Contents

1 Complex Number and the Complex Plane

1.1 Complex Numbers and Their Properties

We define **the imaginary unit** i by $i^2 = -1$. A **complex number** is any number of the form

$$z = a + ib, \quad a, b \in \mathbb{R}$$

where a is real part of z, and b is the imaginary part. That is,

$$\text{Re}(z) = a, \text{Im}(z) = b$$

In addition, $z_1 = z_2$ are equal if $a_1 = a_2$, $b_1 = b_2$.

Arithmetic Operations

let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$

- Addition: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
- Subtraction: $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$
- Multiplication: $z_1 z_2 = (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + (a_1 b_2 + a_2 b_1)i - b_1 b_2$
- Division: $\dfrac{z_1}{z_2} = \dfrac{a_1 + b_1 i}{a_2 + b_2 i} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} i$

The commutitive, associative, and distributive laws also hold for complex numbers:

- Commutitive laws: $\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$
- Associative laws: $\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1(z_2 z_3) = (z_1 z_2) z_3 \end{cases}$
- Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Complex Conjugate

The **complex conjugate** of z, denoted by $\overline{z}$ or z^*, is obtained by changing the sign of its imaginary part:

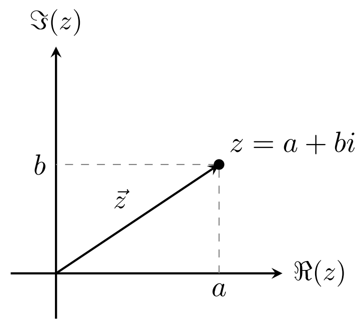
$$\overline{z} = z^* = a - bi.$$

The complex conjugate satisfies the following properties:

- $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}, \quad (z_2 \neq 0)$

1.2 Complex Plane

A complex number $z = a + bi$ can be uniquely represented by an ordered pair of real numbers (a, b). In this way, we associate each complex number with a point in the coordinate plane, or equivalently, with the position vector $\vec{z} = \langle a, b \rangle$ originating from the origin.



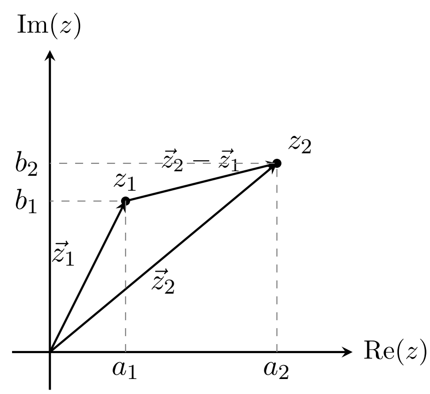
The **modulus** (or **absolute value**) of a complex number is the length of its vector representation:

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z\,\overline{z}}.$$

Thus, each complex number corresponds both to a point (a, b) in the plane and to a vector $\vec{z}$ from the origin to that point.

The difference between two complex number

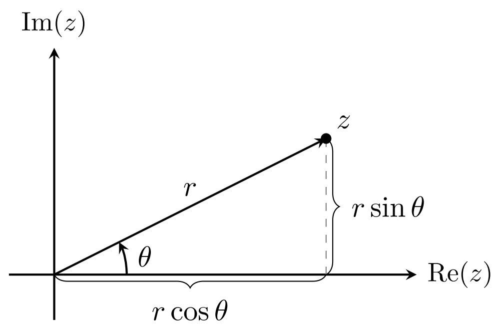
The difference between two complex numbers $z_2 - z_1$ represents the vector pointing from z_1 to z_2:



Inequalities

1.3 Polar Representation of Complex Numbers

Complex numbers can also be represented in terms of polar coordinatess.



This is called the **polar form** of a complex number:

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$, called the argument of z

Principle Argument

θ is called the **principle value** or **principle argument** of z, denoted by $\mathrm{Arg}(z)$, if

$$-\pi < \theta \leq \pi$$

In general, $\arg(z)$ and $\mathrm{Arg}(z)$ are related by

$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$