Contents

1 Definition and The First Principle

1.1 Definition

Let f be a function defined on an open interval containing a. The **derivative** of f at the point a, denoted by $f'(a)$, is defined as

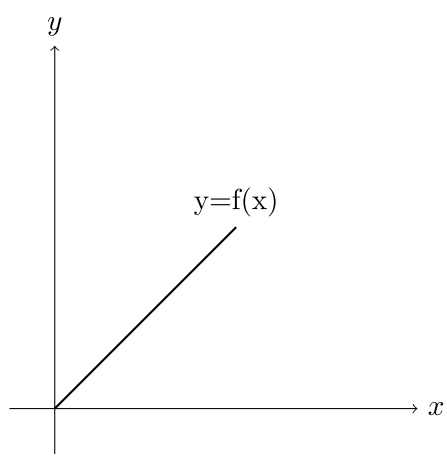
$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \end{aligned}$$

1.2 The First Principle

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

1.3 Geometric Meaning: Slope of Tangent



1.4 Symbols for the Derivative

$$D_x f,\ \frac{d}{dx}f(x),\ y',\ \dot{y}$$

2 Rules and Derivatives of Elementary Functions

2.1 Derivative Rules

1. Constant Rule: $\frac{d}{dx}c = 0$
2. Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
3. Sum/Difference Rule: $\frac{d}{dx}[f \pm g] = \frac{d}{dx}f \pm \frac{d}{dx}g$
4. Product Rule: $\frac{d}{dx}[f \cdot g] = \frac{d}{dx}f \cdot g + f \cdot \frac{d}{dx}g$

5. Quotient Rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\frac{d}{dx}f \cdot g - f \cdot \frac{d}{dx}g}{g^2}$

2.2 Trigonometric Functions

$$\tfrac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

2.3 Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

2.4 Exponential and Logarithhmic Functions

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x},\ x > 0$$
$$\frac{d}{dx}(a^x) = a^x \ln a,\ a > 0\,\&\, \neq 1 \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a},\ a > 0\,\&\, \neq 1$$

2.5 Derivative of Inverse Function

Let f be a one-to-one differentiable function with inverse f^{-1}, and suppose $f'(f^{-1}(x)) \neq 0$. Then,

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Example:

Let $f(x) = e^x$, so $f^{-1}(x) = \ln x$. Then,

$$\frac{d}{dx}(\ln x) = \frac{1}{\frac{d}{dx}(e^x)|_{x=\ln x}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

2.6 Chain Rule

If $h(x) = f(g(x))$ where both f and g are differentiable, then

$$h'(x) = \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x).$$

3 Advanced Differentiation

3.1 Implicit Differentiation

If a function y is given implicitly by an equation involving both x and y, such as

$$F(x, y) = 0.$$

To find the derivative $\frac{dy}{dx}$, we differentiate both sides of the equation with respect to x, treating y as a function of x. This means when differentiating terms involving y, we use the chain rule and multiply by $\frac{dy}{dx}$.

Example:

If

$$x^2 + y^2 = 25,$$

then differentiating both sides gives

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$

3.2 Higher-Order Derivatives

The second derivative, third derivative, and beyond are called higher-order derivatives. These describe how the rate of change itself changes

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^ny}{dx^n}$$
$$f'(x),\ f''(x),\ f'''(x),\ f^{(n)}(x)$$
$$\dot{y},\ \ddot{y},\ \dddot{y}$$

3.3 Parametric Derivatives

Given a parametric curve:

$$x = x(t) \quad y = y(t)$$

the derivative of y w.r.t x is given by

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{(provided } \frac{dx}{dt} \neq 0)$$

4 Theorems

4.1 Rolle's Theorem

Let f be continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that

$$f'(c) = 0.$$

4.2 Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

4.3 Cauchy's Mean Value Theorem

Let f and g be functions continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists at least one point $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4.4 Extreme Value Theorem

If f is continuous on $[a, b]$, then there exist points $c, d \in [a, b]$ such that

$$f(c) \leq f(x) \leq f(d), \quad \forall x \in [a, b].$$

4.5 Darboux's Theorem

Let f be a differentiable function on an interval $I \subset \mathbb{R}$. Then the derivative f' satisfies the Intermediate Value Property: for any $a, b \in I$ with $a < b$, and any λ between $f'(a)$ and $f'(b)$, there exists some $c \in (a, b)$ such that:

$$f'(c) = \lambda$$

This means that even if f' is not continuous, it cannot have jump discontinuities — it must take on all intermediate values.

5 Behavior of Functions

5.1 Critical Points and Extrema

5.2 Concavity and Inflection Points

5.3 Derivative Tests

5.3.1 First Derivative Test

5.3.2 Second Derivative Test

6 Applications

6.1 Related Rates

6.2 Optimization Problems

6.3 Linear Approximation (First-Order Taylor Expantion)

If f is differentiable at $x = a$, then near a, the function $f(x)$ is approximated by

$$f(x) \approx f(a) + f'(x)(x - a)$$

Example:

for all x near 0, $\sin x$ can be approximated by $\sin x \approx \sin(0) + \cos(0) \cdot x = x$