

# Contents

<b>1</b>	<b>Logic and Proofs</b>	<b>1</b>
1.1	Propositional Logic . . . . .	1
1.1.1	Logical Connectives . . . . .	1
1.1.2	Variations of Conditionals . . . . .	1
1.2	Application of Propositional Logic . . . . .	2
1.2.1	Classification of Proposition . . . . .	2
1.2.2	Logical Equivalence $p \equiv q$ . . . . .	2
1.2.3	Laws of Logical Equivalence . . . . .	2
1.2.4	Determine Logical Equivalence: . . . . .	3
1.3	Predicate and Quantifier . . . . .	3
1.3.1	Predicate . . . . .	3
1.3.2	Quantifier . . . . .	3
1.4	Rule of Inference . . . . .	4
1.5	Introduction to Proofs . . . . .	5
1.5.1	Some Mathematical Terminology . . . . .	5
1.5.2	Types of Proof . . . . .	5

# 1 Logic and Proofs

## 1.1 Propositional Logic

Proposition is a statement that is **either** true or false, but not both at the same time. We usually represent it with variables like  $p$ ,  $q$ , and  $r$ .

e.g. "The sky is blue." is a proposition, but "Listen to me" is not.

### 1.1.1 Logical Connectives

- Negation:  $\neg p$ . It is not the case that  $p$ .
- Conjunction:  $p \wedge q$ . "and"
- Disjunction:  $p \vee q$ . "or"
- Implication:  $p \rightarrow q$ . If  $p$  then  $q$ ,  $q$  if  $p$ ,  $q$  is a consequence of  $p$ ,  $p$  only if  $q$
- biconditional:  $p \leftrightarrow q$ .  $(p \rightarrow q) \wedge (q \rightarrow p)$ ,  $p$  if and only if  $q$

### 1.1.2 Variations of Conditionals

- Implication:  $p \rightarrow q$
- Converse:  $q \rightarrow p$
- Inverse:  $\neg p \rightarrow \neg q$
- Contrapositive:  $\neg q \rightarrow \neg p$ . This is logically equivalent to Implication

### Truth Table

$p$	$q$	$p \vee q$	$p$	$q$	$p \wedge q$	$p$	$q$	$p \rightarrow q$
T	T	T	T	T	T	T	T	T
T	F	T	T	F	F	T	F	F
F	T	T	F	T	F	F	T	T
F	F	F	F	F	F	F	F	T

### Example

Find the truth value of  $(p \vee q) \rightarrow \neg r$

$p$	$q$	$r$	$p \vee q$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

## 1.2 Application of Propositional Logic

### 1.2.1 Classification of Proposition

- Tautology: Always true. e.g.  $p \vee \neg p$
- Contradiction: Always false. e.g.  $p \wedge \neg p$
- Contingency: Depends on variable. e.g.  $p \rightarrow q$

### 1.2.2 Logical Equivalence $p \equiv q$

Two statements are logically equivalent if they always have the same truth value in every possible scenario.

e.g.  $p$  and  $q$  are biconditional, i.e.  $p \leftrightarrow q$ , means that  $p$  and  $q$  are logically equivalent.

### 1.2.3 Laws of Logical Equivalence

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$	Conditional
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	Biconditional

**1.2.4 Determine Logical Equivalence:**

1. Verify with Truth Table
2. Apply Known knowledge

Show that  $p \rightarrow q$  is logically equivalent to  $\neg q \rightarrow \neg p$

$p$	$q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Show that  $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$$\begin{aligned}
 (p \rightarrow r) \vee (q \rightarrow r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) \\
 &\equiv (\neg p \vee \neg q) \vee (r \vee r) \\
 &\equiv \neg(p \wedge q) \vee r \\
 &\equiv (p \wedge q) \rightarrow r
 \end{aligned}$$

**1.3 Predicate and Quantifier****1.3.1 Predicate**

A predicate is a statement with variables that becomes true or false only once specific values are substituted.  $P(x)$  denotes a predicate involving  $x$ .

e.g. Let  $P(x)$  be the statement " $x > 4$ ." We read  $P(x)$  as " $x$  is greater than 4."

- $P(x)$  is true if  $x = 5$
- $P(x)$  is false if  $x = 3$

**General Form**

$P(x_1, x_2, x_3, \dots, x_n)$  where each  $x_i$  is a variable from the domain of discourse.

**1.3.2 Quantifier**

- Universal quantifier  $\forall$ : "for all", "every".
- Existential quantifier  $\exists$ : "there exists", "some", "at least one".

**Rules for Quantifier**

Negation of quantifier

$$\begin{cases} \neg \forall x P(x) \equiv \exists x \neg P(x) \\ \neg \exists x P(x) \equiv \forall x \neg P(x) \end{cases}$$

Nested quantifier

$$\forall x \exists y P(x, y) \neq \exists y \forall x P(x, y)$$

Negation of nested quantifier

$$\neg(\forall x \exists y P(x, y)) \equiv \exists x \forall y \neg P(x, y)$$

$$\neg(\exists x \forall y P(x, y)) \equiv \forall x \exists y \neg P(x, y)$$

## 1.4 Rule of Inference

An **argument** is an implication of the form:

$$\bigwedge_{i \in D} p_i \rightarrow q$$

where  $D$  is domain of discourse,  $p_i$  is a premise, and  $q$  is a conclusion

**Notation:**

$$(p \rightarrow q) \wedge p \quad \therefore q \quad \Rightarrow \quad \frac{p \rightarrow q \quad p}{\therefore q} \quad \Rightarrow \quad \frac{p \rightarrow q}{p \therefore q}$$

Name	Expression	Name	Expression
Modus Ponens	$\frac{p \rightarrow q \quad p}{\therefore q}$	Modus Tollens	$\frac{p \rightarrow q \quad \neg q}{\therefore \neg p}$
Hypothetical Syllogism	$\frac{p \rightarrow q \quad q \rightarrow r}{\therefore p \rightarrow r}$	Conjunction	$\frac{p \quad q}{\therefore p \wedge q}$
Disjunctive Syllogism	$\frac{p \vee q \quad \neg p}{\therefore q}$	Addition	$\frac{p}{\therefore p \vee q}$
Simplification	$\frac{p \wedge q}{\therefore p \quad \therefore q}$	Resolution	$\frac{p \vee q \quad \neg p \vee r}{\therefore q \vee r}$

## Rules for Quantified Statement

**Universal Instantiation:** From a universal statement  $\forall x P(x)$ , we may infer  $P(c)$  for any particular  $c$ .

$$\forall x P(x) \Rightarrow P(a)$$

**Universal Generalization:** If  $P(c)$  holds for an arbitrary element  $c$ , then we may conclude  $\forall x P(x)$ .

$$P(c) \Rightarrow \forall x P(x)$$

**Existential Instantiation:** From  $\exists x P(x)$ , we may introduce a new symbol  $c$  such that  $P(c)$  holds.

$$\exists x P(x) \Rightarrow P(c)$$

**Existential Generalization:** From  $P(c)$  for a particular  $c$ , we may conclude  $\exists x P(x)$ .

$$P(c) \Rightarrow \exists x P(x)$$

## 1.5 Introduction to Proofs

### 1.5.1 Some Mathematical Terminology

- **Theorem:** A major, important mathematical statement that has been proved true.
- **Lemma:** A proved result used mainly as a stepping stone toward a larger theorem.
- **Corollary:** A statement that follows directly and easily from a theorem or proposition.
- **Proposition:** A proved statement that is true but of smaller or less central importance than a theorem.
- **Conjecture:** A mathematical claim believed to be true but not yet proved.
- **Proof:** A logical argument that establishes the truth of a mathematical statement beyond doubt.

### 1.5.2 Types of Proof

- **Direct Proof:** Show a statement is true by straightforward logical reasoning from definitions and known results.

**Example:** Prove that if  $n$  is odd, then  $5n + 3$  is even.

*Proof:*

Let  $n = 2k + 1, k \in \mathbb{Z}$ , then

$$5n + 3 = 2(2k + 1) + 3 = 10k + 8 = 2(5k + 4)$$

So,  $5n + 3 = 2(5k + 4)$  and  $5k + 4 \in \mathbb{Z}$

$\therefore 5n + 3$  is even<sub>#</sub>

- **Proof by Contrapositive:** Show a statement is true by proving that if the conclusion is false, then the premise must also be false.

**Example:** Prove that if  $n$  is odd, then  $5n + 3$  is even.

*Proof:*

- **Proof by Contradiction:** Show a statement is true by assuming the opposite and deriving a contradiction.

**Example:** Prove that  $\sqrt{2}$  is irrational

*Proof:*

Let

$$\begin{aligned}\sqrt{2} &= \frac{p}{q}, \quad q \neq 0, \quad p, q \in \mathbb{N}, \quad \gcd(p, q) = 1 \\ \Rightarrow 2 &= \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2\end{aligned}$$

Thus,  $2 \mid p^2$ , so  $2 \mid p$ . We have

$$p = 2k, \quad k \in \mathbb{N}, \quad 4k^2 = 2q^2 \Rightarrow 2k^2 = q^2$$

Similarly,  $2 \mid q$ . Concluding

$$2 \mid \gcd(p, q) \Rightarrow \Leftarrow$$

Thus  $\sqrt{2}$  is irrational.

- **Proof by Cases:** Show a statement is true by dividing into cases and proving it holds in each case.

**Example:**

*Proof:*

- **Methametrical Induction:** Show a statement is true by proving a base case and then proving the inductive step from  $n$  to  $n+1$ .

**Example:**

*Proof:*

- **Existence and Uniqueness Proof** Show a statement is true by first proving that at least one object with the required property exists (existence), and then proving that no more than one such object can exist (uniqueness).

**Example:** Prove that if  $r \in \mathbb{Q}^c$ , then  $\exists! n \in \mathbb{Z} \mid r - n < \frac{1}{2}$ .

*Proof:*

*Existence:*

Let

$$n = \lfloor r + \frac{1}{2} \rfloor$$

By definition we have

$$n \leq r + \frac{1}{2} < n + 1$$

Hence

$$|r - n| < \frac{1}{2}$$

, so such an integer  $n$  exists.

*Uniqueness:*

Suppose

$$\exists m, m \neq n, |r - m| < \frac{1}{2}$$

Consider

$$\begin{aligned} |n - m| &= |(n - r) + (r - m)| \\ &\leq |n - r| + |r - m| \quad (\text{Triangle Inequality}) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

Since  $n$  and  $m$  are integers, the inequality  $|n - m| < 1$  implies  $n = m$ . Therefore, the integer  $n$  is unique.