Contents

1 Logic and Proofs

1.1 Propositional Logic

Proposition is a statement that is **either** true or false, but not both at the same time. We usually represent it with variables like p, q, and r.
e.g. "The sky is blue." is a proposition, but "Listen to me" is not.

1.1.1 Logical Connectives

- Negation: $\neg p$. It is not the case that p.
- Conjunction: $p \wedge q$. "and"
- Disjunction: $p \vee q$. "or"
- Implication: $p \rightarrow q$. If p then q, q if p, q is a consequence of p, p only if q
- biconditional: $p \leftrightarrow q$. $(p \rightarrow q) \wedge (q \rightarrow p)$, p if and only if q

1.1.2 Variations of Conditionals

- Implication: $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$
- Contrapositive: $\neg q \rightarrow \neg p$. This is logically equivalent to Implication

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

Find the truth value of $\boldsymbol{(p \vee q) \rightarrow \neg r}$

p	q	r	$p \vee q$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

1.2 Application of Propositional Logic

1.2.1 Classification of Proposition

- Tautology: Always true. e.g. $p \vee \neg p$
- Contradiction: Always false. e.g. $p \wedge \neg p$
- Contingency: Depends on variable. e.g. $p \to q$

1.2.2 Logical Equivalence $p \equiv q$

Two statements are logically equivalent if they always have the same truth value in every possible scenario.
e.g. p and q are biconditional, i.e. $p \leftrightarrow q$, means that p and q are logically equivalent.

1.2.3 Laws of Logical Equivalence

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$p \to q \equiv \neg p \vee q$ $p \to q \equiv \neg q \to \neg p$	Conditional
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	Biconditional

1.2.4 Determine Logical Equivalence:

1. Verify with Truth Table
2. Apply Known knowledge

Show that $p \to q$ is logically equivalent to $\neg q \to \neg p$

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Show that $(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

$$\begin{aligned} (p \to r) \lor (q \to r) &\equiv (\neg p \lor r) \lor (\neg q \lor r) \\ &\equiv (\neg p \lor \neg q) \lor (r \lor r) \\ &\equiv \neg (p \land q) \lor r \\ &\equiv (p \land q) \to r_{\#} \end{aligned}$$

1.3 Predicate and Quantifier

1.3.1 Predicate

A predicate is a statement with variables that becomes true or false only once specific values are substituted. $P(x)$ denotes a predicate involving x.
e.g. Let $P(x)$ be the statement "x¿4." We read $P(x)$ as "x is greater than 4."

- $P(x)$ is true if x = 5
- $P(x)$ is false if x = 3

General Form

$P(x_1, x_2, x_3, \ldots, x_n)$ where each x_i is a variable from the domain of discourse.

1.3.2 Quantifier

- Universal quantifier $\forall$: "for all", "every".
- Existential quantifier $\exists$: "there exists", "some", "at least one".

Negating Quantifier

$$\begin{cases} \neg \forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg \exists x\, P(x) \equiv \forall x\, \neg P(x) \end{cases}$$

Nested Quantifier

$$\forall x \exists y\, P(x, y) \neq \exists y \forall x\, P(x)$$