# Contents

# 1 An Eigenvalue Approach to Linear Recurrences and Sequences

## 1.1 General Eigenvalue Method

For a Matrix $A \in \mathbb{R}^{2\times 2}$ with two distinct eigenvalues and two corresponding eigenvectors, we know that any vector is a linear combonation of $v_1$ and $v_2$, i.e.

$$\begin{cases} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \end{cases}, \text{ and } v = av_1 + bv_2$$

Applying $A$ repeatedly to $v$ and using the eigenvalue property gives,

$$\begin{aligned} Av &= a\lambda_1 v_1 + b\lambda_2 v_2, \\ A^2 v &= a\lambda_1^2 v_1 + b\lambda_2^2 v_2, \\ &\vdots \\ \Rightarrow\ A^n v &= a\lambda_1^n v_1 + b\lambda_2^n v_2. \end{aligned}$$

## 1.2 Fibonacci Sequence

### 1.2.1 Introduction

The Fibonacci Sequence is a one of the most famous sequence in mathematics. It is defined by the recurrence relation:

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, \text{ for } n \geq 2 \\ F_0 = F_1 = 1 \end{cases}$$

Each term is the sum of the two preceeding terms: $1, 1, 2, 3, 5, 8 \ldots$

### 1.2.2 Matrix Representation of the Fibonacci Sequence

Let

$$x_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}, x_1 = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}, \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

By repeatedly applying the matrix $A$, we can express each term of the sequence as a power of $A$ acting on $x_0$:

$$\begin{aligned} &x_1 = Ax_0, \\ &x_2 = Ax_1 = A(Ax_0) = A^2 x_0 \\ &\Rightarrow\ x_n = A^n x_0 \end{aligned}$$

### 1.2.3 Application to the Fibonacci Matrix

Let us now consider the Fibonacci matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Its eivenvalues are given by the **characteristic polynomial**

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = 0 \Rightarrow \boxed{\lambda^2 - \lambda - 1 = 0}$$

, and a quick computation yields $\lambda = \varphi$ or $-\dfrac{1}{\varphi}$.

**Notice that this is exactly the same as the equation obtained from assuming $F_n = \lambda^n$ in the Fibonacci recurrence:**

$$F_n = F_{n-1} + F_{n-2} \Leftrightarrow \lambda^n = \lambda^{n-1} + \lambda^{n-2} \Rightarrow \boxed{\lambda^2 = \lambda + 1}$$

### 1.2.4 Deriving the Closed Form

We can now express $x_n = A^n x_0$ explicitly in terms of $\lambda_1$ and $\lambda_2$. Let us consider

$$F_n = p \cdot \varphi^n + q \cdot (-\frac{1}{\varphi})^n$$

By initial contidion $F_0 = F_1 = 1$,

$$\begin{cases} p + q = 1 \\ p \cdot \varphi + q \cdot (-\frac{1}{\varphi}) = 1 \end{cases} \Rightarrow \begin{cases} p = \frac{1}{\sqrt{5}}\varphi \\ q = -\frac{1}{\sqrt{5}}\frac{1}{\varphi} \end{cases}$$

Thus,

$$F_n = \frac{1}{\sqrt{5}}\left[\varphi^{n+1} - (-\frac{1}{\varphi})^{n+1}\right] \quad \square$$

## 1.3 Non-homogeneous Recurrence Equation

### 1.3.1 Problem

Given $a_n = 3a_{n-1} + 2$ and $a_1 = 2$, $a_2 = 8$. Find the general formula for $a_n$.

#### Solution

We start by homogeneous linear equation

$$a_n = 3a_{n-1} \Rightarrow x^2 = 3x$$

Quick calculation gives $x = 0$ or 3, then we assume the general formula plus a displacement $r$.

$$a_n = p \cdot 3^n + q \cdot 0^n + r$$

By initial condition $a_1 = 2$, $a_2 = 8$

$$\begin{cases} 3p + r = 2 \\ 9p + r = 8 \end{cases} \Rightarrow \begin{cases} p = 1 \\ r = -1 \end{cases}$$

Thus the general formula for $a_n$ is

$$a_n = 3^n - 1 \quad \square$$

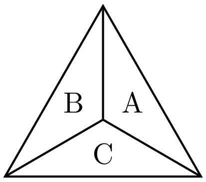
## 1.4 Five-Color Planar Graph Coloring

### 1.4.1 Problem

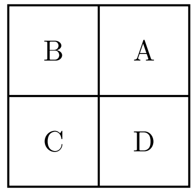
Given a polygon with $n$ sides divided into $n$ regions by drawing lines from the centroid to each vertex, find a general formula for the number of proper colorings of the regions using 5 colors, where adjacent regions must have different colors.

**Solution**

**For Triangle $A_3$ and Square $A_4$**



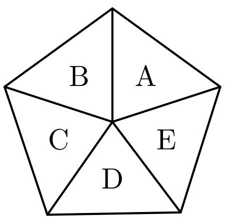

$$C_1^5 \cdot C_1^4 \cdot C_1^3 = 60$$



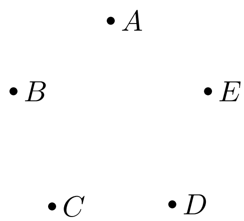

$$\begin{cases} \text{A} = \text{C} : C_1^5 \cdot C_1^4 \cdot C_1^4 = 80 \\ \text{A} \neq \text{C} : C_1^5 \cdot C_1^4 \cdot C_1^3 \cdot C_1^3 = 180 \end{cases}$$

$$80 + 180 = 260$$

**For Pentagon $A_5$**

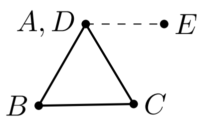



$\Rightarrow$

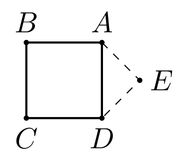



$=$

A $=$ D : $A_3 \times 4\,(E)$




A $\neq$ D : $A_4 \times 3\,(E)$




**Recurrence Formula**

Now we've obtained the recurrence formula with initial conditions $a_3 = 60$, $a_4 = 260$

$$a_{n+2} = 3a_{n+1} + 4a_n$$

Solving the equation yields

$$a_n = 4^n + 4(-1)^n$$

### 1.4.2 General Case

Given a polygon with $n$ sides divided into $n$ regions by drawing lines from the centroid to each vertex, the general formula for the number of proper colorings of the regions using $k$ colors is

$$(k-1)^n + (-1)^n(k-1)$$

## 2 Eigenvalues of General Tridiagonal Toeplitz Matrices

Consider the $n \times n$ **general tridiagonal Toeplitz matrix**:

$$T_n = \begin{pmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ 0 & a & b & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & c \\ 0 & 0 & 0 & a & b \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$.

## 2.1 Characteristic Polynomial

The characteristic polynomial is defined as

$$p_n(\lambda) := \det(\lambda I - T_n),$$

It satisfies the recurrence relation

$$\begin{cases} p_{n+2}(\lambda) = (\lambda - b)p_{n+1}(\lambda) - ac\, p_n(\lambda) \\ p_0 = 1,\ p_1 = \lambda - b \end{cases}$$

## 2.2 A Special Case

Let $a = c = 1, b = 0$. We have an adjacency matrix corresponding to Path $P_n$

$$A(P_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

Let $p_n$ denote the characteristic polynomial of $A_n$. The recurrence formula is given by

$$\begin{cases} p_{n+2} = \lambda p_{n+1} - p_n \\ p_0 = 1,\ p_1 = \lambda \end{cases} \xRightarrow{\text{Ansatz } r^n = p_n} \quad r^2 = \lambda r - 1$$

Solving $r^2 = \lambda r - 1$ gives

$$r = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}$$

Observe that $|\lambda| \leq 2$
Let

$$\lambda = 2\cos\theta \Rightarrow r = \cos\theta \pm i \sin\theta = e^{\pm i\theta}$$

Therefore,

$$p_n(\lambda) = \alpha e^{in\theta} + \beta e^{-in\theta}$$

By initial condition $p_0 = 1,\ p_1 = \lambda$

$$\begin{cases} \alpha + \beta = 1 \\ \alpha e^{i\theta} + \beta e^{-i\theta} = \lambda = 2\cos\theta \end{cases}$$

A quick calculation yields

$$\alpha = \frac{e^{i\theta}}{2i\sin\theta},\ \beta = \frac{-e^{-i\theta}}{2i\sin\theta}$$

Now $\lambda = 2\cos\theta$ and

$$\begin{aligned} p_n(\lambda) &= \frac{e^{i\theta}}{2i\sin\theta} \cdot e^{\text{in}\theta} + \frac{-e^{-i\theta}}{2i\sin\theta} \cdot e^{-\text{in}\theta} \\ &= \frac{e^{i(\text{n}+1)\theta} - e^{-i(\text{n}+1)\theta}}{2i\sin\theta} \\ &= \frac{\sin((n+1)\theta)}{\sin\theta} \end{aligned}$$

$p_n(\lambda) = 0 \Leftrightarrow \sin((n+1)\theta) = 0$ and $\sin(\theta) \neq 0$

$$\begin{aligned} &(n+1)\theta = k\pi,\ k = 1, 2, 3... \\ &\Rightarrow \theta_k = \frac{k\pi}{n+1} \end{aligned}$$

Therefore,

$$\lambda_k = 2\cos\theta_k = 2\cos\left(\frac{k\pi}{n+1}\right) \quad \square$$

### 2.3 General Tridiagonal Toeplitz Matrices

## 3 Trigonometric Solution to Cubic Equations

### 3.1 The Cubic Equation and The Depressed Form

A general cubic equation is given by:

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

**Depressed Cubic Form:**

$$t^3 + pt + q = 0$$

Any cubic equation may be reduced to the depressed cubic form by a simple change of variable

$$x = t - \frac{b}{3a}$$

The roots therefore are:

$$x_i = t_i - \frac{b}{3a}$$

### 3.2 Trigonometric Solution

Recall the cosine triple-angle formula:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

This can be rearranged to:

$$4\cos^3\theta - 3\cos\theta - \cos 3\theta = 0$$

Let $x = 2\sqrt{-\frac{p}{3}}\cos\theta$. We have

$$4\cos^3\theta - 3\cos\theta - \frac{3q}{2p}\sqrt{-\frac{3}{p}} = 0$$

where

$$\cos 3\theta = \frac{3q}{2p}\sqrt{-\frac{3}{p}}$$

Thus,

$$\theta = \frac{1}{3}\left(\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) + 2k\pi\right), k = 0, 1, 2$$

Therefore,

$$x_k = 2\sqrt{-\frac{p}{3}}\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{3q}{2p}\sqrt{-\frac{3}{p}}\right) + \frac{2k\pi}{3}\right), k = 0, 1, 2$$

## 3.3 Example

Find the roots of

$$x^3 - 3x - 2 = 0$$

Let $x = 2\cos\theta$

$$\begin{aligned} & x^3 - 3x - 2 \\ &= 8\cos^3\theta - 6\cos\theta - 2 \end{aligned}$$

Thus,

$$\cos(3\theta) = 1 \Rightarrow \theta = \frac{k\pi}{3},\ k = 0, 1, 2$$

Therefore,

$$x_k = 2\cos(\frac{k\pi}{3}),\ k = 0, 1, 2$$