Contents

1 Limit

1.1 Limit of a function

1.1.1 Definition

Let f be a function defined on an open interval containing a, except possibly at a itself. Then

$$\lim_{x\to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x-a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

1.1.2 Property

Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, and let c be a constant. Then the following limit properties hold:

1. $\lim_{x\to a} [f(x) + g(x)] = L + M$
2. $\lim_{x\to a} [f(x) - g(x)] = L - M$
3. $\lim_{x\to a} [c \cdot f(x)] = cL$
4. $\lim_{x\to a} [f(x) \cdot g(x)] = L \cdot M$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. $\lim_{x\to a} [f(x)]^n = L^n \quad$ for any $n \in \mathbb{N}$
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad$ if $L \geq 0$ for even n

1.1.3 One-sided Limit and Existence of a Limit

Let $f(x)$ be a function defined near $x = a$.

Left-hand limit: $\lim_{x\to a^-} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < a - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Right-hand limit:
$\lim_{x\to a^+} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Existence of Limit
The limit of a function $f(x)$ as x approaches a exists if and only if the left-hand and right-hand limits exist and are equal:

$$\lim_{x\to a} f(x) \text{ exists} \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

1.1.4 Evaluating Limit

1. Substitute directly
2. Factoring and simplifying
3. Multiply by the conjugate of numerator or denominator
4. Use graph/table of a given function

1.1.5 Squeeze Theorem

Theorem 1.1 (Squeeze Theorem)**.** *Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on an open interval containing a, except possibly at a itself. Suppose that for all x in this interval (with $x \neq a$),*

$$f(x) \leq g(x) \leq h(x),$$

and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then,

$$\lim_{x\to a} g(x) = L.$$

For example:
for all $x \neq 0$,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying all parts by $x^2 \geq 0$, we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2,$$

by the ***Squeeze Theorem***,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

1.2 Limit with Infinities

1.2.1 Infinite Limits

If f is a function defined at every number in some open inverval containing a , except possibly at a itself, then

- $\lim\limits_{x\to a} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches a.
- $\lim\limits_{x\to a} f(x) = -\infty$ means that $f(x)$ increases without bound as x approaches a.

Limit Theorems

1. If n is a positive integer, then

(a) $\lim_{x\to 0^+} \dfrac{1}{x^n} = \infty$

(b) $\lim_{x\to 0^-} \dfrac{1}{x^n} = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$

2. if the $\lim_{x\to a} f(x) = c, c > 0,$ and $\lim_{x\to a} g(x) = 0$, then
$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \begin{cases} \infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ -\infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

3. if the $\lim_{x\to a} f(x) = c, c < 0,$ and $\lim_{x\to a} g(x) = 0$, then
$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \begin{cases} -\infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ \infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

1.2.2 Limit at Infinities

Limit at Infinity ($x \to \infty$)

- If f is a function defined at every number in some open inverval (a, ∞), the $\lim_{x\to\infty} f(x) = L$ means that L is the limit of $f(x)$ as x increases without bound.
- If f is a function defined at every number in some open inverval $(-\infty, a)$, the $\lim_{x\to-\infty} f(x) = L$ means that L is the limit of $f(x)$ as x decreases without bound.

Limit Theorems
If n is a positive integer, then

(a) $\lim_{x\to\infty} \dfrac{1}{x^n} = 0$

(b) $\lim_{x\to-\infty} \dfrac{1}{x^n} = 0$

1.2.3 Vertical and Horizontal Asymptotes

Vertical Asymptotes
A function $f(x)$ has a **vertical asymptote** at $x = a$ if at least one of the following holds:

$$\lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty.$$

This means that $f(x)$ grows without bound as x approaches a from the left or the right.

Horizontal Asymptotes
A function $f(x)$ has a **horizontal asymptote** at $y = L$ if:

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

This means that $f(x)$ approaches the constant value L as x tends to positive or negative infinity.

1.3 Continuity

2 Derivatives

3 Integrals