Contents

1 Logic and Proofs

1.1 Propositional Logic

Proposition is a statement that is **either** true or false, but not both at the same time. We usually represent it with variables like p, q, and r.
e.g. "The sky is blue." is a proposition, but "Listen to me" is not.

1.1.1 Logical Connectives

- Negation: $\neg p$. It is not the case that p.
- Conjunction: $p \wedge q$. "and"
- Disjunction: $p \vee q$. "or"
- Implication: $p \rightarrow q$. If p then q, q if p, q is a consequence of p, p only if q
- biconditional: $p \leftrightarrow q$. $(p \rightarrow q) \wedge (q \rightarrow p)$, p if and only if q

1.1.2 Variations of Conditionals

- Implication: $p \rightarrow q$
- Converse: $q \rightarrow p$
- Inverse: $\neg p \rightarrow \neg q$
- Contrapositive: $\neg q \rightarrow \neg p$. This is logically equivalent to Implication

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example

Find the truth value of $\boldsymbol{(p \vee q) \rightarrow \neg r}$

p	q	r	$p \vee q$	$\neg r$	$(p \vee q) \rightarrow \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	T

1.2 Application of Propositional Logic

1.2.1 Classification of Proposition

- Tautology: Always true. e.g. $p \vee \neg p$
- Contradiction: Always false. e.g. $p \wedge \neg p$
- Contingency: Depends on variable. e.g. $p \rightarrow q$

1.2.2 Logical Equivalence $p \equiv q$

Two statements are logically equivalent if they always have the same truth value in every possible scenario.
e.g. p and q are biconditional, i.e. $p \leftrightarrow q$, means that p and q are logically equivalent.

1.2.3 Laws of Logical Equivalence

Equivalence	Name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws
$p \rightarrow q \equiv \neg p \vee q$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$	Conditional
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$	Biconditional

1.2.4 Determine Logical Equivalence:

1. Verify with Truth Table
2. Apply Known knowledge

Show that $p \to q$ is logically equivalent to $\neg q \to \neg p$

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Show that $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

$$\begin{aligned}(p \to r) \vee (q \to r) &\equiv (\neg p \vee r) \vee (\neg q \vee r) \\ &\equiv (\neg p \vee \neg q) \vee (r \vee r) \\ &\equiv \neg(p \wedge q) \vee r \\ &\equiv (p \wedge q) \to r_{\#}\end{aligned}$$

1.3 Predicate and Quantifier

1.3.1 Predicate

A predicate is a statement with variables that becomes true or false only once specific values are substituted. $P(x)$ denotes a predicate involving x.
e.g. Let $P(x)$ be the statement ”x¿4.” We read $P(x)$ as ”x is greater than 4.”

- $P(x)$ is true if x = 5
- $P(x)$ is false if x = 3

General Form

$P(x_1, x_2, x_3, \ldots, x_n)$ where each x_i is a variable from the domain of discourse.

1.3.2 Quantifier

- Universal quantifier $\forall$: ”for all”, ”every”.
- Existential quantifier $\exists$: ”there exists”, ”some”, ”at least one”.

Rules for Quantifier

Negation of quantifier

$$\begin{cases}\neg\forall x\, P(x) \equiv \exists x\, \neg P(x) \\ \neg\exists x\, P(x) \equiv \forall x\, \neg P(x)\end{cases}$$

Nested quantifier

$$\forall x\, \exists y\, P(x, y) \neq \exists y\, \forall x\, P(x, y)$$

Negation of nested quantifier

$$\neg(\forall x\, \exists y\, P(x,y)) \equiv \exists x\, \forall y\, \neg P(x,y)$$
$$\neg(\exists x\, \forall y\, P(x,y)) \equiv \forall x\, \exists y\, \neg P(x,y)$$

1.4 Rule of Inference

An **argument** is an implication of the form:

$$\bigwedge_{i \in D} p_i \to q$$

where D is domain of discourse, p_i is a premise, and q is a conclusion
Notation:

$$(p \to q) \land p \quad \therefore q \quad \Rightarrow \quad \begin{array}{rl} & p \to q \\ \land & p \\ \hline \therefore & q \end{array} \quad \Rightarrow \quad \begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Name	Expression	Name	Expression
Modus Ponens	$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$	Modus Tollens	$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$
Hypothetical Syllogism	$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	Conjunction	$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$
Disjunctive Syllogism	$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$	Addition	$\begin{array}{c} p \\ \hline \therefore p \lor q \end{array}$
Simplification	$\begin{array}{l} p \land q \\ \hline \therefore p \\ \therefore q \end{array}$	Resolution	$\begin{array}{r} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$

Rules for Quantified Statement

Universal Instantiation: From a universal statement $\forall x\, P(x)$, we may infer $P(c)$ for any particular c.

$$\forall x\, P(x) \quad \Rightarrow \quad P(a)$$

Universal Generalization: If $P(c)$ holds for an arbitrary element c, then we may conclude $\forall x\, P(x)$.

$$P(c) \quad \Rightarrow \quad \forall x\, P(x)$$

Existential Instantiation: From $\exists x\, P(x)$, we may introduce a new symbol c such that $P(c)$ holds.

$$\exists x\, P(x) \quad \Rightarrow \quad P(c)$$

Existential Generalization: From $P(c)$ for a particular c, we may conclude $\exists x\, P(x)$.

$$P(c) \quad \Rightarrow \quad \exists x\, P(x)$$

1.5 Introduction to Proofs

1.5.1 Some Mathematical Terminology

- **Theorem:** A major, important mathematical statement that has been proved true.
- **Lemma:** A proved result used mainly as a stepping stone toward a larger theorem.
- **Corollary:** A statement that follows directly and easily from a theorem or proposition.
- **Proposition:** A proved statement that is true but of smaller or less central importance than a theorem.
- **Conjecture:** A mathematical claim believed to be true but not yet proved.
- **Proof:** A logical argument that establishes the truth of a mathematical statement beyond doubt.

1.5.2 Types of Proof

- **Direct Proof:** Show a statement is true by straightforward logical reasoning from definitions and known results.

  Example: Prove that if n is odd, then $5n + 3$ is even.
  Proof:
  Let $n = 2k + 1, k \in \mathbb{Z}$, then

  $$5n + 3 = 2(2k + 1) + 3 = 10n + 8 = 2(5k + 4)$$

  So, $5n + 3 = 2(5k + 4)$ and $5k + 4 \in \mathbb{Z}$

  $$\therefore 5n + 3 \text{ is even}_{\#}$$

- **Proof by Contrapositive:** Show a statement is true by proving that if the conclusion is false, then the premise must also be false.

  Example: Prove that if n is odd, then $5n + 3$ is even.
  Proof:

- **Proof by Contradiction:** Show a statement is true by assuming the opposite and deriving a contradiction.

  Example: Prove that $\sqrt{2}$ is irrational
  Proof:
  Let

  $$\sqrt{2} = \frac{p}{q},\ q \neq 0,\ p, q \in \mathbb{N},\ \gcd(p, q) = 1$$
  $$\Rightarrow 2 = \frac{p^2}{q^2} \Rightarrow 2q^2 = p^2$$

Thus, $2 \mid p^2$, so $2 \mid p$. We have

$$p = 2k,\ k \in \mathbb{N},\ 4k^2 = 2q^2 \Rightarrow 2k^2 = q^2$$

Similarly, $2 \mid q$. Concluding

$$2 \mid \gcd(p, q) \Rightarrow\Leftarrow$$

Thus $\sqrt{2}$ is irrational.

- **Proof by Cases:** Show a statement is true by dividing into cases and proving it holds in each case.

  Example:
  Proof:

- **Methematical Induction:** Show a statement is true by proving a base case and then proving the inductive step from n to n+1.

  Example:
  Proof:

- **Existence and Uniqueness Proof** Show a statement is true by first proving that at least one object with the required property exists (existence), and then proving that no more than one such object can exist (uniqueness).

  Example: Prove that if $r \in \mathbb{Q}^c$, then $\exists!\, n \in \mathbb{Z}\ |r - n| < \dfrac{1}{2}$.
  Proof:
  Existence:
  Let

  $$n = \lfloor r + \frac{1}{2} \rfloor$$

  By definition we have

  $$n \leq r + \frac{1}{2} < n + 1$$

  Hence

  $$|r - n| < \tfrac{1}{2}$$

  , so such an integer n exists.
  Uniqueness:
  Suppose

  $$\exists m,\ m \neq n,\ |r - m| < \frac{1}{2}$$

  Consider

  $$\begin{aligned} |n - m| &= |(n - r) + (r - m)| \\ &\leq |n - r| + |r - m| \text{ (Triangle Inequality)} \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

  Since n and m are integers, the inequality $|n - m| < 1$ implies $n = m$. Therefore, the integer n is unique.

2 Basic Structures

2.1 Sets