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1 Limit

1.1 Limit of a Function

1.1.1 Definition

Let f be a function defined on an open interval containing a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

1.1.2 Property

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, and let c be a constant. Then the following limit properties hold:

1. $\lim_{x \to a} [f(x) + g(x)] = L + M$
2. $\lim_{x \to a} [f(x) - g(x)] = L - M$
3. $\lim_{x \to a} [c \cdot f(x)] = cL$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. $\lim_{x \to a} [f(x)]^n = L^n \quad$ for any $n \in \mathbb{N}$
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad$ if $L \geq 0$ for even n

1.1.3 One-sided Limit and Existence of a Limit

Let $f(x)$ be a function defined near $x = a$.

Left-hand limit: $\lim_{x \to a^-} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < a - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Right-hand limit: $\lim_{x \to a^+} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Existence of Limit
The limit of a function $f(x)$ as x approaches a exists if and only if the left-hand and right-hand limits exist and are equal:

$$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

1.1.4 Evaluating Limit

1. Substitute directly
2. Factoring and simplifying
3. Multiply by the conjugate of numerator or denominator
4. Use graph/table of a given function

1.1.5 Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on an open interval containing a, except possibly at a itself. Suppose that for all x in this interval (with $x \neq a$),

$$f(x) \leq g(x) \leq h(x),$$

and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then,

$$\lim_{x \to a} g(x) = L.$$

For example:

for all $x \neq 0$,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Multiplying all parts by $x^2 \geq 0$, we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} x^2,$$

by the ***Squeeze Theorem***,

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

1.2 Limit with Infinities

1.2.1 Infinite Limits

If f is a function defined at every number in some open inverval containing a , except possibly at a itself, then

- $\lim_{x \to a} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches a.
- $\lim_{x \to a} f(x) = -\infty$ means that $f(x)$ increases without bound as x approaches a.

Limit Theorems

1. If n is a positive integer, then

(a) $\lim_{x \to 0^+} \frac{1}{x^n} = \infty$

(b) $\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$

2. if the $\lim_{x \to a} f(x) = c, c > 0,$ and $\lim_{x \to a} g(x) = 0$, then
$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ -\infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

3. if the $\lim_{x \to a} f(x) = c, c < 0,$ and $\lim_{x \to a} g(x) = 0$, then
$\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} -\infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ \infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

1.2.2 Limit at Infinities

Limit at Infinity ($x \to \infty$)

- If f is a function defined at every number in some open inverval (a, ∞), the $\lim_{x \to \infty} f(x) = L$ means that L is the limit of $f(x)$ as x increases without bound.
- If f is a function defined at every number in some open inverval $(-\infty, a)$, the $\lim_{x \to -\infty} f(x) = L$ means that L is the limit of $f(x)$ as x decreases without bound.

Limit Theorems
If n is a positive integer, then

(a) $\lim_{x \to \infty} \frac{1}{x^n} = 0$

(b) $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

1.2.3 Vertical and Horizontal Asymptotes

Vertical Asymptotes
A function $f(x)$ has a **vertical asymptote** at $x = a$ if at least one of the following holds:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

This means that $f(x)$ grows without bound as x approaches a from the left or the right.

Horizontal Asymptotes
A function $f(x)$ has a **horizontal asymptote** at $y = L$ if:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

This means that $f(x)$ approaches the constant value L as x tends to positive or negative infinity.

1.2.4 L'Hôpital's Rule

Suppose $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\pm\infty$, and that

- f and g are differentiable near a,
- $g'(x) \neq 0$ near a,
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

1.3 Continuity of a Function

Continuous at a Point a
A function f is said to be continuous at a number a if the following conditions are met:

- $f(a)$ exists
- $\lim_{x \to a} f(x)$ exists
- $f(a) = \lim_{x \to a} f(x)$

Continuous Over a Interval
A function is continuous over an interval if it is continuous at every point in the interval.

Theorems on Continuity

1. If the function f and g are continuous at a, then the functions $f + g$, $f - g$, $f \cdot g$, and f/g, $(g \neq 0)$ are also continuous at a.
2. A polynomial function is continuous everywhere.
3. A rational function is continuous everywhere except at points where the denominator is 0.
4. Intermetiate Value Theorem: Let f be a function that is continuous on the closed interval $[a, b]$. Suppose N is a number such that:

$$f(a) < N < f(b) \quad \text{or} \quad f(b) < N < f(a).$$

   Then, there exists at least one $c \in (a, b)$ such that:

$$f(c) = N.$$

2 Derivative

2.1 Definition

Let f be a function defined on an open interval containing a. The **derivative** of f at the point a, denoted by $f'(a)$, is defined as

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

provided the limit exists.

The First Principle

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Symbols for the Derivative

$$D_x f,\ \frac{d}{dx} f(x),\ y',\ \dot{y}$$

2.2 Rules for Derivatives

1. Constant Rule: $\dfrac{d}{dx} c = 0$
2. Power Rule: $\dfrac{d}{dx} x^n = n x^{n-1}$
3. Sum/Difference Rule: $\dfrac{d}{dx}[f \pm g] = \dfrac{d}{dx} f \pm \dfrac{d}{dx} g$
4. Product Rule: $\dfrac{d}{dx}[f \cdot g] = \dfrac{d}{dx} f \cdot g + f \cdot \dfrac{d}{dx} g$
5. Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{d}{dx} f \cdot g - f \cdot \frac{d}{dx} g}{g^2}$

2.3 Chain Rule and Implicit Differentiation

Chain Rule

If $h(x) = f(g(x))$ where both f and g are differentiable, then

$$h'(x) = \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Implicit Differentiation

If a function y is given implicitly by an equation involving both x and y, such as

$$F(x, y) = 0.$$

To find the derivative $\frac{dy}{dx}$, we differentiate both sides of the equation with respect to x, treating y as a function of x. This means when differentiating terms involving y, we use the chain rule and multiply by $\frac{dy}{dx}$.

Example: If

$$x^2 + y^2 = 25,$$

then differentiating both sides gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$

2.4 Derivatives of other Elementary Functions

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Exponential and Logarithhmic Functions

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$

$$\frac{d}{dx}(a^x) = a^x \ln a,\ a > 0 \,\&\, \neq 1 \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a},\ a > 0 \,\&\, \neq 1$$

2.5 Derivative of Inverse Function

Let f be a one-to-one differentiable function with inverse f^{-1}, and suppose $f'(f^{-1}(x)) \neq 0$. Then,

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Example: Let $f(x) = e^x$, so $f^{-1}(x) = \ln x$. Then,

$$\frac{d}{dx}(\ln x) = \frac{1}{\frac{d}{dx}(e^x)|_{x=\ln x}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

2.6 Higher-Order Derivatives

The second derivative, third derivative, and beyond are called higher-order derivatives. These describe how the rate of change itself changes

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^ny}{dx^n}$$
$$f'(x),\ f''(x),\ f'''(x),\ f^{(n)}(x)$$
$$\dot{y},\ \ddot{y},\ \dddot{y}$$

2.7 Rolle's Theorem, Mean Value Theorem, and Extreme Value Theorem

Rolle's Theorem

Let f be continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Extreme Value Theorem

If f is continuous on $[a, b]$, then there exist points $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

2.8 Behavior of Functions

Let f be a function defined on an interval I.

- f is **increasing** on I if for all $x_1, x_2 \in I$, with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.
- f is **decreasing** on I if for all $x_1 < x_2$, $f(x_1) > f(x_2)$.

First Derivative Test for Monotonicity

- If $f'(x) > 0$ for all $x \in I$, then f is increasing on I.
- If $f'(x) < 0$ for all $x \in I$, then f is decreasing on I.

2.8.1 Tests for Relative Extrema

First Derivative Test

Let f be continuous on an open interval containing c, and differentiable except possibly at c.

- If $f'(x)$ changes from positive to negative at $x = c$, then f has a local maximum at c.
- If $f'(x)$ changes from negative to positive at $x = c$, then f has a local minimum at c.
- If $f'(x)$ does not change sign at c, then f has no local extremum at c.

Second Derivative Test

Let f be twice differentiable near $x = c$, and suppose $f'(c) = 0$:

- If $f''(c) > 0$, then f has a local minimum at c.
- If $f''(c) < 0$, then f has a local maximum at c.
- If $f''(c) = 0$, the test is inconclusive.

2.8.2 Tests for Concavity and Inflection Points

Concavity

Let f be twice differentiable on an interval.

- If $f''(x) > 0$ for all x in the interval, then f is concave up on that interval.
- If $f''(x) < 0$, then f is concave down on that interval.

Point of Inflection

A point $x = c$ is a **point of inflection** if the concavity of f changes at c, i.e., $f''(x)$ changes sign at c.

2.9 Application of Derivatives

3 Integral

3.1 Antiderivatives

A function $F(x)$ is called an **antiderivative** of a function $f(x)$ on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

The set of antiderivatives of f is called the indefinite integral of f

Notation:
The indefinite integral of $f(x)$ is denoted by

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant called the **constant of integration**.

3.2 Rules and Formulas

Rules

$$\int f(x)\,dx = F(x) + C \iff F'(x) = f(x)$$

$$\int af(x)\,dx = a\int f(x)\,dx$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

The Basics

$$\int dx = x + C$$

$$\int a\,dx = ax + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$$

Exponential and Logarithhmic Functions

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

Trigonometric Functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C \qquad \int \csc x\,dx = \ln|\csc x - \cot x| + C$$

Inverse Trigonometric Functions
Let $a > 0$ be a constant.

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{-1}{a^2 + x^2}\,dx = \frac{-1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right) + C, \quad |x| > a \qquad \int \frac{-1}{x\sqrt{x^2 - a^2}}\,dx = \frac{-1}{a}\csc^{-1}\left(\left|\frac{x}{a}\right|\right) + C, \quad |x| > a$$

3.3 Definite Integral

Definition of Riemann Sum

Let $f(x)$ be defined on the closed interval $[a, b]$, and divide the interval into n subintervals: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. Let $\Delta x_i = x_i - x_{i-1}$ be the width of the i-th subinterval, and $c_i \in [x_{i-1}, x_i]$ be any point in that subinterval.Then the **Riemann Sum** is:

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

Definition of a Definite Integral

Let f be defined on $[a, b]$ with the Riemann Sum for f over $[a, b]$ written as $\sum_{i=1}^{n} f(c_i)\Delta x_i$.

If

- $\max \Delta x_i$ is the length of the largest subinterval in the partition, and
- the limit $\lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists

Then the definite integral of f from a to b is defined as:

$$\int_a^b f(x)\,dx = \lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

Property of Definite Integrals

1. If f is defined on $[a, b]$, and $\lim_{max\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then f is integrable on $[a, b]$.
2. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

3. If $f(x)$, $g(x)$, and $h(x)$ are integrable on $[a, b]$, then
   (a) $\int_a^a f(x)\,dx = 0$
   (b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
   (c) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
   (d) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even
   (e) $\int_{-a}^a f(x)\,dx = 0$, if $f(x)$ is odd
   (f) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$
   (g) $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$, provided that $g(x) \le f(x) \le h(x)$ on $[a, b]$
   (h) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

3.4 Fundamental Theorems of Calculus

First Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$ at every point x in $[a, b]$

Second Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a)$$

Thus,

$$\boxed{\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x))\cdot u'(x) - f(v(x))\cdot v'(x)}$$

3.5 Integration Techniques

3.5.1 U-Substitution

The **u-substitution method** is used to evaluate integrals by making a change of variables. If an integral contains a composite function, we can simplify it using a substitution.
Let $u = g(x)$, then:

$$\frac{du}{dx} = g'(x) \quad \Rightarrow \quad du = g'(x)\,dx$$

So:

$$\int f(g(x))\,g'(x)\,dx \Rightarrow \int f(u)\,du$$

After integration, substitute back $u = g(x)$ to return to the original variable.

3.5.2 Trigonometric Substitution

Trigonometric substitution is a technique used to evaluate integrals involving square roots of quadratic expressions. The key idea is to use a trigonometric identity to simplify the integrand. Consider using trigonometric substitution when integrand contains expressions of the form:

- $\sqrt{a^2 - x^2}$ — use $x = a \sin x$
- $\sqrt{a^2 + x^2}$ — use $x = a \tan x$
- $\sqrt{x^2 - a^2}$ — use $x = a \sec x$

Trigonometric Identities Used

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $\sec^2 x - 1 = \tan^2 x$

3.5.3 Partial Fraction Decomposition

Partial fraction decomposition is a method used to break a rational function into simpler fractions that are easier to integrate. Given a rational function:

$$\frac{P(x)}{Q(x)} \quad \text{where } \deg P(x) < \deg Q(x),$$

we can express it as a sum of simpler rational expressions depending on the factorization of $Q(x)$.

Types of Decompositions

Let $Q(x)$ be factored as:

$$Q(x) = (x - r_1)^{k_1}(x - r_2)^{k_2} \cdots (x^2 + bx + c)^m \cdots$$

Then:

- For each distinct linear factor $(x - r)^k$, include terms:
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_k}{(x - r)^k}$$
- For each irreducible quadratic factor $(x^2 + bx + c)^m$, include:
$$\frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2} + \cdots + \frac{Yx + Z}{(x^2 + bx + c)^m}$$

Process

1. If improper ($\deg P(x) \geq \deg Q(x)$), perform long division first.
2. Factor the denominator $Q(x)$.
3. Set up partial fractions based on the types above.
4. Multiply both sides by $Q(x)$ to eliminate denominators.
5. Solve for constants by plugging in values or equating coefficients.
6. Integrate each term individually.

3.5.4 Integration by Parts

Integration by parts is based on the product rule for differentiation and is given by:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Where:

- u = part to differentiate (becomes du)
- dv = part to integrate (becomes v)

Mnemonic: LIATE Rule
Choose u based on the following priority:

1. **L**ogarithmic (e.g., $\ln x$)
2. **I**nverse trig (e.g., $\tan^{-1} x$)
3. **A**lgebraic (e.g., x^2)
4. **T**rigonometric (e.g., $\sin x$)
5. **E**xponential (e.g., e^x)

3.5.5 The DI Method

1. Choose $f(x)$ to differentiate, and $g(x)$ to integrate.
2. Alternate the signs starting with $+$.
3. Multiply diagonally (Derivative term $\times$ Integral term just below) and alternate the signs.
4. Stop the process when:
   - The derivative reaches zero ($f^{(n)}(x) = 0$)
   - Repeated derivatives cycle or become too complex
   - Cyclic or repeating patterns
   - The remaining integral is simpler to evaluate directly

Sign	**Derivative(D)**	**Integral(I)**
$+$	$f(x)$	$g(x)\,dx$
$-$	$f'(x)$	$\int g(x)\,dx$
$+$	$f''(x)$	$\iint g(x)\,dx$
$-$	$f^{(3)}(x)$	$\iiint g(x)\,dx$
$\vdots$	$\vdots$	$\vdots$

Final Expression: Combine diagonals with alternating signs:

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)\,dx + \cdots$$

3.5.6 The King's Rule for Definite Integral

King's Rule is a clever substitution technique in which we let $u = a + b - x$, and thus

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

By averaging both expressions:

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\,dx$$

This is useful when $f(x) + f(a+b-x)$ is a constant or simplifies significantly.

3.5.7 Feynman's Integration Technique for Definite Integral

Leibniz Integral Rule, or Differentiation under the Integral Sign is a powerful technique used to evaluate integrals that depend on a parameter. This method became widely known through physicist Richard Feynman, who used it extensively in both theoretical and applied contexts. It allows us to compute an integral by introducing a parameter, differentiating with respect to that parameter under the integral sign, simplifying the expression, and then integrating the result.

Leibniz Integral Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(v(x),t)\cdot\frac{dv}{dx} - f(u(x),t)\cdot\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)\,dt$$

If we take $u(x)$ and $v(x)$ as constants a and b, then:

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt$$

Conditions for Validity
To apply this technique, we generally require:

- $f(x,t)$ and $\partial f/\partial x$ are continuous in a region around the domain of integration.
- The limits $u(x), v(x)$ are differentiable functions of x.
- The integral $I(x)$ converges.

Introducing a Parameter α to Simplify a Complex Integral
One of the most clever applications of this technique is to evaluate a complicated integral by introducing a parameter α that does not initially exist in the original integral. The idea is to construct a new, easier-to-handle integral:

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$

such that:

- The original integral is recovered by evaluating $I(\alpha)$ at some specific value of α.

- Differentiating with respect to α simplifies the integrand.

Steps:

1. Embed the difficult integral into a parameterized family $I(\alpha)$.
2. Compute $\dfrac{dI}{d\alpha}$ under the integral sign.
3. Integrate $\dfrac{dI}{d\alpha}$ with respect to α to recover $I(\alpha)$.
4. Evaluate $I(\alpha)$ at the desired value (e.g., $\alpha = 0$) to obtain the original result.

Example 1 Evaluate:

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx$$

Step 1: Parameterize the Integrand
Let

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx$$

Note that:

$$I(0) = \int_0^1 \frac{x^0-1}{\ln x}\,dx = 0, \text{ and } I(2) \text{ is the original integral}$$

Step 2: Now we compute

$$\begin{aligned} \frac{d}{d\alpha} I(\alpha) &= \int_0^1 \frac{\partial}{\partial\alpha}\frac{x^\alpha-1}{\ln x}\,dx = \int_0^1 x^\alpha dx \\ &= \frac{1}{\alpha+1} x^{\alpha+1}\bigg|_0^1 \\ &= \frac{1}{\alpha+1} \end{aligned}$$

Step 3: Recover $I(\alpha)$

$$\begin{aligned} \int I(\alpha) &= \int \frac{1}{\alpha+1}\,d\alpha \\ &= \ln(\alpha+1) + C \end{aligned}$$

Recall $I(0) = 0 \Rightarrow C = 0$. So:

$$0 = \ln(\alpha+1) + C \Rightarrow C = 0$$

Hence:

$$I(\alpha) = \ln(\alpha+1)$$

Step 4: Evaluate $I(\alpha)$ at $\alpha = 2$

$$I(2) = \ln(2+1) = \ln 3$$

Answer:

$$\int_0^1 \frac{x^2-1}{\ln x}\,dx = \ln 3$$

Example 2 Evaluate:

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

Step 1: Introducing an auxiliary exponential factor
Let

$$I(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx, \quad \alpha > 0$$

Step 2: Now we compute

$$\frac{dI}{d\alpha} = -\int_0^\infty e^{-\alpha x}\sin x\,dx$$

This integral is elementary:

$$\int_0^\infty e^{-\alpha x}\sin x\,dx = \frac{1}{1+\alpha^2} \Rightarrow \frac{dI}{d\alpha} = -\frac{1}{1+\alpha^2}$$

Step 3: Recover $I(\alpha)$

$$I(\alpha) = -\int \frac{1}{1+\alpha^2}\,d\alpha = -\tan^{-1}(\alpha) + C$$

As $\alpha \to \infty$, $I(\alpha) \to 0$. So:

$$0 = -\tan^{-1}(\infty) + C = -\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2}$$

Hence:

$$I(\alpha) = -\tan^{-1}(\alpha) + \frac{\pi}{2}$$

Step 4: Evaluate $I(\alpha)$ as $\alpha \to 0$

$$\lim_{\alpha\to 0} I(\alpha) = -\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$$

Answer:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

3.6 Improper Integral

In some cases, definite integrals involve infinite intervals or integrands that become unbounded. Such integrals are called **improper integrals**. We define these using limits.

Infinite Interval of Integration

Let $f(x)$ be a function defined on $[a, \infty)$. Then the improper integral of f from a to ∞ is defined as:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty} \int_a^b f(x)\,dx$$

Similarly, if f is defined on $(-\infty, b]$, we define:

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

If f is defined on $(-\infty, \infty)$, then:

$$\int_{-\infty}^\infty f(x)\,dx := \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx$$

for some finite number $c \in \mathbb{R}$.

Note: Both limits must exist and be finite for the integral to converge.

Discontinuous Integrand

Suppose f is continuous on $(a, b]$ but has an infinite discontinuity at a. Then:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to a^+} \int_\epsilon^b f(x)\,dx$$

Similarly, if f has an infinite discontinuity at b, then:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to b^-} \int_a^\epsilon f(x)\,dx$$

If the discontinuity is at an interior point $c \in (a, b)$, split the integral:

$$\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Each part must be interpreted as a limit, and the total integral converges if both one-sided integrals converge.

Absolute vs Conditional Convergence

- If $\displaystyle\int_a^\infty |f(x)|\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ is said to be **absolutely convergent**.
- If $\displaystyle\int_a^\infty f(x)\,dx$ converges but $\displaystyle\int_a^\infty |f(x)|\,dx$ diverges, it is **conditionally convergent**.

3.7 Area

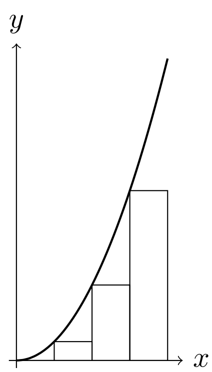
3.7.1 Approximating the Area Under the Curve

Rectangular Approximation

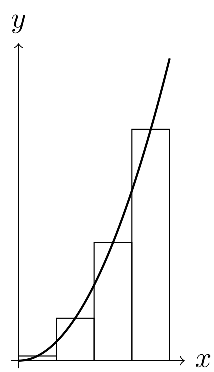
The area under the curve using n rectangles of equal length is approximately:

$$\sum_{i=1}^{n}(\text{area of rectangle}) = \begin{cases} \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ left-endpoint rectangles} \\ \sum_{i=1}^{n} f(x_i)\Delta x \text{ right-endpoint rectangles} \\ \sum_{i=1}^{n} f(\frac{x_i + x_{i+1}}{2})\Delta \text{ midpoint rectangles} x \end{cases}$$

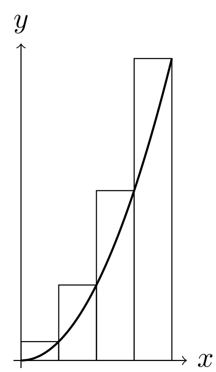
where $\Delta x = \dfrac{b-a}{n}$ and $a = x_0 < x_1 < x_2 < \cdots < x_n = b$



Left Endpoint

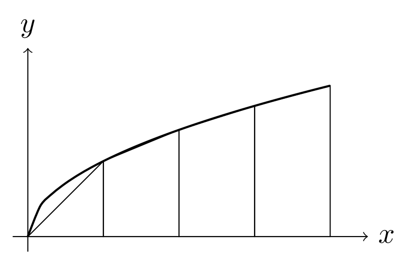


Midpoint



Right Endpoint

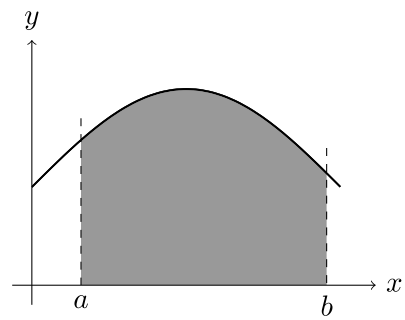
Trapezoidal Approximation



If f is continuous, the area under the curve of f from $x = a$ to $x = b$ is:

$$\text{Area} \simeq \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

3.7.2 Area Under a Curve



The area under the graph of a continuous function $f(x)$ over the interval $[a, b]$ is given by the definite integral:

$$A = \int_a^b f(x)\,dx$$

If $f(x) \geq 0$ on $[a, b]$, this integral gives the area between the curve and the x-axis. If $f(x)$ takes negative values, the integral represents **signed area**.

3.7.3 Area Between Two Curves

Proofs

3.8 testing