Contents

1 Sequences

1.1 Definition of Sequences

A sequence is an ordered list of numbers written in the form

$$a_1, a_2, a_3, \ldots, a_n, \quad \textbf{or} \quad a_n \quad \textbf{or} \quad \{a_n\}_{n=1}^{\infty}$$

It is usually represented as a function whose domain is the set of positive integers:

$$a_n = f(n)$$

1.2 a_n as $n \to \infty$

The **limit of a sequence** as $n \to \infty$ describes the long-term behavior of the sequence:

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence get arbitrarily close to L as n becomes large. If such a number L exists, we say the sequence **converges** to L. Otherwise, it **diverges**.

1.3 Convergence/Divergence

A sequence $\{a_n\}$ **converges** to $L \in \mathbb{R}$ if

$$\lim_{n\to\infty} a_n = L \quad \Longleftrightarrow \quad \forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n > N,\ |a_n - L| < \varepsilon.$$

Otherwise, the sequence **diverges**.

2 Series

2.1 Notation

A series is the sum of the terms of a sequence $\{a_n\}$. Formally, the n-th partial sum of the series is

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{S_n\}$ converges to a finite limit S as $n \to \infty$, then we write

$$\sum_{i=1}^{\infty} a_i = L$$

and say that the series converges to L. Otherwise, it diverges.

2.2 Partial Sum

A partial sum of a series given by

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

2.3 Types of Series

p-Series

The p-series is a series of the form

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

The series converges when $p > 1$, and diverges when $p < 1$

Harmonic Series

The harmonic series is a p-series with p = 1

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$$

It diverges

Geometric Series

A geomeric series is a series of a form

$$\sum_{n=1}^{\infty} ar^{n-1}$$

The series converges when $|r| < 1$. The sum of the first n terms of the series is

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

The sum of the series is

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$$

Decimal Expansion

The rational number equal to the repeating decimal is the sum of the geometric series that represents the repeating decimal.

$$\begin{aligned}
3.8\overline{76} &= 3.8 + .76 + .0076 + \ldots \\
&= \frac{38}{10} + \frac{76}{10^3} + \frac{76}{10^5} + \ldots \\
&= \frac{38}{10} + \sum_{n=1}^{\infty} \frac{76}{10^{2n+1}} \\
&= \frac{38}{10} + \frac{76}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}} \\
&= \frac{1919}{495}
\end{aligned}$$

Telescoping

A telescoping series is a series in which most terms cancel out when expanded, leaving only a few terms that determine the sum. Suppose we have a series of the form

$$\sum_{n=1}^{\infty} \left(a_n - a_{n+1}\right)$$

If the sequence $\{a_n\}$ converges to a limit L as $n \to \infty$, then the partial sum becomes

$$S_N = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_N - a_{N+1}) = a_1 - a_{N+1}$$

Taking the limit as $N \to \infty$, we find

$$\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 - L$$

Example: Consider:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

The partial sum is

$$S_N = \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \left(\tfrac{1}{N} - \tfrac{1}{N+1}\right)$$

All intermediate terms cancel, leaving

$$S_N = 1 - \frac{1}{N+1}$$

Taking the limit as $N \to \infty$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$$

3 Convergence Tests

The necessary condition for a series $\{a_n\}$ to converge is that $\lim_{n\to\infty} a_n = 0$.

3.1 The Informal Principle

$$\sum \frac{4n^3 - n + 1}{n^5 + 7n^2 - 6} \approx \sum \frac{4\cancel{n^3}}{\cancel{n^5}} \approx 4 \sum \frac{1}{n^2}$$

3.2 Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, Then the series **diverges**.

3.3 Integral Test

If $a_n = f(n)$ where f is continuous, decreasing, and positive on $(c, \infty]$, then $\sum_{n=1}^{\infty}$ converges $\iff$ $\int_c^{\infty} f(x)\, dx$ exists

Example:

$$\text{Let } f(n) = \frac{1}{n^2}\sin\left(\frac{\pi}{n}\right)$$

$f(n)$ is continuous, decreasing, positive on $[2, \infty)$. Then by Integral Test:

$$\int_2^\infty \frac{1}{x^2}\sin\left(\frac{\pi}{x}\right)dx = \frac{1}{\pi}$$

Thus,

$$\sum_{n=1}^\infty \frac{1}{n^2}\sin\left(\frac{\pi}{n}\right) \textbf{ converges}$$

3.4 Comparison Test

If $0 \le a_n \le b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

Example:

$$\sum_{n=1}^\infty \frac{1}{n^2+5} \approx \sum_{n=1}^\infty \frac{1}{n^2}$$

and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ converges. Thus

$$\sum_{n=1}^\infty \frac{1}{n^2+5} \textbf{ converges}$$

3.5 Limit Comparison Test

If $0 < a_n$ and $0 < b_n$, if $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = L$, then either both **converges** or **diverges**.

Example:

$$\sum_{n=1}^\infty \frac{1}{4n+3}$$

By approximating

$$\sum_{n=1}^\infty \frac{1}{4n+3} \approx \sum_{n=1}^\infty \frac{1}{n}$$

We choose $\dfrac{1}{n}$ as b_n

$$\lim_{n\to\infty}\frac{\frac{1}{4n+3}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\cancel{n}}{4\cancel{n+3}} = \frac{1}{4}$$

Since $\displaystyle\sum \frac{1}{n}$ diverges,

$$\sum_{n=1}^\infty \frac{1}{4n+3} \textbf{ diverges}$$

3.6 Ratio Test

Given a_n and a_{n+1}, we find the limit of their absolute ratio, i.e. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} < 1, & \textbf{Converges} \\ = 1, & \textbf{Inconclusive} \\ > 1, & \textbf{Diverges} \end{cases}$$

Example:

$$\sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!}$$

We find the limit of their absolute ratio

$$\begin{aligned} &\lim_{n\to\infty}\left(\frac{2^{n+1}(n+2)}{(n+1)!}\cdot\frac{n!}{2^n(n+1)}\right) \\ =&\lim_{n\to\infty}\left(\frac{2^{\cancel{n}+1}(n+2)}{(n+1)\cancel{!}}\cdot\frac{\cancel{n!}}{\cancel{2^n}\cancel{(n+1)}}\right) \\ =&\lim_{n\to\infty}\frac{2(n+2)}{(n+1)^2} \\ =&\,0<1 \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!} \textbf{ converges}$$

3.7 Root Test

Given $\sum a_n$, we find the limit of the n-th root of a_n, i.e. $\lim_{n\to\infty}\sqrt[n]{|a_n|}$. The limit measures the asymptotic size of the terms by looking at their n-th root.

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \begin{cases} < 1, & \textbf{Converges} \\ = 1, & \textbf{Inconclusive} \\ > 1, & \textbf{Diverges} \end{cases}$$

Example:

$$\sum_{n=1}^{\infty} \frac{n^3}{5^n}\left(1+\frac{1}{n}\right)^n.$$

Apply Root Test:

$$\begin{aligned} &\lim_{n\to\infty}\sqrt[n]{|a_n|} \\ =&\lim_{n\to\infty}\sqrt[n]{n^3}\cdot\sqrt[n]{\frac{1}{5^n}}\cdot\sqrt[n]{\left(1+\frac{1}{n}\right)^n} \end{aligned}$$

Evaluate:

$$\sqrt[n]{n^3} \to 1 \text{ because } n^{3/n} \to 1.$$

$$\sqrt[n]{\frac{1}{5^n}} = \frac{1}{5}.$$

$$\sqrt[n]{\left(1 + \frac{1}{n}\right)^n} \to e^{1/n} \to 1$$

$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \cdot \left(\frac{1}{5}\right) \cdot 1 = \frac{1}{5} < 1$$

Thus,

$$\sum_{n=1}^{\infty} \frac{2^n(n+1)}{n!} = \textbf{ converges}$$

3.8 Alternating Series Test

3.9 Absolute vs Conditional Convergence

4 Power Series

4.1 Definition

4.2 Radius and Interval of Convergence

4.3 Differentiation and Integration

5 Taylor and Maclaurin Series

5.1 Taylor Series

A Taylor series is an infinite sum that represents a function as a power series centered at a point a. If a function $f(x)$ is infinitely differentiable at $x = a$, then its Taylor series is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

This expansion approximates the function near $x = a$

5.2 Maclaurin Series

A Maclaurin Series is a special case of Taylor Series centered at $x = 0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

5.3 Common Maclaurin Series

- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}$

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

- $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

- $\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$

- $\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$

5.4 Lagrange Error Bound for Taylor Series

The Lagrange error bound provides a way to estimate how close the Taylor polynomial $T_n(x)$ is to the actual function $f(x)$. Let f be a function with $(n+1)$ continuous derivatives on an interval containing a and x. The Taylor polynomial of degree n centered at a is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The remainder/error term in Lagrange form is

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some $c \in [x, a]$.

Error Bound

$$|R_n(x)| \leq \frac{\mathrm{M}}{(n+1)!}|x-a|^{n+1}, \text{ where } M \text{ is } \max(|f^{n+1}(c)|)$$

for some $c \in [x, a]$

5.5 Limits and Approximations

6 Applications

6.1 Numerical Approximation

6.2 Solving ODEs

6.3 Non-elementary Integrals