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L'Hôpital's Rule **1**

L'Hôpital's Rule

Statement of the Rule

Let f and g be functions that are differentiable on an open interval I containing a, except possibly at a itself. Suppose:

- $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ **or** $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$,
- $g'(x) \neq 0$ for x near a but not equal to a,
- and the limit $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists or is $\pm\infty$,

Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Proof of L'Hôpital's Rule

Case 1: $f(a) = g(a) = 0$

Since $g'(x)$ is non-zero near $x = a$, there is an interval (a, b) such that $g'(x)$ is positive or negative for $x \in (a, b)$. Then by Cauchy's Mean Value Theorem for the interval $[a, x]$ there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Since $f(a) = g(a) = 0$, this reduces to

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

Now, as $x \to a^+$, we also have $c \to a^+$. If the limit $\lim_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L$ exists, then by continuity of the limit,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)} = L.$$

Similarly, if we consider $x \to a^-$, then $c \to a^-$ and

$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{c\to a^-} \frac{f'(c)}{g'(c)} = L.$$

If both one-sided limits exist and are equal, then the two-sided limit exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This proves L'Hôpital's Rule in the case where $f(a) = g(a) = 0$.

Case 2: $f(a) = g(a) = \pm\infty$

We assume that

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$$

By the formal definition of right-hand limits, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$L - \epsilon < \frac{f'(x)}{g'(x)} < L + \epsilon, \quad \forall a < x < a + \delta$$

Case 3: $x \to \pm\infty$