Contents

1 An Eigenvalue Approach to Linear Recurrences and Sequences

1.1 General Eigenvalue Method

For a Matrix $A \in \mathbb{R}^{2\times 2}$ with two distinct eigenvalues and two corresponding eigenvectors, we know that any vector is a linear combonation of v_1 and v_2, i.e.

$$\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}, \text{ and } v = av_1 + bv_2$$

Applying A repeatedly to v and using the eigenvalue property gives,

$$\begin{aligned} Av &= a\lambda_1 v_1 + b\lambda_2 v_2, \\ A^2 v &= a\lambda_1^2 v_1 + b\lambda_2^2 v_2, \\ &\vdots \\ \Rightarrow\ A^n v &= a\lambda_1^n v_1 + b\lambda_2^n v_2. \end{aligned}$$

1.2 Fibonacci Sequence

1.2.1 Introduction

The Fibonacci Sequence is a one of the most famous sequence in mathematics. It is defined by the recurrence relation:

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2 \\ F_0 = F_1 = 1 \end{cases}$$

Each term is the sum of the two preceeding terms: $1, 1, 2, 3, 5, 8 \ldots$

1.2.2 Matrix Representation of the Fibonacci Sequence

Let

$$x_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}, \ x_1 = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

By repeatedly applying the matrix A, we can express each term of the sequence as a power of A acting on x_0:

$$\begin{aligned} &x_1 = Ax_0, \\ &x_2 = Ax_1 = A(Ax_0) = A^2 x_0 \\ &\Rightarrow\ x_n = A^n x_0 \end{aligned}$$

1.2.3 Application to the Fibonacci Matrix

Let us now consider the Fibonacci matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Its eivenvalues are given by the **characteristic polynomial**

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda^2 - \lambda - 1 = 0}$$

, and a quick computation yields $\lambda = \varphi \vee -\dfrac{1}{\varphi}$.

Notice that this is exactly the same as the equation obtained from assuming $F_n = \lambda^n$ in the Fibonacci recurrence:

$$F_n = F_{n-1} + F_{n-2} \Leftrightarrow \lambda^n = \lambda^{n-1} + \lambda^{n-2} \Rightarrow \boxed{\lambda^2 = \lambda + 1}$$

1.2.4 Deriving the Closed Form

We can now express $x_n = A^n x_0$ explicitly in terms of λ_1 and λ_2. Let us consider

$$F_n = p \cdot \varphi^n + q \cdot (-\frac{1}{\varphi})^n$$

By initial contidion $F_0 = F_1 = 1$,

$$\begin{cases} p + q = 1 \\ p \cdot \varphi + q \cdot (-\frac{1}{\varphi}) = 1 \end{cases} \Rightarrow \begin{cases} p = \frac{1}{\sqrt{5}}\varphi \\ q = -\frac{1}{\sqrt{5}}\frac{1}{\varphi} \end{cases}$$

Thus,

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^{n+1} - (-\frac{1}{\varphi})^{n+1}\right]_{\#}$$

1.3 Non-homogeneous Recurrence Equation

1.3.1 Problem

Given $a_n = 3a_{n-1} + 2$ and $a_1 = 2$, $a_2 = 8$. Find the general formula for a_n.

Solution

We start by homogeneous linear equation

$$a_n = 3a_{n-1} \Rightarrow x^2 = 3x$$

Quick calculation gives $x = 0 \vee 3$, then we assume the general formula plus a displacement r.

$$a_n = p \cdot 3^n + q \cdot 0^n + r$$

By initial condition $a_1 = 2$, $a_2 = 8$

$$\begin{cases} 3p + r = 2 \\ 9p + r = 8 \end{cases} \Rightarrow \begin{cases} p = 1 \\ r = -1 \end{cases}$$

Thus the general formula for a_n is

$$a_n = 3^n - 1_{\#}$$

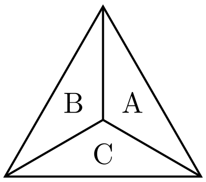
1.4 Five-Color Planar Graph Coloring

1.4.1 Problem

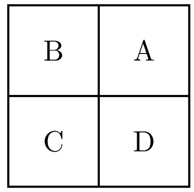
Given a polygon with n sides divided into n regions by drawing lines from the centroid to each vertex, find a general formula for the number of proper colorings of the regions using 5 colors, where adjacent regions must have different colors.

Solution

For Triangle A_3 and Square A_4



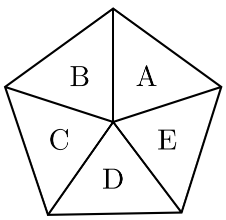
$$C_1^5 {\cdot} C_1^4 {\cdot} C_1^3 = 60$$



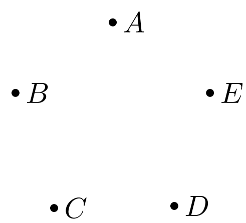
$$\begin{cases} \text{A} = \text{C} : C_1^5 \cdot C_1^4 \cdot C_1^4 = 80 \\ \text{A} \neq \text{C} : C_1^5 \cdot C_1^4 \cdot C_1^3 \cdot C_1^3 = 180 \end{cases}$$

$$80 + 180 = 260$$

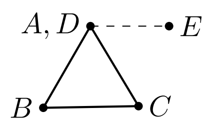
For Pentagon A_5

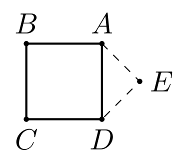


$$\Rightarrow$$



$$= \begin{cases} \text{A} = \text{D} : A_3 \times 4\,(E) \\ \text{A} \neq \text{D} : A_4 \times 3\,(E) \end{cases}$$





Recurrence Formula

Now we've obtained the recurrence formula with initial conditions $a_3 = 60$, $a_4 = 260$

$$a_{n+2} = 3a_{n+1} + 4a_n$$

Solving the equation yields

$$a_n = 4^n + 4(-1)^n$$

1.4.2 General Case

Given a polygon with n sides divided into n regions by drawing lines from the centroid to each vertex, the general formula for the number of proper colorings of the regions using k colors is

$$(k-1)^n + (-1)^n(k-1)$$

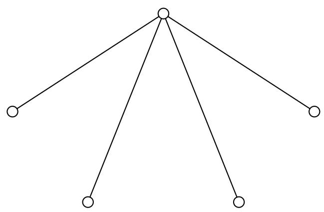
2 Zero Forcing Game

2.1 The game itself

The set of linear equation $\begin{cases} ax + by = 0 \\ a \neq 0, y = 0 \end{cases}$ implies that $x = 0$. We can generalize these condition to:

$$\begin{cases} a_1x_1 + a_2x_2 + \cdots + a_nx_n \\ a_1 \neq 0 \,\&\, x_i = 0 \text{ for } i \geq 2 \end{cases}$$

2.2 Trun into Graph



Coloring Rules

1. If a black vertex has exactly one white neighbor, then the white neighbor is forced to be black.
2. Repeat until no more changes occur.

2.3 The Adjacency Matrix

Let $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$. The **Adjacency Matrix** $A = (a_{ij})$ of G is

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

e.g. For a path graph $G \in P_n$, the adjacency matrix is

P_4

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2.4 Appendix

2.4.1 Eigenvalue of path graph P_n

Let p_n denote the characteristic polynomial of path P_n The recurrence formula is given by

$$\begin{cases} p_{n+2} = \lambda p_{n+1} + p_n \\ p_0 = 1,\ p_1 = \lambda \end{cases} \quad \overset{\text{Ansatz } r^n = p_n}{\Rightarrow} \quad r^2 = \lambda r - 1$$

Solving $r^2 = \lambda r - 1$ gives

$$r = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}$$

Observe that $|\lambda| \leq 2$
Let

$$\lambda = 2\cos\theta \Rightarrow r = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

Therefore,

$$p_n(\lambda) = \alpha e^{\mathrm{in}\theta} + \beta e^{-\mathrm{in}\theta}$$

By initial condition $p_0 = 1$, $p_1 = \lambda$

$$\begin{cases} \alpha + \beta = 1 \\ \alpha e^{i\theta} + \beta e^{-i\theta} = \lambda = 2\cos\theta \end{cases}$$

A quick calculation yields

$$\alpha = \frac{e^{i\theta}}{2i\sin\theta},\ \beta = \frac{-e^{-i\theta}}{2i\sin\theta}$$

Now $\lambda = 2\cos\theta$ and

$$\begin{aligned}
p_n(\lambda) &= \frac{e^{i\theta}}{2i\sin\theta} \cdot e^{\mathrm{in}\theta} + \frac{-e^{-i\theta}}{2i\sin\theta} \cdot e^{-\mathrm{in}\theta} \\
&= \frac{e^{i(\mathrm{n}+1)\theta} - e^{-i(\mathrm{n}+1)\theta}}{2i\sin\theta} \\
&= \frac{\sin((n+1)\theta)}{\sin\theta}
\end{aligned}$$