Contents

1 Limit of a Function

1.1 Definition

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that L is the **limit** of $f(x)$ as $x \to a$, and write

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

1.2 Property

Let $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, and let c be a constant. Then the following limit properties hold:

1. $\lim_{x \to a} [f(x) + g(x)] = L + M$
2. $\lim_{x \to a} [f(x) - g(x)] = L - M$
3. $\lim_{x \to a} [c \cdot f(x)] = cL$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = L \cdot M$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, if $M \neq 0$
6. $\lim_{x \to a} [f(x)]^n = L^n \quad \text{for any } n \in \mathbb{N}$
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \quad \text{if } L \geq 0 \text{ for even } n$

1.3 One-sided Limit and Existence of a Limit

Let $f(x)$ be a function defined near $x = a$.

Left-hand limit: $\lim_{x \to a^-} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < a - x < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Right-hand limit: $\lim_{x \to a^+} f(x) = L$
if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < x - a < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Existence of Limit
The limit of a function $f(x)$ as x approaches a exists if and only if the left-hand and right-hand limits exist and are equal:

$$\lim_{x \to a} f(x) \text{ exists} \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

2 Limit at Infinities

2.1 Infinite Limits

If f is a function defined at every number in some open inverval containing a , except possibly at a itself, then

- $\lim_{x \to a} f(x) = \infty$ means that $f(x)$ increases without bound as x approaches a.
- $\lim_{x \to a} f(x) = -\infty$ means that $f(x)$ increases without bound as x approaches a.

Limit Laws

1. If n is a positive integer, then

   (a) $\lim_{x \to 0^+} \frac{1}{x^n} = \infty$

   (b) $\lim_{x \to 0^-} \frac{1}{x^n} = \begin{cases} \infty & \text{if } n \text{ is even} \\ -\infty & \text{if } n \text{ is odd} \end{cases}$

2. if the $\lim_{x \to a} f(x) = c, c > 0,$ and $\lim_{x \to a} g(x) = 0$, then
   $\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ -\infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

3. if the $\lim_{x \to a} f(x) = c, c < 0,$ and $\lim_{x \to a} g(x) = 0$, then
   $\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} -\infty & \text{if } g(x) \text{ approaches 0 through positive values} \\ \infty & \text{if } g(x) \text{ approaches 0 through negative values} \end{cases}$

2.2 Limit as $x \to \infty$

Limit at Infinity ($x \to \infty$)

- If f is a function defined at every number in some open inverval (a, ∞), the $\lim_{x \to \infty} f(x) = L$ means that L is the limit of $f(x)$ as x increases without bound.
- If f is a function defined at every number in some open inverval $(-\infty, a)$, the $\lim_{x \to -\infty} f(x) = L$ means that L is the limit of $f(x)$ as x decreases without bound.

Limit Laws
If n is a positive integer, then

(a) $\lim_{x \to \infty} \frac{1}{x^n} = 0$

(b) $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

2.3 Vertical and Horizontal Asymptotes

Vertical Asymptotes

A function $f(x)$ has a **vertical asymptote** at $x = a$ if at least one of the following holds:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

This means that $f(x)$ grows without bound as x approaches a from the left or the right.

Horizontal Asymptotes

A function $f(x)$ has a **horizontal asymptote** at $y = L$ if:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

This means that $f(x)$ approaches the constant value L as x tends to positive or negative infinity.

3 Evaluation Techniques

3.1 Direct Substitution

$$\lim_{x \to 2} (3x^2 + 2) = 3(2)^2 + 2 = 14$$

3.2 Factorization

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

Factorizing:

$$\frac{x^2 - 9}{x - 3} = \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = x + 3$$

Then,

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} (x + 3) = 6$$

3.3 Rationalization

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$$

Multiply by the conjugate of numerator:

$$\frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1} = \frac{1 + x - 1}{x(\sqrt{1+x} + 1)} = \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x} + 1)} = \frac{1}{\sqrt{1+x} + 1}$$

Then,

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{2}$$

3.4 Use graph/table of a given function

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2.001	2.01	2.1

, find $\lim_{x\to 1} f(x)$

Solution:

$$\lim_{x\to 1} f(x) = 2$$

3.5 Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on an open interval containing a, except possibly at a itself. Suppose that for all x in this interval (with $x \neq a$),

$$f(x) \leq g(x) \leq h(x)$$

and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

Then,

$$\lim_{x\to a} g(x) = L$$

For example:

for all $x \neq 0$,

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Multiplying all parts by $x^2 \geq 0$, we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since

$$\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2$$

by the **Squeeze Theorem**,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

3.6 Small-Angle Approximation

When $x \to 0$ in radian,

$$\sin x \approx \tan x \approx x, \quad \cos x \approx 1 - \frac{x^2}{2} \Rightarrow 1 - \cos x \approx \frac{x^2}{2}$$

For example:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

Use the identity:

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

So:

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2} = \frac{2\left(\sin\left(\frac{x}{2}\right)\right)^2}{x^2}$$

Apply small-angle approximation:

$$\frac{2\left(\sin\left(\frac{x}{2}\right)\right)^2}{x^2} \approx \frac{2\left(\frac{x}{2}\right)^2}{x^2} = \frac{2}{x^2} \cdot \frac{x^2}{4} = \frac{2}{\cancel{x^2}} \cdot \frac{\cancel{x^2}}{4} = \frac{1}{2}$$

Thus,

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

3.7 L'Hôpital's Rule

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\pm\infty$, and that

- f and g are differentiable near a,
- $g'(x) \neq 0$ near a,
- $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

For example:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$$

Apply L'Hôpital's Rule since it's $\frac{0}{0}$:

$$= \lim_{x\to 0} \frac{e^x - 1}{2x}$$

Still $\frac{0}{0}$, apply L'Hôpital's Rule again:

$$= \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

4 Famous Limits

1. $\lim_{x \to 0} \dfrac{\sin(x)}{x} = 1$

2. $\lim_{x \to 0} \dfrac{1 - \cos(x)}{x^2} = \dfrac{1}{2}$

3. $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$

4. Euler's Number:

   $e = \lim_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x$

5 Continuity of a Function

5.1 Continuous at a Point

A function f is said to be continuous at a number a if the following conditions are met:

- $f(a)$ exists
- $\lim_{x \to a} f(x)$ exists
- $f(a) = \lim_{x \to a} f(x)$

5.2 Continuous Over a Interval

A function is continuous over an interval if it is continuous at every point in the interval.

Theorems on Continuity

1. If the function f and g are continuous at a, then the functions $f + g$, $f - g$, $f \cdot g$, and f/g, $(g \neq 0)$ are also continuous at a.

2. A polynomial function is continuous everywhere.

3. A rational function is continuous everywhere except at points where the denominator is 0.

4. **Intermediate Value Theorem:** Let f be a function that is continuous on the closed interval $[a, b]$. Suppose N is a number such that:

   $$f(a) < N < f(b) \quad \text{or} \quad f(b) < N < f(a).$$

   Then, there exists at least one $c \in (a, b)$ such that:

   $$f(c) = N.$$

5.3 Types of Discontinuity

A function $f(x)$ is said to be **discontinuous** at a point $x = a$ if the limit $\lim_{x \to a} f(x)$ does not exist or does not equal $f(a)$. Discontinuities can be classified into several types:

1. **Removable Discontinuity:** The limit $\lim_{x \to a} f(x)$ exists and is finite, but either $f(a)$ is not defined, or $f(a) \neq \lim_{x \to a} f(x)$.
$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$$
Here, $\lim_{x \to 1} f(x) = 2$, but $f(1) = 0 \neq 2$.

2. **Jump Discontinuity:** The left-hand limit $\lim_{x \to a^-} f(x)$ and right-hand limit $\lim_{x \to a^+} f(x)$ both exist but are not equal.
$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x \geq 0 \end{cases}$$
Then $\lim_{x \to 0^-} f(x) = 1$ and $\lim_{x \to 0^+} f(x) = 2$.

3. **Infinite Discontinuity:** The limit $\lim_{x \to a} f(x)$ diverges to infinity or negative infinity. That is, $f(x)$ increases or decreases without bound near $x = a$.
$$f(x) = \frac{1}{x} \quad \text{has an infinite discontinuity at } x = 0.$$

4. **Oscillatory Discontinuity:** The function oscillates infinitely near $x = a$, so the limit does not exist due to wild fluctuations.
$$f(x) = \sin\left(\frac{1}{x}\right) \text{ has an oscillatory discontinuity at } x = 0.$$