Contents

1 Antiderivatives and Indefinite Integrals

1.1 Definition

A function $F(x)$ is called an **antiderivative** of a function $f(x)$ on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

The set of antiderivatives of f is called the indefinite integral of f.

Notation: The indefinite integral of $f(x)$ is denoted by

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant called the **constant of integration**.

1.2 Property of Indefinite Integrals

$$\int f(x)\,dx = F(x) + C \iff F'(x) = f(x)$$

$$\int af(x)\,dx = a\int f(x)\,dx$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

2 Integration Formulas

2.1 Polynomials

$$\int dx = x + C$$

$$\int a\,dx = ax + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq -1$$

2.2 Exponential and Logarithmic Functions

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

2.3 Trigonometric Functions

$$\int \sin x\,dx = -\cos x + C \qquad \int \csc x\,dx = \ln|\csc x - \cot x| + C$$
$$\int \cos x\,dx = \sin x + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

2.4 Inverse Trigonometric Functions

Let $a > 0$ be a constant.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{-1}{x\sqrt{x^2-a^2}}\,dx = \frac{-1}{a}\csc^{-1}\left(\left|\frac{x}{a}\right|\right) + C,\ |x| > a$$
$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\left|\frac{x}{a}\right|\right) + C,\ |x| > a$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C \qquad \int \frac{-1}{a^2+x^2}\,dx = \frac{-1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$$

3 Definite Integral

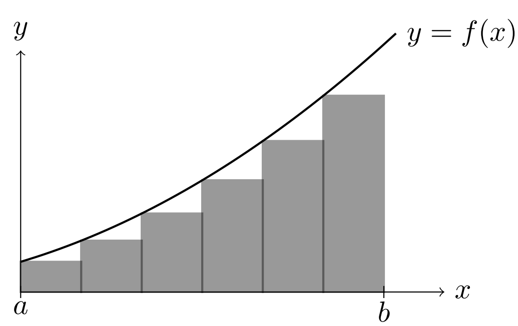
3.1 Approximating Area with Rectangles

The area under a curve can be approximated by a sum of rectangle areas. Let $f(x)$ be a continuous function on the interval $[a, b]$. Divide $[a, b]$ into n subintervals of equal width:

$$\Delta x = \frac{b-a}{n}$$

Choose a sample point $x_i^* \in [x_{i-1}, x_i]$ in each subinterval, and build rectangles of height $f(x_i^*)$. The total approximate area is:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$



3.2 Definition of Riemann Sum

Let $f(x)$ be defined on a closed interval $[a, b]$, and let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of the interval. For each subinterval $[x_{i-1}, x_i]$, define:

$$\Delta x_i = x_i - x_{i-1}, \quad c_i \in [x_{i-1}, x_i]$$

Then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called a **Riemann sum** of f over $[a, b]$.

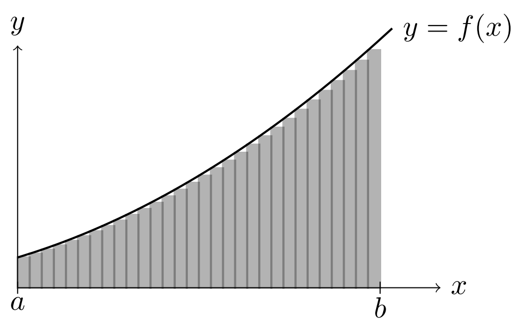
3.3 Definition of a Definite Integral

If the limit of the Riemann sums exists as the maximum subinterval width approaches zero:

$$\max \Delta x_i \to 0$$

and gives the same value regardless of how the sample points c_i are chosen, then the function f is said to be integrable on $[a, b]$, and the **definite integral** is defined by:

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$



Thus, the definite integral $\int_a^b f(x)\,dx$ represents the exact area under the curve $y = f(x)$ from a to b, as the number of rectangles increases and their width approaches zero.

3.4 Property of Definite Integrals

1. If f is defined on $[a, b]$, and $\lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then f is integrable on $[a, b]$.

2. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

3. If $f(x)$, $g(x)$, and $h(x)$ are integrable on $[a, b]$, then

   (a) $\int_a^a f(x)\,dx = 0$

   (b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   (c) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   (d) $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(x)$ is even

(e) $\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd

(f) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(g) $\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$, provided that $g(x) \le f(x) \le h(x)$ on $[a, b]$

(h) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

4 Fundamental Theorems of Calculus

4.1 First Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$ at every point x in $[a, b]$

4.2 Second Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)|_a^b = F(b) - F(a)$$

Thus,

$$\boxed{\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x)}$$

5 Integration Techniques

5.1 U-Substitution

The **u-substitution method** is used to evaluate integrals by making a change of variables. If an integral contains a composite function, we can simplify it using a substitution.
Let $u = g(x)$, then:

$$\frac{du}{dx} = g'(x) \quad \Rightarrow \quad du = g'(x)\,dx$$

So:

$$\int f(g(x))\,g'(x)\,dx \Rightarrow \int f(u)\,du$$

After integration, substitute back $u = g(x)$ to return to the original variable.

Example:

Evaluate:

$$\int 2x\cos(x^2)\,dx$$

Let

$$u = x^2 \Rightarrow du = 2xdx$$

Thus

$$\begin{aligned}
&\int 2x\cos(x^2)\,dx \\
&= \int \cos(u)\,dx \\
&= \sin(u) \\
&= \sin(x^2)
\end{aligned}$$

5.2 Trigonometric Substitution

Trigonometric substitution is a technique used to evaluate integrals involving square roots of quadratic expressions. The key idea is to use a trigonometric identity to simplify the integrand. Consider using trigonometric substitution when integrand contains expressions of the form:

- $\sqrt{a^2 - x^2}$ — use $x = a\sin x$
- $\sqrt{a^2 + x^2}$ — use $x = a\tan x$
- $\sqrt{x^2 - a^2}$ — use $x = a\sec x$

Trigonometric Identities Used

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$

Example:

Evaluate:

$$\int \frac{dx}{\sqrt{a^2 - x^2}}$$

Let

$$x = a\sin\theta \Rightarrow dx = \cos\theta\,d\theta$$

And

$$\sqrt{a^2 - x^2} = \cos\theta$$

Thus

$$\begin{aligned}
&\int \frac{dx}{\sqrt{a^2 - x^2}} \\
&= \int d\theta \\
&= \theta \\
&= \arcsin(\frac{x}{a}) + C
\end{aligned}$$

5.3 Partial Fraction Decomposition

Partial fraction decomposition is a method used to break a rational function into simpler fractions that are easier to integrate. Given a rational function:

$$\frac{P(x)}{Q(x)} \quad \text{where } \deg P(x) < \deg Q(x),$$

we can express it as a sum of simpler rational expressions depending on the factorization of $Q(x)$.

Types of Decompositions

Let $Q(x)$ be factored as:

$$Q(x) = (x - r_1)^{k_1}(x - r_2)^{k_2} \cdots (x^2 + bx + c)^m \cdots$$

Then:

- For each distinct linear factor $(x - r)^k$, include terms:
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_k}{(x - r)^k}$$
- For each irreducible quadratic factor $(x^2 + bx + c)^m$, include:
$$\frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2} + \cdots + \frac{Yx + Z}{(x^2 + bx + c)^m}$$

Process

1. If improper ($\deg P(x) \geq \deg Q(x)$), perform long division first.
2. Factor the denominator $Q(x)$.
3. Set up partial fractions based on the types above.
4. Multiply both sides by $Q(x)$ to eliminate denominators.
5. Solve for constants by plugging in values or equating coefficients.
6. Integrate each term individually.

Example:

Evaluate:

$$\int \frac{5x + 7}{(x - 1)(x + 2)} \, dx$$

Decompose

$$\frac{5x + 7}{(x - 1)(x + 2)} = \frac{A}{(x - 1)} + \frac{B}{(x + 2)}$$

Match coefficients

$$A + B = 5, 2A - B = 7 \Rightarrow A = 4\,, B = 1$$

Thus

$$\begin{aligned} &\int \frac{5x + 7}{(x - 1)(x + 2)} \, dx \\ &= \int \left(\frac{4}{(x - 1)} + \frac{1}{(x + 2)} \right) dx \\ &= 4 \ln |x - 1| + \ln |x + 2| + C \end{aligned}$$

5.4 Integration by Parts

Integration by parts is based on the product **rule for differentiation** and is given by:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Where:

- $u =$ part to differentiate (becomes du)
- $dv =$ part to integrate (becomes v)

Mnemonic: LIATE Rule
Choose u based on the following priority:

1. **L**ogarithmic (e.g., $\ln x$)
2. **I**nverse trig (e.g., $\tan^{-1} x$)
3. **A**lgebraic (e.g., x^2)
4. **T**rigonometric (e.g., $\sin x$)
5. **E**xponential (e.g., e^x)

Example:

Evaluate:

$$\int xe^x\,dx$$

LIATE:

$$u = x,\ dv = e^x$$
$$\Rightarrow du = dx,\ v = \int e^x\,dx = e^x$$

Thus

$$\int xe^x\,dx$$
$$= xe^x - \int e^x\,dx$$
$$= xe^x - e^x$$
$$= (x-1)e^x + C$$

5.5 The DI Method

This is a graph variation of Integration By Part

1. Choose $f(x)$ to differentiate, and $g(x)$ to integrate.
2. Alternate the signs starting with $+$.
3. Multiply diagonally (Derivative term $\times$ Integral term just below) and alternate the signs.

4. Stop the process when:
   - The derivative reaches zero ($f^{(n)}(x) = 0$)
   - Repeated derivatives cycle or become too complex
   - Cyclic or repeating patterns
   - The remaining integral is simpler to evaluate directly

Final Expression: Combine diagonals with alternating signs:

Sign	**Derivative(D)**	**Integral(I)**
$+$	$f(x)$	$g(x)$
$-$	$f'(x)$	$\int g(x)\,dx$
$+$	$f''(x)$	$\iint g(x)\,dx$
$-$	$f^{(3)}(x)$	$\iiint g(x)\,dx$
$\vdots$	$\vdots$	$\vdots$

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - f'(x)\iint g(x)\,dx + \cdots$$

5.6 The King's Rule for Definite Integral

King's Rule is a clever substitution technique in which we let $u = a + b - x$, and thus

$$\boxed{\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx}$$

By averaging both expressions:

$$\int_a^b f(x)\,dx = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\,dx$$

This is useful when $f(x) + f(a+b-x)$ is a constant or simplifies significantly.

Example:

Evaluate:

$$I = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

Apply the King's rule

$$I = \int_0^{\frac{\pi}{2}} \sin^2(\frac{\pi}{2} - x)\,dx = \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$

Thus

$$\begin{aligned}
2I &= \int_0^{\frac{\pi}{2}} \sin^2 x\,dx + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx \\
&= \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)\,dx \\
&= \int_0^{\frac{\pi}{2}} 1\,dx \\
&= \frac{\pi}{2} \\
I &= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

5.7 Feynman's Integration Technique for Definite Integral

Leibniz Integral Rule, or Differentiation under the Integral Sign is a powerful technique used to evaluate integrals that depend on a parameter. This method became widely known through physicist Richard Feynman, who used it extensively in both theoretical and applied contexts. It allows us to compute an integral by introducing a parameter, differentiating with respect to that parameter under the integral sign, simplifying the expression, and then integrating the result.

Leibniz Integral Rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(x,t)\,dt = f(v(x),t)\cdot\frac{dv}{dx} - f(u(x),t)\cdot\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)\,dt$$

If we take $u(x)$ and $v(x)$ as constants a and b, then:

$$\boxed{\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt}$$

Conditions for Validity
To apply this technique, we generally require:

- $f(x,t)$ and $\partial f/\partial x$ are continuous in a region around the domain of integration.
- The limits $u(x), v(x)$ are differentiable functions of x.
- The integral $I(x)$ converges.

Introducing a Parameter α to Simplify a Complex Integral
One of the most clever applications of this technique is to evaluate a complicated integral by introducing a parameter α that does not initially exist in the original integral. The idea is to construct a new, easier-to-handle integral:

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$

such that:

- The original integral is recovered by evaluating $I(\alpha)$ at some specific value of α.
- Differentiating with respect to α simplifies the integrand.

Steps:

1. Embed the difficult integral into a parameterized family $I(\alpha)$.
2. Compute $\dfrac{dI}{d\alpha}$ under the integral sign.
3. Integrate $\dfrac{dI}{d\alpha}$ with respect to α to recover $I(\alpha)$.
4. Evaluate $I(\alpha)$ at the desired value (e.g., $\alpha = 0$) to obtain the original result.

Example 1

Evaluate:

$$\int_0^1 \frac{x^2 - 1}{\ln x}\, dx$$

Step 1: Parameterize the Integrand
Let

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\, dx$$

Note that:

$$I(0) = \int_0^1 \frac{x^0 - 1}{\ln x}\, dx = 0, \text{ and } I(2) \text{ is the original integral}$$

Step 2: Now we compute

$$\begin{aligned} \frac{d}{d\alpha} I(\alpha) &= \int_0^1 \frac{\partial}{\partial\alpha} \frac{x^\alpha - 1}{\ln x}\, dx = \int_0^1 x^\alpha dx \\ &= \frac{1}{\alpha + 1} x^{\alpha+1} \Big|_0^1 \\ &= \frac{1}{\alpha + 1} \end{aligned}$$

Step 3: Recover $I(\alpha)$

$$\begin{aligned} I(\alpha) &= \int \frac{d}{d\alpha} I(\alpha)\, d\alpha \\ &= \int \frac{1}{\alpha + 1}\, d\alpha \\ &= \ln(\alpha + 1) + C \end{aligned}$$

Recall $I(0) = 0 \Rightarrow C = 0$. So:

$$0 = \ln(\alpha + 1) + C \Rightarrow C = 0$$

Hence:

$$I(\alpha) = \ln(\alpha + 1)$$

Step 4: Evaluate $I(\alpha)$ at $\alpha = 2$

$$I(2) = \ln(2 + 1) = \ln 3$$

Answer:

$$\int_0^1 \frac{x^2 - 1}{\ln x}\, dx = \ln 3$$

Example 2

Evaluate:

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

Step 1: Introduce an auxiliary exponential factor
Let

$$I(\alpha) = \int_0^\infty e^{-\alpha x}\frac{\sin x}{x}\,dx, \quad \alpha > 0$$

Step 2: Now we compute

$$\frac{dI}{d\alpha} = -\int_0^\infty e^{-\alpha x}\sin x\,dx$$

This integral is elementary:

$$\int_0^\infty e^{-\alpha x}\sin x\,dx = \frac{1}{1+\alpha^2} \Rightarrow \frac{dI}{d\alpha} = -\frac{1}{1+\alpha^2}$$

Step 3: Recover $I(\alpha)$

$$I(\alpha) = -\int \frac{1}{1+\alpha^2}\,d\alpha = -\tan^{-1}(\alpha) + C$$

As $\alpha \to \infty$, $I(\alpha) \to 0$. So:

$$0 = -\tan^{-1}(\infty) + C = -\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2}$$

Hence:

$$I(\alpha) = -\tan^{-1}(\alpha) + \frac{\pi}{2}$$

Step 4: Evaluate $I(\alpha)$ as $\alpha \to 0$

$$\lim_{\alpha\to 0} I(\alpha) = -\tan^{-1}(0) + \frac{\pi}{2} = \frac{\pi}{2}$$

Answer:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

6 Improper Integral

In some cases, definite integrals involve infinite intervals or integrands that become unbounded. Such integrals are called **improper integrals**. We define these using limits.

6.1 Infinite Interval of Integration

Let $f(x)$ be a function defined on $[a, \infty)$. Then the improper integral of f from a to ∞ is defined as:

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Similarly, if f is defined on $(-\infty, b]$, we define:

$$\int_{-\infty}^b f(x)\,dx := \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

If f is defined on $(-\infty, \infty)$, then:

$$\int_{-\infty}^{\infty} f(x)\,dx := \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx$$

for some finite number $c \in \mathbb{R}$.

Note: Both limits must exist and be finite for the integral to converge.

6.2 Discontinuous Integrand

Suppose f is continuous on $(a, b]$ but has an infinite discontinuity at a. Then:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to a^+}\int_\epsilon^b f(x)\,dx$$

Similarly, if f has an infinite discontinuity at b, then:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to b^-}\int_a^\epsilon f(x)\,dx$$

If the discontinuity is at an interior point $c \in (a, b)$, split the integral:

$$\int_a^b f(x)\,dx := \lim_{\epsilon\to c^-}\int_a^\epsilon f(x)\,dx + \lim_{\delta\to c^+}\int_\delta^b f(x)\,dx$$

Each part must be interpreted as a limit, and the total integral converges if both one-sided integrals converge.

6.3 Absolute vs Conditional Convergence

- If $\displaystyle\int_a^\infty |f(x)|\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ is said to be **absolutely convergent**.
- If $\displaystyle\int_a^\infty f(x)\,dx$ converges but $\displaystyle\int_a^\infty |f(x)|\,dx$ diverges, it is **conditionally convergent**.

7 Application

7.1 Area

7.1.1 Approximating the Area Under the Curve

Rectangular Approximation

The area under the curve using n rectangles of equal length is approximately:

$$\sum_{i=1}^{n}(\text{area of rectangle}) = \begin{cases} \displaystyle\sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ left-endpoint rectangles} \\ \displaystyle\sum_{i=1}^{n} f(x_i)\Delta x \text{ right-endpoint rectangles} \\ \displaystyle\sum_{i=1}^{n} f(\frac{x_i + x_{i+1}}{2})\Delta x \text{ midpoint rectangles} \end{cases}$$

where $\Delta x = \dfrac{b-a}{n}$ and $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

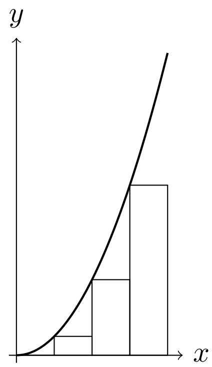


Figure 1: *
Left Endpoint

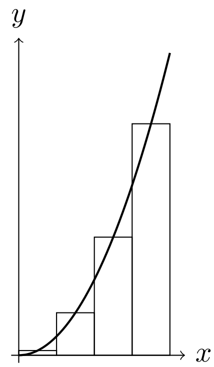


Figure 2: *
Midpoint

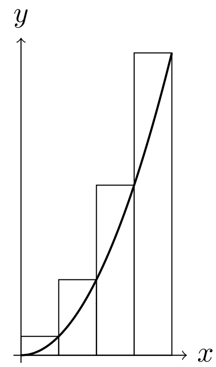
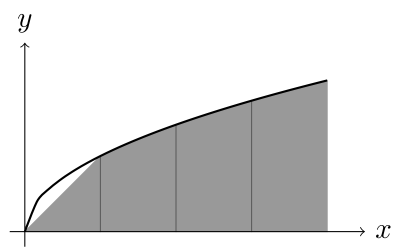


Figure 3: *
Right Endpoint

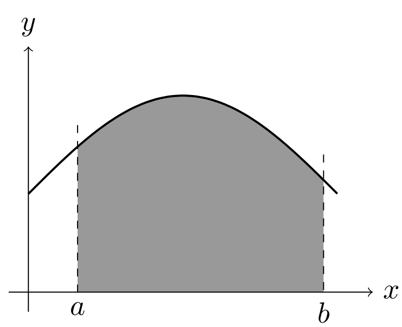
Trapezoidal Approximation



If f is continuous, the area under the curve of f from $x = a$ to $x = b$ is:

$$\text{Area} \simeq \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

7.1.2 Area Under a Curve

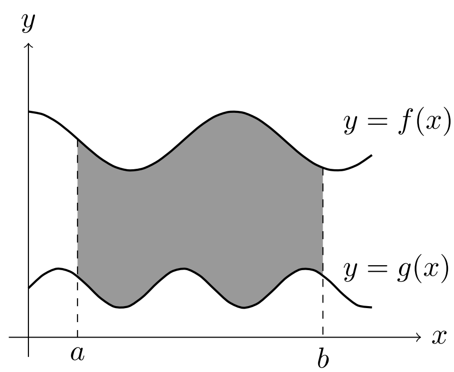


The area under the graph of a continuous function $f(x)$ over the interval $[a, b]$ is given by the definite integral:

$$A = \int_a^b f(x)\,dx$$

If $f(x) \geq 0$ on $[a, b]$, this integral gives the area between the curve and the x-axis. If $f(x)$ takes negative values, the integral represents **signed area**.

7.1.3 Area Between Two Curves



The area between two continuous functions $f(x)$ and $g(x)$ over the interval $[a, b]$, where $f(x) \geq g(x)$, is given by:

$$A = \int_a^b [f(x) - g(x)] \, dx$$

This integral computes the net vertical distance between the top curve $f(x)$ and the bottom curve $g(x)$ at each point x, accumulating the total area between them. It is essential that the functions be continuous on $[a, b]$ and that $f(x) \geq g(x)$ holds throughout this interval to interpret the result as a positive area.

7.2 Volumn

7.2.1 Cross Section

7.2.2 Disk Method

7.2.3 Washer Method

7.2.4 Shell Method

7.3 Arc Length and Surface Area

7.3.1 Arc Length

Let $y = f(x)$ be a smooth curve on the interval $[a, b]$, where f is differentiable and $f'(x)$ is continuous. The length of the curve from $x = a$ to $x = b$ is given by the arc length formula:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

7.3.2 Surface Area

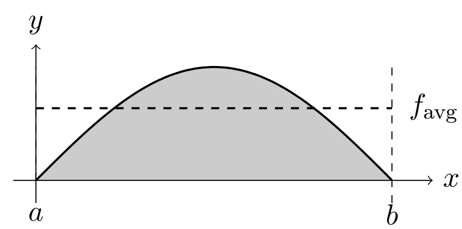
Let $y = f(x)$ be continuous and differentiable on $[a, b]$, and suppose we rotate it about the x-axis. Then the surface area of the resulting solid is:

$$S = 2\pi y \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

7.4 Average Value of a Function

The average value of a continuous function $f(x)$ over the interval $[a, b]$ is given by:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx$$



8 Integration for Parametric

9 Integration for Polar