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## 1 Antiderivatives and Indefinite Integrals

### 1.1 Definition

A function  $F(x)$  is called an **antiderivative** of a function  $f(x)$  on an interval  $I$  if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

The set of antiderivatives of  $f$  is called the indefinite integral of  $f$ .

**Notation:** The indefinite integral of  $f(x)$  is denoted by

$$\int f(x) dx = F(x) + C,$$

where  $C$  is an arbitrary constant called the **constant of integration**.

### 1.2 Property of Indefinite Integrals

$$\int f(x) dx = F(x) + C \iff F'(x) = f(x)$$

$$\int af(x) dx = a \int f(x) dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

## 2 Integration Formulas

### 2.1 Polynomials

$$\int dx = x + C$$

$$\int a dx = ax + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

### 2.2 Exponential and Logarithmic Functions

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

## 2.3 Trigonometric Functions

$$\begin{aligned}\int \sin x \, dx &= -\cos x + C & \int \csc x \, dx &= \ln |\csc x - \cot x| + C \\ \int \cos x \, dx &= \sin x + C & \int \sec x \, dx &= \ln |\sec x + \tan x| + C \\ \int \tan x \, dx &= -\ln |\cos x| + C & \int \cot x \, dx &= \ln |\sin x| + C\end{aligned}$$

## 2.4 Inverse Trigonometric Functions

Let  $a > 0$  be a constant.

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}} \, dx &= \sin^{-1} \left( \frac{x}{a} \right) + C & \int \frac{-1}{x\sqrt{x^2 - a^2}} \, dx &= \frac{-1}{a} \csc^{-1} \left( \left| \frac{x}{a} \right| \right) + C, |x| > a \\ \int \frac{-1}{\sqrt{a^2 - x^2}} \, dx &= \cos^{-1} \left( \frac{x}{a} \right) + C & \int \frac{1}{x\sqrt{x^2 - a^2}} \, dx &= \frac{1}{a} \sec^{-1} \left( \left| \frac{x}{a} \right| \right) + C, |x| > a \\ \int \frac{1}{a^2 + x^2} \, dx &= \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C & \int \frac{-1}{a^2 + x^2} \, dx &= \frac{-1}{a} \cot^{-1} \left( \frac{x}{a} \right) + C\end{aligned}$$

# 3 Definite Integral

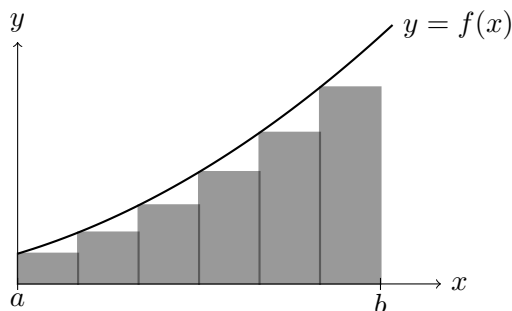
## 3.1 Approximating Area with Rectangles

The area under a curve can be approximated by a sum of rectangle areas. Let  $f(x)$  be a continuous function on the interval  $[a, b]$ . Divide  $[a, b]$  into  $n$  subintervals of equal width:

$$\Delta x = \frac{b - a}{n}$$

Choose a sample point  $x_i^* \in [x_{i-1}, x_i]$  in each subinterval, and build rectangles of height  $f(x_i^*)$ . The total approximate area is:

$$\sum_{i=1}^n f(x_i^*) \Delta x$$



## 3.2 Definition of Riemann Sum

Let  $f(x)$  be defined on a closed interval  $[a, b]$ , and let  $a = x_0 < x_1 < \cdots < x_n = b$  be a partition of the interval. For each subinterval  $[x_{i-1}, x_i]$ , define:

$$\Delta x_i = x_i - x_{i-1}, \quad c_i \in [x_{i-1}, x_i]$$

Then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i$$

is called a **Riemann sum** of  $f$  over  $[a, b]$ .

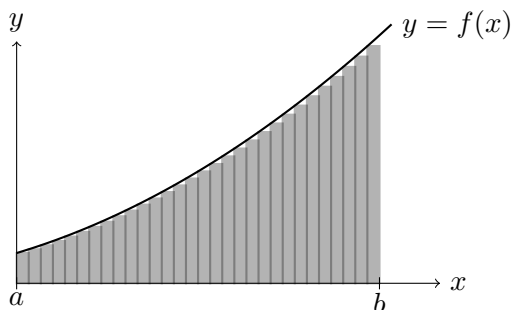
### 3.3 Definition of a Definite Integral

If the limit of the Riemann sums exists as the maximum subinterval width approaches zero:

$$\max \Delta x_i \rightarrow 0$$

and gives the same value regardless of how the sample points  $c_i$  are chosen, then the function  $f$  is said to be integrable on  $[a, b]$ , and the **definite integral** is defined by:

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$



Thus, the definite integral  $\int_a^b f(x) dx$  represents the exact area under the curve  $y = f(x)$  from  $a$  to  $b$ , as the number of rectangles increases and their width approaches zero.

### 3.4 Property of Definite Integrals

1. If  $f$  is defined on  $[a, b]$ , and  $\lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$  exists, then  $f$  is integrable on  $[a, b]$ .
2. If  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .
3. If  $f(x)$ ,  $g(x)$ , and  $h(x)$  are integrable on  $[a, b]$ , then

(a)  $\int_a^a f(x) dx = 0$

(b)  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

(c)  $\int_a^b f(x) dx = - \int_b^a f(x) dx$

(d)  $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$ , if  $f(x)$  is even

- (e)  $\int_{-a}^a f(x) dx = 0$ , if  $f(x)$  is odd
- (f)  $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$
- (g)  $\int_a^b g(x) dx \leq \int_a^b f(x) dx \leq \int_a^b h(x) dx$ , provided that  $g(x) \leq f(x) \leq h(x)$  on  $[a, b]$
- (h)  $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$

## 4 Fundamental Theorems of Calculus

### 4.1 First Fundamental Theorem of Calculus

If  $f$  is continuous on  $[a, b]$  and  $F(x) = \int_a^x f(t) dt$ , then  $F'(x) = f(x)$  at every point  $x$  in  $[a, b]$

### 4.2 Second Fundamental Theorem of Calculus

If  $f$  is continuous on  $[a, b]$  and  $F$  is an antiderivative of  $f$ , then

$$\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a)$$

Thus,

$$\frac{d}{dx} \int_{v(x)}^{u(x)} f(t) dt = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)$$

## 5 Integration Techniques

### 5.1 U-Substitution

The **u-substitution method** is used to evaluate integrals by making a change of variables. If an integral contains a composite function, we can simplify it using a substitution.

Let  $u = g(x)$ , then:

$$\frac{du}{dx} = g'(x) \quad \Rightarrow \quad du = g'(x) dx$$

So:

$$\int f(g(x)) g'(x) dx \Rightarrow \int f(u) du$$

After integration, substitute back  $u = g(x)$  to return to the original variable.

#### Example:

Evaluate:

$$\int 2x \cos(x^2) dx$$

Let

$$u = x^2 \Rightarrow du = 2x dx$$

Thus

$$\begin{aligned}\int 2x \cos(x^2) dx \\&= \int \cos(u) dx \\&= \sin(u) \\&= \sin(x^2)\end{aligned}$$

## 5.2 Trigonometric Substitution

Trigonometric substitution is a technique used to evaluate integrals involving square roots of quadratic expressions. The key idea is to use a trigonometric identity to simplify the integrand. Consider using trigonometric substitution when integrand contains expressions of the form:

- $\sqrt{a^2 - x^2}$  — use  $x = a \sin x$
- $\sqrt{a^2 + x^2}$  — use  $x = a \tan x$
- $\sqrt{x^2 - a^2}$  — use  $x = a \sec x$

### Trigonometric Identities Used

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$

### Example:

Evaluate:

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

Let

$$x = a \sin \theta \Rightarrow dx = \cos \theta d\theta$$

And

$$\sqrt{a^2 - x^2} = \cos \theta$$

Thus

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 - x^2}} \\&= \int d\theta \\&= \theta \\&= \arcsin\left(\frac{x}{a}\right) + C\end{aligned}$$

### 5.3 Partial Fraction Decomposition

Partial fraction decomposition is a method used to break a rational function into simpler fractions that are easier to integrate. Given a rational function:

$$\frac{P(x)}{Q(x)} \quad \text{where } \deg P(x) < \deg Q(x),$$

we can express it as a sum of simpler rational expressions depending on the factorization of  $Q(x)$ .

#### Types of Decompositions

Let  $Q(x)$  be factored as:

$$Q(x) = (x - r_1)^{k_1}(x - r_2)^{k_2} \cdots (x^2 + bx + c)^m \cdots$$

Then:

- For each distinct linear factor  $(x - r)^k$ , include terms:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_k}{(x - r)^k}$$

- For each irreducible quadratic factor  $(x^2 + bx + c)^m$ , include:

$$\frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2} + \cdots + \frac{Yx + Z}{(x^2 + bx + c)^m}$$

#### Process

1. If improper ( $\deg P(x) \geq \deg Q(x)$ ), perform long division first.
2. Factor the denominator  $Q(x)$ .
3. Set up partial fractions based on the types above.
4. Multiply both sides by  $Q(x)$  to eliminate denominators.
5. Solve for constants by plugging in values or equating coefficients.
6. Integrate each term individually.

#### Example:

Evaluate:

$$\int \frac{5x + 7}{(x - 1)(x + 2)} dx$$

Decompose

$$\frac{5x + 7}{(x - 1)(x + 2)} = \frac{A}{(x - 1)} + \frac{B}{(x + 2)}$$

Match coefficients

$$A + B = 5, 2A - B = 7 \Rightarrow A = 4, B = 1$$

Thus

$$\begin{aligned} & \int \frac{5x + 7}{(x - 1)(x + 2)} dx \\ &= \int \left( \frac{4}{(x - 1)} + \frac{1}{(x + 2)} \right) dx \\ &= 4 \ln |x - 1| + \ln |x + 2| + C \end{aligned}$$



## 5.4 Integration by Parts

**Integration by parts** is based on the product **rule for differentiation** and is given by:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

Where:

- $u$  = part to differentiate (becomes  $du$ )
- $dv$  = part to integrate (becomes  $v$ )

### Mnemonic: LIATE Rule

Choose  $u$  based on the following priority:

1. **L**ogarithmic (e.g.,  $\ln x$ )
2. **I**nverse trig (e.g.,  $\tan^{-1} x$ )
3. **A**lgebraic (e.g.,  $x^2$ )
4. **T**rigonometric (e.g.,  $\sin x$ )
5. **E**xponential (e.g.,  $e^x$ )

### Example:

Evaluate:

$$\int x e^x \, dx$$

LIATE:

$$\begin{aligned} u &= x, \, dv = e^x \\ \Rightarrow du &= dx, \, v = \int e^x \, dx = e^x \end{aligned}$$

Thus

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x \\ &= (x - 1) e^x + C \end{aligned}$$

## 5.5 The DI Method

This is a graph variation of Integration By Part

1. Choose  $f(x)$  to differentiate, and  $g(x)$  to integrate.
2. Alternate the signs starting with +.
3. Multiply diagonally (Derivative term  $\times$  Integral term just below) and alternate the signs.

4. Stop the process when:

- The derivative reaches zero ( $f^{(n)}(x) = 0$ )
- Repeated derivatives cycle or become too complex
- Cyclic or repeating patterns
- The remaining integral is simpler to evaluate directly

**Final Expression:** Combine diagonals with alternating signs:

Sign	Derivative(D)	Integral(I)
+	$f(x)$	$g(x)$
−	$f'(x)$	$\int g(x) dx$
+	$f''(x)$	$\iint g(x) dx$
−	$f^{(3)}(x)$	$\iiint g(x) dx$
$\vdots$	$\vdots$	$\vdots$

$$\int f(x)g(x) dx = f(x) \int g(x) dx - f'(x) \iint g(x) dx + \cdots$$

## 5.6 The King's Rule for Definite Integral

King's Rule is a clever substitution technique in which we let  $u = a + b - x$ , and thus

$$\boxed{\int_a^b f(x) dx = \int_a^b f(a + b - x) dx}$$

By averaging both expressions:

$$\int_a^b f(x) dx = \frac{1}{2} \int_a^b [f(x) + f(a + b - x)] dx$$

This is useful when  $f(x) + f(a + b - x)$  is a constant or simplifies significantly.

**Example:**

Evaluate:

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x dx$$

Apply the King's rule

$$I = \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Thus

$$\begin{aligned}
 2I &= \int_0^{\frac{\pi}{2}} \sin^2 x \, dx + \int_0^{\frac{\pi}{2}} \cos^2 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \, dx \\
 &= \int_0^{\frac{\pi}{2}} 1 \, dx \\
 &= \frac{\pi}{2} \\
 I &= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}
 \end{aligned}$$

## 5.7 Feynman's Integration Technique for Definite Integral

**Leibniz Integral Rule, or Differentiation under the Integral Sign** is a powerful technique used to evaluate integrals that depend on a parameter. This method became widely known through physicist Richard Feynman, who used it extensively in both theoretical and applied contexts. It allows us to compute an integral by introducing a parameter, differentiating with respect to that parameter under the integral sign, simplifying the expression, and then integrating the result.

### Leibniz Integral Rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x, t) \, dt = f(v(x), t) \cdot \frac{dv}{dx} - f(u(x), t) \cdot \frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x, t) \, dt$$

If we take  $u(x)$  and  $v(x)$  as constants  $a$  and  $b$ , then:

$$\boxed{\frac{d}{dx} \int_a^b f(x, t) \, dt = \int_a^b \frac{\partial f}{\partial x}(x, t) \, dt}$$

### Conditions for Validity

To apply this technique, we generally require:

- $f(x, t)$  and  $\partial f / \partial x$  are continuous in a region around the domain of integration.
- The limits  $u(x), v(x)$  are differentiable functions of  $x$ .
- The integral  $I(x)$  converges.

### Introducing a Parameter $\alpha$ to Simplify a Complex Integral

One of the most clever applications of this technique is to evaluate a complicated integral by introducing a parameter  $\alpha$  that does not initially exist in the original integral. The idea is to construct a new, easier-to-handle integral:

$$I(\alpha) = \int_a^b f(x, \alpha) \, dx$$

such that:

- The original integral is recovered by evaluating  $I(\alpha)$  at some specific value of  $\alpha$ .
- Differentiating with respect to  $\alpha$  simplifies the integrand.

**Steps:**

1. Embed the difficult integral into a parameterized family  $I(\alpha)$ .
2. Compute  $\frac{dI}{d\alpha}$  under the integral sign.
3. Integrate  $\frac{dI}{d\alpha}$  with respect to  $\alpha$  to recover  $I(\alpha)$ .
4. Evaluate  $I(\alpha)$  at the desired value (e.g.,  $\alpha = 0$ ) to obtain the original result.

**Example 1**

Evaluate:

$$\int_0^1 \frac{x^2 - 1}{\ln x} dx$$

**Step 1:** Parameterize the Integrand

Let

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x} dx$$

Note that:

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln x} dx = 0, \text{ and } I(2) \text{ is the original integral}$$

**Step 2:** Now we compute

$$\begin{aligned} \frac{d}{d\alpha} I(\alpha) &= \int_0^1 \frac{\partial}{\partial \alpha} \frac{x^\alpha - 1}{\ln x} dx = \int_0^1 x^\alpha dx \\ &= \frac{1}{\alpha + 1} x^{\alpha+1} \Big|_0^1 \\ &= \frac{1}{\alpha + 1} \end{aligned}$$

**Step 3:** Recover  $I(\alpha)$ 

$$\begin{aligned} I(\alpha) &= \int \frac{d}{d\alpha} I(\alpha) d\alpha \\ &= \int \frac{1}{\alpha + 1} d\alpha \\ &= \ln(\alpha + 1) + C \end{aligned}$$

Recall  $I(0) = 0 \Rightarrow C = 0$ . So:

$$0 = \ln(\alpha + 1) + C \Rightarrow C = 0$$

Hence:

$$I(\alpha) = \ln(\alpha + 1)$$

**Step 4:** Evaluate  $I(\alpha)$  at  $\alpha = 2$ 

$$I(2) = \ln(2 + 1) = \ln 3$$

**Answer:**

$$\int_0^1 \frac{x^2 - 1}{\ln x} dx = \ln 3$$

**Example 2**

Evaluate:

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

**Step 1:** Introduce an auxiliary exponential factor

Let

$$I(\alpha) = \int_0^{\infty} e^{-\alpha x} \frac{\sin x}{x} dx, \quad \alpha > 0$$

**Step 2:** Now we compute

$$\frac{dI}{d\alpha} = - \int_0^{\infty} e^{-\alpha x} \sin x dx$$

This integral is elementary:

$$\int_0^{\infty} e^{-\alpha x} \sin x dx = \frac{1}{1 + \alpha^2} \Rightarrow \frac{dI}{d\alpha} = - \frac{1}{1 + \alpha^2}$$

**Step 3:** Recover  $I(\alpha)$

$$I(\alpha) = - \int \frac{1}{1 + \alpha^2} d\alpha = - \tan^{-1}(\alpha) + C$$

As  $\alpha \rightarrow \infty$ ,  $I(\alpha) \rightarrow 0$ . So:

$$0 = - \tan^{-1}(\infty) + C = - \frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2}$$

Hence:

$$I(\alpha) = - \tan^{-1}(\alpha) + \frac{\pi}{2}$$

**Step 4:** Evaluate  $I(\alpha)$  as  $\alpha \rightarrow 0$

$$\lim_{\alpha \rightarrow 0} I(\alpha) = - \tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$$

**Answer:**

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

## 6 Improper Integral

In some cases, definite integrals involve infinite intervals or integrands that become unbounded. Such integrals are called **improper integrals**. We define these using limits.

### 6.1 Infinite Interval of Integration

Let  $f(x)$  be a function defined on  $[a, \infty)$ . Then the improper integral of  $f$  from  $a$  to  $\infty$  is defined as:

$$\int_a^{\infty} f(x) dx := \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

Similarly, if  $f$  is defined on  $(-\infty, b]$ , we define:

$$\int_{-\infty}^b f(x) dx := \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

If  $f$  is defined on  $(-\infty, \infty)$ , then:

$$\int_{-\infty}^{\infty} f(x) dx := \lim_{a \rightarrow -\infty} \int_a^c f(x) dx + \lim_{b \rightarrow \infty} \int_c^b f(x) dx$$

for some finite number  $c \in \mathbb{R}$ .

*Note: Both limits must exist and be finite for the integral to converge.*

## 6.2 Discontinuous Integrand

Suppose  $f$  is continuous on  $(a, b]$  but has an infinite discontinuity at  $a$ . Then:

$$\int_a^b f(x) dx := \lim_{\epsilon \rightarrow a^+} \int_{\epsilon}^b f(x) dx$$

Similarly, if  $f$  has an infinite discontinuity at  $b$ , then:

$$\int_a^b f(x) dx := \lim_{\epsilon \rightarrow b^-} \int_a^{\epsilon} f(x) dx$$

If the discontinuity is at an interior point  $c \in (a, b)$ , split the integral:

$$\int_a^b f(x) dx := \lim_{\epsilon \rightarrow c^-} \int_a^{\epsilon} f(x) dx + \lim_{\delta \rightarrow c^+} \int_{\delta}^b f(x) dx$$

Each part must be interpreted as a limit, and the total integral converges if both one-sided integrals converge.

## 6.3 Absolute vs Conditional Convergence

- If  $\int_a^{\infty} |f(x)| dx$  converges, then  $\int_a^{\infty} f(x) dx$  is said to be **absolutely convergent**.
- If  $\int_a^{\infty} f(x) dx$  converges but  $\int_a^{\infty} |f(x)| dx$  diverges, it is **conditionally convergent**.

# 7 Application

## 7.1 Area

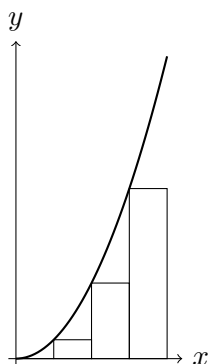
### 7.1.1 Approximating the Area Under the Curve

#### Rectangular Approximation

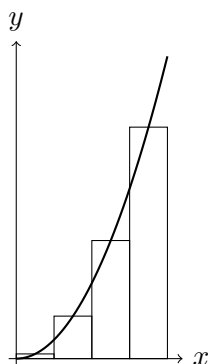
The area under the curve using  $n$  rectangles of equal length is approximately:

$$\sum_{i=1}^n (\text{area of rectangle}) = \begin{cases} \sum_{i=1}^n f(x_{i-1}) \Delta x & \text{left-endpoint rectangles} \\ \sum_{i=1}^n f(x_i) \Delta x & \text{right-endpoint rectangles} \\ \sum_{i=1}^n f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x & \text{midpoint rectangles} \end{cases}$$

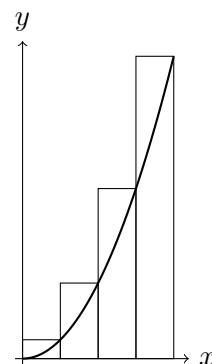
where  $\Delta x = \frac{b-a}{n}$  and  $a = x_0 < x_1 < x_2 < \cdots < x_n = b$



Left Endpoint

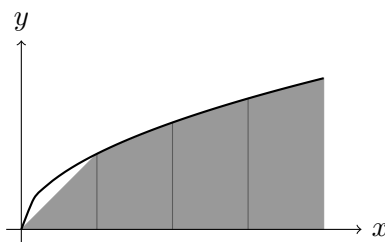


Midpoint



Right Endpoint

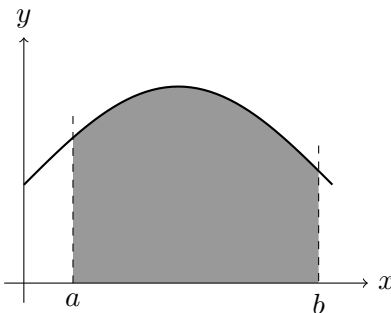
### Trapezoidal Approximation



If  $f$  is continuous, the area under the curve of  $f$  from  $x = a$  to  $x = b$  is:

$$\text{Area} \simeq \frac{b-a}{2n} [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

#### 7.1.2 Area Under a Curve

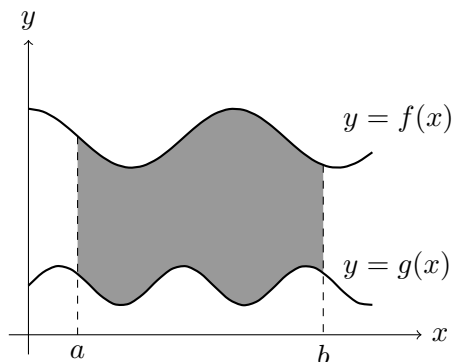


The area under the graph of a continuous function  $f(x)$  over the interval  $[a, b]$  is given by the definite integral:

$$A = \int_a^b f(x) dx$$

If  $f(x) \geq 0$  on  $[a, b]$ , this integral gives the area between the curve and the  $x$ -axis. If  $f(x)$  takes negative values, the integral represents **signed area**.

### 7.1.3 Area Between Two Curves



The area between two continuous functions  $f(x)$  and  $g(x)$  over the interval  $[a, b]$ , where  $f(x) \geq g(x)$ , is given by:

$$A = \int_a^b [f(x) - g(x)] \, dx$$

This integral computes the net vertical distance between the top curve  $f(x)$  and the bottom curve  $g(x)$  at each point  $x$ , accumulating the total area between them. It is essential that the functions be continuous on  $[a, b]$  and that  $f(x) \geq g(x)$  holds throughout this interval to interpret the result as a positive area.

## 7.2 Volumn

### 7.2.1 Cross Section

### 7.2.2 Disk Method

### 7.2.3 Washer Method

### 7.2.4 Shell Method

## 7.3 Arc Length and Surface Area

### 7.3.1 Arc Length

Let  $y = f(x)$  be a smooth curve on the interval  $[a, b]$ , where  $f$  is differentiable and  $f'(x)$  is continuous. The length of the curve from  $x = a$  to  $x = b$  is given by the arc length formula:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

### 7.3.2 Surface Area

Let  $y = f(x)$  be continuous and differentiable on  $[a, b]$ , and suppose we rotate it about the  $x$ -axis. Then the surface area of the resulting solid is:

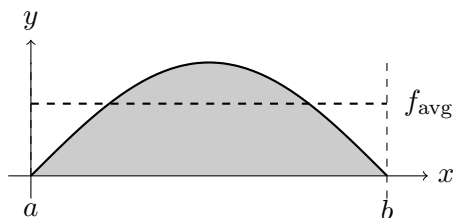
$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$



## 7.4 Average Value of a Function

The average value of a continuous function  $f(x)$  over the interval  $[a, b]$  is given by:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$$



## 8 Integration for Parametric

## 9 Integration for Polar