关函数 $r(s)$

- 自相关函数是协方差的标准化形式：

$$r(s) = Corr\{x_t, x_{t+s}\} = \frac{Cov\{x_t, x_{t+s}\}}{\sqrt{Var\{x_t\}Var\{x_{t+s}\}}} = \frac{R(s)}{R(0)}$$

$S=0 \qquad R(0) = (1+\theta^2)\sigma^2$

$S=\pm 1 \qquad R(1) = \theta\sigma^2$

$$\therefore r(1) = \frac{R(1)}{R(0)} = \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}$$

$$\therefore r(S) = \begin{cases} (1+\theta^2)\sigma^2 & S=0. \\ \theta/1+\theta^2 & S=\pm 1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\{x_t\}$ be a (weakly) stationary time series. If the time series $\{\varepsilon_t\}$ given by

$$\varepsilon_t = x_t - a x_{t-1},\ t \in \mathbf{Z}$$

is a white noise, where $a \neq 0$, show that the time series $\{e_t\}$ given by

$$e_t = x_t - a^{-1} x_{t-1},\ t \in \mathbf{Z}$$

is also a white noise. [Hint: Write down the autocovariance functions of $\{\varepsilon_t\}$ and $\{e_t\}$, each in terms of that of $\{x_t\}$, and compare them. A white noise has an autocovariance function that is proportional to the delta function.]

2. 白噪声的弱平稳性

- 均值 $\mathbb{E}[x_t] = \mu$ 是常数。
- 任意时间点之间的协方差满足：

$$Cov\{x_t, x_{t+s}\} = \begin{cases} \sigma^2, & \text{if } s=0, \\ 0, & \text{if } s \neq 0. \end{cases}$$

这说明，白噪声的协方差只与滞后 s 有关。

3. 自协方差函数

白噪声的自协方差函数可以用 Kronecker delta 函数 δ_s 表示：

$R(s) = \sigma^2\delta_s, \quad s \in \mathbb{Z}.$

其中 $\delta_s = 1$ 当 $s=0$，否则 $\delta_s = 0$。

这反映了白噪声的独立性：只有 $s=0$ 时有协方差，其余滞后时为零。

$R_\varepsilon(s) = Cov\{\varepsilon_t, \varepsilon_{t+s}\}.$

$= Cov\{x_t - a x_{t-1}, x_{t+s} - a x_{t+s-1}\}.$

$= Cov\{x_t, x_{t+s}\} - a\,Cov\{x_t, a x_{t+s-1}\} - a\,Cov\{a x_{t-1}, x_{t+s}\} + a^2\{x_{t-1}, x_{t+s-1}\}.$

$= R(s) - aR(s-1) - aR(s+1) + a^2R(s) \qquad (1)$

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$R_e(s) = \operatorname{cov}\{e_t, e_{t+s}\}$

$= \operatorname{cov}\{X_t - a^{-1}X_{t-1}, X_{t+s} - a^{-1}X_{t+s-1}\}.$

$= \operatorname{cov}\{X_t, X_{t+s}\} - a^{-1}\operatorname{cov}\{X_t, X_{t+s-1}\} - a^{-1}\operatorname{cov}\{X_{t-1}, X_{t+s}\} + (a^{-1})^2\operatorname{cov}\{X_{t-1}, X_{t+s-1}\}$

$= R(s) - a^{-1}R(s-1) - a^{-1}R(s+1) + a^{-2}R(s)$

$= a^{-2}(a^2R(s) - aR(s-1) - aR(s+1) + R(s))$ (2)

(1) = (2) $\Rightarrow$ $R_e(s) = a^{-2}R_\varepsilon(s)$

$\because$ ε_t is the white noise.

$R_\varepsilon(s) = \sigma_\varepsilon^2\delta_s$ IS $s \neq 0$, $\delta_s = 0$

$$\therefore \quad R_\varepsilon(s) = \begin{cases} \sigma_\varepsilon^2 & s=0 \\ 0 & s\neq 0. \end{cases}$$

$$R_e(s) = \begin{cases} a^{-2}\sigma_\varepsilon^2 & s=0 \\ 0 & s\neq 0 \end{cases}$$

Therefore e_t is the white noise with variance $\sigma_e^2 = a^{-2}\sigma_\varepsilon^2$

4. Find the autocovariance function of the (weakly) stationary time series $\{x_t\}$ given by

$$x_t = \varepsilon_t + \sum_{i=1}^{\infty}\phi^i\varepsilon_{t-i}, \ t\in\mathbf{Z},$$

where $|\phi| < 1$ and $\{\varepsilon_t\} \sim \mathrm{WN}(0, \sigma^2)$. [Hint: Use Theorem 2.]

4.2.1 定理 2：线性滤波序列的弱平稳性

假设时间序列 $\{x_t\}$ 是弱平稳的，其自协方差函数为 $R_x(s)$。经过线性滤波生成的序列 $\{y_t\}$ 定义为：

$$y_t = \sum_{i=-\infty}^{\infty} a_i x_{t-i}.$$

那么 $\{y_t\}$ 也是弱平稳的，其自协方差函数为：

$$R_y(s) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a_i a_j R_x(s+i-j), \quad s\in\mathbb{Z}.$$

$R(0) = \sigma^2$. 对 white noise

$R(s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi^i\phi^j R_\varepsilon(s+i-j)$

$= \sum_{i=0}^{\infty}\phi^i\phi^{s+i}\sigma^2$ $(s+i+j=0, \ j=i+s \geq 0)$

$= \sigma^2\phi^s\sum_{i=0}^{\infty}\phi^{2i} = \sigma^2\phi^s / 1-\phi^2$

$\therefore R(s) = \sigma^2 \phi^{|s|} / 1-\phi^2 \qquad s \in \mathbb{Z}.$