1 Minkowski Penalties for Rectangles

Let $A, B \subset \mathbb{R}^2$ and $C = \{x - y \colon x \in A, y \in B\}$. Then $A \cap B \neq \varnothing \Leftrightarrow 0 \in C$. Moreover, value of the signed distance function ϕ of C at 0 gives us suitable energies for the following simple objectives.

Energy	Formula	Objective
e_1	$\max(\phi(0,C),0)$	Intersecting rectangles
e_2	$-\min(\phi(0,C),0)$	Disjoint rectangles
e_3	$\lvert\phi(0,C)\rvert$	Touching rectangles

1.1 Axis-Aligned Rectangles

Consider two axis-aligned rectangles

$$\begin{aligned} A &= \left[x_1^A, x_2^A\right] \times \left[y_1^A, y_2^A\right], \\ B &= \left[x_1^B, x_2^B\right] \times \left[y_1^B, y_2^B\right]. \end{aligned}$$

The new rectangle $C = \left[x_1^C, x_2^C\right] \times \left[y_1^C, y_2^C\right]$ is given by

$$\begin{aligned} x_1^C &= \min_{i,j \in \{1,2\}} \left(x_i^A - x_j^B\right), & y_1^C &= \min_{i,j \in \{1,2\}} \left(y_i^A - y_j^B\right), \\ x_2^C &= \max_{i,j \in \{1,2\}} \left(x_i^A - x_j^B\right), & y_2^C &= \max_{i,j \in \{1,2\}} \left(y_i^A - y_j^B\right). \end{aligned}$$

The signed distance function (see [3]) can be written as $\phi(0,C) = e_1 - e_2$, where

$$\begin{aligned} e_1 &= \left[\max\left(x^P - x^R, 0\right)^2 + \max\left(y^P - y^R, 0\right)^2\right]^{\frac{1}{2}}, \\ e_2 &= -\min\left(\max\left(x^Q, y^Q\right), 0\right), \end{aligned}$$

and

$$\begin{aligned} x^P &= \tfrac{1}{2}\left(x_1^C + x_2^C\right), & x^R &= \tfrac{1}{2}\left(x_2^C - x_1^C\right), & x^Q &= \left|x^P\right| - x^R, \\ y^P &= \tfrac{1}{2}\left(y_1^C + y_2^C\right), & y^R &= \tfrac{1}{2}\left(y_2^C - y_1^C\right), & y^Q &= \left|y^P\right| - y^R. \end{aligned}$$

1.2 General Rectangles

Here, C is an intersection of 8 half-planes (corresponding to sides of A and $-B$). For a given side s of A, the half-plane H_s is given by a signed distance function

$$\phi(x, H_s) = \langle x | n_s \rangle - \max_{v \in V_{-B}} \langle v | n_s \rangle - \alpha_s,$$

where n_s is the outward unit normal vector for the side s, V_{-B} is the set of vertices of $-B$ and $\alpha_s = \langle p | n_s \rangle$ for any point p on s. The signed distance function ϕ can be approximated by a level set function $\tilde{\phi}$ of C given by

$$\tilde{\phi}(x, C) = \max_{s \in S} \phi(x, H_s), \tag{1}$$

where S is a set of all sides of A and $-B$. Note that $\tilde{\phi}|_C \equiv \phi|_C$, but they slightly differ outside. This can be fixed by accounting for the distance to the extreme points of C.

2 Minkowski Penalties for Polygons

2.1 Convex Polygons

The method for approximating ϕ for two arbitrary rectangles derived in Subsection 1.2 is applicable to any convex polygons.

2.2 Simple Polygons

Using algorithms for approximate [2] or optimal [1] convex partitioning of a polygon, we decompose A and B into convex polygons as

$$A = \bigcup_{i \in I} A_i \quad \text{and} \quad B = \bigcup_{j \in J} B_j.$$

The signed distance function of C can then be approximated by

$$\tilde{\tilde{\phi}}(x, C) := \min_{i,j \in I \times J} \tilde{\phi}(x, C_{i,j}), \tag{2}$$

where $\tilde{\phi}$ is defined in (1) and $C_{i,j} := \{x - y \colon\ x \in A_i, y \in B_j\}$.

2.3 General Polygons

General polygons which may include self-intersections can be treated in the same way as simple polygons in the Subsection 2.2. However, the convex partitioning is more involved and the resulting set of convex polygons may include new vertices located at the points of intersection.

2.4 Polygonal Chains

Each line segment of a polygonal chain can be treated as a degenerate case of a convex polygon, i.e. each segment should be regarded as two identical sides with opposite normal vectors. This way polygonal chains can be handled using (2), where the partitioning is just the set of the individual line segments.

2.5 Convex Hull Approximation

In the cases when the computational graph corresponding to (2) or the partitioning algorithm itself become to computationally demanding, one can approximate the original polygon by its convex hull.

For simple polygons, one can iterate through all vertices in order and if two consecutive ones are on the boundary of the convex hull, otherwise skip the next vertex until it is on the boundary. It again becomes more involved when self-intersections are allowed and one may have to resort to iterating through all vertex-vertex combinations.

References

[1] D. H. Greene. “The decomposition of polygons into convex parts.” In: *Computational Geometry* 1 (1983), pp. 235–259.

[2] Stefan Hertel and Kurt Mehlhorn. “Fast triangulation of simple polygons”. In: *Foundations of Computation Theory*. Ed. by Marek Karpinski. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983, pp. 207–218. ISBN: 978-3-540-38682-7.

[3] I. Quilez. *2D distance functions*. Oct. 18, 2021. URL: `https://iquilezles.org/www/articles/distfunctions2d/distfunctions2d.htm`.