Contents

1 Definition and The First Principle

1.1 Definition

Let f be a function defined on an open interval containing a. The **derivative** of f at the point a, denoted by $f'(a)$, is defined as

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \end{aligned}$$

1.2 The derivative at a Point a

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

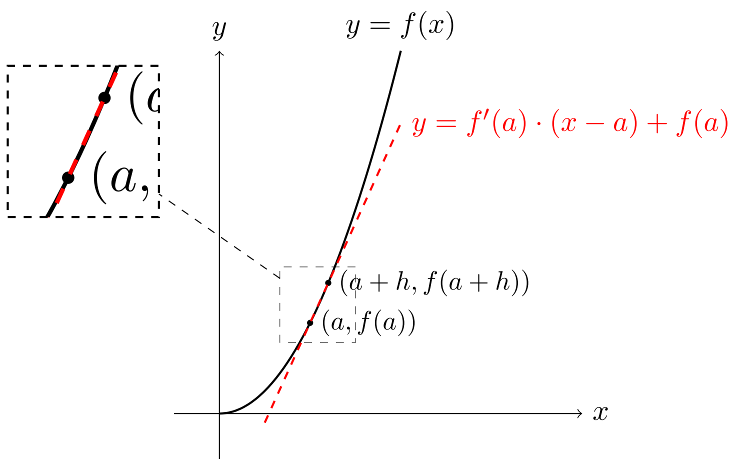
provided the limit exists.

1.3 Geometric Meaning

To find the slope of the tangent line to the curve $y = f(x)$ at a point $x = a$, we consider the slope of the secant line between two points:

$$\frac{f(a+h)-f(a)}{h}$$

As $h \to 0$, this secant slope approaches the derivative $f'(a)$, which is the slope of the tangent line at $x = a$.



1.4 Symbols for the Derivative

$$D_x f,\ \frac{d}{dx}f(x),\ y',\ \dot{y}$$

2 Rules and Derivatives of Elementary Functions

2.1 Derivative Rules

1. Constant Rule: $\dfrac{d}{dx}c = 0$
2. Power Rule: $\dfrac{d}{dx}x^n = nx^{n-1}$
3. Sum/Difference Rule: $\dfrac{d}{dx}[f \pm g] = \dfrac{d}{dx}f \pm \dfrac{d}{dx}g$
4. Product Rule: $\dfrac{d}{dx}[f \cdot g] = \dfrac{d}{dx}f \cdot g + f \cdot \dfrac{d}{dx}g$
5. Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{d}{dx}f \cdot g - f \cdot \frac{d}{dx}g}{g^2}$

2.2 Trigonometric Functions

$$\tfrac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

2.3 Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

2.4 Exponential and Logarithhmic Functions

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$

$$\frac{d}{dx}(a^x) = a^x \ln a,\ a > 0 \,\&\, \neq 1 \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a},\ a > 0 \,\&\, \neq 1$$

2.5 Derivative of Inverse Function

Let f be a one-to-one differentiable function with inverse f^{-1}, and suppose $f'(f^{-1}(x)) \neq 0$. Then,

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Example:

Let $f(x) = e^x$, so $f^{-1}(x) = \ln x$. Then,

$$\frac{d}{dx}(\ln x) = \frac{1}{\frac{d}{dx}(e^x)|_{x=\ln x}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

2.6 Chain Rule

If $h(x) = f(g(x))$ where both f and g are differentiable, then

$$h'(x) = \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

3 Advanced Differentiation

3.1 Implicit Differentiation

If a function y is given implicitly by an equation involving both x and y, such as

$$F(x, y) = 0.$$

To find the derivative $\frac{dy}{dx}$, we differentiate both sides of the equation with respect to x, treating y as a function of x.

Example:

If

$$x^2 + y^2 = 25,$$

then differentiating both sides gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$

3.2 Higher-Order Derivatives

The second derivative, third derivative, and beyond are called higher-order derivatives. These describe how the rate of change itself changes.

$$\begin{cases} \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^ny}{dx^n} \\ f'(x),\ f''(x),\ f'''(x),\ f^{(n)}(x) \\ \dot{y},\ \ddot{y},\ \dddot{y} \end{cases}$$

3.3 Derivative of Parametric Functions

Given a parametric curve:

$$x = x(t) \quad y = y(t)$$

the derivative of y w.r.t x is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad (\text{provided } \frac{dx}{dt} \neq 0)$$

3.4 Derivative of Polar Functions

For polar representations, $r = f(x)$ and that $x = r \cdot \cos\theta$, $y = r\sin\theta$.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta} \cdot \cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta} \cdot \sin\theta + r\cos\theta$$

Thus,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{\frac{dr}{d\theta} \cdot \sin\theta + r\cos\theta}{\frac{dr}{d\theta} \cdot \cos\theta - r\sin\theta}$$

3.5 Derivative of Vector-valued Function

Assume the position of a particle is given by $r = \langle x, y \rangle$, it's velocity vector is given by

$$\frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

and the magnitude **speed** is thus

$$\left|\left|\frac{dr}{dt}\right|\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Similarily, acceleration vector is

$$\frac{d^2r}{dt^2} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$$

and the magnitude of acceleration is

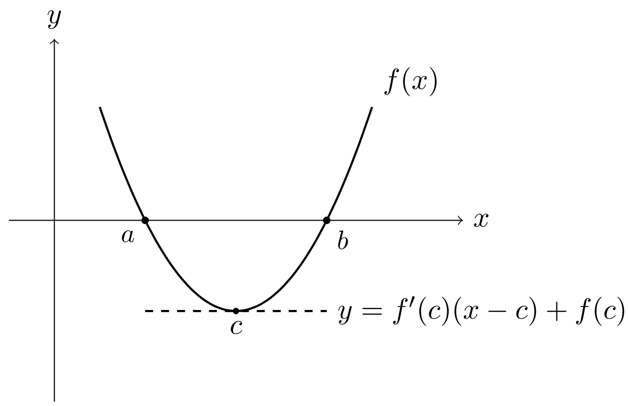
$$\left|\left|\frac{d^2r}{dt^2}\right|\right| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$$

4 Theorems

4.1 Rolle's Theorem

Let f be continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that

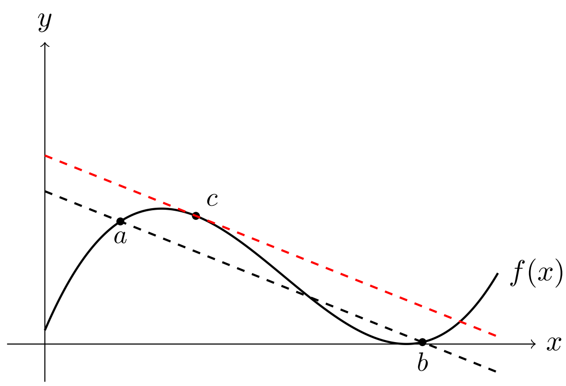
$$f'(c) = 0.$$



4.2 Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



4.3 Cauchy's Mean Value Theorem

Let f and g be functions continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists at least one point $c \in (a, b)$ such that:

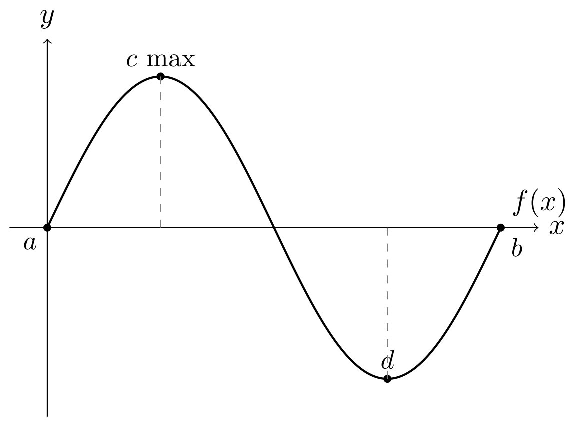
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

For $g(x) = x$, Cauchy's Mean Value Theorem reduces to Mean Value Theorem.

4.4 Extreme Value Theorem

If f is continuous on $[a, b]$, then there exist points $c, d \in [a, b]$ such that

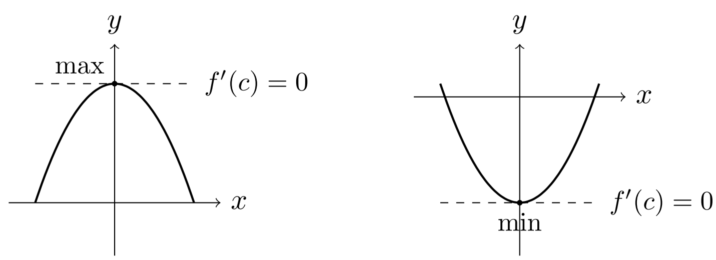
$$f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$



5 Behavior of Functions

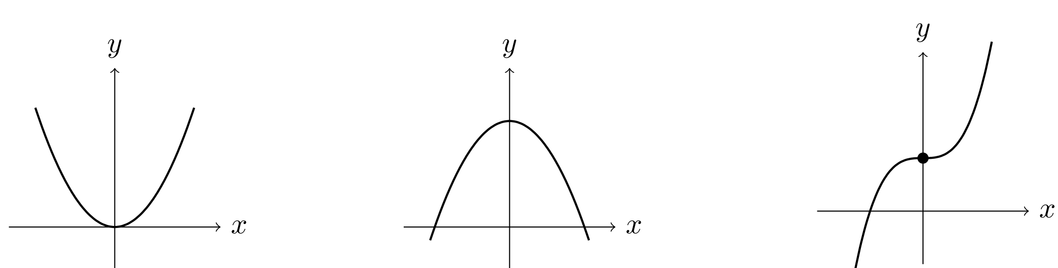
5.1 Critical Points and Extrema

- **Critical Point:** A point c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist.
- **Local Maximum:** $f(c)$ is a local maximum if $f(c) \geq f(x)$ for all x near c.
- **Local Minimum:** $f(c)$ is a local minimum if $f(c) \leq f(x)$ for all x near c.



5.2 Concavity and Inflection Points

- **Concave Up:** $f''(x) > 0$ on an interval $\implies$ graph lies above tangent lines.
- **Concave Down:** $f''(x) < 0$ on an interval $\implies$ graph lies below tangent lines.
- **Inflection Point:** A point where $f''(x)$ changes sign.



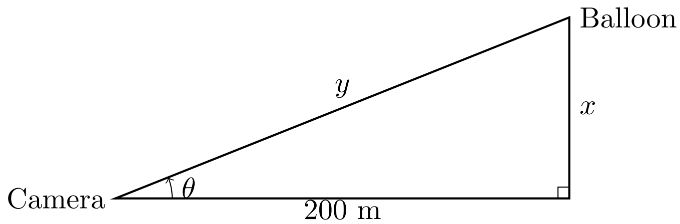
Concave Up $f''(x) > 0$ Concave Down $f''(x) < 0$ Inflection Point $f''(x) = 0$

6 Applications

6.1 Related Rates

6.1.1 Angle of Elevation Problem

A camera on the ground 200 meters away from a hot air balloon, also on the ground, records the balloon rising into the sky at a constant rate of 10 m/sec. How fast is the camera's angle of elevation changing when the balloon is 150m in the air?



Solution

Given $\dfrac{dx}{dt} = 10$ m/sec and $\tan\theta = \dfrac{x}{200}$, we want $\dfrac{d\theta}{dt}$ at $x = 150$. We differentiate both sides with respect to t.

$$\frac{d}{dt}\left(\tan\theta = \frac{x}{200}\right)$$
$$\Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{1}{200}\frac{dx}{dt}$$
$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{200\sec^2\theta}\frac{dx}{dt}\bigg|_{\frac{dx}{dt}=10} = \frac{1}{20}\cos^2\theta$$

We have $\dfrac{d\theta}{dt} = \dfrac{1}{20}\cos^2\theta$, and at $x = 150$, $y = 250$ (by Pythagorean Theorem: $y^2 = 150^2 + 200^2$). Thus,

$$\frac{d\theta}{dt}\bigg|_{x=150} = \frac{1}{20}\cos^2\theta$$
$$= \frac{1}{20}\left(\frac{4}{5}\right)^2$$
$$= \frac{4}{125} = .032$$

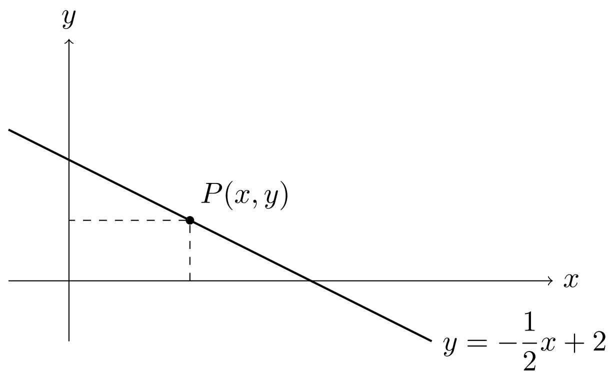
Thus, the angle of elevation changes at .032 radian/sec when the balloon is 150m in the air.

6.1.2 Inverted Cone (Water Tank) Problem

6.2 Optimization Problems

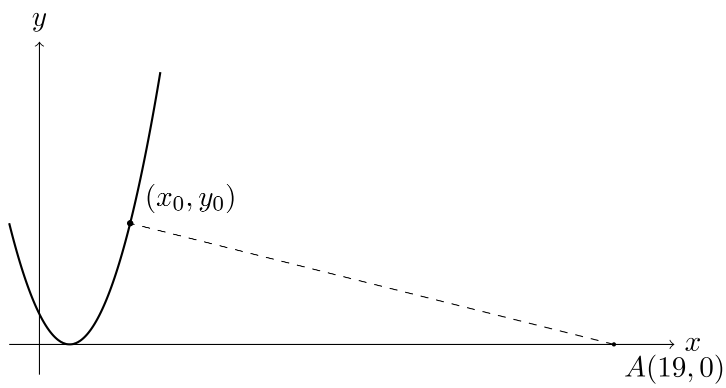
6.2.1 Area Problem

The graph of $y = -\dfrac{1}{2}x + 2$ encloses a region with the x-axis and y-axis in the first quadrant. A rectangle in the enclosed region has a vertex at the origin and the opposite vertex on the graph of y. Find the dimensions of the rectangle so that its area is a maximum.



6.2.2 Shortest Distance Problem

Find the shortest distance between the point $(19, 0)$ and the parabola $y = (x-1)^2$.



Solution

Let D be the distance between the point and the parabola

$$\begin{aligned} D &= \sqrt{(x-19)^2 + (y-0)^2} \\ &= \sqrt{(x-19)^2 + ((x-1)^2 - 0)^2} \end{aligned}$$

To simplify the calculation, we consider $L = D^2$:

$$L = D^2 = (x-19)^2 + (x-1)^4$$

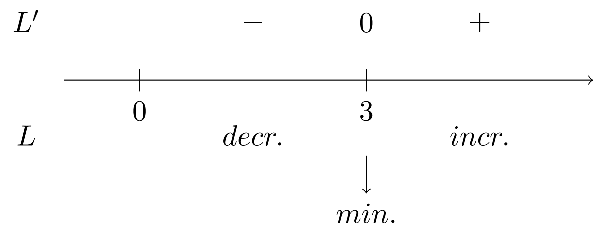
Differentiate L:

$$\begin{aligned} \frac{dL}{dx} &= \frac{d}{dx}\left((x-19)^2 + (x-1)^4\right) \\ &= 2(x-19) + 4(x-1)^3 \end{aligned}$$

Set $L = 0$:

$$\begin{aligned} \frac{dL}{dx} &= 2(x-19) + 4(x-1)^3 \\ &= (x-3)(2x^2+7) \\ \Rightarrow x &= 3 \end{aligned}$$

Apply the First Derivative Test:



Since $x = 3$ is the only relative min point, it is the absolute min.
Thus, the shortest distance is

$$D|_{x=3} = \sqrt{(3-19)^2 + (3-1)^4} = 4\sqrt{17}$$

6.3 Linear Approximation (First-Order Taylor Expantion)

If f is differentiable at $x = a$, then near a, the function $f(x)$ is approximated by

$$f(x) \approx f(a) + f'(a)(x-a)$$

Example:

for x near 0, $\sin x$ can be approximated by $\sin x \approx \sin(0) + \cos(0) \cdot x = x$