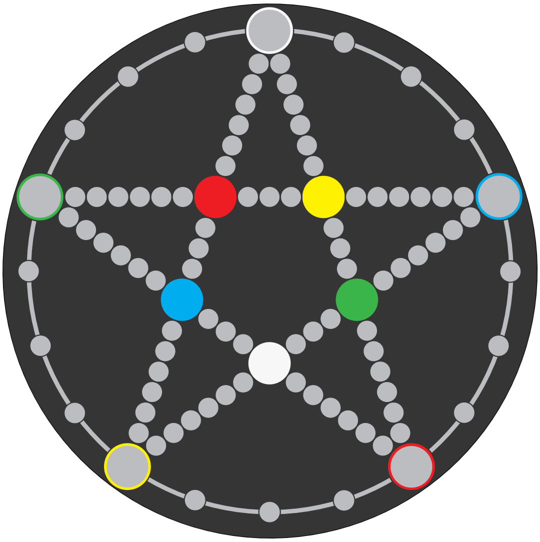
PENTAGAME

Compendium



Jan ‘Penta’ Suchanek

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This is a pre-print, and it needs proof-reading. I apologise for typos etc., please report them to me:
jan@ pentagame.org

pentagame.org

Overview

PENTAGAME is a game with a simple structure, very little material and easy rules. But behind its simplicity lurks complexity: there are more possible games than stars in the sky. And while the image of the board is a simple geometric shape, it nevertheless has a rich history. This book deals with all this, and with more. It grew in parallel to the development of the game and hence is testament to it.

The first part explains the game as it is today, an extensive form of the rule sheets, if you like.

The second part explains how the idea was born and how we (re-)created Pentagame.

The third part shows that there might have been a pentagram shaped board before, and investigates why it was presumably lost.

The forth part is an in-depth analysis of the game at hand, starting from general observations, and moving from the opening theory via strategy and tactics all the way to end game theory.

The fifth part deals with multi player setups and possible extensions.

The sixth part deals with tournaments and how to score and compare games in general.

The seventh part dives into topics of advanced game theory and complexity theory. This part serves to proof some important qualities: complexity, but also drama, of the game.

Part eight is nothing else than a conclusion: we have a game that has all it takes to become a classic, and probably the reconstruction of an ancient game.

The book ends with an **Annex**: A bibliography, a list of figures, a list of tables, an index, and a detailed list of content.

What this book cannot possibly convey is how and how much we enjoy playing this game. Having played hundreds of times, Pentagame continues to puzzle. Every single match was, is and will be a rich, unique and personal experience.

Contents

Part I.

The Game

1. Setting up

The board The board consists of a pentagram in a circle. On the ring, the five *corners* (the points of the star) are coloured white, blue, red, yellow and green. In the middle, there are the five *crossings* (where the lines forming the star bisect), these are also coloured white, blue, red, yellow and green. For each corner of the star, there is a corresponding crossing of the same colour—exactly opposite and furthest from it.

On the lines of the pentagram are additional *round stops:* two times six linking each corner to each of its two adjacent crossings and three linking each crossing to each of its neighbours. On the ring, there are *three stops* between adjacent corners. The three stops on the ring between corners are equal; the middle one is rounded only for decorative reasons. We'll call corners, crossings and all other stops together *stops.* In total, this makes 100 stops of which five corners and five crossings are coloured.

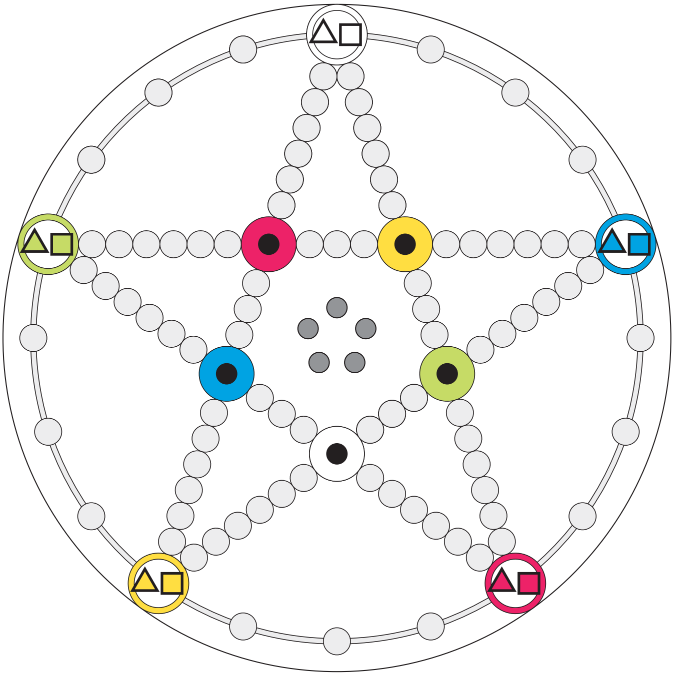
The pieces Each player commands a *set* of five *pieces.* Pieces are shaped differently according to players, or distinguished by having the same shape or ornament. However, a player's set contains a white, a blue, a red, a yellow and green piece.

For examples: the first player may be 'stars' and have five stars—each a different colour, the second plays using five moons of different colours (and so on for circles, triangles, or squares etc.). It is simply necessary for a player's set to be clearly distinguished.

In addition to the player's pieces there are five *black blocks* placed on the five crossings in the middle of the board. The blocks are passive pieces, moved by any player when they land on them (page 5).

Also there are five *grey blocks*; these are also passive pieces. Set them in the centre pentagon for later.

Figure 1.1.: Setup for two players



2. Objective

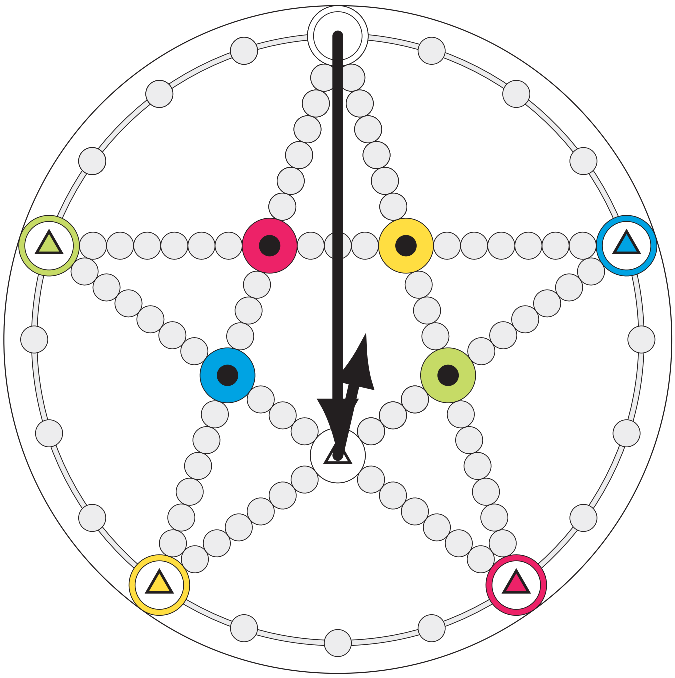
All coloured pieces start at the corner of their colour; each player will have a white piece starting at the white corner, a blue piece that starts at the blue corner, and so on.

Pieces strive to reach the *crossing* on the central pentagon that matches their colour (Fig. 2.1).

The corresponding crossing is always exactly opposite the corner from which pieces of a colour originate.

Who first manages to bring *three* of their pieces to these goals wins.

Figure 2.1.: Objective



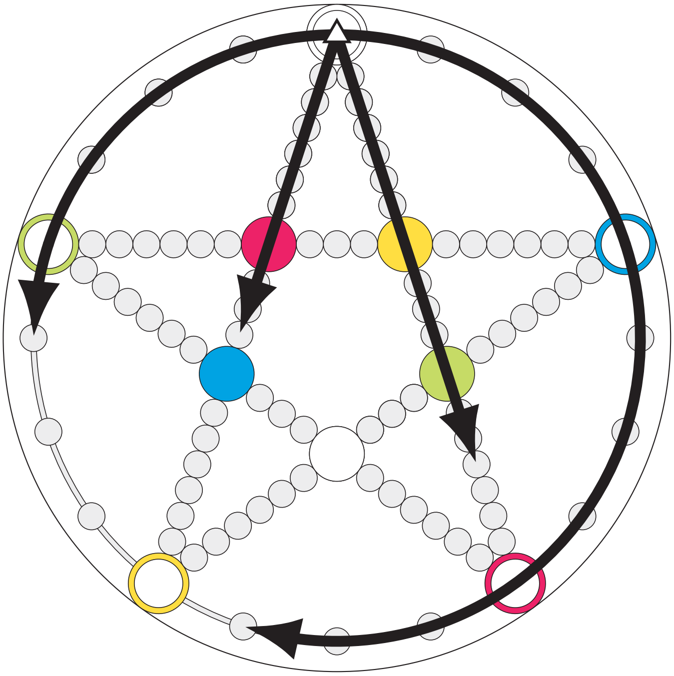
You have five pieces.
Bring them to their **goals opposite their origins:**
White to white, blue to blue etc.
Three out wins.

3. Moving and winning

Moving Move in any direction on the star or the ring as far as you please, as long as the path is free (Fig. 3.1). On a free path you may turn at any corner without stopping. But you cannot jump over anything (Fig. 3.2).

You can, however, take the place of an obstructing piece or block.

Figure 3.1.: Move **in any direction**



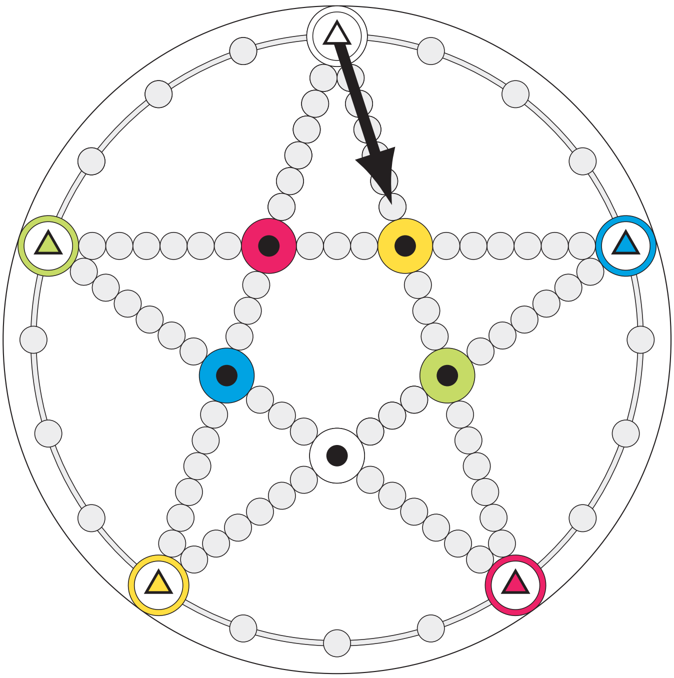
You can move as far as ways are free.
You can move on the ring and the star.

Replacing blocks If you take the place of a black block, take its place and replace it on a free stop of your choice.

You will want to set the block so that you disallow the best option your opponent would have without it. Placing a block is a bet on the best answer of your opponent, so think carefully—this is a majorly strategic part of the game!

Swap your own pieces If you take the place of another of your own pieces, you put that second piece where you have just come from (Fig. 3.4). This allows you to move two of your pieces in one ply and improve your standing.

Figure 3.2.: No jumping allowed



You can not jump over anything.

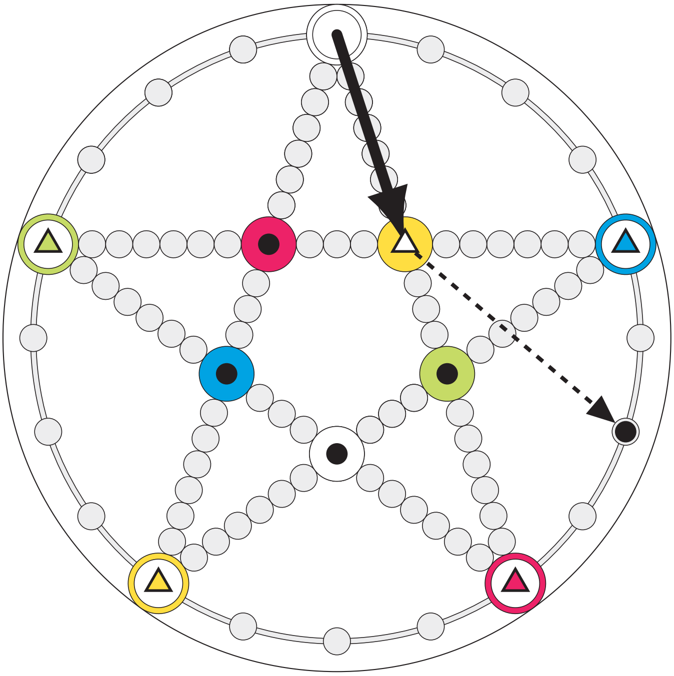
Swap with your opponent's piece If you take the place of an adjacent piece of another player, that second pieces goes to the stop the first piece has just left, swapping the position of your own piece with the adjacent foreign piece.

In most situations, your opponent can just re-swap the two pieces, which nullifies your move; you are not allowed to try the same swap twice **(Ko-rule)**.[1]

But if your opponents occupies one of your goals, this allows you to reach your goal while swapping your opponent's piece (Fig. 3.5).

[1] The reason is discussed in Section 3.4).

Figure 3.3.: Replace move



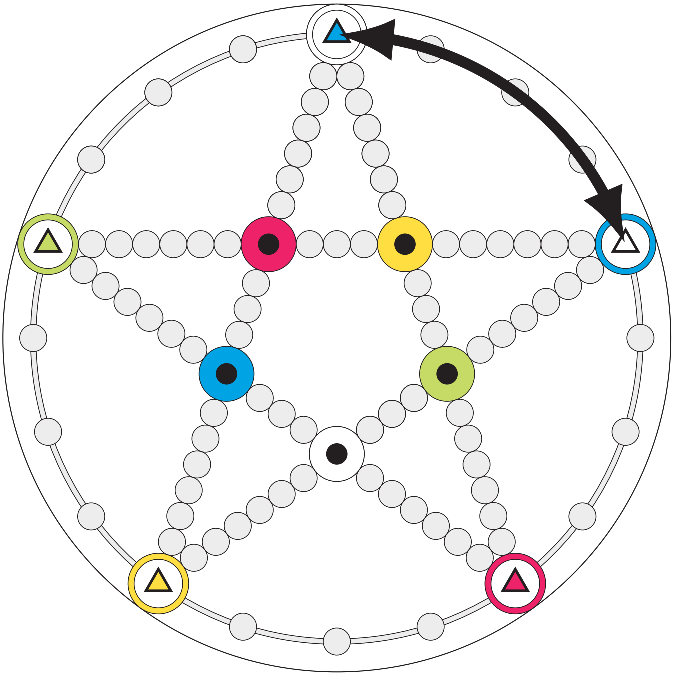
You can **hit a black block.**
You then **replace** it on another empty space.

Move to an empty stop Of course, you can also simply move to an unoccupied stop.

These are the ways you can move. Notice that paths can become quite long (Fig. 3.6).

Initial swaps Only at the start of the game can there be more than one piece on a (corner) stop.—If you move to a corner occupied by multiple pieces, you must choose one of these pieces to swap positions with (Fig. 3.8). This would practically always be one of your own pieces.

Figure 3.4.: Swap move

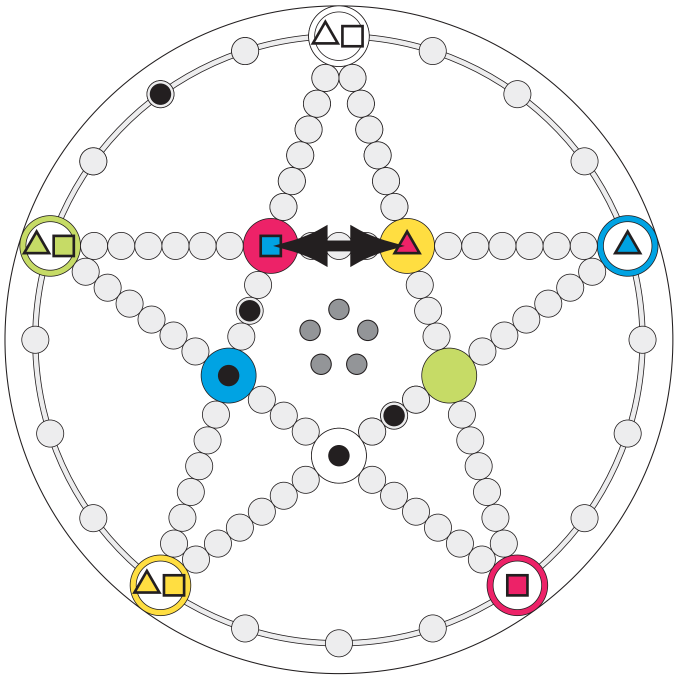


You can **swap** two neighbouring pieces
(at least one of which must be yours).
Of course the way must be free!

Leaving the board When a piece reaches its destination, rejoice! – Then, you remove that winning piece from the board, parking it in the central pentagon of the board.

Grey blocks For moving out a piece you receive a grey block, which you may place on any free stop of your choice (Fig. 3.9). These grey blocks are similar to the black blocks, they are passive pieces.—Remember the black blocks? When you 'beat' them, you replace them somewhere else... But when you take the place of a grey block, it simply leaves the board again (park it in the pentagon again). They are 'one time blocks'.

Figure 3.5.: Swap your opponent



In the rare case that all grey blocks are already in the game and you gain the right to place one more, move any of the existing ones to the stop of your choice.

Winning Whoever gets three pieces to their corresponding coloured stop and off the board is the winner with a score of three.

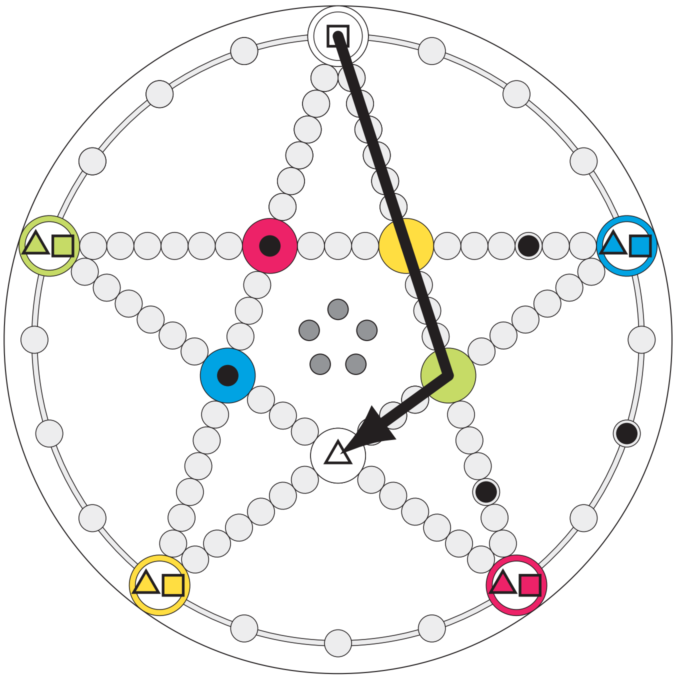
Continue the game to complete the last round so all players have had the same number of moves. In other words, if one player has moved their third piece out, the others can still complete the round and see if they can tie.

A player's score is the number of their pieces out at the end of that last round.

That is all there is to it, this is the whole of the law.

There are only very few necessary clarifications for uncommon

Figure 3.6.: Turn corners on free paths



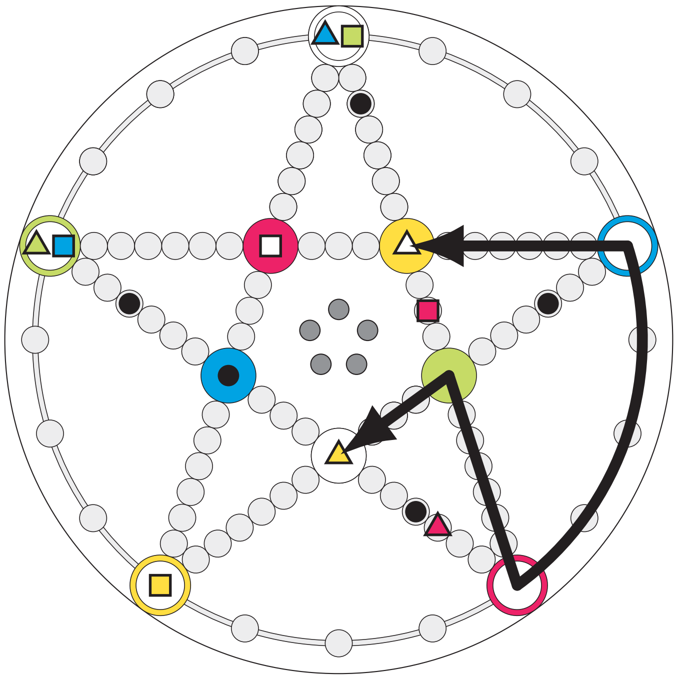
Ways can be long!

situations; there are some tips for special situations, too: these we will briefly go through in the following section.

4. Clarifications

Abracadabra If you can reach a goal and find it occupied by a black block, you remove that black block, move out and gain a gray block. This move is called 'Abracadabra'. It allows you to position both

Figure 3.7.: A long swap move



You can swap on long paths.

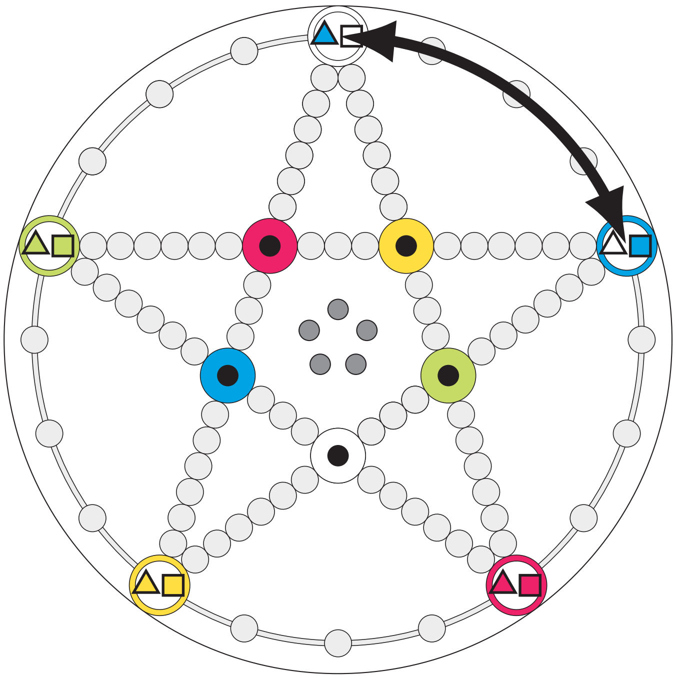
a back and a gray block in one single move, which is great and a challenge at once.

You can moves a piece of another player to its destination When you swap a piece with another player's, this foreign piece may end up on its final destination. That is, a piece has passively been brought to its goal; a player who is not yet to move has reached a goal. What then?

The piece that was passively brought to its goal remains on that goal until it is the turn of its owner. When it is the owner's turn, she *must* remove that piece—as in reaching a goal and winning a point. She receives a grey block as usual.

The player thus profiting from someone else's help in reaching her

Figure 3.8.: Initial swap



Swapping with yourself is often sensible

goal has no other option but to remove that piece. She removes it once it is her turn, not before. Neither has she another choice than to move out with that piece, nor does she get an extra additianl move.

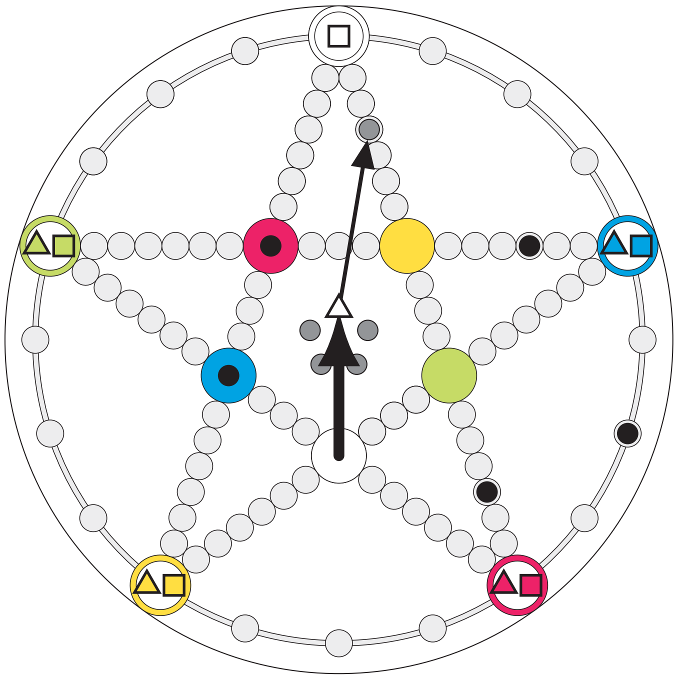
This is so that the player passively brought to her goal does not win another extra advantage.

Forcing an opponent's piece out in this way is sometimes of strategic value.

The last round The last round usually is played out, so that all players have the exact same number of plies to make. In other words, when Alice has started, Bob is the last to move. This makes good for the first mover advantage.

It can be proven that this in fact bestows a slight advantage to the

Figure 3.9.: Moving out



When you **reach a goal,** your piece moves **out.**
For this you **place a grey block** anywhere.
Grey blocks are **one-time blocks.**

second mover. Thus it is only polite if the strongest player starts the game.

Play twice If you want a perfectly fair game, two players must of course play twice where every player begins once.

More than two players The rules for two, three and four players are exactly identical. Just give the additional player her own set of five coloured figures and put these her pieces also on the five corners to start.

Games with two, three and four players have some particular qualities that set them apart; these will be discussed elsewhere. Here it suffices to state that the rules remain unaltered for any number of players.

Three players Three player games are very entertaining. A three player game should take about 45 minutes.

Because the last round in any game always is played out, there is no huge first mover advantage.

In contrast, it makes a very large difference in which *order* players sit and move. The direction—clockwise or counter-clockwise—makes there relations between all players un-equal.

Thus make sure that you play equally often clockwise and widdershins.—You can play twice with three players, then (in case of a draw) have the leading two play a two-player. (See also Chapter 2.)

Four players

- The simplest way is to play with four players is to take the normal, two-player, rules and form two teams. This is quick and a good way to introduce the game.
- Or you play everyone against everyone, each player commanding a set of pieces. This takes maybe 85 minutes.
- You can also have each player have her own set of pieces, but nevertheless form teams. Players sitting opposite each other should form a team. Then every player has one partner (the player sitting opposite) and two rivals. The team that has a total of 5 pieces out first wins. (See also Chapter 3)

Five players Five players *can* play, even though you only have four sets of pieces in your box. To do this, simplify the standard rules as follows. Give each player *three* pieces *of one colour* regardless of shape, so one player player only red, another only blue, another only white, another only green or only yellow pieces. Have them place their pieces on the corners of their colour. Then all players can start their pieces at the corner where they sit. All other rules remain unchanged.

The winner is who brings out two pieces (out of three). (See also Chapter 4)

Scoring The number of pieces a player has brought out when the game has ended is her primary score.

In some settings, you will wish to convert this into zero-sum scoring; then you just note the difference in score to the other player.

In multi-player settings, consider that a score won seated after a strong player must count more than the same score won seated after a weak player.

Time limits You may want to limit the time per move with an egg timer, where one minute per ply has turned out a good measure.

If you move before your minute runs out, your rival can use that extra time before you can turn the egg timer on her.

A player who does not move in time loses turn.

In four-player games, players may need more than one minute.

If you play in teams, you can use a more sophisticated clock and give each party total consideration time.

Silence Etiquette demands not to distract the player who has to move, and never to tell anyone about any possibilities you spot on the board. When you play with three or more players, those who are not to move should seek a topic and conversation, leaving the person to move have her thoughts.

Tournaments Pentagame is a quick game. Thus, players regularly play multiple games. Thus it is very apt for competitions and tournaments. Players should record their scores a zero-sum games. (You can find a detailed discussion of various scoring and tournament from Page 219).

Part II.

The Road to the Game

Saepe notavimus, nusquam homines quam in ludicris ingeniosiores esse: atque dieo lusos Mathematicorum curam mereri, non per se, sed artis inveniendi causa.

—G.W. Leibnitz

On one hand, a good game may be as esoteric as a theorem of pure mathematics, and on the other it is as human as anthropology.

—J. Mark Thompson

1. Motivation

It all started when I wondered why nobody seems to use the pentagram shape for a board game, while there are linear, circular, triangular, square and hexagonal board games.

This was in 1996. I started trying a variety of board designs and rule sets, but ran into dead ends everywhere. Raising to the challenge, I studied both the mathematical properties and the cultural history of the pentagram, games and gaming and the mathematics of play. I only picked up on developing a proper game in London at about 2007. Solving the geometry of the board took about as long as finding a proper rule set: years. Everything got finalised 2015 in Berlin.

This part tells the story of the invention and sketches the reasoning behind it.

1.1. Geometry of classics

Probably most of us have grown up with a collection of classib board games; they usually consist of boards printed on both sides and typically contain classics such as Chess, Nine men's morris, Halma and Ludo.

Just optically, these are different to all the modern author games. They are theme free; but they are played on very simple geometric boards.

These boards are either lines of regular polygons, or on surfaces that are tiled by regular polygons.

For each of these polygons we can find classic board game examples, or abstract board game examples: see Table (1.1).

Table 1.1.: Geometric boards of strategy games

	Form	Examples
—	linear, circular	Pachisi (Ludo), Mancala, Backgammon, …
▷	triangular	Halma (Chinese Checkers), Gipf, …
◇	square	Chess, Go, Othello, Draughts …
⬠	**pentagonal**	**– ??? –**
⬡	hexagonal	Abalone, Hex, Settlers …

We find games for all fundamental geometric shapes—with the notable exception of the pentagon. There is no pentagonal or pentagram shaped game, apparently, yet.

Why this gap? Is there a reason? Can you not possibly also play on the five-fold (pentagram) shape? What would that game be?

1.2. Qualities of classics

New games come out every year. And besides this constant stream of novelties some classics stand as solid rocks. What sets them apart? Is it possible to create another 'classic'?

A concise answer to this question was provided by Thompson in his seminal paper 'Defining the Abstract' [90], and we follow his definitions. We can state:

1. Working games share certain qualities, which are:

   complexity there is no trivial solution or simple method to win;

   drama who is in the lead may change over the game duration;

decisiveness they always end with winner and loser.

2. Games qualify as abstract classic when they also have

   clarity extremely simple rules that offer choice,

3. Games where the action takes place on a board are board games.
   - 3.1. There are just a few board game *classics*, clearly distinct from the large number of themed games, which appear every year.
   - 3.2. These are classics because the above quality of simplicity expands to their boards: they have

     symmetry a simple geometric board design.

So the task will be to *find such a game, but based on a five-fold geometry;* and then, by analysing it, proof that it has these qualities.

1.3. Research Outline

From the above we got the following research programme: Based on the observation that the five-fold abstract board game is unknown:

1. Create the missing pentagonal classic 'Pentagame':
   - 1.1. Construct a pentagram shaped game board;
   - 1.2. Find few and simple rules that potentially create complexity, clarity, drama and decisiveness.
2. Seek potential ancestors:
   - 2.1. Has there been such a game before?
   - 2.2. Can its absence be explained?
3. Analyse Pentagame:
   - 3.1. Give advice on strategy and tactics with respect to the opening, mid-game and end-game phase;
   - 3.2. Analyse the complexity,
   - 3.3. Proof that it has complexity, clarity, drama and decisiveness.

4. Discuss extensions and find it the proposed rule set is minimalistic.
5. Conclude whether or not Pentagame is a member of the set of (potentially) classic board games, and whether or not it has likely existed before.

2. The shape

2.1. Polygons and tiling

We have observed that classic board games are simple games played on simple geometric patterns. The most simple geometric patterns are the regular polygons Fig. 2.1. So let us talk about these first.

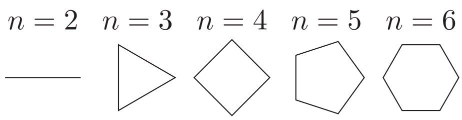


Figure 2.1.: Regular polygons

These polygons are not equally simple. This has two reasons.

The first reason is the difficulty of the construction. You don't just 'plug in' the number of sides and then just 'get' the respective polygon (unless you use software). What you want to do is to draw the polygon free hand, or construct the polygon with ruler and compass. This is easy for the line (or a cirlce), for the triangle, and the square; and it is also relatively easy for the hexagon as it follows from two triangles (since $6 = 2 \cdot 3$). Constructing these polygons with ruler and compass is so trivial that every child can find the construction just by trial and error. The pentagon is somehow surprisingly far harder to construct. Try if you want: if you don't know the construction, you will likely be frustrated very soon. But once you have succeeded, some interesting fractal properties of the pentagram become obvious; view Fig. 2.2.

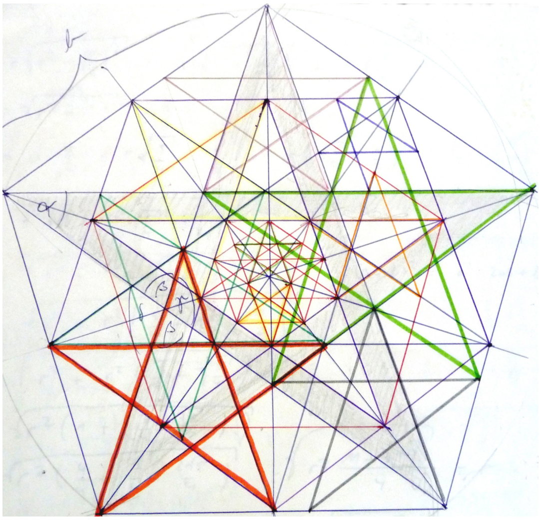


Figure 2.2.: My first study (1996)

The second reason why the pentagon is differnet lies in the somehow surprising fact that you cannot tesselate a surface with regular pentagons. Triangles, squares and hexagons in contrast can easily be used to tesselate surfaces, which explains why such tesselated boards (like the Chess board based on squares) are very common. A tesselation using pentagons requires particular shapes of irregular pentagons; but these fall out of our criteria of simplicity. But we want a *simple* pattern for our game. There also is a way to create non-periodic tesselations using Penrose-tiles (Fig. 2.3). These always come in pairs, can produce pentagonal patterns, but they never sum up to a perfect pentagon [46] at finite size. Thus, these too are not suitable for a simple enough game.

This discards the option to have a simple game on a tiled pentagonal board. We are left with the option to find a simple board based not on penta*gons,* but on a penta*gram* with segmented lines. Mathematically, the pentagram is the most simple regular star polygon. But here too we find challenges. One, as mentioned before, to construct a pentagram is possible, but not easy. Two, to segment it into stops

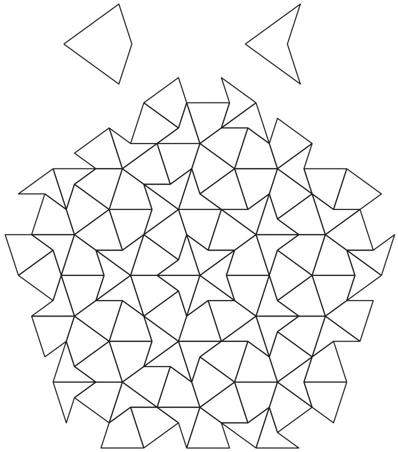


Figure 2.3.: Penrose-tiles ‘kite’ and ‘dart’ tesselation

precisely is also possible, but actually very hard. This is because the lines in a pentagram are incommensurable: they relate to one another in the Golden Proportion, and this is an irrational ratio. It follows that not one size of stops can segment all these lines. In what follows we will show what sizes such stops must have to segment a pentagram perfectly.

2.2. The Golden Proportion in the pentagram

Fig. 2.4 shows one possible way to construct a pentagram with ruler and compass; this is probably the most popular method, and it can be traced back to Ptolemy [76]. It can be used to prove that the lengths within the pentagram relate to one another in the Golden Section. If you can construct the Golden Proportion, you can draw a pentagram, and vice versa.

Shorter and longer lines in a pentagram are incommensurable; it follows that is not possible that all stops on a board like Pentagame have the exact same size. We will demonstrate this in the following

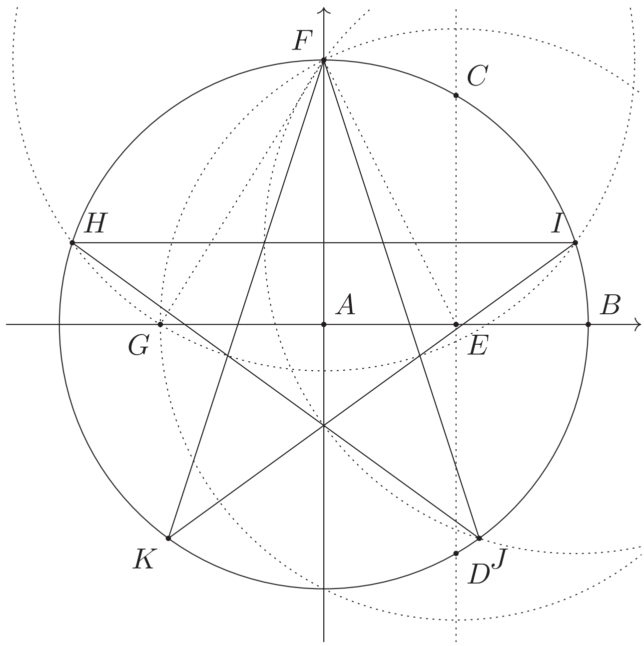


Figure 2.4.: Pentagram construction (Ptolemy)

section; here, we will recall the common knowledge on the Golden Proportion (or Golden Section) first.

Two lengths, a longer one $a = \overline{AB}$ and a shorter one $b = \overline{BC}$, are in Golden Proportion if and only if their sum $a + b = \overline{AC}$ relates to the longer length a like a to b .

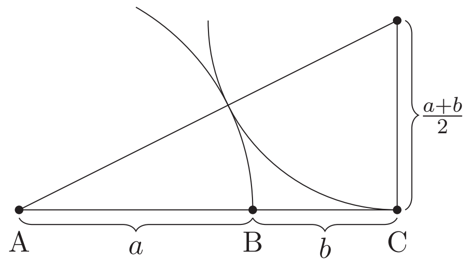


Figure 2.5.: Golden Section
The sides of the triangle are $1, 2$, and $\sqrt{5}$.
$a + b = 2$ and $a = \varphi = 1.618 \ldots b$.

The Golden Proportion is defined by the relation

$$\frac{a+b}{a} = \frac{a}{b} \tag{2.1}$$

Solving this for a yields a quadratic polynomial; you get the algebraic solution $a = \varphi b$ with φ *phi* (named after PHIDIAS) being

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.61803\ldots \tag{2.2}$$

If you now subtract b from a, the Proportion reoccurs:

$$\frac{b}{a-b} = \frac{a}{b} \tag{2.3}$$

And this goes on and on and on. φ is really a 'fractal' number. It can also be expressed in form of a chain fraction, which demonstrates nicely its fractal quality and proves its irrational nature directly. You may compare Fig. 2.2.

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1+\ldots}} \tag{2.4}$$

As a result, the Golden Proportion is somehow self-referential:

$$\frac{1}{\varphi} = \varphi - 1 = 0.61803\ldots \tag{2.5}$$

The Golden Section is the marriage of simplicity with complexity in its most simple form. It is a proportion at once familiar and strange.

Thus equipped we can now attempt to find the desired solution for the stops on our board.

2.3. Pentagame board geometry

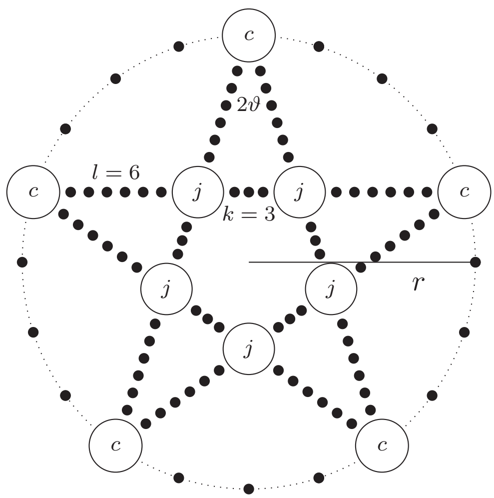


Figure 2.6.: Pentagame stops
c corner, j junction, l 'leg', k 'arm', 2ϑ angle

To draw a Pentagame board precisely requires knowledge of the exact proportions of the stops on the board. We know that these can neither be of identical size, nor can they be chosen at random. There is only one exact solution, as we will demonstrate now.

We begin with some definitions, so that we know what we are talking about in Fig. 2.6.

- Call the long lines within the pentagram 'legs' L and the short ones 'arms' K.
- There are l stops on each leg (usually $l = 6$) and k stops on each arm (usually $k = 3$). (Other lengths are conceivable but impractical.)[1]
- Call the radius of **s**tops on the lines s,
- call the radius of a **c**orner stop c,
- call the radius of a crossing (or **j**unction) j.
- call the **r**adius of the pentagram r.

The most important formula will be the golden ratio:

$$\varphi = \frac{\sqrt{5}+1}{2} \tag{2.6}$$

In the course of our investigation we will deal with the angle $18° = \vartheta = \pi/10$. This angle is half the angle of the points of the pentagram. This angle has a property that will come in extremely handy:

$$\sin(\vartheta) = \frac{\varphi}{2} \tag{2.7}$$

We begin by setting $s = 1$, so the size of the stops on the star be unit. Thus, we are left with the task to identify only two sizes: corner size c and junction size j. As a result, we need, and will find, two conditions. The radius of the board r will follow as a corollary.

First Condition: The lines in a pentagram are in golden proportion. Thus, we know the following, where the left hand side (l.h.s.) of the equation is the 'leg' and the right hand side (r.h.s.) is the 'arm':

$$c + j + 2l = \varphi(2j + 2k) \tag{2.8}$$

[1] We will see that $l \approx \varphi k$ so that $c \approx j$.

This can be solved for j by insertion of $\varphi = (\sqrt{5}+1)/2,\ l = 6,\ k = 3$:

$$j = \frac{c + 9 + \sqrt{5}}{\sqrt{5}} \tag{2.9}$$

Second Condition: Take a closer look at the corner (Fig. 2.7). There we have three circles: the actual corner circle c and two adjacent stops $s_{1,2} = 1$. All three are tangential to each other. In addition both stops s are tangential to our angle $\vartheta = \pi/10 = 18°$. That makes four conditions for three circles. The proportion $s \mapsto c$ is fully determined by the angle ϑ, and it is independent of the number of stops on the board.

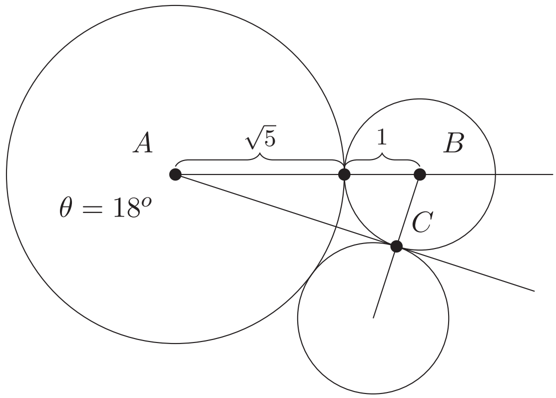


Figure 2.7.: Corner Condition

ABC is a triangle with a right angle $\angle BCA$; thus, we can make use of trigonometric functions. We see that by definition

$$\frac{BC}{AB} = \sin(\vartheta) \tag{2.10}$$

This is now easy to solve since $\overline{BC} = s = 1$ and $\overline{AB} = c + 1$ and since we remember $\sin(\vartheta)$ from Eq. (2.7) as $\sin(\vartheta) = \varphi/2$; thus this becomes

$$\frac{1}{c + 1} = \frac{\varphi}{2} \tag{2.11}$$

which is equivalent to

$$c = \frac{2}{\varphi} + 1 \tag{2.12}$$

And with the golden ratio (Eq. (2.6)) that even reduces to

$$c = \sqrt{5} = 2.23606\ldots s. \tag{2.13}$$

In words: the corner stop is 2.23606... times as large as the stops on the star. Or if the stop on the star is of unit size, the corner is of size $\sqrt{5}$, independent of the number of stops.

That allows us to also solve Eq. (2.9) through insertion and find the radius of the junctions j:

$$j = \frac{9 - 2\sqrt{5}}{\sqrt{5}} = 2.02606\ldots s. \tag{2.14}$$

In words: each junction stop is 2.02606... times as large as the stops of the star.

Corollary 1 *There are necessarily three different sizes of stops, no matter how many k and l; thus $c \neq j \neq s \quad \forall\, k, j \in \mathbb{N}$.*

Proof: *If we either set $j = \sqrt{5}$ or $j = 1$ then we can solve Eq. (2.8) for $\sqrt{5}$ so that the r.h.s. contains only natural numbers, fractions and $k, l \in \mathbb{N}$; it would then follow that $\sqrt{5} \in \mathbb{Q}$ which is a contradiction: if $\sqrt{5} = \frac{p}{q}$ then $p^2 = 5q^2$, where the l.h.s. is even and the r.h.s. is odd $\forall\, p, q \in \mathbb{N}$.* $\square$

These three sizes—corner, junction and unit stop—determine the size of the board in its entirety.

We call the distance from the tip of one corner to another in a pentagram its *diameter*. It can be proven that the diameter d of a pentagram informs its radius r thus:

$$r = \frac{d}{\sqrt{\varphi + 2}} \tag{2.15}$$

In our case, diameter d consists of two long lines L (‘legs’) with l

stops and one short line K ('arm') with k stops:

$$d = 2L + K \tag{2.16}$$

$$L = c + 12 + j \tag{2.17}$$

$$K = 2j + 6 \tag{2.18}$$

Consequently, we can calculate the diameter d

$$d = 2(c + 12 + j) + (2j + 6) \tag{2.19}$$

and insert this into Eq. (2.15) for the overall radius r. In the resulting term, we can insert the terms for j and c from above. The result is a rather complicated term that does not reduce much. It can be slightly simplified to this:

$$r = \frac{2}{5}\sqrt{1570 + 698\sqrt{5}} \tag{2.20}$$

If you compute this, you gain the numerical solution

$$r = 22.38133\ldots s. \tag{2.21}$$

In words: the pentagram from centre to tip is $22.38133\ldots$ times the size of a stop on the star.

The entire board will of course have to be slightly larger, since the corner stops are protruding; their centre point lies on the pentagram, they protrude just by their radius. Thus the absolute board size is

$$R = r + \sqrt{5} \tag{2.22}$$

The numeric result is

$$R = 24.61740\ldots s. \tag{2.23}$$

Table 2.1 collects all the values we have found. With these, you can draw the board precisely. If you want to draw the board, these are the sizes you must set your compass to. Draw a circle with $22.38mm$

radius, in this construct a pentagram. Set your compass to the values of corners and junctions, adding those. Then set your compass to unit length and complete the stops on the star. Et voilà! Fig. 2.8.

Figure 2.8.: The board today

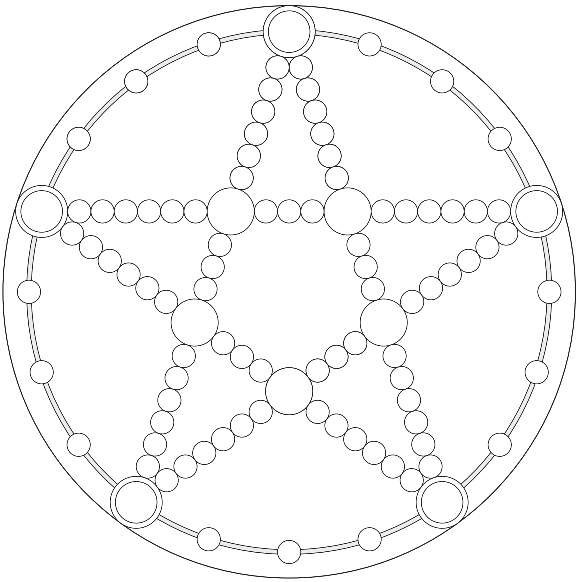


Table 2.1.: The sizes on the board

stop on star	$s = 1$
corner stop	$c = 2.23606\ldots$
junction stop	$j = 2.02606\ldots$
pentagram	$r = 22.38133\ldots$
entire board	$R = 24.61740\ldots$

2.4. Ruler and Compass

Not all geometric shapes can be constructed using a ruler and compass.

That a pentagram can be constructed using ruler and compass alone was known since antiquity. But nobody could ever find a way to construct a polygon with seven sides.

The riddle was finally solved by Carl Friedrich Gauss in 1796. Using complex number and delivering the general solution he managed to proof that constructing a heptagon precisely with ruler and compass is impossible.

The Pentagame board—not just a pentagram, but all the stops—can be constructed using ruler and compass alone, because all trigonometric functions of the angles have algebraic solutions and all roots are square roots.

The complexity of a ruler and compass construction can be measured by the number of steps necessary to complete it. For Pentagame, at the time of writing the ruler and compass construction is unknown. It must be complex, since Eq. (2.20) is a highly complicated term.

2.5. Coordinates

For all practical reasons, we would like to assign a coordinate value to any stop, since the notation we will be using is a relative one. So we wish to have polar and Cartesian coordinates for a given point.

Complex numbers are particularly helpful regarding circular structures, and hence also when it comes to constructing the board. With their help the (cartesian) coordinates of the five corners are easy to find.

We already have the coordinates of the five corner stops. If we set the size of the circle to one, hence operating within a unit circle, the five points of the pentagram correspont to the *five roots of unity* in the complex plane (Fig. 2.9). The terms in Fig. 2.9 are the solutions to the equation

$$z^5 = 1 \; z \in \mathbb{C} \tag{2.24}$$

Allow me to quickly explain the concept of "‘roots of unit"’. The

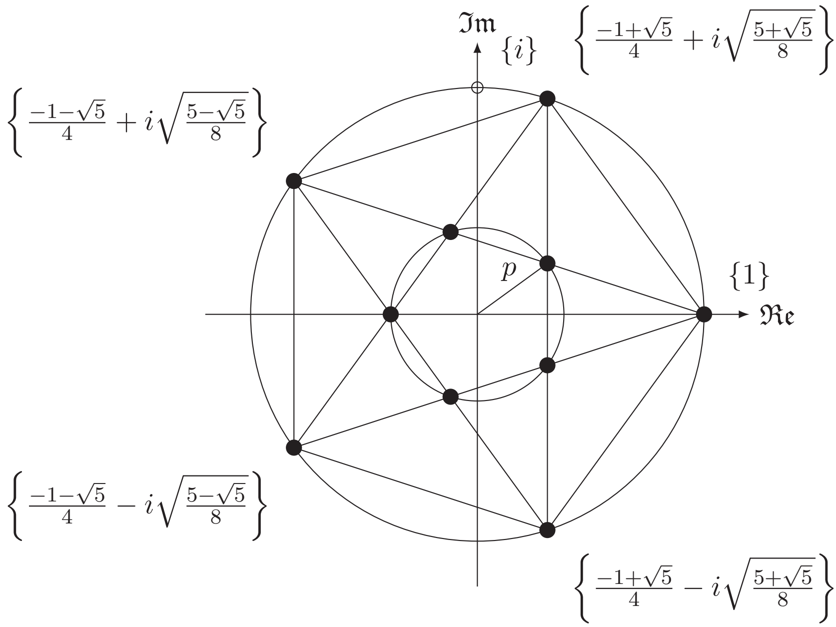


Figure 2.9.: Pentagram in the complex plane

unit of the ordinate in the complex plane is the real unit 1, while the unit on the abscissa is the imaginary unit i defined as $i^2 = -1$.

Now it is easy to see that just *two* numbers z fulfill $z^2 = 1$; these numbers are, of course, $z_1 = 1$ and $z_2 = -1$. These are the 'two second roots of unity'.

It is probably equally obvious that *four* numbers z fulfill $z^4 = 1$; these are the numbers $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $z_4 = -i$. These are the 'four fourths roots of unity'. In the complex plane these construct the points of a square.

Similarly, there are three complex solutions for $z^4 = 1$ and there are five for $z^5 = 1$. These constitute the corners of a regular polygon, and these five points are therefore what we seek.

Every coorinate can be expressed either as a polar coordinate or as a Cartesian coordinate. Within the complex plane it is very easy to 'swap back and forth' between these, simply by applying Euler's identity:

$$e^{i\varphi} \equiv \cos\varphi + i\sin\varphi \tag{2.25}$$

Thus, we find the coordinates in a polar or cartesian coordinate system, as shown in Fig. 2.9. All we have to do is 'plug in' the values of the angle, which are multiples of 72 degrees or $\varphi = \frac{\pi}{5}$. Fortunately, the values for all trigonometric functions of these angles can be written using only square roots (of five).

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4} \tag{2.26}$$

$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5-\sqrt{5}}{8}} \tag{2.27}$$

So we get the coordinates for the Corners by plugging in the above angles; since their distance to the centre is one, we are already there.

To find the coordinates for the five Junctions is straightforward. We see that each Corner corresponds a Junction just opposite; thus, the corresponding angles are just $\varphi'_n = \varphi_n + \pi$. But what we need now also is, of course, their distance to the centre. This is the radius of the circle encompassing the central pentagon in Fig. 2.9.

There are a number of ways to find the solution, only requiring the standard formulae for the pentagram. The result is:

$$p = \sqrt{\frac{25-11\sqrt{5}}{5-\sqrt{5}}} \approx 0.381966 \tag{2.28}$$

We have compiled the main values in Table 2.2. The exact positions of the individual stops between these nodes can from here be calculated in a straightforward manner, taking the radii of Corners, Junctions, and individual stops into account, of course.

2.6. A solid game

When we look closer at the board that we have found, we can also observe that (irrespective of the rules which define a start and end

Table 2.2.: Coordinates of Corners and Junctions

Stop	distance	angle	X-Coordinate	Y-Coordinate
A	1	0	1	0
B	1	$\frac{\pi}{5}$	$\frac{-1+\sqrt{5}}{4}$	$\sqrt{\frac{5+\sqrt{5}}{8}}$
C	1	$2 \cdot \frac{\pi}{5}$	$\frac{-1-\sqrt{5}}{4}$	$\sqrt{\frac{5-\sqrt{5}}{8}}$
D	1	$3 \cdot \frac{\pi}{5}$	$\frac{-1-\sqrt{5}}{4}$	$-\sqrt{\frac{5-\sqrt{5}}{8}}$
E	1	$4 \cdot \frac{\pi}{5}$	$\frac{-1+\sqrt{5}}{4}$	$-\sqrt{\frac{5+\sqrt{5}}{8}}$
a	p	π	$-p$	0
b	p	$\frac{\pi}{5}+\pi$	$-\frac{-1+\sqrt{5}}{4} \cdot p$	$-\sqrt{\frac{5+\sqrt{5}}{8}} \cdot p$
c	p	$2 \cdot \frac{\pi}{5}+\pi$	$-\frac{-1-\sqrt{5}}{4} \cdot p$	$-\sqrt{\frac{5-\sqrt{5}}{8}} \cdot p$
d	p	$3 \cdot \frac{\pi}{5}+\pi$	$-\frac{-1-\sqrt{5}}{4} \cdot p$	$\sqrt{\frac{5-\sqrt{5}}{8}} \cdot p$
e	p	$4 \cdot \frac{\pi}{5}+\pi$	$-\frac{-1+\sqrt{5}}{4} \cdot p$	$\sqrt{\frac{5+\sqrt{5}}{8}} \cdot p$

points) all nodes are somehow equally distant to one another; there is isomorphy. Indeed, the board can also be represented in three dimensions in form of an antiprism (Fig. 2.10). You can see in the antiprism how a player moving from e.g. node A to node a exactly opposite must traverse two short and one long lines (at least). This is the same in the two-dimensional version.

Thus the planar Pentagame board can be seen as the projection of such an antiprism (Fig. 2.10). This means that there the board has no ‘edge’; all nodes are equally connected. With this Pentagame actually transcends the class of *board* games, since it can actually be seen as a ‘solid’ game. Thus what we have found out here about planar games—vulgo board games—have actually lead us into the realm of three-dimensional games, a field not yet much studied, probably because a truly three-dimensional playing surface is somehow cumbersome to handle.

We leave to the mathematically inclined reader to consider all the

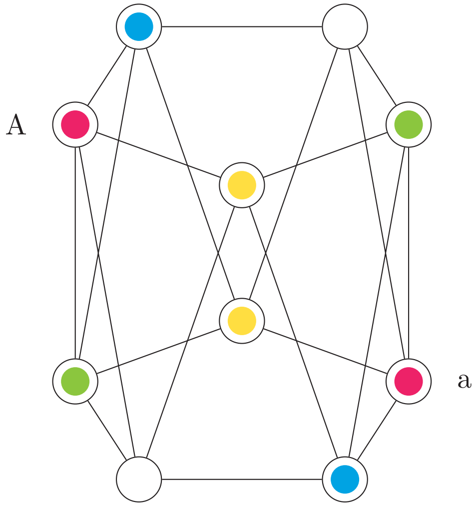


Figure 2.10.: Pentagame as antiprism

geometric qualities of such an object: the angles; the sizes of all spherical nodes and stops; the circumscribed sphere, and so forth.

2.7. Discussion

We have demonstrated (in passing) that a tesselated board using irregular pentagons or Penrose tiles is too complex to create to allow for a game that is simple in form. We have also shown that a pentagram shaped board, where the lines of the pentagram be segmented, is also not easily constructed.

This does, however, not proof that there cannot be a pentagram shaped board game simple enough to be(come) a classic in the sense of Thompson, because to draw such a board—a Pentagame board—in an approximation is easy. It is well possible to improvise a Pentagame board with chalk. In Part III we will return to the question of a possible ancestor.

The Pentagame board we have constructed shows some interesting additional qualities that are worth mentioning since it set this board apart:

- The board does not limit the number of players. This is noteworthy, since almost all other pure strategy games are for two players only, and it is sometimes erroneously assumed that games of pure strategy be limited to two players.
- The board is radially symmetric rather than axially, thus players can sit at any angle to the board.
- Because it is symmetric for all number of players, the game is fair for all number of players. (Few notice that for example Ludo is unfair when played with three players.)
- The board shows isomorphy; there are e.g. always three stops between a red and a yellow node. Consequently, the game can be played outside-in or inside-out.
- Because of this isomorphy, another—and maybe even more appropriate—way to depict it is the shape of a antiprism (Fig. 2.10).
- This demonstrates nicely that this board has no 'borders' as chess and go boards have; all points are equally well connected to the whole.

For now we have found a simple pentagram shaped board. We can now move on to find rules that would make the desired simple game.

3. Finding the rules

3.1. Considerations

Board games, by definition, are games with boards. Now that we have found the desired board, what rules could there be that work on such a board? Modern literature has classified board games by a number of qualities, chiefly by the winning condition (e.g. [71]), and we see categories such as ‘connection games’, ‘race games’ and so forth. So what exactly *is* a game is open to discussion. Let us discard *play* for a moment, thus everything that has to do with psychology, playfulness, human development, and ‘trying out things’ in general; we seek a definition for *rule based play*, which is what we call *game.*

Definition 1 *A* game *is the challenge to achieve a defined goal with defined material as quickly as possible given a starting position by actions limited by rules.*

So first of all, there must be a starting position and a goal or ‘winning condition’. It makes sense to classify games by this winning condition; it could be ‘eradicate the opponent’, it could be ‘survive’, or it could be to achieve a particular position with one’s pieces; these could be anywhere (‘four in a row’), or be bound to the board (‘reach the goal’). But whatever the winning condition, the aim is to reach it *fast.*

A game can be solitary. There are many possible solitary games, even some that work well on our Pentagame board. Any two or multi player game is an extension; here the winner is the one who achieves the goal *first*; and in a particularly interesting class of games, those which are interactive, the move of each player alters the challenge for the other player(s). But nevertheless, there remains a race element: *every game is a race game.* The quest is to find the most efficient, shortest or quickest, path to the winning condition or goal.

The second part of the definition contains the word *rules,* and it might not be entirely clear what this is. Let us discard the two obvious material rules ‘by only using this material’ (pieces) and ‘within this particular space’ (board), and focus on the immaterial rules. Such can be as ‘you may move only in such and such a way’, ‘you may only move these pieces’ and so forth.

We see that the space (board), the material at hand (pieces, devices), the winning condition, and the immaterial rules that constitute a game are somehow interlinked. We have also argued that ‘good’ games are those that are simple; this means a simple board, little material, a single and simple winning condition, and few but clear rules with little exceptions.

The board of Pentagame was not developed out of the air. The particular shape that we have now, the number and colours of stops on it are interlinked to the history of us trying to find suitable rules, and suitable material.

By material we mean pieces and other devices. Ideally a board game is only a board game: everything happens on the board. Some games, however, use both boards and cards; or boards and dice; or other equipment. These are already more complex than the class of games we are interested in. Thus, for classic games, there should not be any material other than pieces.

Pieces are regularly two-dimensional in that they display exactly two qualities: their *shape* and their *colour.* Typically one of these qualities is associated to one of the players; most commonly, that is the colour; however—as in Pentagame—it can also be the other way round, with a player commanding a shape. To some degree these two qualities are exchangeable. There are, however, more possible shapes than discernible colours; and colours 'spring to the eye' more strongly than shapes, most of the time.

Immaterial rules are the rules proper. They can be classified:

1. the definition of start and goal,
2. the restriction of action to certain possibilities,
3. definitions of mechanisms, that is of special situations and necessary consequential actions,
4. definitions of situations and special rules concerning these, that is of exceptions.

It is obvious by the definition of what a game is that Type 1 rules are necessary and well defined.

Type 2 rules often include many non-written rules, for example 'use only the game material'. Rules are often formulated positively, such as 'you may move along a line', because everything not explicitly allowed in a game is generally out of the game. Nevertheless, rules *are* restrictions in that their function is to restrict the options of the players. However, a good game will be game where this restriction is minimal; we cherish games in which simple rule sets allow us many options.

Type 3 rules define mechanisms; an example is ‘if you reach this stop, this and that happens’. Here again players prefer few rules.

Type 4 rules define exceptions; an example is the casteling rule in Chess. Sometimes such exceptions or clarifications are necessary as the ordinary rules may allow situations to occur which cannot be solved by the ordinary rules. Such rules are usually perceived as fixes to flaws in the rule set; a good rule set should not have nor need many additional rules or exceptions.

While it is always easy to come up with complicated rules, it is far harder to find simple ones. Indeed, if you come up with rules that are in part contradictory, the temptation to ‘fix’ this by introducing more rules is large; to actually curb rules is apparently a more difficult intellectual exercise. But once you find simpler rules, they prevail over more complicated rules by virtue of social choice.

Thus the quest to find appropriate rules in itself can be seen as a game: find the most simple rule set that works. This is a meta-game; the winning condition is to find ‘the simplest rule set’. In terms of social choice theory those games prevail that have the most convincing rules: where the number of rules is small and the resulting game space is large; where we thus have maximum fun with minimum restrictions.

We shall report the process of finding the rules in the following paragraphs and sections in more detail. This account is naturally linked to the actual history of development, so I will recall some of the main stages of the engineering process.

3.2. Developing the rules of Pentagame

Once the idea of a pentagram shaped board was born (that was about 1996), the obvious next step was to extrapolate existing rules. The most prominent race game on a segmented path is, of course, the classic Indian game ‘Patcheesi’, known in English as ‘Ludo’.

Accordingly, the first idea was to start at a corner and return to it, with moves being restricted by the cast of number dice. But because we have an intersected graph, this required complicated extra rules regarding allowed and forbidden short cuts; and game play was lengthy. In addition, the use of dice was a bit cumbersome, and tedious much was to be calculated by the players.

Thus I put the idea aside for some years.

I only came back to the project over then years later, once I had learned a good deal about the cultural significance of the pentagram, and that its corners were often associated to Elements, a topic I will deal with at greater length in a different part of this book. These observations inspired me to colour the corners, the junctions, and indeed all the stops on the board.

It turned out that there is only one way to continue a scheme of five colours all through the lines: there must be exactly 3 stops between the crossings in the centre, and 6 between a centre and a corner (Figure 3.1).

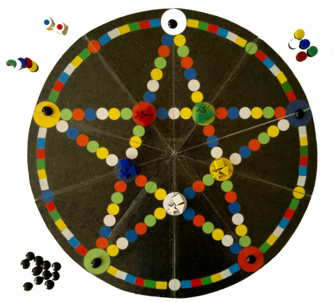


Figure 3.1.: An early Pentagame prototype
The colour scheme repeats throughout.

This colour scheme allowed to get rid of the number dice and replace them by colour dice. This made playing far more intuitive. The rule was then: throw dice; move to the nearest stop of the colour shown by the dice. This was about 2007.

The resulting game still was a bit dull and lengthy. Nevertheless, we enjoyed playing this in London for a while. It was good, but not good enough; there was potential, but the rules were not yet satisfying; and the geometry of the board needed solving (we have shown the solution in the previous section).

At this point I wish to express my gratitude to Billy SMITH for his hospitality in the fog.

We tried to swap the colour (signifier for origin and goal) and shape (signifier for player), and found out that we shouldn't, since it is much more important to see which pieces are close to their goals than it is to see whose pieces these are. Meanwhile I worked on the geometry trying to solve the board, all the while learning more about classics games and game theory.

In the ancient version of Chess, pieces were only allowed a few steps per move; for example, the bishop could only move one or two steps diagonally per move. This was only changed during the Renaissance [64] [71]; the result was a massive gain in complexity, since the number of options per ply increased, and in speed.

Applying this principle to Pentagame as it was at that stage would allow players to move not just to the nearest, but to any stop of the colour shown by the colour dice.

This was nice - but would also allow any player to win immediately, so another restriction needed to be added. This was the rule that players may not jump over anything, which worked with the the addition of the passive pieces, the black blocks.

The idea of purely passive blocks is borrowed from another classic game called Malefiz; Malefiz is a variation of Ludo with multiple paths and blocks popular in Germany. Nevertheless, the game was a little lengthy, and still rules on allowed and disallowed directions were needed to hinder shortcuts.

Another issue became clear at this stage: if you may not jump over anything including the pieces of other players, how can you then overcome them? Another player's piece would indeed be a perfect obstacle when placed on a goal; the result was that a player could always force a draw or the game never end. The solution was the swapping rule, which was invented in Glastonbury (John MARTINEAU).

So now we had quite a playable game—albeit still with some confusing traffic rules regarding the directions allowed.

In today's Backgammon, pieces start their way already half way into the board. That halves the time the game takes. This idea appealed much to us, as we could alter the winning condition: a piece would no longer need to return to its origin, but have an goal on half of this way.

Comfortably, due to the colour scheme, every corner has a junction of the same colour opposite in the middle of the pentagram. Thus the new winning condition made the game at once shorter and allowed us to abandon all those complicated traffic rules; suddenly, you could choose which ever path you wanted. This was a true breakthrough (Tim Grünewald né Suchanek).

This was fun, but there was an obvious problem: once pieces leave the board, the board becomes more empty. Thus the game would suddenly 'flush' and be over too quickly. Also, to move a piece out is actually a disadvantage for the player who moves out, since she then commands one piece less.

The solution to both these issues was the introduction to have the player add another block somewhere on the board.

At that stage we played the game with two colour dice, so players would have to find the best combination. This makes already an entertaining game. However, not everyone is a friend of dice. We tried cards, so you would draw a colour rather than roll a colour. You could collect these cards in your hand and play them later. This was fun but required more material.

Then it finally dawned on us that the colour restriction *in general* (be it by dice or cards) was only *limiting* player's options without any true justification, and that we actually need none of that at all. Instead, a player can now move on *any* stop within reach.

Thus, we arrived at rules much simpler than before; and it was always our intention to create a *simple* game, happy to eliminate and reluctant to introduce rules.

2016 marked the end of the main development of the rules; this was a process that took 20 years. Since we reached this rule set, only very few special cases had to be decided and added.

One is that the winning condition got simplified, so that you would only have to bring three pieces out rather than all five. The reason is that this just cuts short an otherwise cumbersome end game and makes the game more strategic at once.

The last and final addition was the introduction of grey blocks. At first we simply used more black blocks, and this is a feasible way to play the game. However, the more pieces move out, the more blocks appear, and since blocks are passive the game becomes 'blocky' towards the end; this is particularly the case in multi-player setups.

The grey blocks are an elegant solution that allows to have the exact same rules for two, three and four players. As a nice side effect they add an extra strategic element.

Finally, there are two 'edge cases' that need clarification or extra rules. This concerns passive moves and the danger of endless loops. The solutions for both follow from the existing rules and the logic of the game.

But since these are a little more complex to explain, I will deal with them in their own two subsequent sections.

3.3. Passive moves

Every swap move is also a passive move of the piece that is swapped. Some issues arise from this, in particular when the swapped piece belongs to another player; so a piece of a player is moved whose turn only comes later. In the general case this poses no problem at all.

But what happens when Alice moves the piece of *Bob* to its destination?

One can argue that Bob would then immediately move out. However, he would gain the right to set a grey block. The trouble is that it is not Bob's turn yet; he would thus gain the right to do something even though it is not his turn yet.

This problem becomes more obvious when Alice moves out *and* moves Bob out at the same move; both would gain grey blocks: who is to set their block first? Can Alice set her block onto the goal which Bob has now reached, or does he only move out *after* she has set her grey block? Or is it maybe that Alice, who has moved out her own and Bobs piece, may now place two grey blocks?

Thus, the situation needs some clarification. We have discussed this issue with a number of experts from the legal field (Diana MERCER). What is the rule most consistent with the existing rules and the spirit of the game?

Remember the rules that players take turn playing. This means that a player can only make an action once it is her turn. If he would suddenly have the right to do something without it being his turn, he would have an advantage *in addition* to the advantage that he has been brought out by someone else; which does not seem right.

Instead, he is brought to his stop *and remains there* until it is his turn.

But this cannot yet be the complete answer to the riddle. Now that one of Bob's pieces sits on one its goals when Bob is to move—what is happening then? Does he have to move out, or is he only allowed to move out? Does he have the right to move any other piece instead, or even in addition?

The key here is that Bob has already gained an advantage from Alice's action. If he would still have another option, he would benefit even further; this seems not in line with the spirit of the game. We remember that the swapping rule was originally introduced to allow players to overcome other player's pieces that otherwise may serve as unsurmountable blocks. Thus it seems appropriate that the swap with another player shoudl be encouraged. An extra advantage to the passive player would contradict this intention of the original law.

Thus, when it is then Bob's turn, he *must* move out—gaining a grey block, and he *does not* have any other or additonal choice.

This rule regarding passive moves has a rare, but nevertheless interesting additional consequence: when a player who has for example a red piece on the white goal and a white piece on a red goal can swap these two, he effectively brings two of them out in one single move. The argument from above states that in effect he only moves one of the two out, while the other one remains on its goal until it is her turn again.

3.4. The Ko-rule

Sometimes players can reach a situation where the game runs into a loop. This is a stalemate, and stalemates are nobody's favour. In Pentagame this can happen when a player swaps with the piece of another player, who then swaps the same two pieces again, and so forth. So we need to rule out such loops. Rules forbidding loops are called 'Ko'-rules, a Japanese term taken from the game Go.

There are two commonly proposed Ko-rules for Pentagame. We will discuss them briefly and show why Variant 2 is the correct one to use.

Variant 1 Do not undo the move of another player, creating the same position that there was before.

Variant 2 Do not make the same move twice, creating the same position that there was before.

Now consider Alice and Bob both have two pieces on the same origin still, for example their white pieces. Alice can move her piece to a neighbouring node, and Bob can now move his piece from the same origin to the same node and swap Alice back. The question is: can Alice re-do this move?

Under Variant 1, she could not, since this would re-create the same position that there was before. But this is not fair, since it was her who first tried to go to that node. Her advantage has turned into a disadvantage, and that is not appropriate.

Under Variant 2, Alice can re-swap, since this is a move she has never yet tried. But Bob could not try the same move again as before. Consequently, his attempt to swap with Alice only succeeds if Alice 'agrees' by not reversing it.

Thus Variant 2 is more in line with the spirit of the game. It does not turn an an advantage into a disadvantage. Instead, Variant 2 only considers the pieces of the player on whom the rule applies. It also means that the swap with a foreign piece is something that the passively moved player must generally accept, as long as it has not lead to a fundamental change in the position. (A case with such a fundamental change would for example be if Bob's piece reaches a goal upon such a swap, which changes the position so that Alice cannot possibly re-swap at all.)

Effectively this rule makes swapping pieces with those of other players an unattractive option unless either the passive player accepts that move or the swapping piece moves out.

The situation is slightly different in a multi-player setup, but the rule remains the same. Variant 2 *do not try the same twice* applies as well; players need not to discuss if the situation has changed—because it always has. But still players are not allowed to try the same move again. 'The same move' is to move the same piece from the same one stop to the same other stop occupied by the same piece as before. If the move is not exactly the same, it is a legal move. An example is given in Section (2) in Part (IV).

Figure 3.2.: Set-up for two players
Rabbits vs. hedgehogs. Image: Marten Suhr
3 or 4 playes would begin with 3 or 4 pieces per corner.

4. The design

Since the inception of the idea, I have built about 19 different prototypes; since each game as a product consists of the following parts, each of which has to be considered:

The board which should be its absolute size? Should it be round or square? And how should it be folded? What materials can be used?

The box what colour should it have? What lettering should be used? What picture(s) should it show? What languages be present? What size should it have? What are legal requirements?

The pieces what shapes and sizes should they have?

The instructions should they be a minimal rule set? Should they include background information, and tips? What languages should they be in?

I abstain from accounting the entire history of the product development here; suffice to say that after the many prototypes, I started manufacturing small series of games, rubber stamping them as I went along. Very prominent are the four magnetic games that I have made in 2015, which allowed me to introduce the game and play it regularly in spaces such as the hacker space c-base in Berlin. I have also made a 70cm tournament fold-up table.

I presented the game on the Board Game Inventor's Meeting at Göttingen twice and in Haar once, and went to the game fair in Essen once. There was much more interest by the general public, who played and enjoyed and bought the game, than from publishers. Most of these walked past the game since they thought it to either be a Ludo variant, or too abstract, or too occult, for their programmes.

I fathomed that my long development of this single one game has not done me well, since I lack the record or reputation of being a (frequent) game inventor; the question "have you invented another game before" comes up often. (I have, but in childhood, and for the last 20+ years I only worked on this one game.) Thus, I had no name in the scene. The same applied for the academic scene studying board games. I am, I confess, a maverick. But then the development of great ideas can be a process, but also an event.

Proven difficult in the development of a final product was of course the rule book. One idea was that the rule should be separate from a little leaflet presenting some of the background information and results of my research in a nutshell. This Idea I somehow finally dropped, and the brochure grew into the book you hold in your hands instead.

Since prototypes became more or less finalised, I started selling paper back copies to anyone interested without any further advertisement than regular playing in publicity. This created an amazing network of players, many of whom contributed by translating the rules in a plethora of foreign languages, one of which was Latin. These translations can be found in the Annex.

The game got considerably more attention on the Chaos Communication Camp 2019, where it was a major past time activity in the c-base assembly. Shortly afterwards I had sold the 100$^{\text{th}}$ copy and organised a proper tournament. That tournament in September 2019 was only the second of its kind, and was won by Manja Gärtner. On that day, fans of the game encouraged me to start the crowd funding campaign, which ran over a course of 45 days.

Within this time, and also thanks to the discussion with a rapidly growing community, it became clear that animal shaped pieces are more fun than abstract shapes; but that it is also a lot of fun to make one's own pieces: hence, that there need not be one definitive set of pieces. It also became clear that simple and beautiful as the game is, the box and the instructions should be.

During this time Nikky started programming a proper first online version. This will also allow us at some point to gather empirical data about the game, which is likely to shift the level of what we know about it considerably up; since most of what is dealt with in this book—the entire Part IV—is based on theoretical considerations, not on statistical study.

These days of the crowd funding campaign were incredibly intense. Honestly, the campaign was not that well prepared. But nevertheless, I succeeded to collect funds to get the game produced, set up a web shop, and bring it to wider attention. One half of this sum was brought together by only five founders who we call the 'First Five': Andreas Grübel, Christian Jantz, John Martineau, Gerhard Suchanek, and Nathan Toups. From now on, these shall be addressed as 'Honourable Member of the First Five' and be given a special ceremonial role in every Pentagame tournament.

The final design of box and game was only possible with the support of Marten Suhr. I am also thankful to Michael Bucknell, Veit Busch, c-base, Jan Fels, Daniel Franke, Manja Gærtner, Jonas Gollard, Christoph Kollmann, Ingo Krallmann, NikkyAI, Thomas Nielsen, Dietrich Pank, Alper Peker, Anna Redlich, Elias Scheideler, Lovis Schulze, Billy Smith, Marten Suhr and Daniel Swærd.

At the time of writing the publication date is set for March 14th 2020. From this day on, and on a shoestring of marketing really, the game will become available to the wider public.

5. Conclusion: A new but simple game

We set out to find a simple game played on a simple pentagonal or pentagram shape, and we have succeeded: The Pentagame board is easy to draw roughly (though hard to draw precisely). The rules fit on a single sheet, and it can be explained in a minute. Thus Pentagame has Thompson's quality of clarity.

We know from experience that is has the other qualities from Thompson's list: depth, drama and decisiveness. It has depth, because it doesn't seem to have a trivial solution and become boring soon. It has drama, because it can obviously ‘swing’ and a player once in the lead may loose in the end. It has decisiveness in that it always ends (and most time not in a draw). We will in Part IV proof that is has these qualities indeed.

Part III.

Ancient Sources

> Wenn wir nun auch die Idee des Spieles als eine ewige und darum längst vor ihrer Verwirklichung schon immer vorhandene und sich regende erkennen, so hat ihre Verwirklichung in der uns bekannten Form doch ihre bestimmte Geschichte, von deren wichtigsten Etappen wir kurz zu berichten versuchen wollen.
>
> —H. Hesse [40]

1. Introduction

This part shows that there was indeed a forerunner game in history—a game played on five lines, or a pentagram—in Antiquity. This was lost at some unknown time in history. We have no archaeological evidence of it, at least no convincing evidence, but only a number of fragment text sources. The research into this matter is thus piecemeal, comparable to the reconstruction of data from a scrambled hard drive. We will try to be a systematic as possible. The likely reasons why the original game was lost will be investigated afterwards.

2. The Greek Game Petteia

2.1. A word and a game

People have been playing board games since times of yore. We have archaeological finds from Egypt and Mesopotamia, where race game with dice were played. Enters the Greek civilisation. The Greek civilisation was, as Huizinga has shown in his seminal work [43], very much based on the idea of play, and clearly venerated any kind of playfully competitive behaviour.

The Greek had a word 'pessoi' which seems to mean 'game pebbles', a verbal form thereof 'pettein', which is usually translated as 'to play a board game', and 'petteia' which renders 'board game'. Nevertheless, we note: it does not mean 'board', but 'pebble'.

Of central interest for our investigation is the following quote by Sophocles:

> καὶ πεσσὰ πεντέγραμμα και κύβων βολαί
>
> —Sophocles Fr. 429
>
> 'of both the five-lined *petteia* and of dice-throwing'

So what exactly is this 'petteia'? Often the use of *petteia* is so that it seems to refer to a very specific game known to the reader. In other places it seems to denote *the* board game as a generic term. We cannot have certainty, but will discuss this in depth below.

Since Sophocles—and others—mention a shape of five lines, some have come to the conclusion that all mentions of *petteia* rely to this one mysterious five-line game; others hold that those passages refer to a *petteia* of the name 'Five lines' or *Pente grammai*.

To sum up: there are three ways to interpret *petteia*:

(a) *petteia* was *one* game; all mentiones of *petteia* relate to one game, with one rules; it be called *Petteia*, capital P.

(b) *petteia* is a *generic* term for board game, so we can *not* sum up what we find; we write *petteia* to keep things open;

(c) *petteia* means 'to pebble' and may not even be restricted to board games at all, but also include other activities such as using calculation boards.

If the first is true, then we can sum up all that is said in different places with regard to *petteia* and reconstruct the game. This game we name *Petteia*, with a capital P. We shall refer to this as 'unionist position'.

If the second is true, then we can only take what is written in one place or source about *petteia* as a context; in other words, if some ancient source tells something about a particular *petteia*, that means that there was *at least one game* for which this description is valid. For this we will use the neutral term *petteia*, with lowercase p. Only some mentions of *petteia* refer to five lines, and this game is then referred to as *Pente grammai*. This we shall refer to as 'separatist position'.

And finally, if the third is true, we should get a different picture altogether; this we will call the 'scepticist position'. We will discuss this option separately.

The trouble with the above is that even the ancient seem to not have stuck to one or the other position. It is well conceivable that *petteia* could have been used sometimes as a general term, and sometimes a specific term; for there are some ancient games who are named explicitly, while *petteia* seems sometimes to be used in a pars-pro-toto fashion, just like later 'playing board' (*tavli*) worked as a reference to Backgammon. Thus whatever conclusions we draw must be taken *cum grano salis*.

2.2. Interpretations

From the middle ages on no game played on a pentagram shape was known: no such game is listed in any of the treatises of the subject ([62] [18] [23]), and when the game *Pente grammai* or *Petteia* is discussed, it is discussed as an antique and lost game.

When speaking of antiquity, we are talking about a period stretching from maybe 1600 BC to about 400 AD, so about a period of 2000 years that is bygone for almost two 1,500 years.

Much change and turmoil happened within that long period, probably more than afterwards! Much has been obliterated, much purported second or third hand only. Consequently, the study of antiquity is piecemeal. All we have are suspicions, fragmented literature, shards and some archaeological leftovers.

> The study of the Greek games is, in fact, a journey into complete darkness.
>
> —Austin [8].

Any reconstruction of antiquity resembles the attempts to reconstruct a complex text from a scrambled hard drive.

Estimates about the amount of books that were available in antiquity—e.g. in the famed library of Alexandria—bear in themselves large degrees of uncertainty; but clearly, today we only possess a tiny fraction of what there was.

Almost all scholars in the early modern times have interpreted *petteia* as name of one specific game, often equated to whatever their own favourite, mostly Draughts, Backgammon or Chess. They were mostly unclear on their method, and sometimes take the 'unionist' and sometimes the 'separatist' position in regard to their sources. The most prominent authors were the Italian CALCAGINI (ca. 1500), [14], Dutch DE SOUTER (1622) [23] and French DE FOUQUIÈRES (1869) [22]). In this tradition classic translations of antique texts often translate *petteia* with 'draughts'. Since this seems unjustified, modern translations often render *petteia* with 'board game'; we have in the translations given in this text replaced that with *petteia*.

Later scholars have been more careful in their methodology. Still a brilliant overview on the natter is LAMER (1962) [52], who remained inconclusive about the nature of *petteia* and the exact relation to *Pente grammai* or *Petteia*.

A careful investigation was done by AUSTIN (1940), now a classic text in its own right due to its eloquence. He sums the *Petteia*-riddle up worth quoting [8]:

> "[...] it is clear [. . .] that each player had five lines, but do they mean five vertical and five horizontal lines, or two sets of five lines running in the same direction? If the latter, why the name πεντεγραμμα ? Finally, what was the 'sacred line'? Did it run between two sets of five lines (thus making 11 altogether), or was it the middle one of each set, or can we infer from Eustathius that the board did in fact have five lines each way, and that the 'sacred line' was the middle one in each direction? [...] No answer seems possible to any of these problems."
>
> —AUSTIN [8]

Interesting is in this passage how AUSTIN rules out a pentagram shape, without reason given.

The ancient five-line *Petteia* may have been a thoroughly *lost* game of its own right, relatively unrelated to the games we play today and not resembling any of the surviving classics.

Only Murray (1952) assumed that *Pente grammai* is a lost game of the pentagram shape. He also shows the famous roof slabs of Kurna, some of which clearly show board designs of Nine Men's Morris, some of which show pentagrams, and some of them show other pretty random designs [63]. His treatment of the subject is rather succinct; he quotes another game apparently still sometimes played on a pentagram shape, in which only the nodes are possible stops.

Today's standard works by Parlett [71] and Lhôte [55] in French brush the subject of *Pente grammai* off quickly by brief mentions.

Fresh wind has been brought into this age-old discussion recently by a seminal paper of Ulrich Schädler [78]. His starting point is not the interpretation of the literature, but an interpretation of archaeological finds; his conclusion is that *Petteia* or *Pente grammai* was a specific game, but played on five parallel lines. Since we differ, we will discuss his position in greater length below.

2.3. Antique authors

Homer (ca. 840 BC) mentions *petteia* in the Odyssee, where the suitors play *petteia* in front of Odysseus' house:

> οἱ μὲν ἔπειτα πεσσοῖσι προπάροιθε θυράων θυμὸν ἔτερπον ἥμενοι ἐν ῥινοῖσι βοῶν, οὓς ἔκτανον αὐτοί
>
> 'they were taking their pleasure at *petteia* in front of the doors, sitting on the hides of oxen which they themselves had slain'
>
> —Homer, Odyssey I, 106-112

Heraclitus (535–c. 475) compared *petteia* to the world:

> Αἰών παίς ἔστι παίζων, πεσσεύων· παιδὸς ἡ βασιλήη
>
> 'Time is a child at play, moving pieces in a board game (πεσσεύων); the kingly power is a child's.' —Heraclitus [82, Fr. 79/52].

The reader may allow us to mention the similar concept in Hinduism, that of *līlā* : the world is play of Brahma; because it cannot be conceived that the world has been made with an intention by an already perfect being, it must have been created as play.

SOPHOCLES (c. 497–406) is our next, and in this context most important witness. We have a fragment we will discuss in greater length in the context of IULIUS POLLUX below. SOPHOCLES provides us with information about the shape of the *petteia* board.

> 'καὶ πεσσὰ πεντέγραμμα και κύβων βολαί'
> [Palamedes the inventor] 'of both petteia's five lines [pentegramma] and of dice-throwing.'
>
> —SOPHOCLES [Fr. 429] [74]

The term 'pentegramma' literally means 'five lines'. Later the term is used explicitly for what we know today as pentagram, by LUCIAN [57]. We note that in SOPHOCLES, *petteia* is put in contrast to the throw of dice. This is a fragment from the lost play 'Nauplios'; Nauplius was the father of inventive Palamedes. We shall discuss this in more detail below. (SOPHOCLES' Fr. 479R is similar.)

According to PROCLUS [87], the *Kypria* tells us that PALAMEDES played *petteia* while waiting for ODYSSEUS and the wind to sail to Troy. According to EUSTATHIOS, the stone upon which PALAMEDES used to play was kept at Troy long after the Troyan war [28].

HERODOTUS (490–420) calls the Lydians (I.94.1-7) the inventors of all games, including dice, with the explicit exception of *petteia*[50].

PLATO (c. 428–c. 348) mentions *petteia* often in many of his dialogues.

In *Nomoi,* the Athenian stranger speaks about the importance of education, in particular mathematical education. He accounts how children are taught arithmetic in a playful manner, and that the ability to calculate is later handy to organise both warfare and households. He then laments that the Athenians are generally ignorant about irrational numbers, and says that to study those things would be better than to pass time playing *petteia.* To this his dialogue partner replies:

> τε πεττεία καὶ ταῦτα ἀλλήλωντὰ μαθήματα οὐ πάμπολυ κεχωρίσθαι.
> 'But playing *petteia* and these studies do not seem that far from each other'
>
> —PLATO Nomoi 7:820d

The link between board games and mathematics is clearly made. What is interesting here is: most simple race games are indeed played on boards that resemble calculation devices, if not *on* calculation devices; games like Patcheesi or the Egyptian Senet are indeed based on getting the arithmetics right. But there is nothing in those games that has any conncetion to the incommensurability. But this is the topic the Athenian stranger speaks about in this dialogue. Thus we can suspect that *petteia* here is a game that has some qualities that link to irrational numbers.

In *Phaidros*, PLATO has SOCRATES discuss the vanity of rhetorics and then discuss the nature and quality of the written word or alphabet. Recounting a myth, he credits THOT, the Egyptian HERMES, TRISMEGISTOS, as inventor of numbers and calculation, geometry and astrology, *petteia* and the play with dice, and the alphabet.

> τοῦτον δὴ πρῶτον ἀριθμόν τε καὶ λογισμὸν εὑρεῖν καὶ γεωμετρίαν καὶ ἀστρονομίαν, ἔτι δὲ πεττείας τε καὶ κυβείας, καὶ δὴ καὶ γράμματα
>
> 'He it was who invented numbers and arithmetic and geometry and astronomy, also *petteia* and dice, and, most important of all, letters.'
>
> —PLATO, Phaidros 274c-d

This shows two things: *petteia* was held in high esteem, in the rank of a science; and it was different to the play with dice. Thus we can suspect that *petteia* was not a game with dice.

In *Politeia* PLATO mentions *petteia* as an activity that requires skill and practice:

> πεττευτικὸς δὲ ἢ κυβευτικὸς ἱκανῶς οὐδ᾽ ἂν εἷς γένοιτο μὴ αὐτὸ τοῦτο ἐκ παιδὸς ἐπιτηδεύων, ἀλλὰ παρέργῳ χρώμενος·
>
> 'though no man in the world could make himself a competent expert at *petteia* or the dice who did not practise that and nothing else from childhood'
>
> —PLATO, Politeia 32.74c

This *petteia* was a game not just confined to luck, but quite obviously a more sophisticated affair. Again we note that it is mentioned along with, but not in unity with the play with dice.

ARISTOTLE (384–322) seemd less concerned with games, albeit he mentiones *petteia* once:

> ἅμα γὰρ φύσει τοιοῦτος καὶ πολέμου ἐπιθυμητής, ἅτε περ ἄζυξ ὢν ὥσπερ ἐν πεττοῖς.
>
> 'for one by nature unsocial is also 'a lover of war' inasmuch as he is solitary, like an isolated piece at *petteia*'.
>
> —ARISTOTLE, Politeia 1.1253a10–11

The interesting information here is that pieces obviously needed to be somehow connected to be powerful; a non-connected piece would best go berserk.[1]

POLYBIUS (c. 220–c. 120), a historian, writes about Hannibal's father:

> πολλοὺς μὲν γὰρ αὐτῶν ἐν ταῖς κατὰ μέρος χρείαις ἀποτεμνόμενος καὶ συγκλείων ὥσπερ ἀγαθὸς πεττευτὴς ἀμαχεὶ διέφθειρε
>
> 'For in many partial engagements, [Hamilcar], like a good *petteia*-player, by cutting off and surrounding large numbers of the enemy, destroyed them without their resisting.'
>
> —POLYBIUS, Hist. I, 84;7

Here *petteia* clearly has a strategic element; it can thus not have been a simple race-game. In particular we note the element of cutting off and surrounding.

This is about all information we have in the ancient literature prior to the begin of the common era.

SUETONIUS (69–c. 120) was a Roman historian, and we know that he also wrote a book on Greek games. This would of course shed light into all of the above, but unfortunately this book is lost. There are, however, a number of later writers who apparently had access to that book, and when they describe *petteia*they seem to refer to his account. The way they do this let us suspect that they had no first-hand information about *petteia*, which is why we conclude that the game was fallen into obscurity by then [48].

IULIUS POLLUX (2nd century AD) is probably our most important source. He wrote a 'reference book' called 'Onomasticon'. This is extant in heavily truncated form, but it mentions *petteia* in two places.

In the first passage (passage 7) he simply includes *petteia* in a list of other games.

[1] On the topic of ἄζυξ see [48] and [42].

The second passage is more detailed. It is this passage which has preserved the fragment from SOPHOCLES we have quoted above.

> ἐπεὶ δὲ ψῆφοι μέν εἰσιν οἱ πεττοί, πέντε δ' ἑκάτερος τῶν παιζόντων εἶχεν ἐπὶ πέντε γραμμῶν, εἰκότως εἴρηται Σοφοκλεῖ "καὶ πεσσὰ πεντέγραμμα καὶ κύβων βολαί." τῶν δὲ πέντε τῶν ἑκατέρωθεν γραμμῶν μέση τις ἦν ἱερὰ καλουμένη γραμμή· καὶ ὁ τὸν ἐκεῖθεν κινῶν πεττὸν παροιμίαν "κινεῖ (Βετηε: κίνει) τὸν ἀφ' ἱερᾶς."
>
> Since pettoi are pebbles, and each of the players had five (pettoi) on five lines, it has been reasonably said by Sophocles (fr. 429 Radt): 'Both five-line game-pieces and throws of the dice.' And of the five lines from either side was a certain middle one called the 'holy line': and the person moving the game-piece from there [according to] proverb 'moves the piece from the holy line'. (Tr: KIDD [48]).
>
> —IULIUS POLLUX 9,97

This passage, both obscure and lucid, has been subject to much discussion through the ages (see above). As we will see below, most later scholars assume the five lines to be parallel. We can and must, however, read it in conjunction with an important passage in LUCIAN. LUCIAN was a contemporary of IULIUS POLLUX and apparently not very fond of him [51, 8, 84].

LUCIAN (c. 125-180) dedicated a paragraph on the Pythagoreans in his essay on greeting forms. [57]. This is worth quoting in full:

> ὁ μέν γε θεσπέσιος Πυθαγόρας, εἰ καὶ μηδὲν αὐτὸς ἴδιον ἡμῖν καταλιπεῖν τῶν αὑτοῦ ἠξίωσεν, ὅσον Ὀκέλλῳ τῷ Λευκανῷ καὶ Ἀρχύτᾳ καὶ τοῖς ἄλλοις ὁμιληταῖς αὐτοῦ τεκμαίρεσθαι, οὔτε τὸ χαίρειν οὔτε τὸ εὖ πράττειν προὔγραφεν, ἀλλ' ἀπὸ τοῦ ὑγιαίνειν ἄρχεσθαι ἐκέλευεν: ἅπαντες γοῦν οἱ ἀπ' αὐτοῦ ἀλλήλοις ἐπιστέλλοντες ὁπότε σπουδαῖόν τι γράφοιεν, ὑγιαίνειν εὐθὺς ἐν ἀρχῇ παρεκελεύοντο ὡς καὶ αὐτὸ ψυχῇ τε καὶ σώματι ἁρμοδιώτατον καὶ συνόλως ἅπαντα περιειληφὸς τἀνθρώπου ἀγαθά, καὶ τό γε τριπλοῦν αὐτοῖς τρίγωνον, τὸ δι' ἀλλήλων, τὸ <u>πεντάγραμμον,</u> ᾧ συμβόλῳ πρὸς τοὺς ὁμοδόξους ἐχρῶντο, ὑγίεια πρὸς αὐτῶν ὠνομάζετο, καὶ ὅλως ἡγοῦντο τῷ μὲν ὑγιαίνειν τὸ εὖ πράττειν καὶ τὸ χαίρειν εἶναι, οὔτε δὲ τῷ εὖ πράττειν οὔτε τῷ χαίρειν πάντως καὶ τὸ ὑγιαίνειν.
>
> Pythagoras the mystic has vouchsafed us no writings of his own; but we may infer from his disciples, Ocellus the Lucanian and Archytas, for instance, that he headed his letters neither with Joy nor Prosperity, but recommended beginning with Hail. At any rate all the Pythagoreans in writing to one another (when their tone is serious,

> that is) started with wishing Health, which they took to be the prime need of soul and body alike, and to include all human blessings. The pentagram, that interlaced triple triangle which served them as a sort of password, they called by the name Health. They argued that Health included Joy and Prosperity, but that neither of those two was coextensive with Health.
>
> Lucian [57] 5

What is important here and what we can conclude from this is:

1. Lucian explains the term πεντάγραμμον explicitly as what we know today as pentagram (Fig. 2.1). In the light of this it is difficult to interpret the quote by Sophocles in the above section in any other way. We conclude that *Pente grammai* was played on a pentagram shape.

2. The pentagram was equated to health. In the light of this it must have been seen depicting a human in balance, or with five elements in balance. It thus had some cosmological meaning. This makes it holy or sacred.

3. The pentagram served as 'a sort of password', thus secred 'trademark' of the Pythagoreans.

4. In the light of the prosecution of the Pythagorean sect and those in their tradition this will have resulted in destruction of many, if not all, pentagram-shaped objects.

The reason why I assume that 'pentegramma' indeed means 'pentagram' in Iulius Pollux, and that *Pente grammai* was thus played on a pentagram star, are the following:

1. Board games on parallel lines can have any number of lines, though some number may become standard. Only a pentagram shaped board *must* have exactly five lines.

2. Lucian explains the term The term 'pentegramma' explicitly as what we call a pentagram today. Why should the term 'pentagram' that we have now not have the exact same meaning back then and today, why not take Pollux literally?

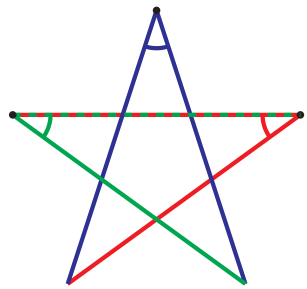


Figure 2.1.: Pentagram composed of three angles
Hence also called 'pentalpha'

3. Pollux is clear that there are five pieces associated each with one of these lines. On race games such as Backgammon, Patcheesi and so forth the number of pieces is not intrinsically tied to the number of lines.

Eustathios (1115 – 1195) finally, a Byzantine scholar who seem to have had access to some more antique sources, quotes Diodoros (1^{st} century CE) in a worthwile passage:

> Διοδώρου δέ φησι τοῦ Μεγαρικοῦ ἐνάγοντος τὸν τοιοῦτον λίθον εἰς ὁμοιότητα τῆς τῶν ἄστρων χορείας, Κλέαρχος τοῖς πέντε φησί πλάνησιν ἀναλογεῖν.
>
> 'And he [i.e., the one who writes about games] says that Diodorus the Megarian introduced such a stone for likeness to the dances of the stars, while Clearchus makes them as an analogy to the five planets.' (Tr.: Kidd [48])
>
> —Eustathios [28] 1397, 40–48

In all other accounts Eustathios follows Iulius Pollux; only this information is an addendum. This hints at a circular structure of the board, a quality not quite fitting to parallel lines.

This is about how far the study of the ancient literature can lead us—admittedly not very far. Another thing is to look at the archaeological record, which is relatively scant as well: we do not find many game boards; we find some boards which may or may not have been game boards; we are uncertain which type of games these were. We shall return to that further down in the text.

3. Petteia re-engineered

We shall now take the 'unionist' stance and assume that all mentiones of *petteia* above refer to *one* game: *Petteia*. We thus synthesize all the statements above.

Please note that there is one feature mentioned in the sources which our reconstruction cannot attempt for or only with difficulty—the 'holy line'. We shall look at that issue below.

Board:

1. Petteia was played on five lines [83] [74] [28].
    - Any game playable on 5 parallel lines will equally function on any odd number of lines.
    - For a game to be *distinguished* by being played on five lines inasfar as being named 'game of five lines' (pessa pentegramma), five lines would mean the pentagram star.
2. Thus Petteia was played on a pentagram.
3. The travelling of the pieces was comparable to the motion or 'dance' of stars [28].
    - There must have been a circular feature.
    - An ancient Greek calendar was based on Venus cycles, which draw a pentagram in the Zodiac a pentagram.
4. Thus it must have been **a pentagram in a circle.**

Setup:

1. There were five pieces per player [74] [28].
    - For a game on five lines to necessarily require exactly five pieces per player, each must correspond to one of the five lines.

2. Thus the pieces of each player must have had different origins on the pentagram.

3. Due to isomorphy, the game can be played inside-out or outside in.

4. **The pieces started at corners** (or junctions).

Rules:

1. 'Pessa pentegramma and the throwing of dice' are exclusive (Sophocles, Plato).

2. The game compares to mathematics and requires reason (Plato).

3. Thus Pesseia was played without dice. At least chance did not take centre stage.

4. It was a race game of some sort, blocking was key, pieces had to be coordinated in a strategic fashion with pieces not be isolated (Aristotle, Polybius).

5. There was an element of 'taking' independent pieces (ἄζυγες; Polybius).

    - These could have been 'unconnected' pieces of a player (as in Backgammon), or special stones which are not 'tied' to any particular player, like blocks [48].

6. It was complex but simple enough to survive for a long time.

    - Only 'classics' survive a long time: simple rules, complex play.

7. Thus **it was a race game of pure strategy.**

Thus is the conclusion:

> Players of *Petteia* had to move their pieces from points on the pentagram, racing with each other without the aide of dice, taking independent pieces and blocking one another in a coordinated fashion according to simple rules.

This agrees rather beautifully with Pentagame.

However, stating that Pentagame *must be Petteia* re-engineered would be jumping to conclusions. First we need to be aware that our 'unionist' assumption is weak. Second, there is an additional feature mentioned in the antique literature none of this accounts for, namely the 'holy line. Thirdly, there is the stunning fact of a total absence of any archaeological record of a pentagram shaped game. The latter two issues are discussed in the next two sections.

4. Difficulties

4.1. The mysterious 'holy line'

Both Iulius Pollux and Eustathios seem to copy from Suetonius when writing about Petteia; and both jump subject within their explanatory paragraph, since they first start explaining a game, and then use the game (or a feature therein) to explain a proverb [48].

This proverb:

> κίνει τὸν ἀφ᾽ ἱερᾶς
> 'to move [a] pebble from the holy line'

must have been common, but was also obscure in both origin and meaning while still in use, so that it needed explanation. (That proverbs continue to be used without anyone remembering their origin or the exact meaning of the metaphor can be observed in any language.)

Alcaeus (7TH-6TH century BC, a friend of Sappho), is the oldest and most quoted source of this proverb:

> νῦν δ᾽ οὗτος ἐπικρέτει ͺ κινήσαις τὸν ἀπ᾽ ἴρας †πύκινον λίθον
> moving the pebble from the holy line, she gains the upper hand.
>
> Alcaeus [3, Fr. 351]

Plato also uses this proverb in *Nomoi:*

> ἡ δὴ τὸ μετὰ τοῦτο φορά, καθάπερ πεττῶν ἀφ᾽ ἱεροῦ, τῆς τῶν νόμων κατασκευῆς, ἀήθης οὖσα, τάχ᾽ ἂν θαυμάσαι τὸν ἀκούοντα τὸ πρῶτον ποιήσειεν
>
> The next move in our settling of the laws is one that might at first hearing cause surprise because of its unusual character—like the move of a *petteia*-player who quits his 'sacred line'.
>
> —Plato, Nomoi 5.738-9

This seems to rather signify 'make the second step before the first'.

Theocritus (ca. 270 BC) is another source:

> καὶ φεύγει φιλέοντα καὶ οὐ φιλέοντα διώκει,
> καὶ τὸν ἀπὸ γραμμᾶς κινεῖ λίθον: ἦ γὰρ ἔρωτι
> πολλάκις ὦ Πολύφαμε τὰ μὴ καλὰ καλὰ πέφανται.
>
> [she] flees when she is loved and pursues when she is not loved and moves the rock from the line. —Theocritus [89, 6.19]

This seems to hint to a rather contradictory behaviour. Also, we observe that Alcaeus and Thoecritus use λίθος *lithos* which means 'stone' rather than pebble.

Pollux and later Eustathius explain this proverb so that the five-line *Petteia* had featured a line called 'holy line', and further that this was in the middle and between the other lines [74] [28].

The problem here is that this is self-contradictory: how can a game be called Five Lines, when it has an additional middle line and thus six lines? Thus one can suspect that there is some kind of error on the side of the antique authors.

Firstly, these passages seem not intent to meticulously explain a board geometry, but to give an explanation of a proverb.

Secondly, it seems that Pollux wrote from hearsay.

Thirdly, this interpretation of the proverb in itself is not quite convincing, since that move 'from the holy line' seems sometimes desperate, sometimes advantageous (Plato).

The negative picture that his contemporary Lucian and the later Philostratus [73]) paint of Iulius Pollux is fitting.

The proverb may as well have completely different origins. It could refer to cheating the starting line in a race, or it could refer to some action on a calculation board, like spending the last penny. In short: it refers to cheating. 'To move the goal posts' could be similar expression.

Nevertheless, the 'holy line' and Petteia could indeed have been the same thing or the 'holy line' a feature of Petteia. It is well possible that the whole design was dubbed 'the holy line'. As we have shown the pentagram, which can be drawn in one line, has some 'holy' attributes. The same applies for the circle, which has often be used as a theological metaphor, most prominently by Plato in *Timaios*, but also later [79].

4.2. Pebbles on lines

Ulrich Schädler claimes that *Petteia* (that he calls Five Lines) must have been played on five parallel lines [78], and the evidence he calls are archaelogical finds of boards containing such five lines. However, we beg to differ, and now reach out to explain why.

Our perception and comprehension of games have changed over the millennia, and are still changing. The old languages mainly draw a line between that which is precise (without luck factor) and that which is a gamble (with luck factor). In ancient Greek there was no word that exactly means 'game', but there are two words: ἀγών 'agon', which means something our present understanding of competition;[1] and there is παίζων 'paidson', which is to behave like a child, child's play. Latin also knows two words: 'alea', which means gambling or dice throwing[2] and the betting on chances, and 'ludus', which is, well, pretty much what we today understand as 'game': a rather childish activity separate from the dealing of adults in the real word.

The most primitive form of board games is to move counters of some kind on a board; in the antique world these were mainly pebbles, which can be found in great number on Greek beaches. Playing with pebbles, counting, and moving counters is at its root much the same. The curious human mind must at an early stage have observed

[1] competere is Latin 'to strive together'; thus here too a certain rule guaranteeing equality in the competition is necessary.

[2] The actual dice were known as *tesserae.*

and learnt that not only can one use pebbles as representations of something else (like people, cattle, money or anything else), but that different pieces and positions can have different meanings. There is an early point where playing with pebbles on a line, making simple arithmetic operations, and play both have their origin. Fig. 4.1 shows a typical antique calculation board, the 'computer' that was used exclusively until the discovery of the logarithm rules (John NAPIER [65]) and the invention of the slide rule.

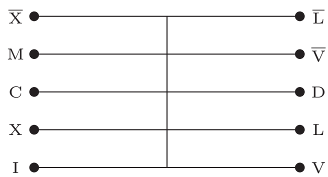


Figure 4.1.: A calculation board (abakus)

With such a counting board it is easy to add and subtract numbers, typically in a base 5 system: a base 5 (or 6) system needs much less pebbles than a base 10 system, which explains the middle line. Of course counting in dozens was popular, but the antique often used a base 5 system, as can still be seen easily in the Roman numerals which has symbols for 1, 5, 10, 50 (I, V, X, L) etc. where the counting system resembles the counting with fingers (fittingly called 'digits'). These boards were called ἄβαξ'abax', which simply means 'tablet' [20].

The most famous calculation board of this style (clearly marked as such) is known as the 'Salamis tablet' with both 5 and 11 lines, each crossed by a middle line [20]. Such 'table' designs were common in antiquity and compare to modern spreadsheets and calculators. Note that 'calculus' is Latin for 'pebble'.

Now SCHÄDLER proposes a re-engineered dice game as the original 'petteia' for such boards. But even that these boards were often found with dice is no proof for their (sole) purpose as games: dice can served as counters (e.g. for the floating-point or exponent), while gambling was done with the much cheaper astagaloi, which are knuckle bones.

Calculation devices of this type were used in commercial business

Figure 4.2.: Achilles & Aiax move counters.
Exekias (ca. 540 bc)

and in military campaign. They are a common find, though understandably sometimes (mis-)interpreted as board game tables.

A particular popular image deserves our attention here: The image of Achilles and Aiax using such a device. Fig. 4.2 shows the most famous version by Exekias, which is just one among hundreds of such images. They are usually catalogued as 'Achilles and Aiax *playing a board game*'. However, such a description jumps to conclusions, since we can only see that the heroes are moving counters on a board. Thus more cautiously these images should be described showing 'Achilles and Aiax moving counters'. The following famous passage in Homer's Illias corresponds:

> He may offer me ten or even twenty times what he has now done, nay—not though it be all that he has in the world, both now or ever shall have; he may promise me the wealth of

> Orchomenus or of Egyptian Thebes, which is the richest city in the whole world, for it has a hundred gates through each of which two hundred men may drive at once with their chariots and horses; he may offer me gifts as the sands of the sea or the dust of the plain in multitude, but even so he [Agamemnon] shall not move me till I have been revenged in full for the bitter wrong he has done me.
>
> —Achilles
> Homer, Ilias IX, 379-386 (Tr.: Butler)

We have an extant treaty of Archimedes where he actually calculates a figure for the number of grain that could fill the universe [6]. The ancient clearly knew exponentiation.

An illuminating passage is found in Plato's *Nomoi* again, where the Athenian stranger explains the connection between children learning how to do calculations in a playful manner as useful preparation for the organisation of both warfare and house-holding.

To conclude, the images generally considered to show gaming partners may actually show business partners.

Of course one could still also play games on computation boards, and likely Backgammon developed therefrom. Any computer is always good for gaming, as we know. 'Abacus' was often also used as word for 'game board'.

Eusathius explains *petteia* as somehow similar to *tavli* or Backgammon [28]. However, the fact that today's Greeks call Backgammon 'tavli', which cleary is a Latin lean word from 'tabula', speaks for Roman origins of that game. The archaeological record shows that Backgammon has been played throughout the Empire on *twelve* lines (hence its roman name 'duodecim scripta')[3].

This also implies that maybe the whole interpretation of *petteia* as 'playing with pebbles' i.e. 'playing a board game' may be faulty. It actually means 'to pebble' which is nothing else but 'to move counters'. This can be done according to strict mathematical rules or with some element of luck. It can be doubted if the ancient have

[3] In later Latin texts Backgammon became called 'Nerdiludum' in Latin, from Persian 'nerd' for dice.

strictly distinguished one from the other. That would also mean that SOPHOCLES' line quoted above can be interpreted as

> [Palamedes, inventor of] *both* [calculation using] *pebbles on five lines and* [generating random numbers through] *the throwing of dice.*

If this is true—if *petteia* simply means 'to pebble' and would encompass all board games and the use of abacis—everything deduced above is doubtful, since this disallows to combine what different authors have said about 'pebbling' for the reconstruction of one specific game.

5. Summary

1. The available sources from Greek antiquity ascribe various qualities to *petteia*, where *petteia* could either be a general term for board game or relate to one specific game, *Petteia*.

2. All descriptions of those particular *petteia* which are called *Pente grammai* (POLLUX) fit Pentagame as well, except for the somehow shady 'holy line'.

3. The sum of all that is said in the sources about *petteia* except what relates to the 'holy line' can lead to a description of a game *Petteia*: they do not contradict.

4. The description of *Petteia* (the sum of all that is said about *petteia* in various places) fits Pentagame as well.

There is nothing in *petteia* that is not in Pentagame (except the holy line). The opposite is not true. All that has been said about the number of lines and pieces and some characteristics of the rule set of *Petteia* is also true for Pentagame. This does not say anything about the shape.

We have also substantiated the following statement:

Pente grammai was played on a pentagram shape. It was likely a pure strategy game.

If this is the case, then

$$\{\textit{Pente grammai} \sim \text{Pentagame}\} \tag{5.1}$$

In words:

Pollux' game *Pente grammai* is similar to Pentagame.

And if

$$\begin{aligned}\{\textit{Pente grammai} \sim \text{Pentagame}\} \wedge \{\textit{petteia} = \textit{Petteia}\} \\ \rightarrow \{\textit{petteia} = \textit{Petteia}\} \sim \text{Pentagame}\end{aligned} \tag{5.2}$$

In words: If all mentions of *petteia* relate to the same game *Petteia*, then Pentagame is simliar to all of them.

Had a pentagram shaped board existed, we would expect archaeological finds in addition to the literary sources. This is not the case.

But the absence of evidence is not evidence of absence.

$$\begin{aligned}\{\text{pentagram}\} \rightarrow \{\text{evidence}\} \\ \neg\{\text{evidence}\} \nrightarrow \neg\{\text{pentagram}\}\end{aligned} \tag{5.3}$$

Lack of archaeological evidence can only disapprove the theory of the pre-existence of a pentagram shaped board (the game *Pente grammai*) if there is no convincing other explanation for the absence of such evidence.

If we find such an explanation, the equation

$$\{\textit{Pente grammai} \sim \text{Pentagame}\} \tag{5.4}$$

may still be true. Put differently: the thesis that Pentagame is the ancient game resuscitated can only hold if there is some very substantial reason why this ancient game got wiped out of the archaeological record completely.

The possible explanation for this lack of physical evidence is of course the cultural history of the pentagram, to which we will turn now.

6. The Loss of the Ancient Game

We have shown that there has been a game in antiquity that was played on five lines (*Pente grammai*). Theories on how these lines were arranged vary. Our reconstruction of a playable game on a pentagram star, initially motivated by purely theoretical considerations, has resulted in a game that in all aspects except one agrees what the ancient said about their game. This substantiates the suspicion that in Pentagame, we actually have a reconstruction of *Pente grammai.*

Our main caveat lies in the fact that we have no single pentagram shaped board in the archaeological record. So if it has existed, there must be a reason for why it vanished. And that reason must have been the fact that we are dealing with a pentagram shape here.

We have already mentioned the difficulty to come to conclusions about antiquity, a period of extraordinary length and with extraordinary violent turmoil, particularly towards the end of the period; turmoil that in part explains why so little is extant, causing our reconstruction problems.

So far the literature studied was exclusively that which is concerned with board games and with the pentagram. Now we will look at another aspect which is closely related: cosmology related to the pentagram. Any cosmology that accounts for n Elements will use n-polygons for illustration. In a world where religion and science were not yet clearly separated, such illustrations were deemed holy or sacred. The holy or sacred can likely become object of concurring faiths. This would have happened to pentagram shaped objects.

In our development of Pentagame we have taken inspiration from how Alchemy associated colours with the corners of a pentagram. Indeed, only colouring the board made it the playable specimen we have now. The colours, and thus the cosmology concerned, plays a role in the reconstructed game.

And the history of the theory of Elements will provide a reason to why such a game, if it ever existed, has vanished.

Caveat: We beg the reader to please read the following sections in the understanding that this section is a broad and synthetic overview, rather than a critical analytic examination. The topics covered in this chapter are extremely wide, and many issues bear large degrees of uncertainty.

Much that has been written by earlier authors—explicitly including those from antiquity—must be read highly critically, and critical reading of these texts often lead to results contradicting those author's statements. Due to the scope of this work we cannot discuss every aspect and detail in depth. All we can do is offer substantiated conclusions.

With the possible exception of some dubious inscriptions on the roof slabs on the temple of Kurna in Egypt[1] and some coins and amulets, not a single pentagram shaped antique object exists today.

Fig. 6.1 plots possible causes for the absence of such findings.

1. There may not be such objects because they have simply never existed.

2. Such objects have existed but they have completely disappeared. This could be for a number of reasons, which are not mutually exclusive:

    a) they were of such poor material that they have vanished;

[1]There seem to be a number of game boards, among them Nine Men's Morris, but also some general glyphs. There are pentagrams which have been interpreted as board games [64]. They may also be calculating boards, mason's marks or have served other or no purpose.

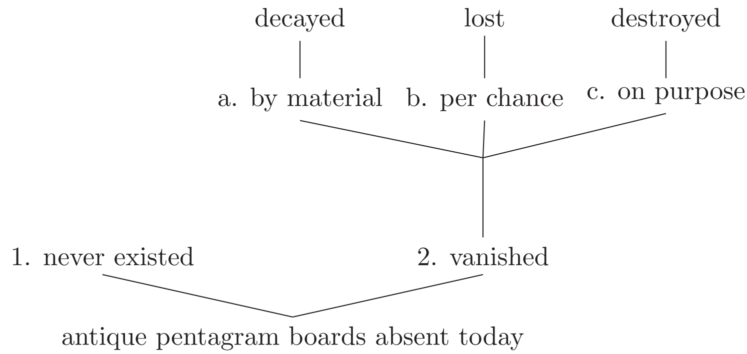


Figure 6.1.: Why is there no trace?

b) they vanished for no reason, by pure chance; or

c) they were destroyed on purpose.

1. They never existed There are no pentagram shaped boards, but according to Schädler there are at least eleven objects with five *parallel* lines from at least two sites (Vari, Kerameikos, etc.), and some additional boards which have been expanded from five to eleven lines; along with these were found dice which may have been used with them. Schädler concludes that the Five-line game was played on parallel lines. We are of the opinion that these are calculation boards, not board games.

If we buy into the argument that a pentagram shaped game has never existed, then of course all further speculations are futile. The absence of evidence is not evidence of absence. In other words: only if we assume that had a game on a pentagram shape existed then there would have to be physical evidence, then we must conclude that it has not existed. Nevertheless, there are some good reasons why such boards and pictures of them may have been lost.

2. They vanished Admittedly, *'a pentagram shaped game has existed'* is unproven, speculative, and not even likely. What we say is that *'it was possibly there and all specimen got destroyed'* is a possibility of a likelihood greater than nil. Suppose that 1 out of 1000 artefacts were a game, and suppose further that today less than 1 out of 1000 of all artefacts be games. Then we would assume that the rate of diminishing was not random, but that games have been particularly destroyed; this could be for material reasons (games were mostly drawn on the floor and not durable [80]) or for ideological reasons (destruction focused on games more than on other artefacts). You could also state that there were multiple different games (one of which in pentagram form), but only specific types exist today, and start to argue why this could be the case (pentagrams were especially destroyed or pentagram shaped games particularly perishable).

2a. Decay The pentagram is difficult to produce precisely on hard material for geometric reasons (see Section 2.3). But it is easy to sketch with chalk etc. on the occasion, which would explain why no board has been found. Pentagrams are also particularly difficult to draw in perspective. Thus it makes sense to not depict it in an artistic setting where lines drawn with rulers are uncommon; that game boards, where depicted on ancient frescoes, paintings, mosaics or vases, are mostly given sidewise (cf. Fig. 4.2) could thus be explained.[2] Further, boards may have been make-shift; the antique word *petteia* means 'pebbles', not 'board'; the game may actually have consisted of pebbles (in a box) with a board drawn on the occasion.

2b. Chance What we have from antiquity is a tiny sample of what there was. Since the finding of the Anitkythera mechanism, we are more aware how advanced ancient science was, and also how much was lost, and that surprising new evidence can surface. The filter 'chance' should not be underestimated.

[2]These images are generally interpreted to show warriors playing a board game. They may though show warriors making calculations using an abacus or planning a campaign with a model.

2c. Destruction The pentagram and golden section were important in ancient culture; however, not only have we no pentagram shaped board game, but we have nothing in the form of a pentagram at all. In the special case of the pentagram, the surprising lack of archaeological findings in this shape *can be explained* by what went on later.[3] A possible cause is not a necessary condition.

As we will show below (Section 2.2), the golden proportion and the pentagram played a major role in ancient Greek culture [39]. Nevertheless, nothing pentagonal has survived. The surprising lack of the five-fold game among today's existing classics (Table 1.1) in conjunction with the name *'pente grammai'* plus the rich and somehow wild history of the pentagram (Chapter 6) lead us to suspect that a pentagram-shaped game may have existed and was suppressed or forgotten. We know that the ancients knew enough about geometry to have studied and used the pentagram; there is, however, not only no pentagram shaped board game, but there are no ancient pentagrams left over *at all.* Our 'evidence' is the stunning *absence* of evidence. Albeit, of course, such evidence is not a proof.

Summary There are two equally plausible explanations, or alternative models, of why there is no ancient pentagram shaped board to be found: (1) that it has never existed; (2) that it got lost. Consequently, the cause for its present non-existence cannot be finally identified from the evidence at hand [72]. If there would be no reason to assume that it got lost, then it must have never existed. If we find compelling reason for why it may have been lost, we can be undecided whether it has existed or not.

[3] We mean 'can be explained' in a non-casual way: we wish to express 'there is A that may be the reason for B'. This should not be confused with 'thus it is proven that B is only because of A'.

7. Elements, Atoms and Geometry

The concept that all matter is composed of a limited amount of fundamental Elements—we will write Elements with a capital 'E' to distinguish from today's chemistry—is common to all mankind from East Asia to the West. To list Fire, Water, Earth and Air seems also canonical for a long time, though the actual lists of Elements in the ancient sources often differ in Elements, and different in number; this was shown by Sachs (Table 7.1) for the antique European and by von Glasenapp for the Indian sources [93].

The idea seems to predate the invention of alphabets and is thus not quite traceable. One can suspect that it originated somewhere in Persia or Mesopotamia, since such lists appear in early Zarathustrian texts, in ancient Indian texts, in ancient Egyptian texts, in ancient Mesopotamian and in ancient Greek texts. For a more complete text, see [9] and [47].

In the light of an abundance of studies on this subject we can only attempt a broad overview. What we like to demonstrate is that Five Elements was a common count in the schools of Pythagoras, to some extent in Plato but, probably most decisively, Mani. Only later the count of just four Elements started to become common place and a canonical order became established [77].

The theory has continued to be used until the development of modern Chemistry by Robert Boyle, Antoine Lavoisier and finally Dimitry Mendeleev, and is still in use in folklore and pseudosciences.

Antiquity is a time long gone by—1,500 years and counting. It is also a very long period in itself, within which much has happened. Literature from that period often narrates much earlier history, and much that we learn is not only second, but third and fourth hand information. Before we can reconstruct the past, we must deconstruct the history of the (past) discourse (Foucault). Prior to history comes the study of traditions, and their often twisted ways.

This applies for Ancient Greek Philosophy.

Table 7.1.: Elements in antique sources (Sachs) [77]

Pythagoraens	No.	Order	Plato	No.	Order
Hermias	5	FAWQE			
Olympiodor	5		Olympiodor	4	
Simplikios	5		Simplikios	5	
Proklos	5		Philoponos	4	
Epiphanios	5		Proklos	5	
Jamblichos	5		Jambliochos	5	EWAFQ
Porphyry	5		Porphyry	4	
Irenaeus	4		Plotinos	4	
Aetios	5	EFAWQ	Hippolytos	4	
Nikomachos	5		Anatolios	4	
Ps. Timaios	4		Diogenes L.	4	
Vitruv	4		Albinos	4	
Okkelos	5	QFWEA	Attikos	4	
Poseidonios	4		Tauros	4	
Alex. Polyhist.	4	FWEA	Apuleios	4	
Theophrast	4		Plutarch	5?	
Aristotle	-		Aetios	5	var.
Speusippos	5		Xenokrates	5	QFWEA
Philolaos	4	FWEA	Philippos Opus.	5	var.
			Platon (Tim.)	4	FAWE

The history of Ancient Greek Philosophy is deeply entangled with the history of mathematics, a discipline first advanced to a scientific level at that time, which bore magnificent results.

The history of Ancient Greek Philosophy also includes early theoretical inquiry into nature. It was early observed that substance appears in many different forms. These forms were termed ‘Earth’, ‘Water’ and ‘Air’—solid, liquid and gaseous. These three were observed to morph into one another. It was quickly also discovered that there are types of being which cannot be explained as composed of these Elements: heat, for instance: Fire. The mind itself seems to belong to neither of these categories, just as little as numbers and abstract ideas, which seem to exist in some entirely different manner, but nevertheless exist.

Empedocles, of whom we have some original text, proposed four elements: Fire, Water, Earth, and Air. He is vivid about these four changing into another. The idea was that these elements possess different weights, so that the spheres of the Earth, of Water, of Air and the Heavens exist.

Democritos proposed the idea that all substance be composed of Atoms; and that these Atoms be of different nature: heavier Earth Atoms and lighter Water Atoms would behave just in the way required by Empedocles.

But quickly the obvious contradiction between unchangeable Atoms and changeable Elements became apparent. For both Plato and Aristotle the Elements were what we call today *phases* or states of matter, not elements in the modern sense.

At that time mathematics advanced from planimetry (called Geometry) to stereometry: mathematicians started studying polygons.

One of the prime results of planimetry was the proof of the existence of irrational numbers (alogos): the diagonal of a square cannot be measured in the same measure as its sides. Probably the most easy object this can be demonstrated with is the pentagram (Fowler). Euclid in *Elements book 13 prop. 8* shows that the lines in the pentagon relate in the golden section, and that the triangles therein are golden triangles, where the sum of two angles equal the third.

It bears some irony that—according to late-ancient Lucian—the pentagram was a symbold and badge of the Pythagoreans. 'Pythagoreans' means quite different groups:

1. Neo-Pythagoreans, self-confessed: Iamblichos and Porphyry, both of whom wrote quite elaborate hagiographies of the order's mystical founder Pythagoras;

2. Early Pythagoreans, those said to have followed the founder's school some centuries earlier;

3. Philosophers heavily influenced by Pythagorean ideas, whose

thoughts have later even been identified with those: namely, Plato and his *Timaios*;

4. Pythagoras himself, of whom we have little evidence, since he left nothing in writing.

Sachs has shown clearly [77] that Neo-Pythagoreans report a Pythagorean theory of Five (!) Elements: Earth, Water, Fire, Air and Ether; a list regularly deployed by Aristotle, who explicitly elaborated a theory of Ether (Table 7.1). Sachs also showed that this canon of Five Elements cannot be original Pythagorean though, but that the older theory must have consisted of just three, if not the four elements of Empedocles.

Sachs has shown by investigating the composition of both Euclid's *Elements* and other sources (Proclus) [77] that likely Theaetitos, a contemporary and friend of Plato, must be credited to first have solved a major problem of the then new field of stereomety: with the first mathematical description of the five Platonic solids. The construction of the icosahedron and the dodecahedron require understanding of the golden section, a ratio most easily to encounter in the study of the pentagram.

To be precise, Theaetitos proved that there are exactly there five regular solid bodies. This caught Plato in the act of composing his *Timaios*. In this he assigns one of the solids to each of the four Empedoclesian Elements. This, so his thought, would reconcile the concept of unchangeable Atoms with the concept of changeable Elements: the actual Atoms are of one kind: triangles; these form the solids, each of which is an Element. When you put them in a box and rattle, they will sort according to size (granular convection). Sachs concludes that Plato came very close to today's concept of Molecules [77], albeit assuming mathematical structures to be the final building blocks of the world, similar to modern ideas of e.g. Wolfram [101].

The schools of both Plato—the Academy—and of Aristotle—the Peripatetic school—counted Four Elements, those established earlier still by Empedocles; and would henceforth assign spirit into a realm of its own, some other-worldly sphere of ideas, in which the mind exists.

History has much distorted these original ideas, and the equation 'Platonic solid'='Element' is an oversimplified interpretation of PLATO. Nevertheless, Neo-Pythagoreans like IAMBLICHOS were adamant that Five Elements exist, one of them the Quintessence, Ether or Spirit; and that PLATO copied that from PYTHAGORAS; and that spirit be an inner-worldly substance of some type. The accounts of IAMBLICHOS have by today in almost any aspects been disproved as non historical.

Whatever the reasons, the fundamental difference between the Four-Element theory vs. the Five-Element theory may be this: whether or not the mind is actually part of the material world, or something remote and different altogether. And this is exactly the point where things become a religious matter; opinions on this issue are rated breaking points, and arguments turn violent.

Such ruptures were plenty in Late Antiquity, where idealism had to succumb to worldly state religion before pagan materialism gave way to religious idealism again, in dramatic turns of events.

One important step in this story is the rise of MANICHAEISM, the synchretistic religious systems of MANI. The accounts we have of this believe come in more than a dozen different tongues, and many works are still discovered and in the process of deciphering. What is clear so far—in that there seems academic consent about this—is that MANI seemed to have a certain predisposition for the number five: there are five inner senses, or Tabernacles of God; there are five outer Elements, the substances that make this world up.

One can easily spot the similarity between Neo-Pythagorean and Manichaean thought: an in-worldly or an outer-worldly soul; transmigration of the soul (reincarnation) or afterlife. And since the Church since ST. AUGUSTINE was in sharp conflict with either believe system, it makes one wonder how much of the ancient texts and thoughts have still survived the passage.

And ever since, the Pentagram was perceived as sign of heresy.

The history of these discussions has not ended, these continue to be divisive issues. The symbol, whatever its interpretation, remains controversial. Lest we forget that any such association of geometry with metaphysics is fundamentally incidental: you can read in a letter any word, and in a geometric shape whatever your cosmology.

8. Pythagoreans

> [S]ie deuteten die Wissenschaft in Mystizismus um, und in dem Mythos sahen sie die Offenbarung der Wahrheit.
>
> —E. Sachs [77]

Pythagoras (570-510) is of course a key figure. We have two extant monograph 'biographies' of Pythagoras from late antiquity: that of Iamblichos (245–325) and that of Porphyry (234–305). Both these biographies should rather be called hagiographies, since they consist more of myth than of history. Notably both authors wrote some 700 years after Pythagoras [60].

The earlier Diogenes Laertius [51] is more succinct, and Aristotle mentions Pythagoras and the Pythagoreans in many places with considerable disdain. We will nevertheless follow these witnesses, and here account what we call the 'Pythagorean tradition'; the section above may serve as significant caveat: much of this is later lore, so more story than history. So here comes the story.

Pherecydes (6$^{\text{th}}$ century) was the teacher of Pythagoras, and it was him who wrote the first European prose book; this was called πέντεμυχός 'on the five angles' and is lost to us. Here the number five appeared prominently.

Pythagoras was an eminent philosopher and mathematician, who left not written text of his own, but a very influential school, the Pythagoreans. Much of what follows here, and indeed in many of the following sections, can be traced back to his and their ideas [45]; it has been argued that among his school mathematics in general, and in particular irrational numbers were first studied.

Pythagoras was the first man to call himself a philosopher, a 'friend of wisdom' (rather than a sage), and the one who coined the term *cosmos* 'that which is adorned'.

He was a believer in the harmony of the spheres, laid the foundations of acoustics, and greatly advanced mathematics (Porph. *Vita Pyth.*).

He taught numbers to be the essence of all things, with apparently included some system of numerology; famous is the design of the

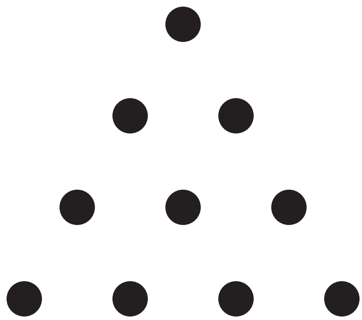


Figure 8.1.: Tetractis

'Tetractis' (Fig. 8.1). It was said (Iamblichos, Porphyry, Diogenes) that some of his disciples he called 'acousmathicoi', those who learn from hearsay, as opposed to the 'mathematicoi', those who study. Only the latter were given 'esoteric', or secret, instructions. Among these was mathematics, a term he is said to have coined: 'that which must be learned'.

It was known by then that the five-pointed star can be drawn using ruler and compass alone (Fig. 2.4). Much harder, however, was to find a measure for its lines. In other words: when you try to measure the lengths within a pentagram with a ruler, you quickly observe that whatever unit length you choose, you cannot measure all lengths precisely. This is because they relate to each other in a proportion called 'Golden' or 'Divine' Proportion, which is an irrational relation. Studying the pentagram reveals this very quickly, and the pentagram can indeed be used for an easy proof of this concept [31].[1]

Once the discovery was made how to construct the Golden Proportion and its unique fractal qualities (see below) were discovered, it became a core feature in Greek aesthetics, particularly in architecture and in sculpture.

For the classic Greek, beauty and truth were a inseparable (καλοκ-

[1]The reason Euclid uses other proofs are clearly for systematic reason: he prefers right angles at that stage in his *Elements* (2:11).

ἀγαθία, the unity of that what is good and that what is beautiful). This explains why we call this proportion 'divine'.

The most prominent example for this importance is, of course, the Parthenon in Athens, which is a temple for the Goddess of Wisdom, Knowledge, and Strategey, Athena, and which was built from about 485 AD. Its architects were Iktinos and Kallikrates; the artistic supervision had Phidias.

★

Iamblichos accounts the story of a disciple of Pythagoras who fell ill on a journey and of an the inn keeper who took care of him regardless of costs [44]. Before meeting his end, the disciple drew a symbol on a table of the inn. Later, after the disciple had died, another follower of Pythagoras saw this and paid the inn keeper his expenses. A 'symbol'could have been a sentence or a figure. If we read this with what Lucian writes about the importance of health and the pentagram for Pythagoreans [57], this can have been a pentagram: symbol of being learned, and of health.

Even today, provide anyone with a ruler and a compass and ask her to draw a pentagram: only those with some geometrical training will be able to comply (Fig. 2.4).

Hippasos was a disciple of Pythagoras. He made an astonishing observation: The existence of 'irrational' numbers. With other words: that there are mathematically well defined proportions which cannot be measured with the same yard stick by natural or rational numbers [15].

This discovery of the 'incommensurability' was of course a milestone on the intellectual journey of mankind. How this discovery was made is not reported, however it is particularly easy to demonstrate it on a pentagram, and may have occured upon study of this (von Fritz [92]). One can imagine Hippasos trying what the author has tried: to find a way to put stops on a pentagram to use it as a board game. Lore goes that Hippasos met death by drowning in conncetion to this discovery [92].

The importance of the Golden Section for the ancient Greek can hardly be stressed enough. This includes Plato [68] [69].

The Golden Proportion appears in many natural structures from microscopic shells to the solar system (page 97). It has been used in architecture and fine arts, and considered a beautiful proportion for centuries [70] [68]. That this particular relation deeply governs all proportions of the pentagram makes it so particularly charming.

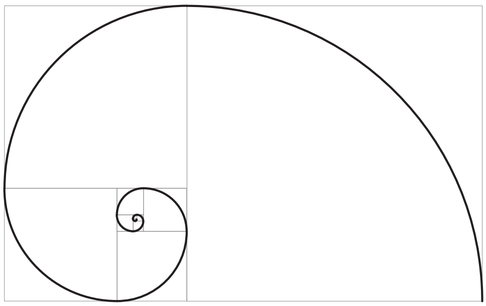


Figure 8.2.: The Fibonacci spiral
A logarithmic spiral in a 'golden' rectangle (appr.)

The Golden Proportion as a ruling principle can be found in Fibonacci sequences[2] and in many growth related structures. It makes for a beautiful picture format,[3] and gives rise to a Golden Spiral (Fig. 8.2), a form of which can also be drawn within a pentagram.[4] Pentagonal forms can be found in fauna and flora; think sea-stars and the five fingers all land chordata share. A five-fold structure is naturally more stable than anything that can be cut more easily in two equal halves.

[2]The Fibonacci sequence is defined as $F_n = F_{n-1} + F_{n-2}$

[3]In contrast, today's common 'A' series paper, among them A4, has the proportions $1 : \sqrt{2}$ with A0 having a surface of $1m^2$.

[4]You can construct one logarithmic spiral based on a golden rectangle and another on a golden triangle as found in the pentagram; these two are not identical, albeit both *are* related to the Golden Proportion. Contrary to popular believe, most logarithmic spirals in nature (galaxies, nautilus...) in fact have nothing to do with the Golden Proportion.

Along with these mathematical aspects, Pythagoras is also credited with the discovery of the connection between the length of a chord and its sound: natural overtones result from rational fractions (a string of half a length produces an Octave, a strong of one third the length produces a musical Fifth, etc). He is further credited with the idea of the music of the spheres; he was a great mystic.

Indeed we have far more information about his mystical side than about actual mathematics, much of which seems to be a later addition. This also includes the knowledge of irrationality; it seems today that the story of the irrationality to be a school secret was a later addition [31] [77]. Who he really was and what he really taught or knew is shrouded in mystery.

Empedocles (495-435) [26] is often cited as the 'inventor' of the classic inventory of Four Elements. Later authors have credited the Pythagorean school to have invented the Five Element theory; it has been shown that this is a later ascription [77]. Rather, Empedocles' theory of Four Elements seems to have been expanded to Five Elements some time after Plato, probably in the later Academy i.e. school of Plato.

Democritus is the 'inventor' of Atomism, in that he stated that the world be composed of smallest particles—one per Element. So here the Elements are no longer states of matter, but matter itself.

Plato (428–348) was clearly influenced by Pythagoras; so far that he has even be accused that his cosmological *Timaios* plagiarise Philolaos (Diogenes, 8.85). Influences of Empedocles are also obvious, as we shall see.

In *Timaios* he he gives four Elements and equates them with a sphere of the world and sorts them by heaviness, or from light to heavy: Fire, Air, Water, Earth. Each of these he equates with a Platonic solid. But there are five Platonic solids, yet only four canonical (Euripidean) Elements. He circumvents the issue by stating the fifth solid, the dodecahedron, to have been the 'building plan of the heavens' [37].

The idea of the Fifth Element or Quintessence is a logical enlargement of the theory that matter consists of four Elements, and Elements be Platonic solids, of which exist five types. However, the

Fifth Element is somehow distinct from the four traditional Elements. The soul is made of some other material or Element which existst in the material plane, but does not partake in the change from one Element into another.

The same applies for the heavenly bodies, which apparently are above all other matter, and were speculated to also be of some fundamental different quality; in *Phaedros* PLATO famously draws a parallel: as fish in their water sphere gaze upon the dry sphere and may wonder what they see, so do we gaze upwards into the heavens without proper understanding of what is above.

ARISTOTLE discusses various aspects of the Element theory in the light of a theory of substrate, interaction, and qualities, mainly referring to the four Elements of EMPEDOCLES; he reduces the four Elements on two contraries (Table 8.1). This focus on the change of phases results in a concept Elements closer to what we today call the phases of matter: liquid, solid, gaseous (though 'gas' was a yet unknown concept), his preferred example evaporation and condensation of water. He is also credited with formally introducing the Fifth Element under the name of Ether, but this Fifth Element was considered to be substantially different to the other four in that it does not participate in the circle of change from one to the other: it is supposedly an unchanging, immortal, heavenly Element.

As a consequence the pentagram can serve as a universal picture for the four Elements plus the soul, while the four material Elements suffice for practical purposes.

To see the four 'physical' Elements at one footing with the one 'spiritual' Element is of course a holistic view. Another is to separate and oppose the physical world (*physis*) and the spiritual world (*logos*) altogether. From this arise philosophical questions as to which has priority, an age old discussion ensuing.

ARISTOTLE ascertains, but also mocks the association of a character or humour with each Element as well as the equation of each to a sense sense: that of smell, of taste and so on. Nevertheless, he also discusses the existence of an inner sense (*sensus communis*, or *common sense*) or inner senses of truth, of justice, of beauty and so forth (De Anima,

Table 8.1.: Peripatetic Scheme of Four Elements

	dry	moist
hot	Fire	Air
cold	Earth	Water

De Sensu; cf. [61]). While the 'outer' senses serve perception, the inner serve apperception.

This idea of inner senses or qualities of the soul as opposed to material Elements should play an important role later, particularly in Manichæism.

Pliny the Elder (23–79) gives the same ordre of the four material Elements in his *Naturalis historia* [88], an extant work that was very popular for a very long time. He again sorts the four Elements by heaviness: at the bottom, water; on this the earth thought floating; above this, fire and air; and finally above all the heavenly bodies or the almost weightless soul.

Ptolemy in his *Almagest* gave a planet to each of these Elements. Thus some kind of orthodoxy in the association of Elements, planets, and Platonic solids emerged, which we have assembled in Table 8.2.

These five Elements can easily be associated with five colours. There is additional trouble here since there seems not to be a very clear and unique way to associate colours and Elements. We have no ancient sources on this. An additional difficulty arises from the fact that the very names of colours seem to not be clearly the same from one language to the next, nor from one time to the other [17].

When we recall the passage from Lucian where the Pentagram is presented as sign of the Pythagoreans and, more importantly, as a sign of Health, interpreting the pentagram as a cosmological chart showing all five Elements is appealing. For them the pentagram was a symbol of harmony of the five Elements and of the human being. It also has a distinctive humanoid shape, which makes it be associated to the human.

This is why in this tradition the pentagram has been called 'sign of

Table 8.2.: Colours, Elements, planets, bodies (trad. Europe)

	colour		Element		planet	shape
0	white		spirit	☿	Mercury	dodecahedron
1	blue		water	♄	Saturn	icosahedron
2	red		fire	♃	Jupiter	tetrahedron
3	green		earth	♂	Mars	hexahedron
4	yellow		air	♀	Venus	octahedron

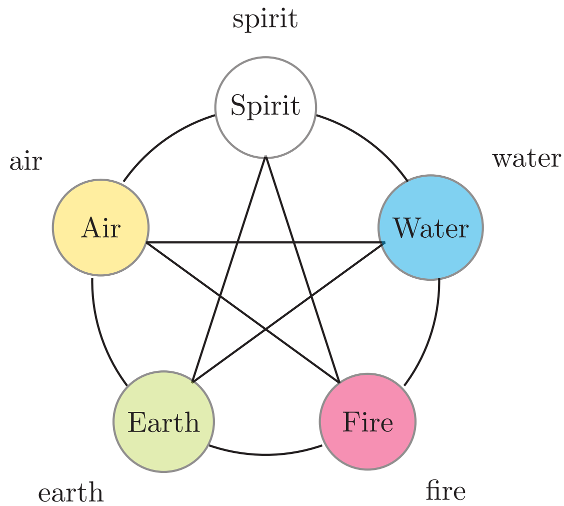
the microcosm'. It is also traditionally called 'Solomon's Seal', and in this meaning it appears still today, for instance in the flag of Morocco (The corresponding 'sign of the macrocosm' is the Star of David).

Associating the Elements with the five corners of the pentagram is not far-fetched. A clear system as to which Element should correspond to which corner has not been developed in the West: there are competing schools, even though today there seems to be some kind of standard. Sometimes the colours have been sorted by intensity or in the order they become visible at dawn, and then it seems the Elements have been ordered in the same way. Sometimes the Elements have been ordered so that their first letters create a word, commonly 'Hygeia', supposedly the name given to the pentagram by the Pythagoreans according to Lucian. This is non-convincing though since there is no ancient word 'aer', so this interpretation is probably of later origin.

The deeper reason for this variation of attempts to associate corners, colours and Elements is that the actual theory of the Elements has not really made any progress in the West, and the relations between these Elements seems of limited explanatory value. We shall see below how the East Asian Wu-Xing-tradition is more advanced in this.

Today exists a canonical form in which we find Five Elements associated to the five corneres of the Pentagram in a fashion sometimes referred to as 'traditional'. This is depicted in Fig. 8.3.

Figure 8.3.: Esoteric Pentagram (modern)



The order of Elements varies between authors.

9. Oriental Traditions

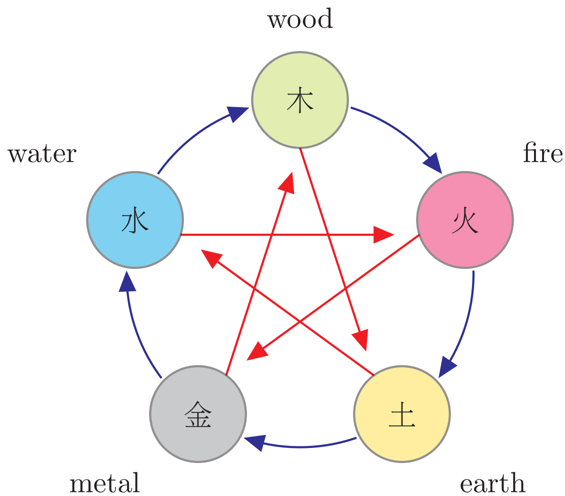
In the Far East a concept of five elements is known as 五行 *Wu Xing*. While in the West 'element' is something substantial (or at best, a *state* of matter), 'Xing' is dynamic, a movement or a shift.[1] Each *xing* is thought to have one that generates it and one which it generates. It is thus instrumental—or better: fundamental—to arrange them in a circular fashion, where all lines bear significance, and so does the pentagram in its centre. In a typical arrangement, the *xing* change into each or 'generate' each other clockwise; if you connect them creating a pentagram, the lines of the star signify 'overcoming' (again clockwise); see Fig. 9.1.

This map is fundamental to Feng shui and Traditional Chinese Medicine practice, inter alia.

Table 9.1 presents the most commonly associated colours, plus

[1] Marcus Aurelius (121–180 AD) had the same conception [7].

Figure 9.1.: Wu Xing



Circle: generating. Star: overcoming.

the five Chinese seasons, which defines the core of the concept. The Elements *('phases')* are **Metal, Water, Wood, Fire** and **Earth**. Colours associated with them vary a little. This table is often extended to include not only planets, but body parts, senses, characters and so forth, similar to the ancient European association tables; and again, there is much variation from school to school.

Table 9.1.: Colours, phases, seasons, planets (trad. East Asia)

colour		行 *xíng* phase	season	planet
white / black		金 *jīn* metal	autumn	Venus
black / blue		水 *shuǐ* water	winter	Mercury
green / blue		木 *mù* wood	spring	Jupiter
red		火 *huǒ* fire	summer	Mars
yellow		土 *tǔ* earth	dry season	Saturn

The Wu Xing is of enormous importance for East Asian culture. A clear difference to the Western tradition is that the order of elements here are truly fixed, and that there is meaning associated to the lines connecting them in a circle and a pentagram; in the West, the association of pentagram corners with elements is incoherent in time and does not convey much meaning.

Since we find the concept of five elements in both the occident and the orient, we can speculate that this is a very old concept. Porphyry accounts that Pythagoras studied in Asia. However the pentagram is first and foremost a very simple geometric shape and a fundamental graph. It will appear wherever one maps the complete relations between five elements of which kind ever.

The colours we have settled on for the use in Pentagame have nothing to do with any of these schemes, but are purely chosen for visual reasons. Of course any five colour scheme—or indeed context scheme—could be used.

Buddhism knows the Five Faculties (pañcendriya, Table 9.2); this can literally mean 'five senses' and refer to the outer senses, but is more often used in a spiritual context as five inner faculties. These Five Faculties, when developed, become Five Powers (pañcabala).

The pentagram is also common in Japan, where it is called 五芒星 gobōsei and is, just like its western counterpart, for some a diagram

Table 9.2.: Five Faculties in Buddhism (pañcendriya)

faith	śraddhā
exertion	vīrya
memory	smṛti
concentration	samādhi
wisdom	prajñā

of cosmic forces and for others a sign of witchcraft. Be as it may, the symbol and cosmological interpretations thereof appear universally.

10. Astronomy

When you aspire to practice astrology, or astronomy for that matter, measuring angles is of course of utmost importance. So first of all you need to segment a circle. As we all know from early play with ruler and compass, to segment a circle in two, three, four or six parts is child's play. As a result, you get a dial of twelve. But only if you manage to draw a pentagram, you can then proceed to the 60 minutes of a circle, and from there on to 360° or seconds., Because you will need the factor 5 as $60 = 2 \cdot 2 \cdot 3 \cdot 5$.

Geometrical knowledge and astronomy / astrology is in this way technically linked. You can simply not draw a horoscope (with ruler and compass) without knowledge of drawing a pentagram. So if you equate astrology and superstition, you can easily mistake the pentagram for a sign of superstition, all the while it actually is a symbol of science.

If one marks the points of largest elongation or of transits of Venus ♀ in the zodiac, in four years a pentagram results. This is because the orbital periods of Earth (♁ = 365.256 days) and Venus (♀ = 224.701

days) relate to each other almost perfectly in the golden proportion [59] ♁ ÷ ♀ = 1.6255... ≈ φ. This is why the pentagram is associated with the planet Venus, at times the brightest star after the Sun and the Moon. The Babylonian goddess Ishtar is depicted with a four or eight pointed star.

The *siderian* Venus period is 224.710 days. The *synodic* Venus period (time between appearance in the same zodiac sign as seen from Earth) is approximately 583,92 days. Now $584 \cdot 5 = 2920 = 365 \cdot 8$. Because of this a Venuvian calendar produces five Venus periods in eight earth years, or a similar constellation every four years.

The most ancient Greek calendars were Venuvian calendars, which is likely the root of the Olympic four-year cycle and another hint on the importance of the pentagram for ancient culture. The Mayan calendar is another prominent Venus based calendar.

Venus as Morning Star is called $\varphi o \sigma \varphi o \rho o \varsigma$ ‘Phosphoros’ in Greek and ‘Luzifer’ in Latin, where both words simply mean ‘light-bringer’.

11. Hermeticism

As shown above, the pentagram was important for the Pythagoreans: it was a symbol of health to them, and probably a symbol for their whole school and philosophy [57].

At least at two times in antiquity Pythagoreans were persecuted as a heretic sect: the original school was long extinct at the begin of the Common Era, when the Neo-Pythagoreans appeared. The Pentagram as symbol of a scientific world view would stand in opposition to religious state cults of both pagan and later Christian flavour.

Hippolytus (c. 170–235) wrote against many schools of thought—or sects—for maintaining pagen believes; in particular those who believed in reincarnation, of which both Pythagoras and Plato were convinced.

Late antiquity saw a renewed interest in Pythagoras and his school.

Plotinus (204–270) sought to reconcile the philosophies of Plato and Pythagoras. Iamblichos (240–320) and Porphyry (233–301) were from his school, and both of whom wrote biographies of Pythagoras still extant.

That they both explicitly presented Pythagoreanism as an alternative to Christianity did certainly not help their case.

In the complex cosmology of the Manichæan world religion founded by Mani (216–274) the number five plays an important role [86]. *Reason, mind, intelligence, thought and understanding* are the five light qualities—or better Shechinatha or tabernacles–of God [98]. They stand in opposition to the dark material sphere of five Elements: *ether, wind, light, fire and water*; thus, the platonic distinctino between *physis* and *logos* persists. One can easily see that pentagrams will have played a role in this cult [86]. Manichæans were prosecuted heavily in late antiquity and the early middle ages by both pagan and Christian emperors.

Notably Augustine (354–430) was a follower of this cult prior to his Christian conversion.

It seems quite reasonable to assume that this period of suppression was when most ancient pentagram shaped objects were destroyed. Today practically no ancient pentagrams survive—neither in architecture, nor in any other form.

The change from paganism to the monotheistic religions accompanied iconoclasm, and there were waves of such in times of political instability which are destructive in their own right. *Damnatio memoriae* includes the process of the destruction just as much as that which has been destroyed [58]. In the Roman empire much was destroyed as pagan artefacts by Church and Christian rulers fighting Gnosis, Manichaeism, Bogumilism, Katharers and so forth, who had kept a good deal of pre-Christian ancient mysticism [84].

About all these movements only little detail is known. But when the pentagram is understood as a symbol of five—not four—elements showing spirit on the same plane as the material elements, then it may appear a heresy to idealist religions. If it was proof of someone practising astrology, it was a proof of heresy. These were indeed the times in which science was considered if not heresy vanity, a deadly sin, since the only source of knowledge was supposed to have been revelation.

Science was mainly confined to the art of collecting, classifying and simply memorising knowledge. Often scholastic classification followed PORPHYRY, the same author we have encountered above. His *Isagoge,* also known as *quinque voces,* was originally an introduction to ARISTOTLE's conception of categories in his 'Topics' (*Isagoge*). A five-fold scheme of categories is also found in PLATO'S *Sophistes.* These classic κατηγορόυμενα *kategorema* or (Latin) *predicabilis* are listed in Table 11.1.

Table 11.1.: Predicalbes (ARISTOTLE, PORPHYRY, BOETHIUS)

	Greek κατηγορόυμενα	Latin predicabilis
1	γένος	genus
2	εἶδος	species
3	ἴδιον	proprium
4	διαφορά	differentia
5	συμβεβηκός	accidens

Such a scheme can of course be very useful for definitions of all kinds: to define a particular plant, a particular action (as a crime) and so on. Like any classification scheme it also has its natural boundaries: when is something *generic,* when is it *specific,* when *accidental*? Nevertheless, PORPHYRY'S scheme has later had a massive influence on collectors and classifiers in history: DIDEROT and D'ALEMBERT [25] and LINNAEUS [95] are prominent names.

With such a scheme objects can be sorted by genus and accidents, and anything 'watery' would be sorted into the same category [54] [34]. Such a classification system naturally leads to large tables, and one may be inclined to abhor empty slots. It can of course also lead to speculative thought. For example PARACELSUS speaks of Elementary spirits, one type per element [1].

Thus it comes to no surprise that geometric diagrams to illustrate

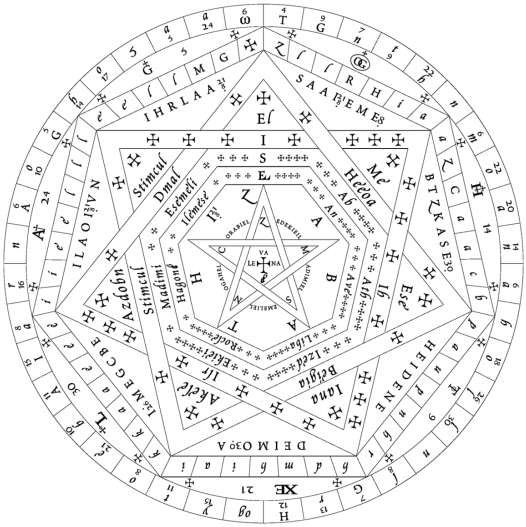


Figure 11.1.: Sigillum Dei
as found in the *Liber iuratis Honorii*, a medieval grimoire

‘deinde scias quod communiter in exemplaribus pentagonus fit de rubeo cum croceo in spaciis tincto et primuseptagonus de azurio, secundus de croceo, tertius de purpureo, et circuli de nigro, et spacium inter circulos ubi est nomen dei maximum.’

such systems were widespread. The pentagram was one of them. It makes a prominent appearance in the 13$^{\text{th}}$ century grimoire *Liber iuratis Honorii* where it appears as part of a larger construction of polygons and letters (Fig. 11.1). Such a device could then symbolise the cosmos, and contain all characteristics of God. The idea that unity unfolds into duality, trinity and so forth is called ‘emanation’ and became a central topic in middle and late neo-platonism from (Plotin) to Proclus [87].

We see that medieval thought was predominantly magical, in that analogy ruled; this heremeutic *‘as above, so below’* [33] is the very essence of magical thinking, which is of course today discredited, although today for other reasons: not because it contradicts revelation, but because it contradicts science. Humans see patterns and construct reasons too easily where there are often only coincidences (apophenia).

The pentagram also symbolised Five Virtues or the Five Wounds of Christ [12]. It can be found on many medieval churches—Marktkirche in Hannover (1388) is an early example—and is prominent in freemasonry, where inter alia it is a symbol of engineering knowledge. It was also known and used as badge by guilds of medicals as symbol of Hygeia, the goddess of health, and PYTHAGORAS was held in high esteem throughout the middle ages.

With the Renaissance individual inquiry became again more prominent, certainly inspired to a good degree by the study of ancient originals such as EUCLIDS *Elements* [30], which still today serves as the ultimate example for a axiomatically inductively structured mathematical text book, and PORPHYRY's *Isagoge* in the translation of BOETHIUS. Only in the 13$^{\text{th}}$ century the first university was founded in Bologna. Nevertheless, it took some further centuries for the scientific methods we use today to become firmely established, with pioneering work from René DECARTES [24], explicitly inspired by EUCLID, and Francis BACON [56]. Until then, scientists were burnt at stake for their use of reason.

Still today, the pentagram is surprisingly rare in elementary mathematical text books, even though it is so basic and perfect for explaining irrational numbers. Books containing pentagrams were certainly often censored, the Vatican denying them Imprimatur. The famous 'De divina proportione' from PACIOLI (1498) [70] does not contain a pentagram, a symbol suspiciously absent as well in EUCLID's *Elements*.

In the early modern times, characters of the like of John DEE, Jakob BÖHME and Athanasius KIRCHER saw the pentagram as a sign of Jesus in the Christian Kabbalah. A popular design can be found in a fine illustration to Agrippa (Fig. 11.2). Back then science was still interested in the supernatural—because that was a time before statistical testing became a bullet proof scientific method. At this time magic was considered a reality worth to be studied.

For long, folk magic has seen the pentagram as a banishing sign. It appears as such in GOETHE's *Faust*, where it has lost half its power due to not been drawn correctly [94].

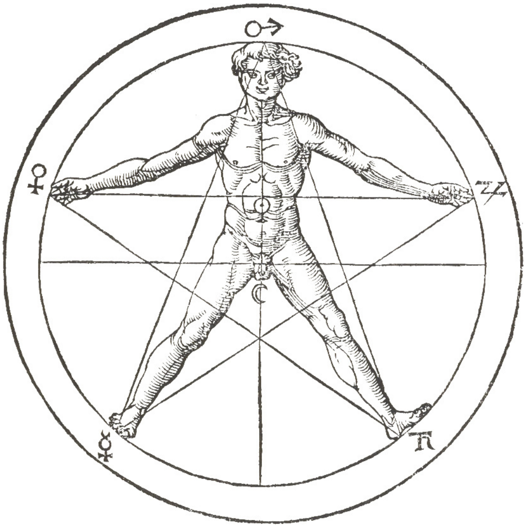


Figure 11.2.: Man in a pentagram

Anonymous illustration in [96].

'...nullum fit in homine mebru, quod non respondeat alicui signo, alicui stellae, allicui intelligentiae, allicuis divino nomini, in ipso archetypo deo.'

'...there is no human limb that does not respond to some sign, some star, some intelligence, or some divine name: the archetype [of us] is God.'

—Agrippa [96] lib. ii cap. xxvii.

The interest in the mathematical properties of the pentagram never vanished, of course. A fine example in architecture is Jan Santini's baroque Church of St. John Nepomuk in Bohemia which is based on a pentagonal plan.

The pentagram regained prominence in the 19th century with the first wave of 'occultism'. This is a counter movement to enlightenment; since then, an up-pointing pentagram is called a symbol of the rule of reason, and a down-pointing of the opposite [53] (1861). This distinction has become popular since, but stems from the heyday of a fresh wave of discussion on materialism vs. idealism [99]. Since then, the pentagram has more often been associated with the occult, often even being perceived as 'sign' of occultism or Satanism per se. Rudolf Steiner recommended it [91] for meditation on the human nature. It appears as sign of black magic in Ottfried Preussler's novel *Krabat* [75]. It is burdened with superstitions indeed [38].

Today ritualists in the neo-pagan, occult or Wicca tradition distinguish the two directions in which it can be drawn, clockwise being held as 'banishing' and counter-clockwise as 'invoking' [19]. It also plays an important role in Heavy Metal and Goth subculture and is also used by both theistic and atheistic Satanists (on theology see Chapter 12). It simply fascinates by its surprising beauty and inner complexity, and it lends itself to manifold interpretations—and superstitions. All this—and a simple geometric shape!

12. Some theological remarks

Pentagame is a parlour game played on a pure geometric shape. Some people interpret this beautiful shape spiritually. Some Christians seem to see the sign of the Antichrist; some neo-pagans accuse me of 'mocking' their holy symbol. Both see something 'occult' in the sign of the pentagram and feel estranged by it published and played in public.

The term occult 'that which is hidden' resonates with Pythagoras' distinction of *acousmaticoi* and *mathematicoi,* those who hear, and those who learn, and between *exoterioi* and *esoterioi:* that what is spoken about in public, and that what is taught individually.

Mathematics seems to some esoteric. Some see the devil in geometry, others speak of 'sacred geometry'. Geometry measures the world (hence the name), and 'the world belongs to Satan' (2 Cor 4:4). But the world is Gods creation. If we look at its mathematical beauty, we see the Creator.

'Every good gift and every perfect gift is from above, coming down from the Father of lights, with whom there is no variation or shadow due to change.' (James 1:17) 'So, whether you eat or drink, or whatever you do, do all to the glory of God.' (1 Cor 10:31).

We should be proud of our godly ability to see the light, to reason. *Sapere aude*—use your own judgement. Test everything, keep what is good (1 Thess 5:21). You shall see the truth, and the truth shall set you free (John 8:32).

Satan is in the Bible the personification of evil. But the figure or concept of the devil as we conceive it today is a mix of Greek, Roman and Pagan myths. 'Lucifer' is originally the Latin name of the planet Venus as morning star, which means 'bearer of light'; and as we have shown above (page 97), Venus is associated to the pentagram. It is a twist that the advent of light has become associated with the advent of darkness. But in the New Testament, the Morning Star, the bringer of light, is JESUS himself (2 Petr 1:19).

Evil is not what goes into a person, but what comes out of them (Mt 15:11). Thus, it is not the pentagram that is evil, but the thoughts it may evoke. If you think of harmony—then it is a good sign. If you think otherwise—you should try to find out why that is so.

Someone suggested 'Pentagame—Satan approved' as a sales slogan! Of course, Satan would *approve* Pentagame; that is because the devil is the *inability* to distinguish good and evil. God, said the serpent in 1 Mos. 2, has the capability to distinguish between good and evil. And since the Fall of Man we share this ability. Humans can and must find their best guess on good and evil all the time and decide for their best knowledge, much like you ought to seek the best move in a game.

The devil, in contrast, though the accuser, is unconcerned with truth: he is the confusion of truth and lie (John 8:44). *Satan is impartial* to good and evil. It is bad not to tell good from evil. Not knowing, not telling good from bad, while having all capability to do, that is evil. Ignorance, relativism, is evil. The very word 'diabolo' means 'the one who throws things (into disarray)'. The very concept of the devil is that he confuses good and evil, and lures you into approving anything. Mixing up a pretty geometric shape with gross superstition is right up this path.

The serpent denied the godly distinction between what is good and what is bad for Man, suggesting either anything to be fine, or everything forbidden (1 Mos 2:16 vs. 2 Mos 3:1). Satan is the approver of *all* things, and denier of all things as well. All the while God lets His sun shine, and let come rain, over good and evil alike (Mt 5:45). He is the enlightening source of distinction between what is good and what is not. Satan, in contrast, who is not in truth, the father of lies, is the voice of relativism. Here is indifference, and at the end short-sighted selfishness; the darkness which confuses.

The very idea of a 'person' Devil is contradictory, because there cannot be 'pure evil'. There is evil in this world, like there is illness; but there cannot be illness without something ill; the earth quake cannot be bad if there is nothing good that is affected.

This game will never harm anybody. That I can clearly see that Pentagame is good: a good game, offering nice encounters, and training your mind.

13. Conclusion: A probable reconstruction

We have shown that the pentagram has astonishing mathematical properties, where the incommensurable, and thus the infinite, it obvious. We have seen how because of this, and also for other reasons, it has a truly wild history as a symbol.

We have seen that a theory of Five Elements was a staple in the schools of Pythagoras and Plato and most likley took centre stage in the believe system of Mani.

It has served as badge and as meditiation device, like a yantra, in Neo-Pythagoreanism and most likely also in Manichaeism. As such it has resurfaced through time, and is still of such appeal in masonry, in neo-pagan and in occult circles. It has thus also faced at times rather violent suppression.

We have also observed that these were persecuted gravely at multiple times in history. Since every pentagram will have to be associated with one of these believe systems, prosecution is likely to have resulted in the destruction of pentagram shaped objects. This would include pentagram shaped boards.

All this explains well why we have no pentagram shaped objects from antiquity today. This may as well explain why the pentagram game, *Pente grammai*, that has very likely existed in antiquity, was finally lost at some point prior to the middle ages.

This serves the argument from above. The result is:

We cannot rule out that *Pente grammai* or, if all *petteia* are *Petteia*, was played on a pentagram shaped.

If it was played on a pentagram shape, it is likely that the Five Elements, represented by colours, played central stage. It was thus quite probably very similar to Pentagame.

The only true validation of this argument would of course be to find an ancient pentagram shaped board; unfortunately chances for this are slim. We can now only rejoice in the fact that the game we have today *fits so well* into the thought of the ancient that chances are that this is a re-discovery of a game played by past sages.

Part IV.

Playing Pentagame

> [P]laying an abstract strategy game is an exercise in logical thought. There is an intimate relationship between such games and puzzles: every board position presents the player with the puzzle, What is the best move?, which in theory could be solved by logic alone. —Thompson [90]

1. Introduction

As it stands, Pentagame is a joyous affair: a quick game, suitable for two, three and four players, easy to learn and rich enough in variation to remain entertaining for many years.

It can be played on many different levels of skill and involvement. This part will be interesting for those who are move invested.

We will begin with tactics. The idea behind tactics is to find the best possible action in a given situation, without taking into account any further future complications such as counter-play, or even one's own future play through combinations. Thus we start with a 'one player game', so to speak, and only later introduce the second player.

This will then allow us to see how the qualities of the game change from its beginning to its end, and look at the specifically interesting aspects of each of these phases, from the opening to the end.

The intention of this part is to analyse game play of Pentagame in a systematic manner, and will be interesting for more experienced, ambitious players. As a side product this investigation yields vocabulary to describe the game and establishes categories that may help discussions and future publications: a notation; a classification of openings; attractive refutation of seemingly good openings; and so forth.

To make the text more readable we will call our fictitious players $\mathcal{A}$ and $\mathcal{B}$ etc. by the names of $\mathcal{A}$lice, $\mathcal{B}$ob, $\mathcal{C}$harlie and $\mathcal{D}$oris. Alice is always the one to move first; or better: the player $\mathcal{A}$ who moves first we call 'Alice', the second to move 'Bob' and so on. As common in game literature, we will call the action of an individual player a

Table 1.1.: Important Conventions

$\mathcal{A}$	Alice; first player
$\mathcal{B}$	Bob; second player
a *ply*	the action by one player
a *move*	a turn where each player plies once
	$\rightarrow$ a move consists of *two* plies

ply (plural: plies); so each *move* consists of a ply by Alice and by one of Bob, to avoid confusion (Table 1.1). Alice always starts, and by convention with white and clockwise, and our illustrations will all have the white corner on the top.

2. Tactics

Tactics asks: which is the best action for a piece in a given position? Thus, the point of view of the tactician is a static one ('ceteris paribus'). The tactical value of a move can thus be considered without taking any counter-action or possible futures into account. Since the development of any strategy requires understanding of the tactical values of moves, this is where we begin.

2.1. Moves

To begin, look at a Pentagame board that has only one party on it—a solitary game, a game where Alice plays without any opponents. This may not be the most exciting setup, but considering it will do us well to prepare for the more complex settings.

As Alice starts the game, she notices that it matters neither which colour she chooses now nor whether she moves to the left or to the right—everything is still very symmetric.

Table 2.1.: Move onto a node vs. onto a line

Move onto node	Move onto line
four ways to go	two ways to go
many options	fewer options
often long path	often short path
difficult to get blocked	easy to get blocked
blocks lightly	blocks very strongly
less committed	highly committed
	great near piece's goal

She can move some stops forward on a free stop on a line, or move on until she hits an obstacle. But the former would leave her on a line, and to be on a line is generally less worth than to be on a node (cf. Table 2.1). So she will move as far as she can until she hits an obstacle.

For now, all obstacles (pieces and blocks) are on nodes. So if Alice is wise, her opening move will be onto a node. This leaves her with two options (**??**): swap and replace. Swap and replace are indeed the two most common and important types of moves. In the opening position swap happens on the ring, and replace in the pentagon.

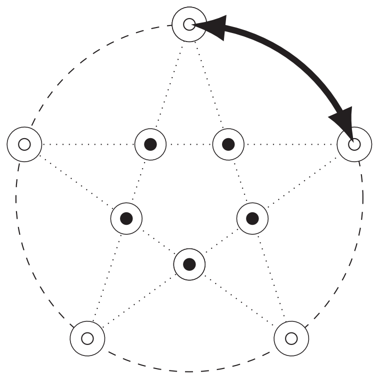
The move onto the pentagon replaces a block, while the move on the ring is a swap with another piece. We will encounter these two most important types of moves a lot more; here they are:

swap swap two own pieces $\leftrightarrow$ (on the ring, or elsewhere)

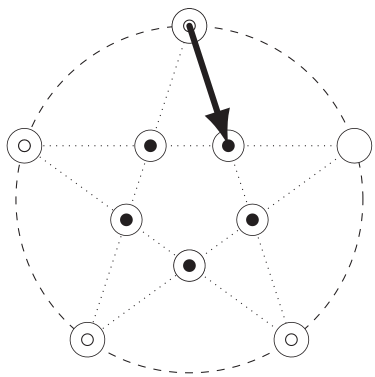
replace beat and replace a block $\rightarrow$ (on the pentagon, or elsewhere).

Swap and *replace* are the two most relevant types of moves.

These two most important types of moves have very different qualities collected in Table 2.2.



(a) ↔ swap



(a) → replace

Both *swap* and *replace* can happen anywhere on the board, as long as the position allows. It is only at the beginning that all swap moves are ring moves and all replace moves are star moves; this later changes.

Swap moves are good because they move two pieces at once, which means a gain in speed.

In contrast to swap move a replace move conquers a stop unclaimed before, which is a great advantage. Later in the game, replace moves always liberate the stop from which the move is made, an important fact.

Table 2.2.: Swap and Replace compared

Swap	Replace
'fast' because two pieces move	'slow' as ony one piece moves
symmetric	aysmmetric
does not affect territories	affects territories
	- abandoning creates area
	- conquering increases territory
	blocking hems opponent
	- over 20 blocking options
	- placement difficult choice
best in absence of threat	best in presence of threat

The second part of any replace move is of course the re-positioning of the black block. Of course such blocks are particularly useful under immediate threat, but still of great strategic importance to structure the game. The placement of the black block is a bet on the presumed best answer move of the opponent, thus a major part of the strategic interaction of players. There are twenty lines on the board, most of which Alice should consider. She will like to use these blocks in a strategic fashion, impairing Bob.

Thus a replace move is in fact some kind of a double move since it consists of the actual movement of the piece plus the setting of the block. At this point it is worth noting that this means that *every replace move is accompanied by a choice from about 20 lines to block.*

Of course, rather than playing swap or replace, Alice could also just simply walk onto a stop on one of the lines. But rarely is it advisable to step on a line when not forced to (cf. Table 2.1). So most of time, in general, her pieces rest on nodes. At least this is true for the first couple of moves.

Now nodes are 'crossroads', where pieces have four ways to walk from. With all her five pieces on nodes, Alice has 20 directions to consider—per move. In other words: usually a player has to choose her favourite move out of about 20 options, not counting block re-position challenges.

So Alice's number of realistic choices per move is relatively small; the hard challenge is to choose the best one. This shows that Pentagame has THOMPOSON's quality of **clarity:** *an ordinary human being, without devoting his career to it, can form a jud[e]gment about what is the best move in a given situation.* [90]

Since every move is one out of twenty, any given position has about twenty possible next-generation-positions. In other words, at every move the game can develop in about twenty different directions. Just 5 moves per 20 options result in a total of $20^5 = 3{,}200{,}000$ possible states after Alice's third move—not taking blocks into account. This will we investigate more precisely below, in Part VII.

2.2. Directions

Within the category of abstract symmetrical board games we see two extremes: one where pieces are added to a board on which every stop is equal; and one where pieces traverse the board. The former games classify as positional games, the latter one as race games. But behold, this is actually a scale (Fig. 2.3). Go, and Nine Men's Morris, have no directionality, they are purely positional. Chess is highly positional, but nevertheless has a direction for the pawns of each player.

Pentagame sits in the middle as a game both positional and directional. Noteworthily all players play in the same direction in Pentagame, and there are no dice.

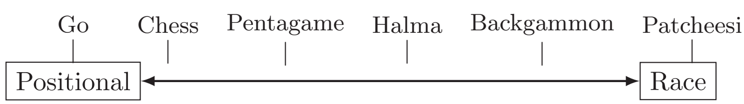


Figure 2.3.: Positional and directional games

The fact that Pentagame is both a positional game and a race game is noteworthy. Any stop is characterised by its distance relative to the centre goals. Any stop is of different value for each piece due to this.

That also means that directions are of different 'value' per colour. It also implies that speed is important: who ever needs the least moves to reaches her goals wins. The quest is to be fast and in being fast creating positions that allow to be faster than the opponent.

While Pentagame is directional, it is also chiral: each piece can go *left* or *right.* If Alice moves a blue piece to the right, Bob might want to go to the left. Alice may choose to prefer right over left for all of her pieces, while Bob may choose to either follow her or opt for the opposite chiral direction. Pieces moving in the same direction (in chirality: to the left or right) create very different game play then do pieces running in opposite directions, in particular when we are talking about pieces of the same colour. The distance to their goal we will call 'temperature' of a piece, it being 'hotter' the further it is away. Consequently, a distance transversed is the hotter the longer it is, as in temperature difference.

Once a piece has been moved from its origin it has in fact decided on a direction: left or right. We speak henceforth of 'left' and 'right' as choices for the path of a piece, and of 'dextral or sinistral' chirality. Imagine all pieces move to the *left*: then the board would display perfectly *dextral* chirality.

2.3. Paths

What Alice can do with a piece depends on what is at the end of any of its potential paths. Paths can involve nodes and lines, and they can be on the central pentagon, on the ring, or between the two on the star.

A path is per se only a connection between two points available for a piece. But one of these points may be closer to the pieces's goal than the other. A piece that has multiple paths available may thus 'prefer' one path over the other, with 'shorter' paths preferable.

Sorted roughly by desirablility, any path may be–

free so that the piece can reach a goal. This is usually the best, since it means the piece can be brought out straight;

open so that the piece can reach a free node, which is great because it means that piece has many ways to go;

connected so that the piece can reach another piece of the same player, which allows for a *swap*; a connection establishes an *area* the other player cannot possibly enter with any of her pieces; a piece that is not thus *connected* is *isolated;*

blocked by a black block. A block can always be overcome by a *replace*. Blocks slow down, but do not hinder movement. Overcoming a block gives the player the right to re-position the block onto one of the 20 lines or an empty node, which is great, breaks symmetry, and adds structure;

stopped by a grey block. Grey blocks can also always be overcome by a *replace*. Nevertheless, they disallow the player to in turn block his opponent;

cut off by another player's piece (on a stop that is not a goal). Because of the Ko rule this is an obstacle that cannot be overcome unless the other player accepts. Cutting off is a very brutal method to stop pieces that bears the disadvantage of tying the agressor.

tied, pinned A piece that cuts off another piece is tied by that piece in that if it is moved, it gives way for the cut off piece. In other words, a piece is tied if it is pinned by an enemy piece. A piece that forces another to remain where it is is said to pin that piece, which is thus tied.

This ranking can already serve as a rule-of-fist where and how to move. But of course, pieces need to be coordinated, which brings us to the topic of space.

2.4. Area

Pentagame is a very spacial game. A player with a large and interconnected territory has an advantage over a player with a fractured territory and isolated pieces.

At any time Alice can reach only a subset of all lines and nodes. This sum of all paths available to her we call her territory. The development of pieces falls into one with the development of space, which is of course of paramount importance.

The first and easiest thing to spot is that a piece 'lost' on a line has just two ways to go, while a piece on a node has four. From this alone follows a general advantage of moving onto a node over moving onto a stop on a line.

The second important thing to observe is that swapping never affects territory, while replacing very much does. Moving into the centre occupies a formerly unoccupied stop. Unfortunately, in replace moves played from the starting position this is compensated in Alice leaving a stop to Bob behind.

Of course, not just area matters, but distance to the goals. This has to do with time, so we will return to that concept (of 'temperature') later.

The ensemble of lines and nodes, thus the entire Pentagame board, is heavily symmetric. Nevertheless, it comprises of an inside (the central pentagon), an outside (the ring) and the connecting lines (the star). Everywhere, movements can happen clockwise, or counter-clockwise.

The space is configured by two actions: the movement of pieces, and the placement of blocks. Since a block placement happens on one of 20 lines (or, if available, on empty nodes), one has more options to consider in blocking than in moving. Of course, a player will set her blocks so to limit the other player's space and to shorten or diminish the value of her paths.

Let us for now look a little closer at the total space available, and see about development independent of counter-play.

2.5. Development and Predictability

Development of one's party is actually a strategic rather than a tactical recommendation, because a well-developed party typically offers more options than advancing with just one piece.

Prior to all development, a piece on its origin must traverse at least three different nodes to reach its goal. It has six such paths. It is worth contemplating Fig. 2.4, which shows just three of these (the other three being symmetrical). Consider the last part of each of these paths: two of three are 'short', as they are on the central pentagon; just one is 'long', approaching the goal from the ring. This shows the eminent importance of the central nodes; which can give us a hint on why control of central nodes may be more important than control of ring nodes.

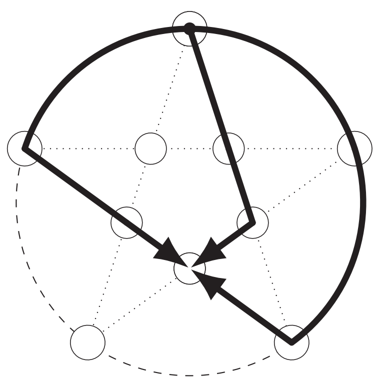


Figure 2.4.: The shortest paths from the origin
There are $2 \cdot 3 = 6$ equally short paths.

Every piece that is still on its origin can take any of these paths. Each of these path traverses two other nodes and 12 stops. The two ending on the pentagon have short paths between nodes at the end; the third a rather long last path at the end. Choosing one of them over the other is thus a gamble on the time in the game's development.

Since the opponent cannot guess which path it may take, a piece still on its origin is rather unpredictable. It is almost as if an un-developed piece has no ‘character’ yet; it has no preferred direction. This also makes un-developed pieces almost ‘invisible’.

Once the piece has progressed by one node, it has at least two nodes to traverse to reach its goal (Section 2.5). On the pentagon it has just one such path; on the ring it has two. Pieces on such stops are more predictable.

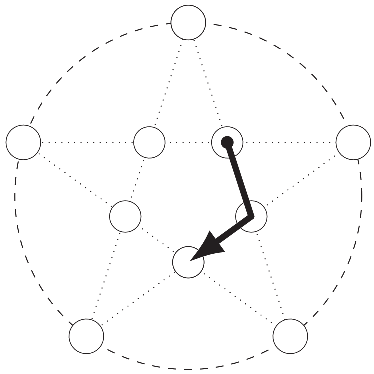


Figure 2.5.: Shortcut on the pentagon

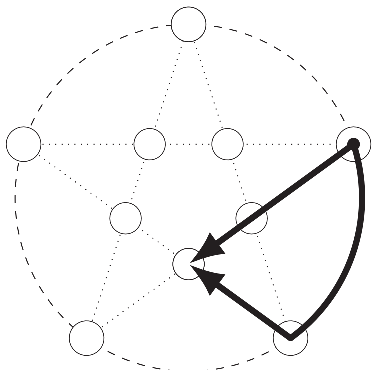


Figure 2.6.: Shortcuts on the ring

And once a piece has progressed twice on nodes, it can next of course move out straight in just one direction.

Thus the advancement of pieces produces commitment and makes predictable, which allows for sharp defensive counter-play. Pieces further back are generally less predictable.

From this follows that it is advisable to consider and advance all pieces in an even development, rather than pacing forward with an individual.

Because two pieces suffice to win, one may focus on just three chosen ones. Pieces remaining will be uncommitted, and thus turn out to be 'jokers'; for which threat they often get 'bricked in'.

Pieces on the ring have slightly better choices in paths than those on the pentagon. Thus development on the ring may be advantageous.

However, pieces on the pentagon claim 'neutral' nodes, throw blocks, dominate the centre, cut off many paths, and open up connections. An aggressive move in that is offers both advancement and restriction.

In practice, both strategies—to develop on the ring, or to dominate the centre—can be successful, because not only connections, but also blocks matter.

2.6. Time, tempo and speed

Pentagame is a race game: who manages to reach the goals first wins. Clearly, how much closer any move brings pieces to their goals is a measure of the value of that move; the more a move reduces the distance to the goals, the better. Time in a sequential games is, of course, measured in moves or plies. Tempo is the number of plies necessary to get from one position to another.

Every player tries to reach the goals as quickly as possible, with less moves than the other. For Alice it is all about needing less moves to reach three goals than Bob. She will play so that she gains tempo relative to him; she will always try to pursue the *hottest* move available to her.

We know that Pentagame is a relatively quick game with few moves ending the game. Just how many moves are at our disposal as we play? Is there a lower limit—the minimum duration of a game—and an upper limit, the maximum number of moves? Is there an average? These are difficult questions, to which we will return.

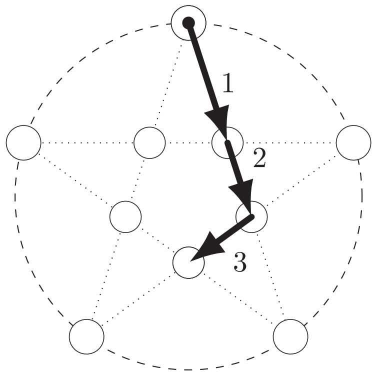


Figure 2.7.: Shortest path *unblocked:* min. 3 moves

Let us look at the *minimum* duration. Consider Alice's goal to bring three pieces out. Take a look at the board, it is fairly obvious: without any counter-play Alice will need at least 3 movements to get her first piece out. Fig. 2.7 shows an example route over the central pentagon; convince yourself that she could rather move over the rim but would also need 3 moves.

Because Alice needs these three moves for her first piece out, and at least two more moves for her two further pieces out, she will need at least 5 moves to win—*in any case.* This means that it is impossible to win Pentagame with less than five moves. We have found the lower bound for the duration of a game!

- Five moves (per player; five *plies* per player) is the absolute theoretical minimum duration of any single game.

To find out whether there is a *maximum* duration, or whether the game always ends at all, is more challenging. For the first, we will now give an estimate; the second we will prove in Chapter 11.

For now imagine a game where Alice is trying to win but Bob, rather than making any effort to win himself, does nothing but try

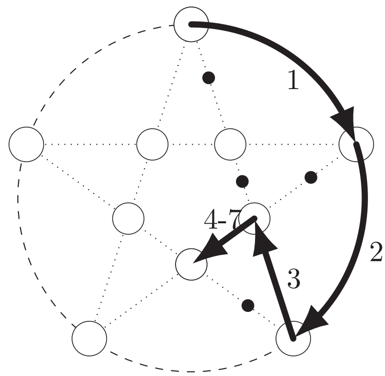


Figure 2.8.: Shortest path when blocked: min. 7 (!) moves

to hinder Alice from winning at all. Alice makes a move, Bob sets a block, and so forth. Let him only use black blocks. (We will drop this assumption in Chapter 11).

Alice quickly finds the shortest way to move out her first piece under opposition. It is drawn in Fig. 2.8. It will take Alice no more than 3 moves to reach the crossing that neighbours her goal; from there she will need another 4 moves to reach it, always advancing one stop and each time replacing a block.

- Against 'perfect opposition' the shortest way to a goal starts with a *swap* on the ring (!);
- it takes 7 movements.

The example shows nicely how ordinarily a piece can circumvent or replace any block. Blocks never freeze a piece, but only slow it down. This gives us an estimate for the likely longest possible duration:

Again Alice needs not just one, but three pieces out to win. If malicious Bob tries to slow her down as much as he possibly can, she will likely not win with less than $3 \cdot 7 = 21$ moves.

- 21 moves is a rough estimate for the longest meaningful game.

Depending on how played, games may of course be shorter or longer; where sharp play between equal players produce the longest games. But as long as at least one player is rational, the game will always end (Chapter 11). So there is limited time to sort oneself out in this game.

Then again it is clear that a move that Alice does *not* yet make may be an option for her one move later—in particular in the first number of moves, where the board is relatively unstructured still.

2.7. Potential, Temperature, Advantage

Every piece is somehow distant from its goal, and this distance can be measured in a variety of ways; there seems not to be a perfect model to measure this distance. The distance of a piece to a goal could be named (and is named in the literature) *temperature*, *weight* or *potential*; the further a piece is away from its goal, the higher its temperature, weight or potential. We will call it temperature.

Thus every piece has a temperature, the position of all pieces of one player has a temperature, and the game as such has a temperature. It starts at a maximum value and diminishes to zero: the game is said to 'cool down'.

The term 'temperature', although common, is slightly misleading, since temperature in thermodynamics is linked to entropy. But the initial game state is highly ordered, and so is the end state; maximum entropy should be somewhere in the middle. In games, 'temperature' is a measure for the attractiveness of a ply. Some games may even have sub-zero temperature; this is when a player would prefer not to move at all.

First thing to observe is that the higher the temperature difference a piece runs though when it is moved, the better, since every player tries to reduce potential.

Measuring temperature is difficult since there are many paths. Do they all add up?—A piece on a straight line just stops away from its goal certainly has only one way to go; but how about pieces that yet have many options to go?

It seems often more simple to ‘calculate’ or estimate *relative* temperature rather than absolute temperature. Every player has a piece of each colour. The question whether Alice’s or Bob’s piece (of a colour) is more advanced is comparatively simple to answer. And pieces cannot overtake each other on the same path, but rather block each other in that respect.

It follows that to play so that each piece of a colour sits in front of the concurring piece of the opponent is a good strategy. Thus when Alice advances in one direction, Bob will be advised to choose a different path for his piece of the same colour.

2.8. Options

We have so far acted as if the board consists of only nodes and lines, abstracting from the stops in between, which makes *some* sense *sometimes*. We shall now briefly but sharply look at the actual number of options available at any ply.

First we need to store in our memory that the initial position is quite different from any later position in two aspects:

1. there are multiple pieces on the corner stops
2. pieces have not yet advanced on any path.

The first quality makes any replace or ordinary move *away* from a corner distinct to any later move since later any such move necessarily *vacates* a move.

The second quality means that all options ‘left or right’ are equally valid. This will change, since a piece that has ‘chosen’ a certain path will ‘prefer’ that now shorter path over all other ones. This does not mean that its total options need to diminish, but if you rank them, there will be a clear(er) hierarchy of how preferable (i.e. short) they are.

Closing this bracket let us now answer the question: how many options does one have at least (infimum) and at maximum (supremum)?

The lower bound (infimum) computes as follows. In the very worst case—a case that will not appear in practice—all Alice's five pieces are next to one another on a line like sardines in a can, cornered by enemy pieces. Then she has only 4 swapping options and two regular moves in which she would swap with the enemy. That means 6 options.

$$F_{inf} = 6 \tag{2.1}$$

The upper bound (supremum) computes as follows. Imagine a situation where it is Alice's turn, and for some miraculous reason all pieces are positioned so that she can truly reach all stops N without being blocked anywhere. Then she has the largest possible number of options. They are listed in Table 2.3, where N_f is the number of *free* stops, and that is for two players $N_f = N - 15$.

Table 2.3.: Options for Alice

with N_f empty stops

option	count		
move onto empty stop	$5 \cdot N_f$		
beat block, replace block	$5 \cdot 5 \cdot N_f$	$+$	25
reach goal, place grey block	$5 \cdot N_f$	$+$	5
swap with self			10
swap with opponent			$5 \cdot 5$
$\sum$	$35 \cdot N_f$	$+$	65

If we treat all stops on the board individually, we have $N = 100$ and $N_f = 85$; the result for $N_f = 85$ is

$$F_{100,sup} = 3040 \tag{2.2}$$

Alice can never have less than 6 options and she can never have more than 3040 **options per ply.** [1]

[1] If Bob has already 2 pieces out, we would get an even higher value. The

A middle value can be gathered if we only account for nodes and lines in the calculation, thus setting $N = 30$. Then

$$F_{30} = 35 \cdot 15 + 65 = 590 \tag{2.3}$$

which is still an impressive number. But yes, most of the time Alice will have less options. The actual number of *reasonable and available* moves must be a fraction of this. But even if only a tenth are available, she still has quite a handful of options.

Note that Alice may have the options to swap and to move onto unoccupied spaces, the largest contribution to her number of options comes from the many options to place blocks.

It takes no wonder then that the potential setting of blocks is often and rightly perceived as the most difficult, challenging, and important part of a move. This makes sense: whenever it is my move, I have only a limited number of the interesting swap and replace moves; but any replace move gives me a plethora of options to position a block.

We now have some boundaries for the number of options per ply F. We note that this factor F will not be constant over the course of the game.

This leads us to see different phases in the game play, which we will turn toward now.

2.9. Phases

Game usually have an opening, an opening phase, a mid-game, and end-game and an end result, and Pentagame is no exception; Table 2.4. Game play differs between the different parts for a number of reasons. Thus below we will dedicate individual chapters to each of these phases.

The opening Only at the very beginning are two pieces (or more) on all the corner stops, and none anywhere else. This means that any

actual value must be slightly larger still, since upon beating a block one more stop is free (when the piece is 'in the air').

Table 2.4.: Phases of game play (n players)

	phase	initial event	moves past	moves left
I.	opening	swap/replace	0	$\lesssim 20$
II.	early game		2	$\lesssim 20$
III.	mid-game	1st piece out	≥ 3	$\lesssim 17$
		all blocks moved	$\geq (5/n)$	$\lesssim 20-(5/n)$
IV.	end-game	have 2 out	≥ 4	$\lesssim 7$
		have 3 out	≥ 5	≤ 1
V.	end	outcome	≥ 5	0

move from such a node (occupied by multiple pieces) is different to all later moves, classifying as ‘opening type move’: moves from a shared origin.

The situation at the beginning of any game of Pentagame differs from later position significantly:

1. all pieces are at the opposite position from their goals;
2. left and right are symmetrical (axial symmetry);
3. all five pieces are equal (radial symmetry);
4. multiple pieces share one stop.

Most of these qualities vanish after just one or two plies.

The opening phase begins with the first move which has already been covered above; it ends once the symmetry has vanished. Notably, the first action of Alice does *not* destroy symmetry. Thus the opening really consists of the first *two* moves (or four plies). Because of the importance of this we will analyse this phae in more depth below Chapter 3

The early game The opening is followed by the early game, which usually lasts roughly until either a first piece has been moved out or until all five black blocks have been removed.

The mid-game The mid game is marked by the fact that there is already someone in the lead. The leader has less pieces at her disposition than the enemies; and that the game may turn one way or the other. The mid-game is the product of the opening. After a balanced opening everything can happen in this phase; after an unbalanced opening, this is the phase of surprises and despair. It should still be unclear who will win and who will loose. Certainly this is the most vivid phase where game complexity is at its maximum. This is the prime realm of counter-play, strategy and tactics, at which we will look in Chapter 5.

The end-game The end game begins when one player has already two pieces out. Thus in this stage the end becomes foreseeable, which gravely changes player's considerations. Since the question of who wins seems almost settled, the number of points each player makes gains weight. We will look at this stage in Chapter 10.

The end The end result, scoring and what to make out of it is then covered from Chapter 12 onward.

The different game phases are marked by events that signal changes in how the game develops (Table 2.4). For all these events we know how many moves must have past (at least) to make them possible; for some we can estimate how many more moves may be left. Of course the delineations between the various phases is not 'clear-cut' but conventional. What is important here is that *there is* time—not every single move is decisive. Indeed, often very crucial moves can be prepared beforehand, be postponed, be built up as a threat. After all, like in any game, the total end result is what finally counts.

3. Openings

The first couple of moves are decisive for the way the game develops. We will now investigate the various openings.

There are just two good opening moves: swap or replace. So this does not help us much. But regularly the answer of Bob to Alice's first ply does not hinder her to make a particular second ply, because if he blocks her to the left, she can do the same ply to the right and vice versa. In Chess, this is called *transposition* of moves: the same positions appears, but the plies leading towards it are 'swapped in time'.

Thus, it makes sense to look at the first two moves of Alice *independently* of what Bob does. This is something that makes sense in practice, too, since if Alice and Bob are versed players, she may open with *two* plies and then have Bob answer with *two* plies as well, after which the game proceeds as usual.

Openings can be grouped by Alice's first two moves as follows:

Type 1, SxS Alice swaps twice: **double swap openings;**

Type 2, SxR Alice swaps, then replaces: **mixed openings**;

Type 3, RxS Alice replaces, then swaps: **mixed openings**;

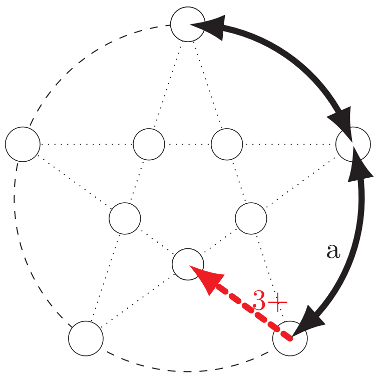
Type 4, RxR Alice replaces twice: **double replace openings**.

We shall now look at each of these families of two-ply openings. Then we will collect the results and find out just how many of these exist. Finally, we will look at how these combine when both Alice and Bob make two plies each.

3.1. Type 1: SxS double swap

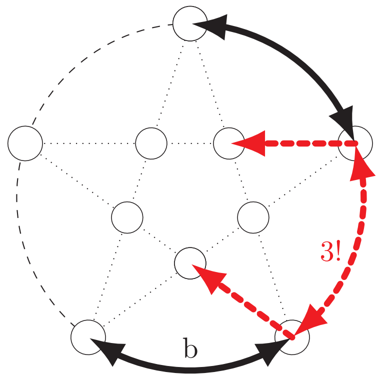
Openings with double swaps are worth studying in particular. There are two types of such double swap openings.

Figure 3.1.: Opening with two swaps



'Running Rabbit'

Figure (3.2) SxS type 1a: adjacent



'Fork Opening'

Figure (3.3) SxS type 1b: distant

Type 1a The Adjacent type. This we dub 'Running Rabbit', since usually the white rabbit starts (Fig. 3.2).

Type 1b The Distant type. This we dub 'The Fork', since it creates a major threat (Fig. 3.3).

The swap opening maintains the axial symmetry, which only vanishes after Bob's move. This means that Alice can, if she wishes, still make any second swap move that she likes.

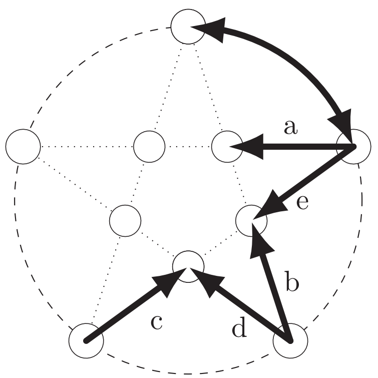
This means that Type 1 openings can always be played. If a player intends this opening, she may as well open with both plies at once, to which Bob can choose two answering plies; then the game proceeds regularly.

3.2. Type 2 and 3: mixed openings

Alice can decide to do one swap and one replace move as her first two moves. This we call the family of 'mixed openings'. Let us have a look whether it matters what she does first.

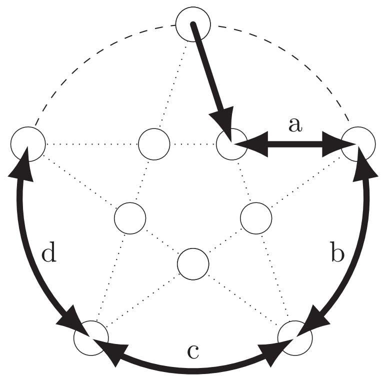
Figure 3.4.: Mixed openings

Figure (3.5) Type 2a-2e: SxR



First swap, then replace

Figure (3.6) Type 3a-3d: RxS



First replace, then swap

Fig. 3.5 shows the options that Alice has when she starts with a swap. These are labeled $a - e$. All the other options are symmetrical. We see that whichever line Bob will block, Alice can always make her desired replace move on the opposite site. (He could though hinder her from some of them by positioning his own piece, so not all these second moves may actually be available for Alice.)

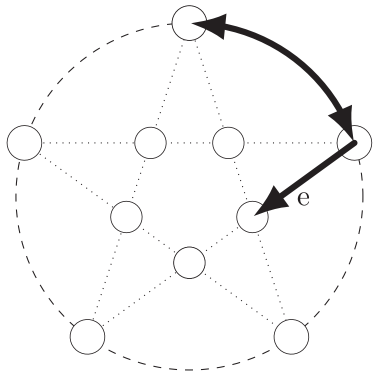
Fig. 3.6 shows the options that Alice has when the starts with a replace move. Here again, all her second mixed options are labelled $a - d$.

When comparing Fig. 3.5 and Fig. 3.6 we can observe that they are fundamentally symmetrical. If Alice decides to make the double move $1, a$ from Fig. 3.5 her pieces end up in the exact same position as if she does $1, a$ from Fig. 3.6. Thus, the type 2b opening results in the same position as the type 3b opening, etc.

But there is one noticable exceptions inthe symmetry in Fig. 3.5 and Fig. 3.6: **e** appears only once.

But we can also observe that Alice has one option in Fig. 3.5 that she does not have in Fig. 3.6. This one option she only has in Fig. 3.5 is the move 1.**e**. She swaps on the ring and then takes one of the

Figure 3.7.: Type 2e: Little Rabbit



swapped pieces to enter the polygon in the direction of the its goal. Type 2e has no equivalent type 3 opening. This particular opening we dub '**little rabbit**' (Fig. 3.7). The 'little rabbit' is of course a popular opening.

Since Alice has one more option when she swaps first and then replaces, *it is better for Alice to start with a swap rather than with a replace.*

3.3. Type 4: double replace

Alice can of course also start with a double replacement opening. There are fundamentally two types of this, which again can be subdivided.

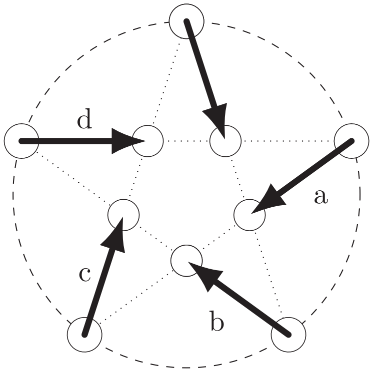
The first family of these openings is the one where she moves two pieces from the ring into the centre first (Fig. 3.9 and Fig. 3.10).

The game is chiral; there are two directions: clockwise, and counterclockwise. Alice can move both pieces in the same chiral direction (Fig. 3.9), or in opposite chiral directions (Fig. 3.10).

Choosing opposite chiral directions is not advisable, since these pieces will need to swap their positions to reach their goal. This can

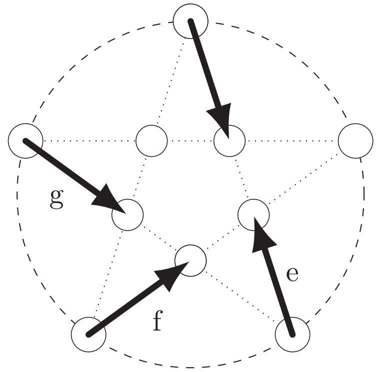
Figure 3.8.: Double replace openings

Figure (3.9) Type 4a-4d: RxR



Uni-directional
'clockwork'-openings.
Possible cLockwise or
counter-clockwise

Figure (3.10) Type 4e-4g: RxR



Bi-directional openings. This
tends to lead to self-conflict

easily be blocked; and even when it is not blocked, it constitutes the loss of one ply.[1]

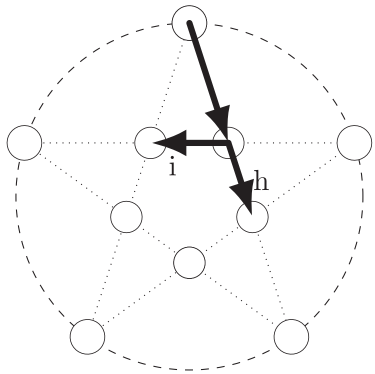
So the entire family of uni-directional openings (type 4a-d) seem better than type 4e-g. This family of openings (Fig. 3.9) has Alice move two pieces into the board, both either clockwise or counter-clockwise. This kind of openings are nicknamed 'Clockwork'-openings.

Finally, Alice can also simply move the same piece twice (Fig. 3.11). One possibility (*i*) changes direction; this 'zig-zag' ply seems not very strong. Instead, it seems natural to move the piece straight on further towards its goal (*h*). This move (*h*) is dubbed 'Rocket' since it is a launch along a straight line.

Since this is apparently the only good option where the same piece moves twice, it is quite foreseeable, and very popular with aggressive new players. Its drawback is that the overall development is neglected for the price of one fast advancing piece.

[1] There is also one less option in the bi-directional family; the missing one equals a mixed opening, since a swap happens.

Figure 3.11.: Type 4h,i: RxR



Two replaces with the same piece.
4*h* is called ‘Rocket’ and 4*i* ‘zig-zag’

3.4. Summary

Table 3.1 shows that Alice’s options for her first two plies are somehow limited to 20 distinctive openings (not taking blocks into account); but of these, only 12 seem recommendable from a tactical point of view. Of these, 10 include a replace move as *second* ply.

This provides us with a scheme to analyse the opening moves in more depth (or to train them). Indeed, players can use the above to trains certain openings or analyse their individual strengths and weaknesses.

3.5. Concluding remarks

The above has listed the most important double openings and discussed their tactical value briefly. We see some very strong openings, and we see others the value of which is uncertain. Even those we have here ruled out as ‘likely not recommendable’ we have ruled out for tactical reasons, since they include contradictory moves. Nevertheless, depending on counter-play, they may have their value. But even with very simplified assumptions we see that there are more than a thousand different possible positions after just two plies per player; a number that would be vastly larger if we took placement of blocks into account as well.

Thus, the opening theory of Pentagame cannot conclude to recommend individual perfect openings, but only serve to demonstrate that the multitude of options at the beginning of a game leads to complicatons beyond what is possible to grasp by pure heuristic, at least for human players.

4. The two player game

4.1. Peculiarities

The two player setup is quite challenging, since there are no coalitions. The board is also quite empty, thus each game relatively short. Nevertheless, two player games are still full of surprises. We will dedicate some more space to this case, since it is the most general case. In particular we will investigate openings and the general form of the game.

Alice and Bob each have 5 pieces. That plus the 5 impediments (black blocks) makes 15 pieces. Thus there are 5 more pieces than there are nodes on the board. When all nodes are occupied by pieces, 5 of the 20 lines will also be blocked. The proportion of pieces to nodes and lines is almost constant through the game.

In this setup a player's piece is a very strong block for the other player due to the Ko-rule: whenever I get removed by a swap move I can simply reverse and retake my initial position. This is different to settings with more than one rival, where another rival may hider me from doing so by e.g. placing a block. Consequently in two player setups it does not make much sense to 'cover one's back' with blocks, while in multi-player setups it does.

A common phenomenon in two-play games is what beginners often

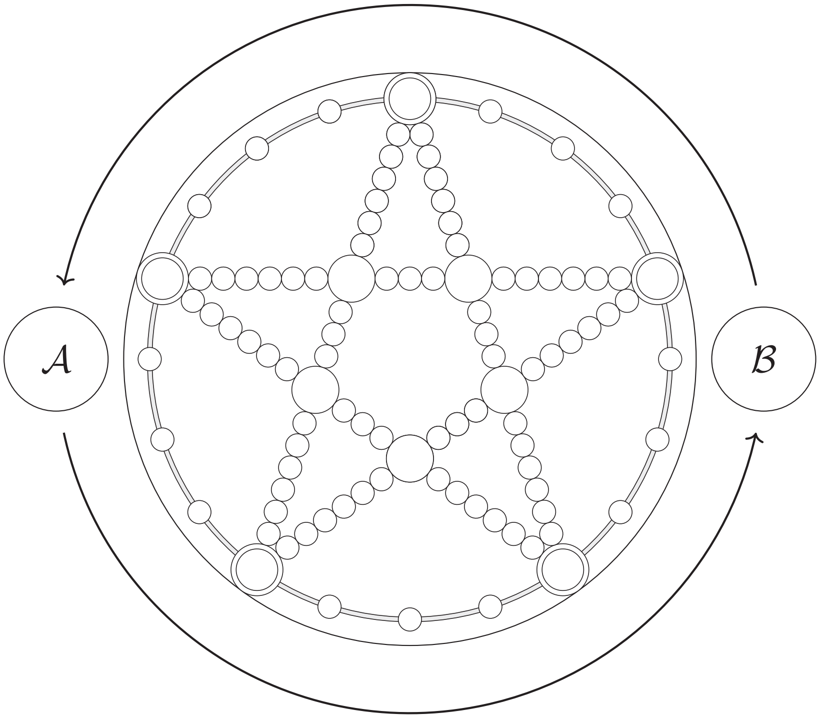
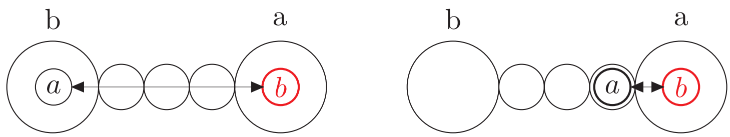


Figure 4.1.: Two players $\mathcal{A}$ and $\mathcal{B}$

perceive as 'standoff', but which has an elegant solution. Here two pieces face each other on the pentagon, each on the goal of the other one. If any player swaps, both reach their goals (Example 1).

Example 1: Standoff resolved
Two pieces of $\mathcal{A}$ and $\mathcal{B}$ face each other on the pentagon so that a swap would bring both pieces to their goals. This is not an impasse: instead of moving out $(\mathrm{a} \leftrightarrows \mathrm{b})$? Alice just moves close to b with $(\mathrm{a}, 3, \mathrm{b})$!



(a) a and b can reach their goals (b) b is tied

Figure 4.2.: A 'standoff' resolved

which brings her into the lead. Indeed any of the the two player to first move in this manner takes the lead (Fig. 4.2).

Example (1) shows two things:

- Often situations perceived as symmetric are in fact not so due to the order in which players move.
- It is advisable to well consider moves close to nodes rather than just moves onto nodes. In a two-player setting a piece on the stop adjacent to its goal is safe and as good as out, but still functions as a very strong block.

4.2. Opening Responses

Above, we have only looked at the first two plies of Alice only. Within the first two moves, Bob also moves twice. Alice (as shown) has about 12 nice options, and how many options does Bob have?

Alice has 'only' 12 options, because many of her actual options are perfectly symmetrical on an empty board; we have not distinguished whether she begins with white or blue or whatever.

For Bob the situation will be different, since the board is not empty anymore once Alice has made her plies. It does make a difference now where exactly he swaps or replaces. Thus, he has many more than just 12 options; for him, all the isomeres are relevant. This means: where Alice has just 5 'SxR' double ply options, Bob has 50. This is listed in Table 4.1.

We can find the number of possible two ply openings by multiplication of Alice's with Bob's options: There must be

$$(2+5+5)\cdot(15+50+50)=1380 \tag{4.1}$$

reasonable combinations of two plies of Alice with two plies of Bob.[1]

So a book discussing all 'reasonable' double opening combinations of both Alice and Bob would require 1380 entries; or if you would like to study the two opening moves of this game by brute force, you would have to look at 1380 combinations. That will probably take a while.

But there is even more. The number 1380 includes only positions of Alice's and Bob's pieces. How about the blocks?

Alice has a certain number of double openings that produce no, one, or two blocks. Bob has such combinations as well. Table 4.2 lists both Alice's and Bob's options.

Thus from Table 4.2 we can calculate the number of possible positions after *four* plies *including* the placement of blocks. It computes to

$$250\binom{20}{4}+500\binom{20}{3}+425\binom{20}{2}+175\binom{20}{1}+30\binom{20}{0}=1{,}865{,}530 \tag{4.2}$$

Thus 1,865,530 different such positions can appear after two such double-moves of two plies each—quite a handful. Note that $250+500+425+175+30=1380$. We must subtract the two symmetrical cases of Fork-Fork; so 1,865,528. And this does not consider individual stops, but just lines.

Note that this does also not take moves into account we have ruled out sub-optimal, but which are nevertheless possible, and maybe even strategically reasonable, like those of type 3. In other words, this is a *minimum* value.

Also note that this is the number of possible positions, some of

[1] Actually two less, because in the cases where both Alice and Bob play the fork, Bob has only 3 rather than 5 distinct options due to persisting symmetry, so 1378.

which may be reachable by different paths. For instance, a line may have been blocked in any of the four plies.

A book listing and analysing the qualities of every such decent two move opening would be a vast volume; any discussion of Pentagame openings will have to be restricted to very few cases.

This shows that the 'combinatorial explosion' of Pentagame is pronounced. Notice that the number above is reached after four plies. Because

$$\sqrt[4]{1865530} = 36.9573\ldots \tag{4.3}$$

it can be reached after four plies with ~ 37 reasonable options on average, or with an Average Branching Factor of ~ 37. If we take into account that Alice starts with just 12 reasonable moves, we can narrow it down to

$$F^* = \sqrt[3]{\frac{1865530}{12}} = 53.7700\ldots \tag{4.4}$$

Average Branching Factor after the opening ply. This is close to the Average Branching Factor that canonically given for Chess, where it is estimated to be around 36 [5]. We will return to this in the complexity section.

4.3. Results, Points and Score

The game ends when one of the players has led three of her pieces out: this is the 'winning rule'. The last round is usually played out, so all player have had the exact same number of movements.

Pentagame can of course end with a win for $\mathcal{A}$ ($\mathcal{A} \succ \mathcal{B}$), a draw ($\mathcal{A} \sim \mathcal{B}$) or a win for $\mathcal{B}$ ($\mathcal{A} \prec \mathcal{B}$). The winner always wins with three pieces out and hence three points, since three pieces out wins the game; the second may draw with three or loose with two, one, or nil pieces out, hence points. Table 4.3 shows what I call the 'result matrix'; you notice the empty slots which are due to to the winning rule.

Writing down who won and who lost is of course the simplest way to record score, and it is of course possible to assign points to each event similar to what happens in other games, be it Chess (with 1 for a win, 0 for a looser, and $\frac{1}{2} : \frac{1}{2}$ for a draw) or Football (3 for a win, 0 for a loos, and 1 for a draw).

Most commonly, however, we simply note the number of points every player has made; this needs to be recorded anyway, since all other score assignments project this. Thus the winner always scores 3 points, and the other 3 to nil points.

The trouble with this simple scoring is that each player's result is actually independent of, or not taking into account, the other player's. This poses no problem on a single evening without swapping of opponents. With other words: as long as you don't plan on comparing the results to other players who have not participated, don't worry.

But if you do play a tournament of any kind, players could collude and try to play 3:3 so that they both would score well in comparison to the players on the other tables.

There is a simple remedy, which is to fix the total amount of points achievable in any game; as a result, players then compete for a give size 'cake' or price. This is called 'zero sum scoring', and it renders the game a 'zero sum game'.

4.4. Zero sum scoring

A game is called a 'zero sum game' when any player's win always is another player's loss. In other words, the 'cake' players compete for is constant; 'constant sum' is an equivalent expression.

Zero sum scoring awards that forcing eg. a 3:0 is harder than playing a draw; while just noting points would award the winner just three points in each case. Thus zero sum scoring only has advantages, the only disadvantage is that it is ever so slightly more complicated.

The simplest rule to make Pentagame (or actually any two-player score game) a zero sum game is:

From the points of any player, subtract the points of the other (Table 4.4).

If you or your fellow players feel uncomfortable with negative numbers, you can of course add 3 to each player's points and score, so that each game bears 6 points in total rather than 0. We call this 'constant sum' scoring, In a two player setting there are only 4 possible outcomes, and for each regular result exists exactly one zero-sum equivalent. So you can simply look the corresponding tournament scores up in Table 4.5.

We will use a similar technique in the three-player setting, where there are many more possible distributions of points.

More on tournaments can be found further down in the text (Page 219).

4.5. Game duration

The duration of a two player game can be estimated. Reasonable players take all possible answers into account when choosing a move. This takes time. Assume we consider our own 20 options and the 20 possible answers, thus 40 moves in total, each for a second; then 40 seconds suffice for each move. In practice we limit the time per player to one minute. Then a two-player game then takes about 20 minutes. Notice that playing time grows somehow quadratically with player number.

5. Strategy

5.1. Rationality and uncertainty

Every player will give her best—tactically, but also strategically. While with some practice it is somehow possible to find the ply with the best tactical value, finding the best strategical move is hard.

Games can include an element of chance or be devoid of chance; there can be hidden information or there can be perfect information. When there is no chance or hidden information only skill and experience determine the outcome [11].

One of the results of game theory is that for each game with perfect information exists a perfect strategy. This is known as ZERMELO'S theorem [102] [67] [66]. But interesting for us human players are exactly those games with perfect information that have a simple enough rule set but are so complex that we cannot see the perfect strategy, but only try our best; in other words, finding the perfect move is a hard problem, and probably intractable.[1]

In such games, we neither know the exact results of our own actions, nor can we truly predict what our rivals will do. Uncertainty takes the place of chance.[2] Players no longer act perfectly rational, but with 'bounded rationality'.

Thus even a game without chance in a technical sense can be a game full of surprises, if only it is complex enough. And because both players, as they play in a non-perfect manner, face unforseeable positions, moves and strategies, the game never wears out.

---

[1] The problem of finding the ideal strategy can be proven to be in the complexety class PPAD (Christos PAPADIMITROU).

[2] With other words, we do not quite know what chance exactly is. Since PASCALE, FERMAT and LAPLACE established probability theory, two views on chance have been proposed: the Bayesian view that chance merely means that we don't understand the reasons for the variables; and the Frequentist view, who actually believe in chance. This issue is largely unsettled.—Investment theory distinguishes between risk and uncertainty so that risk is a random variable with known properties (Frank KNIGHT 1921).

So there are always new positions, and there are always many options to be considered. But what is a reasonable move? Game theory tells us that in games of perfect information, there must actually only be one perfect move (or a number of equally perfect, and thereby symmetric moves). But then there must be a number of *almost perfect* moves. And then there must be a number of cleary silly moves (Fig. 5.1).

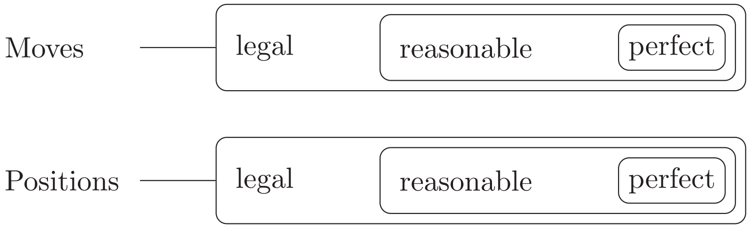


Figure 5.1.: legal $\supseteq$ reasonable $\supseteq$ perfect

In other words, the quest for the player is to narrow her choice from all available to only a few, and of these few choose the best.

This is naturally a process that has a number of steps. Firstly, there may be moves which are, under all circumstance and at first sight, self-destructive or plainly nonsensical. So a player will usually begin with some heuristic that rules out certain moves and recommends others.

Since we don't want to go to deep into this far reaching topic, we must stress at this point that in what comes we will restrict our investigation to a subclass of moves. The reason is that we want to know how many options exist at a given point, but that we are not interested in what are silly moves.

An obvious heuristic is: even if allowed, a piecs should not move away from its goal, but towards it. This rules out half of all legal moves, in general, as unreasonable. But we also see immediately: some of these backwards moves may actually be reasonable if a situation occurs where the total of pieces can advance better. So something tactically silly may strategically be recommendable. But such plies are rare; they are usually called 'sacrifies'.

Nobody really knows where exactly the boundaries are between legal, reasonable, and perfect moves, unless everybody knows what is perfect. If we know what is perfect, only perfect moves are reasonable. Thus the set of reasonable moves (for a player) is an approximation of the set of perfect moves within the space of the legal moves.

Strategical moves take into account that the opponent is also unaware of the perfect set of moves. A strategic move is a move that is reasonable in respect to the anticipated judgment of the opponent on what is reasonable.

Thus all the values we will get in what comes will be relative to our assumptions about what constitutes reasonable moves. We can often give reason about our evaluation; but nevertheless, there remains a degree of subjectivity.

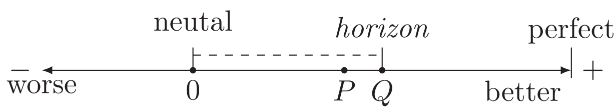


Figure 5.2.: Ranking options

Assume that the player ranks all her options them from worse to better as in Fig. 5.2. Assume further that the perfect move is unkown. She will have a certain *horizon*; some even better moves off her horizon she will not consider (dashed in Fig. 5.2). So option Q is the best option she can find; even though it may not be the (unknown) perfect move, it is clearly a good move, and by all her reasoning a tactically better move than option P. However, $Q \succ P$ tactically does not necessarily imply that $Q \succ P$ strategically.

the tactial value of a ply takes into account the situation of the individual piece after the ply.

the strategical value of a ply takes into account the situation of all pieces (i.e. the position of the game) of all possible future moves.

Strategy is 'hard', in that it is often impossible to take all possible futures into account. So it practically reduces to some heuristic strategy or bounded strategy. Rather than evaluating all possible futures, only some general principles are applied.

There are many futures; and these depend on the decisions Alice and Bob make. Since in games Alice and Bob work against each other, typically any move that gives Alice more and Bob less options will be good for Alice in the longer run.

Recall the difference between tactics and strategy: tactical behaviour seeks the best action in a given situation; strategy takes the future(s) into account.

We have seen that it is humanely impossible to calculate all future ramifications of the game; the solving perfect strategy is humanely intractable due to 'combinatorial explosion'. If we could do that, we would always know the perfect move. This is known as ZERMELO's theorem [102]. But since we cannot, we must try to find the best move applying strategic thinking, resulting in strategy rules.[3]

Strategy requires predictions of the opponent's actions. As already observed by THUCYDIDES good strategists usually assume the opponent to make the best move *we* would find, thus assuming the worst (for us); this prunes the decision tree. Then we weight our options against the possible answers [16].

Our predictions of the other will prove correct when we consider moves deeper than our challenger; we then play with superior strategy, since for every *answer* we know a *move.* In the terms of Fig. 5.2 we play with a larger horizon.

When our challenger never makes moves that truly surprise us, we realise that we consider more contingent options than he does. Thus superior strategy results from better foresight. And our foresight improves with our understanding of the game mechanics.

[3]Bounded rationality results in time efficient heuristics rather than time consuming complete analysis. This explains how time limits improve many complex games.

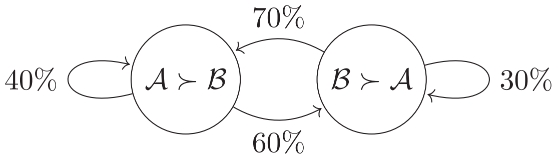


Figure 5.3.: A state diagram of a two-player game
(numbers arbitrary)

5.2. Situation assessment

In games, the past is what is not possible anymore, the present is what is possible now, and the future is what is not possible yet [97].

Effectively from my > 20 options in the present situation I must chose the one where the future looks brightest. From the set of all that will be possible I can healthily subtract that what must be: this I can take as a given; the rest is that what is *contingent:* that what can be, but need not be: the true future.

To all these future contingencies I give weights of probability and weights of desirability. Then my preference of states informs my preference of action.

At any moment one of the players will be in the lead; if Alice leads, we write '$\mathcal{A} \succ \mathcal{B}$' and if Bob leads '$\mathcal{B} \succ \mathcal{A}$'. The game will somehow oscillate between the two states '$\mathcal{A} \succ \mathcal{B}$' and '$\mathcal{B} \succ \mathcal{A}$'. In fact, many situations bear the potential for the game to 'tip' in one direction or the other; but the 'odds' of it tipping in each directions may differ.

Fig. 5.3 shows this in form of a 'Markov-chain'. It shows the two stages $\mathcal{A} \succ \mathcal{B}$' and '$\mathcal{B} \succ \mathcal{A}$'. The numbers on the arrows represent (arbitrarily chosen) probabilities that the game may change from one state into the other. They can be interpreted as 'percentage of moves leading in that direction'.

Considering this we see intuitively that the strategy '*gain more good options and diminish the options of the other*' is the dominant strategy, since this means influencing these 'odds' of winning in our favour.

As a result, players will try a (sub-)strategy and make tactical moves. Such a strategy could be to attack over the ring, to generally choose the opposite direction of the other player, and so forth. Such strategies may be temporary; the switching from one strategy to another (so called mixed strategy) within a game can be a (meta-)strategy.

At the core of strategy lies an evaluation process of possible positions like this:

1. Evaluate the situation,
2. evaluate the situations achievable,
3. of these go for the best.

Any move is exactly as good as the improvement it brings to the situation. We cannot here develop a full or finite algorithm to evaluate situations. But any such algorithm must count the number of moves necessary to win and the number of valuable options at hand, because the player leads (‘$\succ$’) who scores better in these two categories:

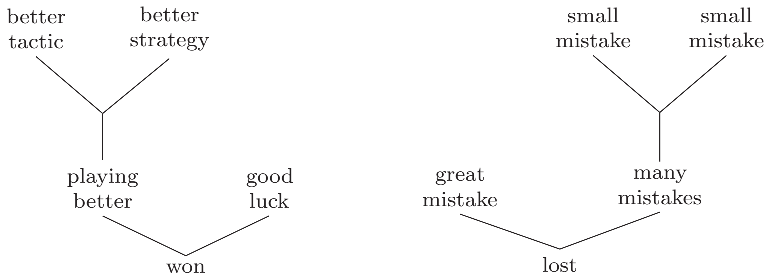
1. Number of moves / distance to win (if all else stays equal)
2. Number of options / room for manoeuvre (in case things change).

- Prefer the situation that scores best in these two categories.

However, the fact that the game is highly asymmetric and that pieces can be connected in multiple ways gives rise to many combinations; and it is combinations that defy simple heuristics.

The only possible strategic (long term, by definition) advantage is the adherence to superior principles; the simple formula for this is to **better the situation.**

With this in mind we can condense some guiding principles for strategic play. Some are so general that they may apply to any game that shares the principles of Pentagame (Table 5.1). These can be used to further assess a given situation, leading to more specific recommendations (Table 5.2).



(a) Why have I won?

(b) Why have I lost?

Figure 6.1.: Hindsight
Positive and negative hindsight differ!

6. Hindsight

So far, we have looked at foresight: *What can follow if I do this or that?*—But we can also reason backwards: *why have I won, why have I lost?*—This is hindsight. Hindsight asks for the reason of a result, the cause of an effect. It typically seeks to single out an individual cause. In this, hindsight clearly differs from foresight. While we perceive the future to be open, we tend to perceive the past as a single line of necessarily connected events. This is a fault.

Not only differs hindsight from foresight, but hindsight is also biased. We attribute causes to wins in a different way than how we attribute causes to losing. We tend to attribute a win to skill, but a loss to chance. We tend to believe that winning is caused by overall superior play, while we tend to attribute loosing to individual events. Compare Fig. 6.1, where the positive tree (to the left) attributes the outcome to general qualities of the player and the negative tree to the right attributes the outcome to individual events.

That what was 'unforseen' easily morphs in to that what is 'unhindseen'. Hence, there is another bias: while we readily admit not to know the future development of a game, we tend to assume that once the game is over we know everything. Thus, while game theorists would say that there is 'perfect information', there still are many open questions we can ask in the interpretation of this information; this 'perfect information' does not include information about the consideration, strategy and tactics of our opponent. These are the real causes, not the individual moves.

Because a better player foresees more, she will experience less 'luck' than a worse player who will often be caught by surprise. Her moves will be 'stronger' as she chooses from a wider range of options.

The concept of 'strengths' of moves: in the ideal case, there is the absolutely best move. But in the face of uncertainty, moves can only be ranked in an order of (perceived) strength. There is no way to know if there is the truly best, or strongest, move, since this verdict always depends on the horizon.

Freedom is the insight into necessity [27]: as I gather of which strengths moves I must do to keep the upper hand, and I have an insight of my chances to win a higher gamble, I can try to push the limits of my fortune. Thus the strong player enjoys freedom, while the weaker player is bound to feel constrained.

Many times we oversee the complexity of what was happening: there must have been multiple causes, or the 'cause' may not clearly be identifiable. This may be the factor 'luck': the unforeseeable.

The reason *for* something is why I have decided to do it, while the causes *of* something is another concept altogether. We often focus more on the negative than on the positive and ask more often for the causes of failure than for the reasons of success. We may be biased in how we think about winning and losing just like we are biased between foresight and hindsight.

In hindsight we tend to see mistakes, while at the time we actually made the best move we could think of. 'Mistake' is thus a problematic term: there is no opposite of 'mistake'. Mistakes make the game.

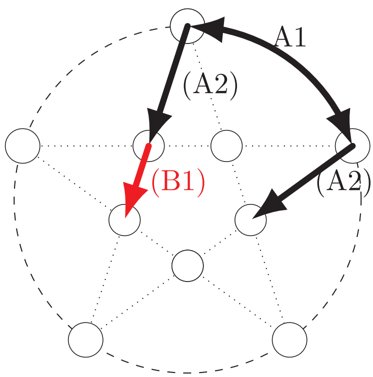
And finally it is not yet clear what advantage hindsight has over foresight, if any. Looking forward is certainly more rewarding and in many ways more natural. Among the most noble lessons that can be learned from games is to never regret, but to play again.

7. Combinations

Individual moves can be assessed by how close they bring a piece to their goals etc.; but often, moves work relative to other moves. The result are tactical moves or combinations. The following is a collection of some very basic ones in respect to the two-player case. Since every multi-player game can be separated into multiple two player games, all of these work (mutatis mutandis) in multi player games as well. There are of course countless other combinations, many of which are too complex to be covered here.

C1. Create a fork

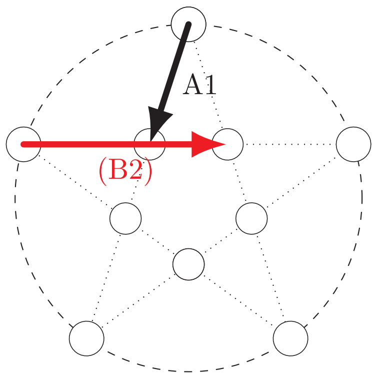
To focus on blocking a piece that is almost out may sometimes be a waste of effort. In particular it may be good to allow your opponent to reach a goal when you can make good for that immediately afterwards. Bob is already very close to his goal and threatens to move out ($\mathcal{B} \succ \mathcal{A}$):



Alice could now either replace $\rightarrow \times$ to block him, but prefers to rather swap $\leftrightarrow$, thus creating a fork. Wherever Bob now sets his grey block, Alice will also get a piece out, which equalises the game again ($\mathcal{A} \sim \mathcal{B}$). Given that $\leftrightarrow$ moves two pieces, this is likely the better move.

C2. Cutting off

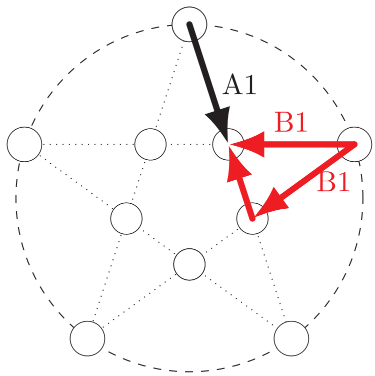
Sometimes the best way to stop your opponent is to use your own piece as a very effective block; this we call 'cutting off' the opponent. We have already encountered an example on page 139, but there are many other cases where this is handy, even though it ties the piece used:



Bob is threatening to move out via a rather long path crossing a node. Rather than playing replace → × at an unfavourable place, Alice makes a simple move (→) onto the open node to block Bob and gain more options.

C3. Anticadabra

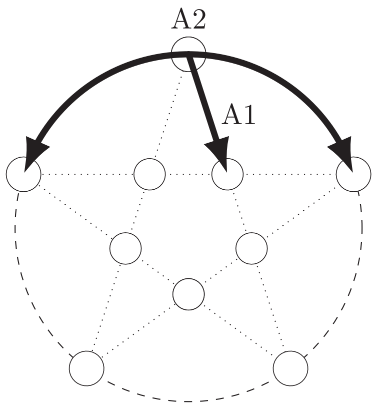
The move know as 'Abracadabra' is when a player can reach a goal and when that goal is occupied by a black block. Abracadabra bestows the lucky opportunity to place *two* blocks in one go: a black and a grey one. This is obviously very detrimental to the other player. So when an Abracadabra looms, it may be an interesting option to beat the black block on the goal in question: then the challenger can still move out gaining a grey block, but can at least not also replace a black one:



Bob has a fork and can now reach his goal in any case. On his goal is still a black block, so not only will he move out, he will gain the right to position *two* blocks—the movement known as 'abracadabra'. Alice cannot stop Bob from reaching his goal; but she can hinder the 'abracadabra' by simply eating that block on Bob's goal. Hence 'anticadabra'.

C4. Make room

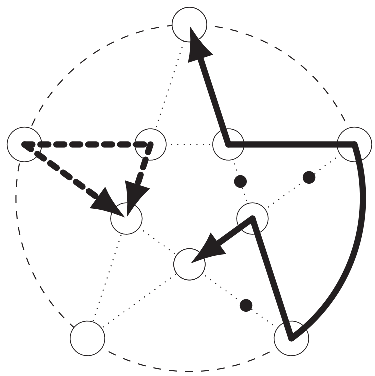
Pieces do not only move *toward* a stop, but in every movement they also move *away* from a stop, which typically opens up a path. This usually bears risks, since the other player will be the first who can use that new path. Nevertheless, sometimes one can also move out of one's own way, thus liberating other pieces and gaining new territory. Our next example shows how you can gain room for manoeuvre by stepping out of your own (and hopefully only your own!) way:



Alice moves out of her own way freeing another one of her pieces. When prepared, done on purpose and in particular when creating a fork at the same time this is strong. But vacating a node opens up that path also for the rival, so attention should be payed to the effects.

C5. Some like it hot

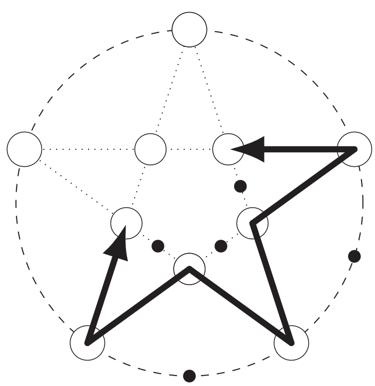
When evaluating space and distance players must consider the likelihood that these may vanish or persist. A piece very close to its goal is almost as good as in. A piece that is further away but has the opportunity to move along a long path should then be preferred for gain of speed and since that opportunity many vanish. Figure shows an example. The more a move improves the situation the more it is 'hot'. Hotter moves should be preferred to less hot moves, as the following illustrates:



The fork from (E) to (b) will persist into the next round when Bob plays → ×, while the long path from (A) to (a) will not. a(A ↔ a) is ‘hotter’ than b(E → b). The shorter move can still be made after the long move, which is ‘hotter’.

C6. In Your Face

This is an example from a real game. I was in the lead, having already moved a piece to its goal, and two other pieces of mine were just steps away from their goals. My opponent (NIKO, a Go player) was all over the board, so much that he actually wanted to give up. "‘No"’, said I, "‘there may still be something possible. Let us look."’ And then we both saw it.



He had a white piece on the red crossing and a red piece on the white crossing, and via a twisted path managed to swap both in one move, thus moving two pieces out at once, and won the game.

Note that I was actually lucky: he could have won *four pieces out* even! And yes, that is possible!

Needless to say, it took me some days to get over that. Epic!

8. Notation

For some of what follows we need to establish a notation.

We have already been introduced to players Alice and Bob, aka first moving and second moving player. We will shorthand them as $\mathcal{A}$ and $\mathcal{B}$.

The action of a single player we call a *ply*. So a move is made off a ply by $\mathcal{A}$ and a ply by $\mathcal{B}$.

We write $\leftrightarrow$ for *swap* moves and $\rightarrow$ for *replace* moves.

With this in mind we can notate play like this (Example 2):

Example 2: A Running Rabbit

1.	$\leftrightarrow$	$\mathcal{A}$ swaps
	$\rightarrow$	$\mathcal{B}$ replaces
2.	$\leftrightarrow$	$\mathcal{A}$ swaps (and threatens to reach a goal)
	$\rightarrow$!	$\mathcal{B}$ is forced to replace to stop $\mathcal{A}$!

This shorthand already describes what type of action players take, and it suffices in some contexts.

We also need to be able to tell *where* the pieces are and whereto they move. Since every position can be expressed relative to some nodes, we begin by labelling these.

The obvious thing to do would be to call them by colour. But the actual colours are arbitrary, and it does not matter from which colour a game begins. Thus, we simply label the nodes A, B, C, D, E. Then each letter 'N' represents a colour (Fig. 8.1).

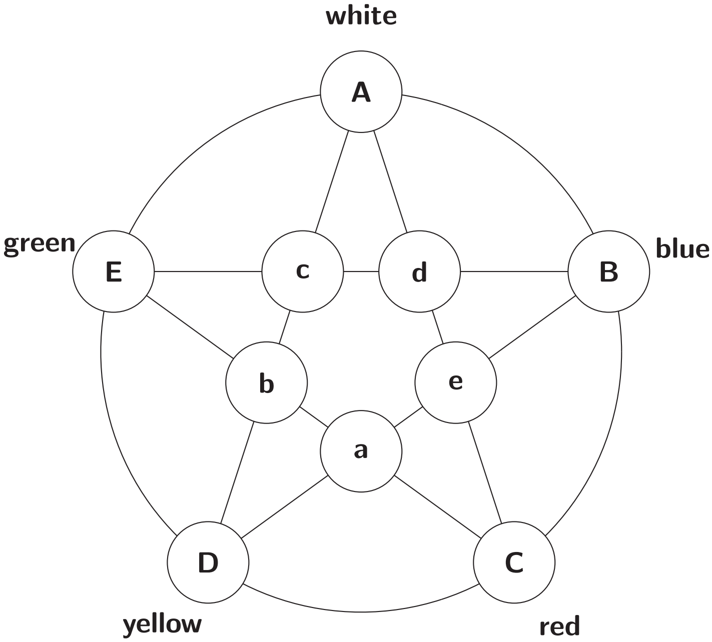


Figure 8.1.: Node label convention

A) white B) blue C) red D) yellow E) green

The advantage of this notation is that we instantly know that B is located between A and C both on the ring and on the pentagon.

By convention the game always starts at node A with piece a, played by player $\mathcal{A}$ (aka 'Alice').

To distinguish between nodes on the ring and nodes in the pentagon we mark those on ring with capital letters and those on the pentagon with small letters. This is fitting because the sizes on the ring are in fact a little larger than those on the pentagon.

Observe that A is neighbour to B, c, d and E. Every node has four neighbours of the other four colours, always.

When a piece simply runs from one node to another, we can note that as $(\mathrm{N}_1 \to \mathrm{N}_2)$. We enclose movements in round brackets.

Most of the time it will be clear which piece moves; nevertheless, we can also note them with small letters in italics, so that a is the white piece etc. A swap on the ring can then be written e.g. $a, b\,(\mathrm{A} \leftrightarrow \mathrm{B})$.

The placement of a block be put into square brackets [...], and we use '×' for 'replacing a block'. When we are not really concerned with the exact position of the block we simply write [E × D] for 'block the line from E to D'.

If we must note a precise stop on a line, we note its number. For example [E-1-D] blocks stop 1 between E and D. (E-2-D) is a movement to the second stop on the arc between E and D (a middle stop). Note that there are always three stops on the paths on the ring or pentagon and there are always six stops on the rays of the star, e.g. (A-3-B) and (a-3-b) but (A-6-c).

This notation has some noteworthy properties. Each node has four paths leading to the four other colours. Thus we can also talk of directions. The directions $N \pm 1$ moves on ring or pentagon (A → B, B → A, b → a etc.), the directions $N \pm 2$ along the star ($A \to c$, $c \to A$ etc.). A number of moves yields a sum; for instance a piece can move $+2 - 1 - 1 = 0$ and reach its goal. An alternative notation notes origin N, direction d and number of stops, for example:

$$\begin{bmatrix} \mathrm{N} \\ d \\ s \end{bmatrix} := \begin{bmatrix} \mathrm{A} \\ 2 \\ 7 \end{bmatrix} \equiv \begin{bmatrix} \mathrm{A+2=c} \\ 0 \\ 0 \end{bmatrix} \Rrightarrow (\mathrm{A} \to \mathrm{c})$$

Because the board is in symmetry this notation must produce redundancies. We can save us from many of them by the simple convention to always begin at node 'A' (hence the rule 'the white rabbit starts'). We also convene that it always starts clockwise.

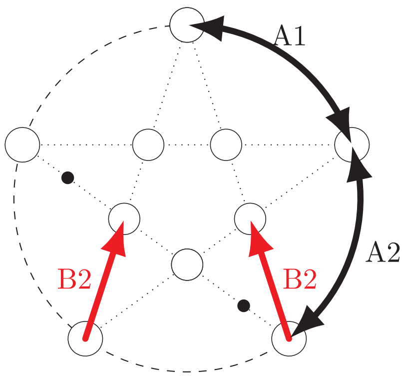
We can expand this convention in the following manner:

- Always label the nodes in the direction of the first move, which is usually $(A \leftrightarrow B)$ or $(A \rightarrow d)$.

This is the convention followed in this book. It is summarised in Table 8.1.

We can now write down Example 2 more precisely as Example 3.

Example 3: A Running rabbit refuted



Bob counters Alice's ring moves invading the centre.

1.	$\mathcal{A}$:	$a, b\,(\mathrm{A} \leftrightarrow \mathrm{B})$		
	$\mathcal{B}$:	$d\,(\mathrm{D} \rightarrow \mathrm{b})$	&	$[\mathrm{E} \times \mathrm{b}]$
2.	$\mathcal{A}$:	$a, c\,(\mathrm{B} \leftrightarrow \mathrm{C})$		
	$\mathcal{B}$:	$c\,(\mathrm{C} \rightarrow \mathrm{e})$	&	$[\mathrm{C} \times \mathrm{a}]!$

By moving to (e) Bob now occupies the path attractive for Alice's piece a, cutting it off from its goal.

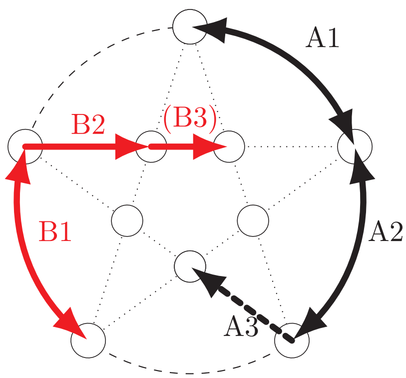
---

9. Refutations

Clearly, some openings are better than others. Every now and then a particular opening becomes fashionable among players who suspect it to be the best opening of them all, and it is sometimes hard to refute this claim. Nevertheless, there are some more or less obvious, and some more or less strong, refutations to all standard openings. We will now show some possible refutuation to those openings that create an immediate threat: the running rabbit; the fork; and the rocket.

9.1. The Running Rabbit

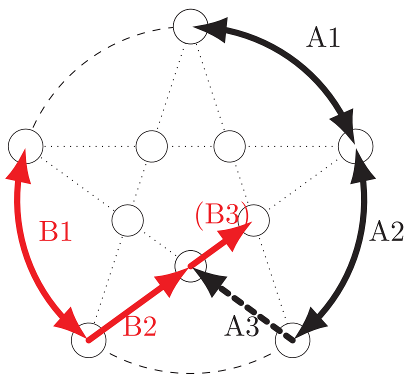
Figure 9.1.: Clockwise answer to Running Rabbit



The Running Rabbit opening is where Alice swaps twice right at the beginning such that one of her pieces threatens to reach its goal next. It is strong, popular and playable.

There are a number of possible refutations or good answers for Bob. He can counter Alice's clockwise movement by moving clockwise

Figure 9.2.: Counter-clockwise answer to Running Rabbit



himself and preparing a counter-threat (Fig. 9.1):

1.	$\mathcal{A}$:	$a, b(\mathrm{A} \leftrightarrow \mathrm{B})$				
	$\mathcal{B}$:	$e, d(\mathrm{E} \leftrightarrow \mathrm{D})$				
2.	$\mathcal{A}$:	$a, c(\mathrm{B} \leftrightarrow \mathrm{C})$			$\triangle$	$a(\mathrm{a}+)!$
	$\mathcal{B}$:	$d(\mathrm{E} \rightarrow \mathrm{c})$	&	$[\mathrm{C} \times \mathrm{a}]!$	$\triangle$	$d(\mathrm{d}+)!$

This cuts off $A : b, c$. The drawback is that $B :\ d(\mathrm{c})$ is also tied by $A :\ b(\mathrm{A})$. Wiser be to move counter-clockwise as in Fig. 9.2:

2.	$\mathcal{B}$:	$e(\mathrm{D} \rightarrow \mathrm{a})!$	&	$[\mathrm{C} \times \mathrm{a}]!$	$\triangle$	$e(\mathrm{e}+)!$

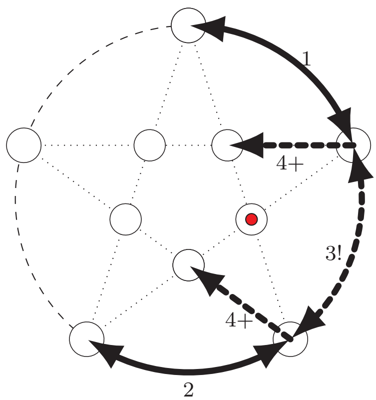
This example shows an interesting quality: pieces can move in two directions on the board only: clockwise or counter-clockwise. The game dynamics possess chirality. When my opponent runs clockwise, I may consider counter-clockwise movement.

9.2. The Fork

The second class of the 'double swap' openings is called 'Fork'. This is the opening where Alice swaps twice, but instead of adjacently, she makes a distant swap. The opening bears its name because it poses a serious threat. It is a true 'opening trap' for inexperienced players, a strong opening which is hard to refute; Fig. 9.3.

1.	$\mathcal{A}:$	$a, b(\mathrm{A} \leftrightarrow (\mathrm{B})$
	$\ldots$	$\ldots$
2.	$\mathcal{A}:$	$c, d(\mathrm{C} \leftrightarrow \mathrm{D})$
	$\ldots$	$\ldots$

Figure 9.3.: The Fork



Alice swaps twice on the ring on two distant arches. Her threat is to 'connect' these two swaps by swapping in between them. That would move two of her pieces close to their goals (Fig. 9.3). To counter this, Bob needs two coordinated moves; therein lays the challenge.

Fig. 9.3 shows the threat: if Alice can do 3!, she can either do of the moves 4+, here dashed lines.

There is nothing that Bob can do in his first move to hinder Alice from playing the Fork. Thus it can be played as a two-move opening: Alice swaps twice, and Bob has two answer moves; then the game proceeds as usual.

Early in the game these two goals may still be occupied by black blocks, so the fork often simultaneously threatens an Abracadabra *twice*. This is of course a grave threat.

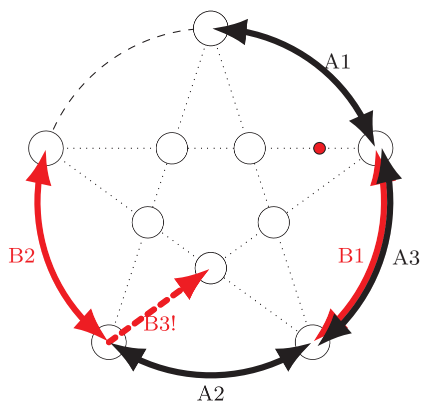
Bob is compelled to block the line [B × C]. So one of the two moves he can make in response to this opening should really be a replace move; one that allows him to close the arc between B and C. A 'running rabbit' seems reasonable, since this simultaneously builds up a counter-threat.

If Bob has missed the initial opportunity to refute the Fork but allows Alice to swap [B ↔ C], his next best bet is to at least hinder the Abracadabra. He should beat one of the challenged blocks (a, d) and replace it so to hinder the other to be beaten. That way Alice will still get a piece out, but at least the Abracadabra was avoided.

Alice can hold up the pressure by moving to e, which threatens the same swap with similar effect a move later; thus, occupying stop e is clever for Bob.

Here comes another refutation. Rather than occupying e, Bob 'mirrors' Alice and creates a fork himself (Fig. 9.4). The gist here is that even though Alice can now complete her fork, Bob catches up afterwards; so the game remains in balance.

Figure 9.4.: The Fork refuted



$$\begin{array}{lll} 1. & \mathcal{A}: & a, b(\text{A} \leftrightarrow (\text{B}) \\ & \mathcal{B}: & b, c(\text{B} \leftrightarrow \text{C}) \\ 2. & \mathcal{A}: & c, d(\text{C} \leftrightarrow \text{D}) \qquad\qquad \triangle \quad a, d(\text{B} \leftrightarrow \text{C})! \end{array}$$

Here comes the surprisingly bold counter:

$$\mathcal{B}: \quad d, e(\text{D} \leftrightarrow \text{E})$$

This is somehow a standoff; both players threatend to build a massive fork. But that is not all that is to it at this point. ‘What could possibly go wrong’, thniks Alice, ‘once I reach a goal my abracadabry will bring me into the lead for sure.’ But now picture this.

$$\begin{array}{lll} 3. & \mathcal{A}: & a, d(\text{B} \leftrightarrow \text{C}) \\ & \mathcal{B}: & e(\text{D} \rightarrow \text{a})! \quad \& \quad [\text{B} \times \text{d}] \\ 4. & \mathcal{A}: & a, d(\text{B} \leftrightarrow \text{C})? \end{array}$$

Not only does Bob hinder Alice from getting an abracadabra. If she now decides to move out with her white piece $a(\text{C} \rightarrow \text{a})$ Bobs green piece will be unstoppable afterwards; and his blue piece is also threatening to move out. Abracadabra for Bob!

This is a fine example that even a strong opening like the Fork can be refuted.

If there is a moral of this story than it is that rather than falling into defensive mode, a bold counter-attack should also be considered.

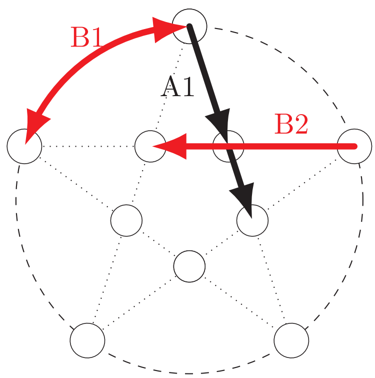
9.3. The Rocket

Since there are so many replace games, we will discuss a single example in more depth, a not unusual two-player (A, B) of type *replace, swap, replace.* This is one of the openings on the branch of Fig. 3.3. We will discuss 3 moves consisting of 6 movements in Example (9.3). A starts with white a.

1.	$\mathcal{A}$:	$a\,(\mathrm{A} \rightarrow \mathrm{d})$	&	[. . .]
	$\mathcal{B}$:	$e, a\,(\mathrm{E} \leftrightarrow \mathrm{A})$		
2.	$\mathcal{A}$:	$a\,(\mathrm{d} \rightarrow \mathrm{e})$	&	[A × d]
	$\mathcal{B}$:	$b\,(\mathrm{B} \rightarrow \mathrm{c})$	&	[e-1-a]

If Alice now continues her berserk attack and advances her white piece again, Bob can move out with either his blue or his red piece.

Figure 9.5.: A Rocket refuted



Some lessons can be learned:

- Advancing one single piece and neglecting development is not advantageous.
- Focus instead not on the pieces that are already near their goals but seek where space can be won.
- Consider not only where to you move, but also which lines you open up when moving away.

10. Endgames

10.1. Last round effects

The rule that the last round always gets played out can lead to interesting opportunities. Sometimes it is advantageous for Bob to not delay the end, but to focus on his own possible points by creating a ‘final fork’, as the following Examples will show.

A final fork can be built using a *swap*, as Example (4) shows:

Example 4: A final swap
It is Bob to move, Alice has started. It is the end game and by now crystal clear that Alice will win: it is only a matter of time that her third piece will arrive at its goal. Bob has two choices:

Option one: Bob continues to *replace* black blocks, slowing Alice down; his suffering will be longer, but he will not gain any more points by doing this;

Option two: Bob can play *swap,* bringing two of his pieces on free paths at once; now Alice, since Bob is not blocking her, can win immediately. But since she gains only one block upon moving out she cannot hinder Bob to make another point in the final move.

A final fork can also be achieved using a *replace*-move, as Example (5) illustrates:

Example 5: A final replace
It is Bob to move. Alice has started and is about to win: she has two pieces on free paths, so either of them will become her winning piece in the next move. Depending on which one she moves a path will be freed:

- if she wins with piece $a_{\mathcal{A}}$, she frees Bobs piece $b_{\mathcal{B}}$;
- if she wins with piece $c_{\mathcal{A}}$, she doesn’t.

Since Bob sees this he will seek a *replace* move with another piece $d_{\mathcal{B}}$ so that (1) his piece $d_{\mathcal{B}}$ is free to move out and (2) he can use the block to block Alice's $c_{\mathcal{A}}$. Once Alice wins with $a_{\mathcal{A}}$ Bob will be left with a fork, which Alice's grey block won't hinder. This secures Bob the final point.

10.2. The score space

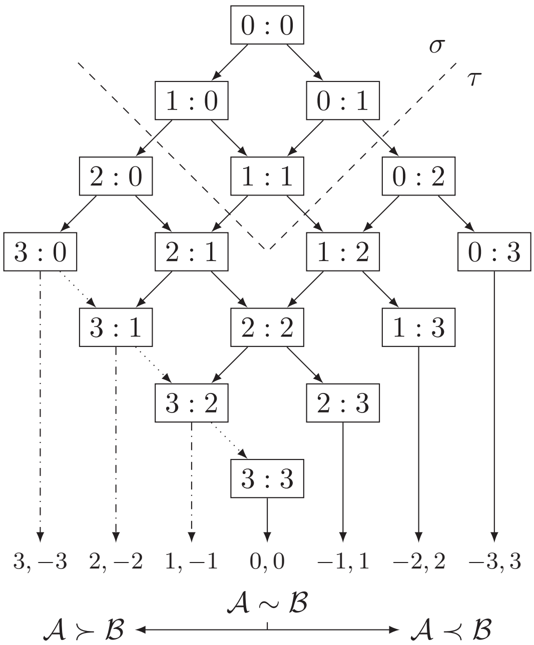
Let us consider a somehow reduced state space of Pentagame, where we only look at the state of the scores; let us call this the 'score space'. It is shown in Fig. 10.1. The game begins with a score of $0 : 0$ and can then, depending on what happens on the board, develop towards $\mathcal{A} \succ \mathcal{B}$ or $\mathcal{B} \succ \mathcal{A}$, in our graph: to the left or to the right. From each state two other states are possible, as indicated by the black lines.

Towards the end of the game, when one player has brought 3 pieces out, things look slightly differently since there is only one ply left (if at all) before the game ends, indicated by the dash dot and dotted lines. On the bottom we have indicated the results when zero sum scoring is used, which we assume is the case.

We can see that the quality of the game in respect to the use of strategy and tactics changes at certain points. Consider the dashed line in Fig. 10.1. At any of the nodes above it is not clear who will win the game. From any node above the dashed line, paths lead down that intersect. This means that in this realm it may make sense to sacrifice; it may make sense for Alice to let a piece of Bob reach a goal, but then catch up later for a higher win. Thus above the dashed line is a realm where strategy prevails; hence the σ.

Below the dashed line no such intersecting path can appear; in no case will it be better for Alice to 'go right' than to go left. The same applies for Bob. Thus the strategy 'allow one piece of the opponent to leave the board for a better position and a final higher score' is not valid in this realm. In this realm all that counts is to score; thus, tactical moves prevail; hence the τ. The game mechanics are quite diffferent once the score stands at $2 : x$ or $x : 2$.

Figure 10.1.: The score space



Firstly, the not leading opponent must block the lead player in every move, or the game will end; she can now only then make swap moves if the lead player gives her time for that. Thus for the non-leading player it is hard to make a swap in this phase; instead, the non-leading player will have to go for blocks often, or even use her own pieces as blocks. This the leader can counter by constantly threatening to go out, and by positioning blocks out of reach of the non-leader.

Secondly, unless she manages to block with one of her own pieces, she will only be able to slow the leader down; this means that *the end comes into sight*, and—depending on player's horizons—the end my even become predictable. Some say the game is effectively over at this point, but of course this depends on the positions of the remaining pieces. It may well be that these have so many options, and are so far from their goals, that the end is still more than a handful of moves away; which means that still many unforseen developments can happen.

The quality of the game changes again once a player has brought three pieces out, of course, because by force of the rules this is the very last move; there may or may not be another ply, depending on who started the game. Very often the end game phase between the crossing of the dashed line and the arrival at the last move is about whether or not Bob will manage to get a piece out in his last move, that is: whether the last event is along the dash dot lines or the dotted-dash dot-lines in Fig. 10.1. We will return to this subject in Section 10.3.

The representation of the score space chosen above in Fig. 10.1 suspiciously resembles a Galton board (also known as quincunx or 'Bean machine'). This is a device where balls are dropped onto a similarly arranged array of nails. It can be proven that the distribution of the balls falling out of such a board takes the shape of a bell curve, i.e. that the results are normally distributed. Because you could overlay Pascale's triangle over the (upper half) of Fig. 10.1, you would expect a binominal distribution. You would expect a draw to happen in slightly more than 31.25% of the cases.

Psychologically, there are two peculiar moments in any end game:

One is the moment where all give up. This is when it becomes obvious to all players how the game *must* end unless players make serious mistakes; how it *will* end when everyone plays perfectly.

But before that comes the moment when it becomes clear who may win, but is not yet clear how many points everyone else will get. This is often the most intense part of the game. In multi-player settings this is where alliances are challenged, and everything may take unexpected turns.

In games with more than two parties end game player behaviour depends much on how you score. A player may realise that she has no chance to win anymore, but can by her behaviour influence who wins or gets how many points. This is the *kingmaker effect* that naturally occurs in all multi-player games and can only be remedied by repeated play.[1]

We can also observe that within Fig. 10.1 more arrows lead to the right than to the left due to the rule that Bob still has one move once Alice has won. This means that rather than being symmetrical, Bob has an advantage. This advantage may just compensate for the first mover advantage of Alice; or it may not. We will look at this more closely now.

10.3. Completing the last round

There is a rule that the last round always gets played out (last move rule), so that both players have the same number of plies. Firstly, it seems fair that both players have the same number of plies. Secondly, it seems fair that Bob has an extra move to make good for the fact that Alice moves first. These two arguments contradict each other. Neither with nor without this rules is it per se clear if any of the two players has a genuine advantage over the other.

Clearly, the choice between 'swap' and 'replace' in ply one is so miniscule that no large advantage can stem from it. On the other hand, to have the last ply may actually be a rather huge advantage for Bob. This is particularly clear in multi-player setups:

- Charlie may think: *I can play so that either Alice or Bob wins. If Alice wins, I make two points; if Bob wins, I make three points just like him. So I am going to play so that Bob wins and not Alice.*
- Alice must think: *I cannot decide to have either Bob or Charlie win; because if I play so that one of them does, I will not have another move, as the game will immediately be over.*

This difference can be seen in Fig. 10.1 as well; there are more arrows leading towards $\mathcal{A} \prec \lfloor$ than vice versa. One effect is that Alice

[1] To be precise it exists in any zero-sum multi-player game. Many games are not zero-sum to avoid it; the result are Eurogames, where effectively nobody can do harm to anyone else. Such games are actually independent races.

must be careful to win in such a way that does not allow Bob to draw, while Bob can be cool about Alice's position; for once he wins, the game is over.

Does this make good for Alice's advantage of the first move, or does this over-compensate? Let us look at this a bit more closely.

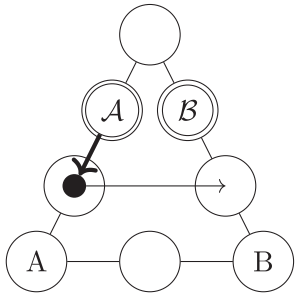


Figure 10.2.: $\mathcal{A} \sim \mathcal{B}$

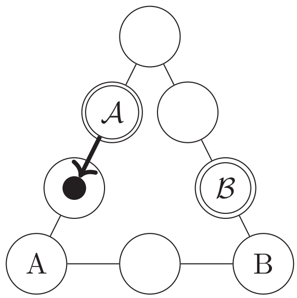
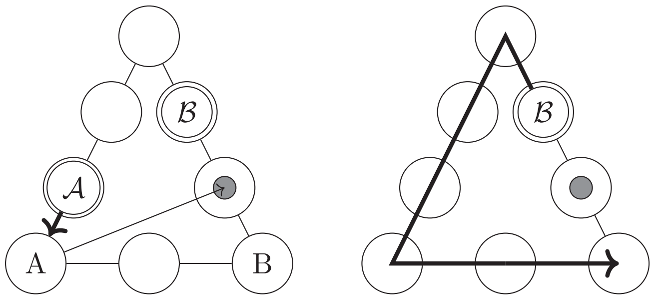


Figure 10.3.: $\mathcal{A} \prec \mathcal{B}$

Figure **??** depicts two cases schematically. In the first case, both Alice and Bob are equally far from their respective goals. Here, Alice has a first mover advantage; she can beat the block, block Bob, who can only advance close to his goal before Alice move out first. So without the last move rule Alice would simply win, which is not fair given that Alice and Bob were actually equally far from their goals. With the rule, the game ends a draw. Thus the rule here converts $\mathcal{A} \succeq \mathcal{B}$ to $\mathcal{A} \sim \mathcal{B}$.

In the second case Bob has a positional advantage. With or without the last move rule he will win; so in any case $\mathcal{A} \prec \mathcal{B}$.

But now consider the case where Alice has a positional advantage: she is one stop closer to her goal than Bob (Fig. 10.4).



(a) (b)

Figure 10.4.: $\mathcal{A} \succ \mathcal{B} \longrightarrow \mathcal{A} \sim \mathcal{B}$

In this interesting case one would expect Alice to finish with a win, but Bob can still achieve a draw that Alice cannot hinder. This is because when she moves out now she gains a grey block to stop Bob; but because her piece has left the board, another long path over the goal of Alice opens up for him. Thus here he can achieve a goal even though he had a clear positional disadvantage. This turns $\mathcal{A} \succ \mathcal{B}$ to $\mathcal{A} \sim \mathcal{B}$. This is an *additional* advantage for the second player Bob!

Thus we have strong reasons to suspect that to go second in the game is an advantage that *may over*-compensate Alice's first mover advantage. Just how large this advantage is is just as hard to estimate as to how large the first-mover advantage actually is.

To conclude: the game cannot be completely fair, even though it may seem so at first. A completely fair competition requires a match and a re-match, of course; and the last move rule does not render this general rule redundant.

Since Bob has an advantage over Alice, it makes sense to have the etiquette that the stronger player begins the game. This fits to the older etiquette that a host always begins when playing a guest.

11. Decisiveness

An *impasse* we call the situation where one player ultimately hinders the other from winning without winning herself—a forced draw with no winner. We will show that it is not a viable outcome in a two-player game, because *who can force and impasse, can also win.*

This is the proof that Pentagame fulfils THOMPSON's quality of **decisiveness.** A game has decisiveness when it always ends with one party winning.

Of course, the game can only end if at least one player wants to win, thus preferring a win over a draw or loss ($w \succ d \sim l$). Now let us also assum that (at least) one player prefers draw over a loss ($d \succ l$). This can lead to situations where a player will prefer to force a draw, as she prefers this over a loss.[1] The result could be an undecisive outcome, a permanent stale mate, or impasse.

Imagine a game where Alice tries to win ($w \succ d$) and a malicious Bob has no other objective than to stop her to finish ($w \succ d \succ l$).

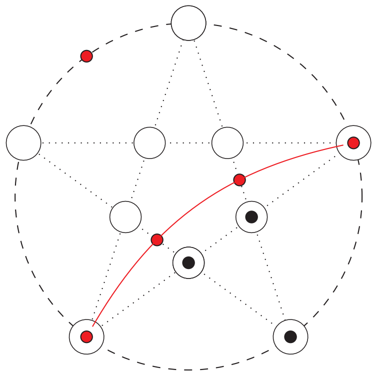
Now in Pentagame, malicious Bob can only do this by using *his pieces* to cut off Alice's pieces from reaching their goals. To cut off all 5 pieces would translate into cutting off all 5 central nodes, which is impossible because Bob has 5 pieces only, but 10 paths lead to the centre. So Alice will get something out.

Since Alice must get 3 out of 5 out to win, the critical moment is when she has just 3 remaining on the board, since this is simultaneously the latest and the easiest moment in play where she could be cut off: now Bob must cut off only 3 goals. If he can, he can force an impasse; if an impasse is achievable at all, it will be achievable now.

He can block Alice's pieces by positioning his pieces so that they divide the board into two areas: one half with Alice's pieces and the other half with her goals. For this Alice's pieces must be on one side of the board; they also must neighbour each other (be connected). Such a position is sketched in Fig. 11.1. Note that this is the strongest

[1] If both players were indifferent ($w \sim d \sim l$), pieces would move at random over the board, but would *eventually* reach a valid win or draw.

Figure 11.1.: Impasse position in a two-player



Alice (black) has to still bring out b, c and d and malicious Bob (red) cuts her off from the (empty) central nodes b, c and d.

position Alice can have (all her pieces are on nodes) in an impasse, albeit it is unlikely to occur.

- Alice's remaining three pieces must be connected to one another and be on one half of the board;
- When they are on nodes, then Bob needs at least four pieces to build such a wall;
- he also needs a fifth piece so that he can leave the wall intact; otherwise, he would have to open one of the paths due to zugzwang.

This impasse can only happen when the present score is 2:0. Thus even though this is some kind of stalemate, there is a clear leader. If the situation were to stall in this position, a referee would end the game 2:0.

But if we apply tournament scoring then the score of the impasse 2:0 would become 5:0. Obviously Alice can do no better in such an impasse.

But Bob can. He necessarily still has a free piece that can roam on his half of the board, clearing up blocks. He can also swap two of his pieces once the paths are free. This means that further or later he will have the opportunity to bring at least one of his pieces to a goal—which would also break his wall and the impasse. In the worst case Alice will then win in the next move; then the final score will be 3:1, which translates to 5:1 in tournament scoring.

And since this makes Bob better off than insisting on the impasse, he has no incentive for keeping the impasse. Instead, he would use the fact that he has paralysed Alice to make more points; at the end of which he will dissolve the impasse and simply win.

This proves that the impasse is not a stable situation if players prefer winning. And that proves that Pentagame really always, always ends with at least one party bringing three pieces to their goals. Pentagame is a decisive game. □

As a corollary we now know as well that perfect two-player play is of finite length. (There may be more than one such perfect game.)

12. Ends

Now that we know that the game always ends and also know some peculiarities of how to play the ends, we are going to look at the end *positions*. In a way this is turning the perspective on its head: rather than studying beginning, we now study ends. And we will see that not all ends (as in: end positions) are equal.

12.1. Superior and inferior wins

For ordinary scoring the actual position *on* the board becomes irrelevant once the game is over: for ordinary scoring it is only relevant how many pieces a player has moved out, and irrelevant which.

Obviously, there are—in the two player case—four types of results in number: 3:3, 3:2, 3:1 and 3:0. But these are not necessarily all the same regarding the end position on the board.

To see this, play a game and collect all winning pieces as usual in the centre. Then, clear the board of all other pieces and re-position the winning pieces on their goals. You will see that there are different patterns which can result in the same numerical end result. With other words: there is more than one way to get three, two, one or nil pieces out. This can be used as a tiebreaker, or even to impose a handicapping on players.

Here, and in the following section, we limit our investigation on the two player case. The insights easily extend to the other cases.

By definition, the winner will have 3 pieces out. Abstracting from colour, there are two arrangements: the three reached goals either form an *acute* or an *obtuse* triangle, as show in Fig. 12.1 and Fig. 12.2. Each of these occur equally often, or are equally hard to achieve. So none of them is per se superior to the other.

There are also two different ways to get two pieces out. The two pieces can either be adjacent, or they can be across (Fig. 12.3 and Fig. 12.4.).

While an acute and an obtuse win is equally likely to be achieved *per se,* acute and obtuse is *not* equally easy to achieve relative to the two pieces out beforehand. Thus a win can be *superior* or *inferior* relative to the history of the game, and a 3:3 result can nevertheless have a winner!

To see this, consider your options once you have two out. You still have three goals ‘open’. These are not all symmetrical; compare the crossed out nodes to the ‘open’ nodes in Fig. 12.3 and Fig. 12.4.

- If your first two out are *across,* you have only 1 way to win in an *obtuse* fashion, but 2 ways to build an *acute* triangle. *After* across, obtuse *is superior to* acute.

- If your first two out are *adjacent,* you have only 1 way to build an *acute* triangle, but 2 ways to win in an *obtuse* fashion. *After* adjacent, acute *is superior to* obtuse.

Fig. 12.5 shows this in form of a tree, and Table 12.1 in form of a lookup table.

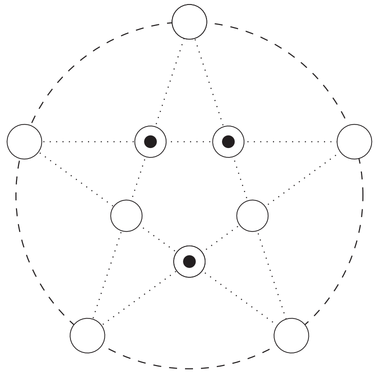


Figure 12.1.: acute

Superiority / inferiority is a quality of a win. Since Alice and Bob can both achieve superior or inferior wins, such a tie-break may, but need not appear. In other words: only paired with an inferior win a superior win will break the tie.

Since the 'likelyhood' of a superior win is half that of an inferior win, this tie break rule occurs roughly in 4 of 9 cases (Table 12.2).

12.2. End constellations

While every game begins with the one and only starting position, it ends in one of many different end positions. Clearly, there are more end positions than we can possibly gather. But the number of combinations of *winning* pieces is relatively small and can be computed as follows.

First recall that both wins (**??**) retain axial symmetry. Suppose Alice won in either the acute or the obtuse shape. How many ways are there to loose for Bob?

Suppose Bob draws with 3 out. Again, these three pieces could be arranged in one of the two shapes shown in **??**. Under axial symmetry there are 3 distinct ways these can be oriented. Thus Bob has 6 ways to draw 3:3.

Suppose Bob has 2 out. These can be adjacent, or be across. Again,

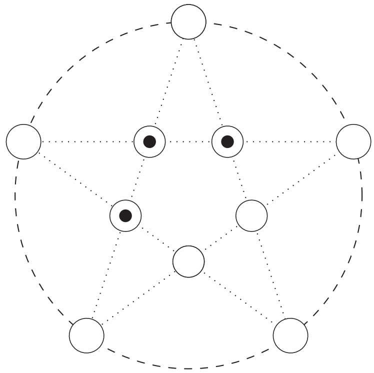


Figure 12.2.: obtuse

under axial symmetry there are 3 distinct ways these can be oriented. Thus Bob has 6 ways to loose to Alice 2:3.

Suppose Bob has only one out. Under axial symmetry there are 3 distinct ways they can be oriented. Thus Bob has 3 ways to loose relative to Alice 1:3.

And he has only one way of loosing 0:3.

Fig. 12.6 shows this in shape of a tree. We see that there are in total 16 different arrangements Bob can achieve, relative to Alice's win.

Each of these 16 'losing arrangements' (Fig. 12.6) of Bob can be combined with either an acute or an obtuse win by Alice; this simply doubles that number to 32 patterns (Table 12.3).

Either Alice wins, or Bob wins, or they tie. Ties have no 'winner' quality and can thus be counted just once. Every position where Alice wins and Bob loses has a corresponding position which looks exactly the same with inverted pieces, thus Alice losing and Bob winning. The twelve ties, in contrast, exist just once. So there are altogether 52 different end constellations.

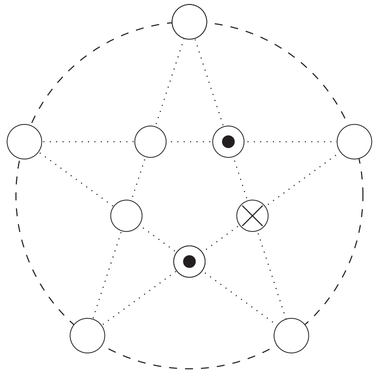


Figure 12.3.: across

12.3. The terminus

There will be some ruptures or steps where pieces leave the board, which are events that significantly reduce complexity (meshedness).

The game continues until the cutoff rule ‘three out wins the game’ is reached, which is when it break off by virtue of this rule.

Without this rule the game would continue to run until *all* pieces had reached their goals. Then ‘temperature’—the sum of the distances of all pieces to their goals—would reach zero. Then the state space would have two ‘roots’: the (extremely hot) start position, and the (absolutely cold) empty board.

But because of the cutoff rule, the state space has more than one terminus. These end branches are the the 52 end constellations known from Section 12.2, each belonging to one of the groups ‘Alice wins’, ‘draw’ or ‘Bob wins’. Thus, we know something about the right-hand side, or end, of the state space: the number of vertices leading to each of the end results. This is recalled in Table 12.4.

We note that there are less combinations for a high win (like 3:0) than there are for draws or close wins (3:3 or 3:2). This indicates how results should be distributed if Alice and Bob play equally skilled, and make equally many mistakes on average. It also shows that in

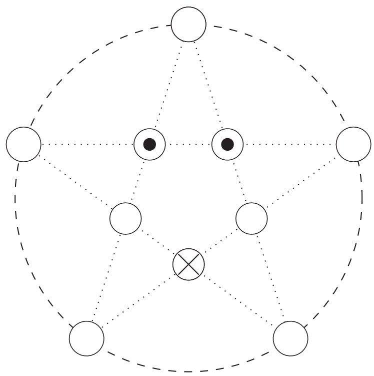


Figure 12.4.: adjacent

respect to possible constellations a 3:0 win is not three times as 'hard' or statistically likely as a draw, but much harder.

The more pieces players have moved out, the lower—by definition—the 'temperature' of the game. A draw will take more moves than a 3:0, thus these ends of the state space will be further to the right. With this in mind we can zoom into the end of the state space we call 'terminus'.

Fig. 12.7 shows the 'final frontier' within the state space that players approach from the left, 'landing' on one of the outcomes. Alice will steer portside (to the above), and Bob will try to influence the game to end in the lower half. The closer they are to a draw, the longer they will play: what seems to become a 3:0 may end up a 3:1; but a 3:1 will never turn into a 3:0. And there are more combinations for 3:3 wins than for 3:0 wins; so when there is a random factor involved, narrow results are more likely. You may want to compare this to the results from Section 10.2. Here it appears that *if* the game is a random walk, draws (3:3) will appear approximately in $12/52 \approx 23\%$ of all cases.

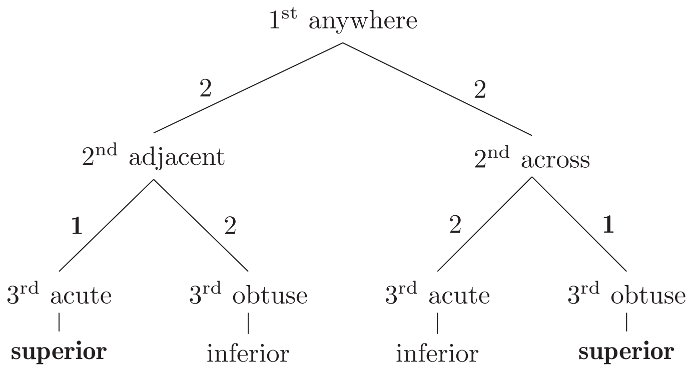


Figure 12.5.: Superior and inferior wins

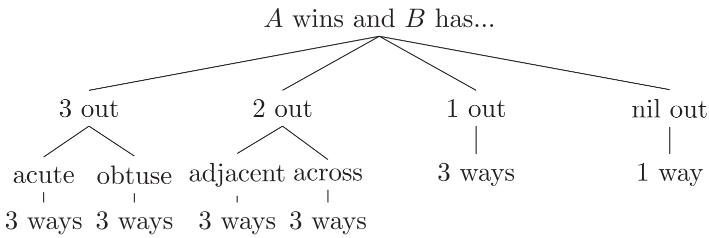


Figure 12.6.: Unique relative end positions

12.4. Summary

We have seen that not all end positions are equal. This can be used as a tie break in four of nine cases. However, if this tie break method is used, it is of course paramount to agree on this method prior to playing.

We have also seen that combinatorially, there are twelve 3:3 positons, twelve 3:2 and twelve 2:3 positions; and only six 3:1 and six 1:3 positions, and just two 3:0 and two 0:3 positions. If all of these appear equally often—which they do not in practice, of course—one could expect 12 of 52 games or 23% to end in a draw. This we can compare to the result form Section 10.2, where we found that random play could result in 31.5% of the cases.

Table 3.1.: Pentagame Openings

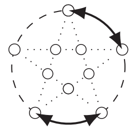
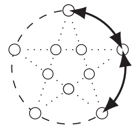
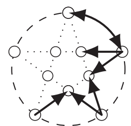
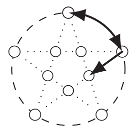
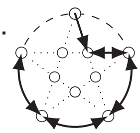
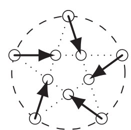
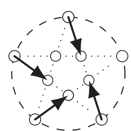
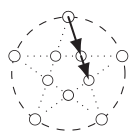
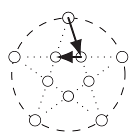
	type			nickname	number	bl
1.	SxS, *double swap* swap twice	a.		Fork	1	
		b.		Running Rabbit	1	
2.	SxR, *mixed* swap—replace	a.-d.		Mixed	4	
		e.		Little Rabbit	1	
3.	RxS, *mixed* replace—swap	a.-d.		Inverse Mixed (equivalent to mixed)	(4)	
4.	RxR, *double repl.* replace twice	a.-d.		Clockwork	4	
		e.-g.		bi-directional (self-conflicting)	(3)	
		h.		Rocket	1	
		i.		Zig-zag (loss of tempo)	(1)	
$\sum$					12 (20)	

Table 4.1.: Opening Responses

	type		nickname	number	blo
1.	SxS, *double swap*	a.	Fork	5	
	swap twice	b.	Running Rabbit	10	
2.	SxR, *mixed*	a.-d.	Mixed	40	
	swap—replace	e.	Little Rabbit	10	
3.	RxS, *mixed*	a.-d.	Inverse Mixed	(40)	
	replace—swap			(equivalent to mixed)	
4.	RxR, *double repl.*	a.-d.	Clockwork	40	
	replace twice	e.-g.	bi-directional	(30)	
				(self-conflicting)	
		h.	Rocket	10	
		i.	Zig-zag	(10)	
				(loss of tempo)	
$\sum$				115 (195)	

Table 4.2.: Openings where Alice and Bob hit 0, 1 or 2 blocks

	Bob	15	50	50
Alice	blocks	0	1	2
2	0	30	100	100
5	1	75	250	250
5	2	75	250	250

Table 4.3.: 'Result matrix' of Pentagame

		$\mathcal{B}$			
		3	2	1	0
$\mathcal{A}$	3	$\mathcal{A} \sim B$	$\mathcal{A} \succ \mathcal{B}$	$\mathcal{A} \succ \mathcal{B}$	$\mathcal{A} \succ \mathcal{B}$
	2	$\mathcal{A} \prec \mathcal{B}$	–	–	–
	1	$\mathcal{A} \prec \mathcal{B}$	–	–	–
	0	$\mathcal{A} \prec \mathcal{B}$	–	–	–

Table 4.4.: Two-player tournament scoring example

	Alice $\mathcal{A}$	Bob $\mathcal{B}$	$\sum$
Game 1			
points	3	2	5
amendment	$\mathcal{A} - \mathcal{B}$	$\mathcal{B} - \mathcal{A}$	
zero sum score	1	-1	**0**
Game 2			
points	3	1	4
amendment	$\mathcal{A} - \mathcal{B}$	$\mathcal{B} - \mathcal{A}$	
zero sum score	2	-2	**0**

Table 4.5.: Two player points and score lookup table

points	vs.	$\sum = 0$	vs.	$\sum = 6$
$3:3$	$\leftrightarrow$	$0:0$	$\leftrightarrow$	$3:3$
$3:2$	$\leftrightarrow$	$1:-1$	$\leftrightarrow$	$4:2$
$3:1$	$\leftrightarrow$	$2:-2$	$\leftrightarrow$	$5:1$
$3:0$	$\leftrightarrow$	$3:-3$	$\leftrightarrow$	$6:0$

Table 5.1.: General strategic principles

1.	I want to need less moves
2.	I want to be as close to the goals as I can
3.	I want to have more options in general
4.	I want space and room to go
5.	For my rivals I want the exact opposite
6.	I choose a strategy that counters the one I suspect my rival to follow
7.	I need to be aware of partisanship.

Table 5.2.: Specific strategic principles

1.	To limit another player's options is as good as to enlarge one's own
2.	Coordination and spacial development is better than rapid progress of a single piece
3.	It is helpful when my pieces neighbour each other so I can swap
4.	A follower is never the first to a goal
5.	It is good to separate the rival's pieces
6.	Never allow an Abracadabra
7.	Consider wisely where to place blocks
8.	Nodes are better than lines: more options
9.	Space is better than nodes for the same reason
10.	Stops next to nodes are stronger than stops further away from nodes
11.	Seek positions where you have strong moves
12.	*replace* is stronger than a blank move
13.	*swap* is stronger than *replace*
14.	*cooperate* is stronger than *swap*
15.	Swapping with a foreign piece is rarely a good idea
16.	A block close to a piece is stronger than a block far from it
17.	A grey block is stronger than a black block
18.	A foreign piece is a very strong block, but ties the piece
19.	Be aware of the two directions clockwise and counter-clockwise

Table 8.1.: Notation shorthand

$\mathcal{A}, \mathcal{B}, \ldots$	players
A, B, C, D, E	outer nodes (corners)
a, b, c, d, e	inner nodes (goals, crossings, junctions)
a, b, c, d, e	pieces
$\rightarrow$	move without swap
$\leftrightarrow$	swap own pieces
$\rightleftarrows$	swap foreign piece
$\Leftrightarrow$	cooperative swap (4-player)
$\times$	replacing a block
$(\ldots)$	a move (of a piece)
$[\ldots]$	block placement
$+$	reaching a goal
!	recommendable
?	questionable
$\triangle$	idea

Table 12.1.: Superior and inferior wins

adjacent-acute	superior
adjacent-obtuse	inferior
across-acute	inferior
across-obtuse	superior

Table 12.2.: Ties and tie breaks

	sup.	inf.	inf.
sup.	tie	break	break
inf.	break	tie	tie
inf.	break	tie	tie

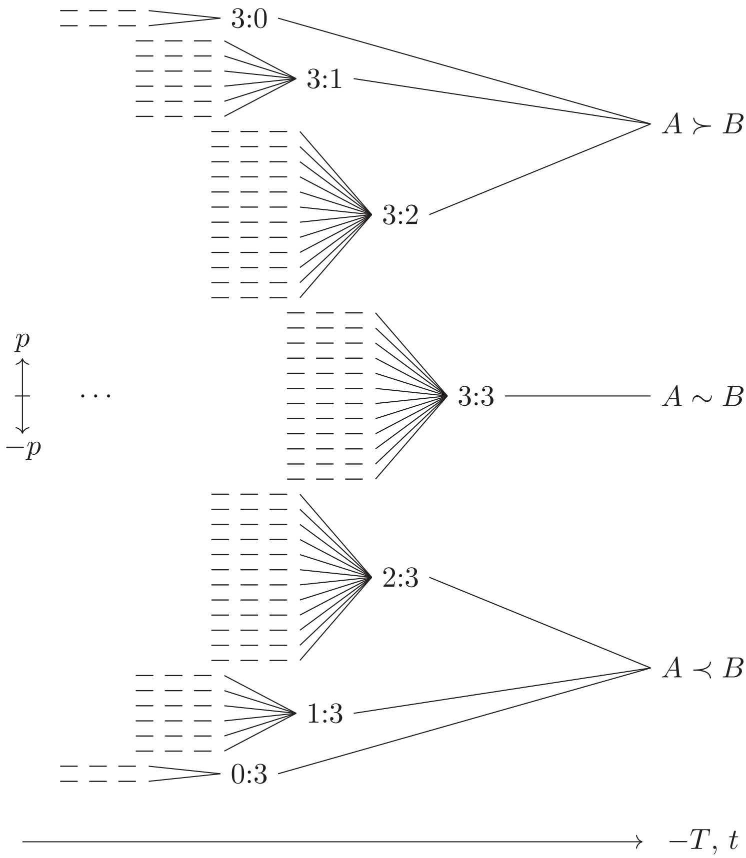
Table 12.3.: Scores and number of arrangements

Alice wins acute		Alice wins obtuse	
Bob scores	# ways	Bob scores	# ways
3	6	3	6
2	6	2	6
1	3	1	3
0	1	0	1
$\sum$	16	$\sum$	16

Table 12.4.: The 52 / 7 / 3 end states

2	3:0	
6	3:1	$A \succ B$
12	3:2	
12	3:3	$A \sim B$
12	2:3	
6	1:3	$A \prec B$
2	0:3	

Figure 12.7.: Terminus



To bring more pieces out takes longer, but is more likely.

Part V.

Extensions

1. More than two players

1.1. Introduction

So far, we have looked at general qualities of Pentagame and at the two-player case, which we perceive as the ‘normal’ case. This is a bit judgmental, since playing with three or four players is equally normal. However, multi player setups have some qualities *in addition* to our previous findings. Even worse, some of the insights we have gathered above cannot strictly be applied to such setups; for example, our proof of decisiveness (i.e. that the game always ends) is only valid for two players. So what is different in these multi-player setups?

In a two-player game, every player faces opposition that is just as strong as she is. In a multi-player game, the opposition commands more pieces than the player; every single player is weaker than her opponents combined. This means that the win is a matter of coalitions and politics. Multi-player setups are hard to win, since loosing parties usually team up. Strategic alliances, tacit collusion, psychology and defensive use of pieces are paramount.

In a two player game, after every ply it is the other player’s ply; the order is always $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{A}$ and so forth. In multi-player setups, a player has one opponent drawing before and one opponent drawing afterwards. This means that the opponents are not equal, and that the order in which they move plays a role. And this in turn influences the politics of the game.

More participating players mean more pieces on the board. Thus, lines will be shorter.

And finally, more players will take longer to play because (a) more people must think before a round ends, (b) there are more pieces on the board for each player to consider, and (c) players must also think about some of the moves and effects that exist in such setups but not in the two-player case. Playing time thus increases with player number more than quadratically.

One important effect every multi-player game shows is known as ‘kingmaker-effect’. In the narrower sense of the word this describes the situation where a party who cannot win anymore can by her action decide who of the other players will win. In a wider sense this means that a player has the opportunity to decide on the final score on any other player; thus even in situations where the winner is clear, *some* player can still decide on another player's score. This could be that the winner may play so that the two losers draw between themselves, or so that one of them scores more than the other. In other words: the Kingmaker effect applies to sub-games as well.

Kingmaker effects are typically unwanted in game design. Games of the 'Euro game' type typically avoid it by limiting the strategic component; in a pure race game, where nobody can really hinder nor advance anyone else much, it does not appear. In a strategy game it appears with necessity, but the actual decision that leads to any particular result may be hidden somewhere so early in the game tree that players may not even notice; or that not all players may have noticed.

The Kingmaker effect is unpleasant only if the order of preferences (or adversary) between players is open. If you play sufficently often and note scores, the effect will eventually cancel itself out. We shall look at some means to this effect further below.

Multi player setups will tend to be won by an edge only, since typically loosing parties will coalition against winning parties. It is hard to win against many, albeit not at all impossible.

Because three-player games are coalition games, it is particularly important to remain quiet and not to mention possible moves. Etiquette demands to leave the player in peace who is to move. If you feel like talking, you should rather speak to another idle player.

2. Three player Pentagame

In the three player case, each player faces two opponents; and two are potentially stronger than one. And any player has a very different

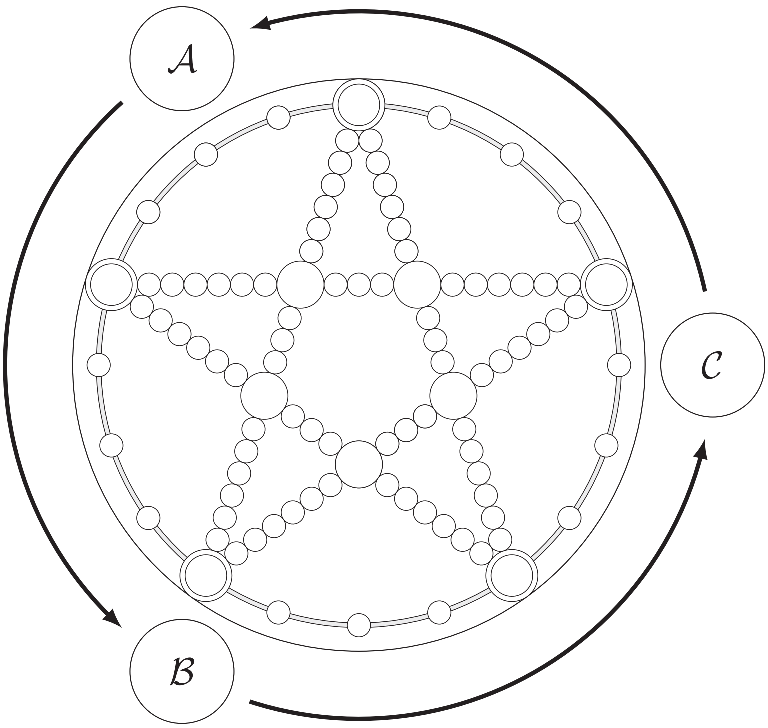


Figure 2.1.: Counter-clockwise

relation to each of the the two other players. Thus partisanship is at the core of such a setup.

Players must agree whether they play clockwise (Fig. 2.2 or counter-clockwise Fig. 2.1).

The effect the order of moving has on the game is tremendous. The player playing prior to a particular player is called vorhand and the one playing afterwards is called hinterhand.

Imagine that Alice is much stronger than both Bob and Charlies. Then clearly Bob has a disadvantage relative to Charlie.

For a fair evening, players should always have a re-match in reverse moving order. Sometimes it is practical to swap seats instead.

We will in the following assume clockwise play, thus $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$.

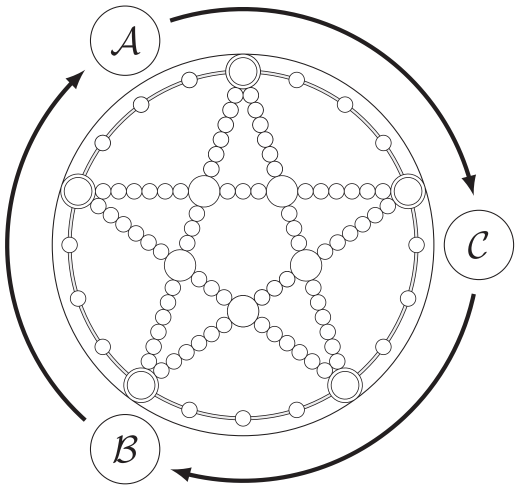


Figure 2.2.: Clockwise

In this setting there are just as many pieces plus blocks as there are lines on the board. One consequence is that there will be less turning corners and taking long paths than in the two-player game. Another is that two players can jointly stop the third from winning.

In a three-player setting it can happen that you run out of grey blocks. Rather than to go fishing for an extra object, allow the player who has gained the right to place a fresh grey block to instead re-position one of the grey blocks already in the game.

The three player game is not always cecessarily decisive, since two players can completely block any third. It seems that some games run into such stalemates, while others develop so that the game necessarily ends.

2.1. Passive moves

The rules permit that a player's piece can passively be brought to its goal by another player in a swapping move. The rule that a piece brought passively to its goal remains there until it is that player's move, which was an insignificant rule in a two-player game, now becomes very important to remember. It exists to not allow any confusion of the order in which players move. Here we observe the chrirality; it makes a difference *by whom* a piece was passively moved.

Example 6: Vorhand swaps mittelhand

1.	$\mathcal{A} \leftrightarrows \mathcal{B}$	$\mathcal{A}$ swaps $\mathcal{B}$ ($\mathcal{B}$ moves passively)
	$\mathcal{B} \leftrightarrows \mathcal{A}$	$\mathcal{B}$ swaps $\mathcal{A}$ back
	$\mathcal{C} \ldots$	$\mathcal{C}$ does whatever
2.	$\mathcal{A} \not\leftrightarrows \mathcal{B}$	$\mathcal{A}$ cannot swap $\mathcal{B}$ again (Ko).

Example 7: Vorhand swaps hinterhand

1.	$\mathcal{A} \leftrightarrows \mathcal{C}$	$\mathcal{A}$ swaps $\mathcal{C}$ ($\mathcal{C}$ moves passively)
	$\mathcal{B} \ldots$	$\mathcal{B}$ does something affecting $\mathcal{C}$
	$\mathcal{C} \ldots$	$\mathcal{C}$ faces a new situation.

Combinations of the type of Example 7, where vorhand swaps hinterhand, can be surprising as they do not appear in two player games.

Under the right circumstances swapping with an opponent's piece may now hinder that from moving out (Example 8):

Example 8: Two players cooperate against one

1.	$\rightarrow$	$\mathcal{A}$ lands on a stop adjacent to her goal!
	$\leftrightarrows$	$\mathcal{B}$ swaps with $\mathcal{A}$'s piece!
	$\times$	$\mathcal{C}$ blocks $\mathcal{A}$'s piece from moving back!
2.	$\ldots$	$\mathcal{A}$ cannot move out.

Even worse, Alice could move Charlie passively to Charlie's goal; but Bob could still reach this goal first in his move, displacing Charlie again, gaining a grey block and hindering Charlie to move out (Example 9):

Example 9: Swallow this

1.	$\mathcal{A} \leftrightarrows \mathcal{C}$	$\mathcal{A}$ bring $\mathcal{C}$ to her goal!
	$\mathcal{B} \leftrightarrows \mathcal{C}$ &$\times$	$\mathcal{B}$ reaches the goal first & blocks $\mathcal{C}$!
	$\mathcal{C}\ldots$	$\mathcal{C}$ cannot move out!

We have seen that for vorhand, swapping hinterhand is far more effective, and thus far more dangerous for hinterhand, than swapping with mittelhand. Thus Alice is advised to seek positions where Charlie cannot move her passively, etc. This she can achive by clever positional play and by intelligent positioning of blocks.

Protecting one's own pieces from being moved passively away from their goals by hinterhand is a major strategic recommendation in three-player settings. In a three-player setting, defensive play pays out: watch your back.

These insights already deliver the most important lessons for a three-player setup: to play cooperative and to play defensive. Cooperative: it is genuinely important to form alliances and not just battle everyone, but to cooperate strategically, a topic we will return to below. Defensive: it is important to isolate one's own pieces in strategic fashion and protect them from both being swapped back and from being blocked by foreign pieces.

2.2. Loops, Ko and impasse

The general Ko rule of Pentagame applies to the three-player setup just as well. We recall that it states *do not undertake the same move twice.*

As can two player setups, three player setups can run into loops; these can be of two types: either a 'sub-loop', where two of the three players continue to swap each other's pieces, or a true loop involving all three players. The 'sub loop' is hindered ruled by the Ko-rule in the ordinary fashion, so such a loop simply means that the two players involved both loose a turn, while the third player will quietly advance.

If three players are involved, the Ko-rule takes a little longer to take effect.

Example 10: Three player loop and Ko rule

1.	$\mathcal{A} : \mathrm{A} \leftrightarrows \mathrm{B}$	$\mathcal{A}$ swaps with $\mathcal{C}$
	$\mathcal{B} : \mathrm{C} \leftrightarrows \mathrm{B}$	$\mathcal{B}$ swaps with $\mathcal{A}$
	$\mathcal{C} : \mathrm{A} \leftrightarrows \mathrm{B}$	$\mathcal{C}$ swaps $\mathcal{B}$
2.	$\mathcal{A} : \mathrm{A} \not\leftrightarrows \mathrm{B}$	$\mathcal{A}$ *may not* swap $\mathcal{C}$ again.

Sometimes the action of the third player can have the effect to change the piece that is on a particular stop that a player wishes to reach gets changed. In this case the move of that player to that stop is not the exact same move:

Example 11: Three player permissiable semi-loop

1.	$\mathcal{A} : \mathrm{A} \leftrightarrows \mathrm{B}$	$\mathcal{A}$ swaps with $\mathcal{B}$
	$\mathcal{B} : \mathrm{A} \leftrightarrows \mathrm{C}$	$\mathcal{B}$ swaps with $\mathcal{C}$
	$\mathcal{C} : \mathrm{A} \leftrightarrows \mathrm{B}$	$\mathcal{C}$ swaps $\mathcal{A}$
2.	$\mathcal{A} : \mathrm{A} \leftrightarrows \mathrm{B}$	$\mathcal{A}$ *may* swap $\mathcal{C}$.

Such a development may be perceived as a true loop, so sometimes the question may come up if this is permitted: it is. Noteworthy the situation has changed, and no exact repetition has taken place, even though the same stops and pieces are involved. If the situation were to repeat a number of moves later—which is a possibility—the Ko-rule would take effect. Thus in three-player setups the Ko-rule may hold not in the immediate move afterwards, but one move later. This can be confusing because of the rather large number of plies in such a loop.

We have proven above that the two-player game always ends under perfect play; that perfect two-player play is decisive (Page 174). The arguments used in said proof is based on the observations that 4 pieces suffice to totally block a player. But a three-player game that has advanced to 2:2:2 means that each player is facing 6 pieces of her two opponents. So you can position three pieces of three parties on an otherwise empty board in such a way that whoever is to move cannot reach a goal. Thus there can be a three-player impasse.

Players can escape from an impasse in three ways:

1. players agree on a draw,

2. players continue until one player makes a mistake,

3. two players cooperate against the third.

Since the impasse usually appears at 2:2:2, the error is likely to result in 3:2:2 or 3:3:2, while the cooperative solution should result in 3:3:2, if not 3:3:3. This is because results are super-additive.

The cooperative solution is a tricky one. Imagine a 2:2:2 situation. Suppose Alice has started the game and is to move next. She has obviously no interest to let anyone win, because if that happens she automatically looses. But Bob could have an interest in making a move that lets both him and Alice win, because *if Alice wins, he still has a move.* The same applies for Bob. Thus in a three player Charlie has an advantage over Bob and both over Alice. Nearer inspection reveals that this advantage is very slight; Charlie would only have Alice or Bob to win if the result would be so that whatever Alice or Bob do with their grey block or otherwise can not hinder Charlie's final winning move. But that requires Charlie to have either Bob win and Charlie have a fork, or Charlie to have Alice win and have a multi-fork; both is very unlikely, and if this is possible, Charlie would just win in an ordinary fashion.

The order of preference for any player depends on the exact scoring method. In absolute scoring—and if a player only bothers for her own score—it looks as shown in Eq. (2.1). This is the complete space of possible wins for Alice.

$$\begin{aligned}
&(\{3:0:0\})\\
&\succ(\{3:0:1\}\sim\{3:1:0\})\\
&\succ(\{3:0:2\}\sim\{3:1:1\}\sim\{3:2:0\})\\
&\succ(\{3:0:3\}\sim\{3:1:2\}\sim\{3:2:1\}\sim\{3:3:0\})\\
&\succ(\{3:1:3\}\sim\{3:2:2\}\sim\{3:3:1\})\\
&\succ(\{3:2:3\}\sim\{3:3:2\})\\
&\succ(\{3:3:3\})
\end{aligned}\tag{2.1}$$

If we were to visualise this in a 3-dimensional coordinate system with x, y, z representing the score of players $\mathcal{A}, \mathcal{B}, \mathcal{C}$ we would see three faces of a cube. This overview can help to identify possible priorities.

This leads us to a core section of this Chapter, as we will now look into the coalitions that are likely to emerge and how to deal with them.

2.3. Adversary and Alliances

In a game of three players—call them Alice, Bob and Charlie—every player faces of course two rivals. Again, Alice must consider her own reasonable moves (which maybe somewhere in the order of 25) and will take some reactions into account. But Bob will react to her move first, and then Charlie will react to Bob, and so forth. Alice will treat Bob and Charlie differently.

If Alice now sees that Bob threatens to move out, she is pretty much forced to block him; if Alice sees that Charlie can move out, she can leave the duty of stopping that to Bob.

Every player will regularly pay more attention to what the player after can do. Furthermore, for any mittelhand player, hinterhand is more to watch also because hinterhand can by swapping cause great danger to mittelhand's play.

Thus from the direction of play results an intransitive 'natural' order of adversary:

$$\cdots \succ \mathcal{A} \succ \mathcal{B} \succ \mathcal{C} \succ \mathcal{A} \succ \cdots$$

This natural order of adversary is relatively unstable. Players will form varying coalitions.[1] As soon as one player shows signs of winning, the others will cooperate. But once Alice, Bob and Charlie are on equal footing again, the game reverts back to the 'natural' order of adversary. This makes it hard to to win a three-player game,[2] and the win is usually by an edge.

Imagine Alice plays strong while Bob and Charlie are weak and moved in alphabetical order; then by the order or natural adversary Alice will play more against Bob then against Charlie. And since she is a stronger player than both, that is to Bob's disadvantage. Thus a single three-player game is unfair!

Furthermore, it is easy to observe that at least in the very last round preferences between players will become unstable. Once Alice comes into the position to choose whether Bob or Charlie are to make one more point, she has no game-intrinsic preference. For this, we can find technical remedies, which we will tackle next.

The considerations above have abstracted from players with different ability; we have shown which alliances will likely occur when all players are more or less of identical strength, and there are no personal preferences among players. The situation looks different though when players have expectations on each other's strengths. This is of course a wide field, and we shall only dip into the problematic here.

Let us now assume that Alice, Bob and Charlie are not equally strong, but that Alice is stronger than Bob is stronger than Charlie. This means that the expected outcome of any non partisan game would be $\mathcal{A} \succ \mathcal{B} \succ \mathcal{C}$; most naturally this would be $3:2:1$. The question is: if all players know this before, would the outcome be different?

[1]Unless they have personal preferences. The technical term for games where players have preferences motivated outside the game is *hedonic games*.

[2]Οὐδὲν Ἡρακλῆς πρὸς δύο. (Suda, omikron 794; also Michael Apostolius, c13.19); Not even Heracles (can win) against two.

- Alice's main concern is to win the game, and she feels most threatened by Bob. Thus she prefers Charlie over Bob ($\mathcal{A} :\ \mathcal{C} \succ \mathcal{B}$).
- Bob sees his main opponent in Alice, since he also wants to win the game. Thus he also prefers Charlie; that is, over Alice ($\mathcal{B} :\ \mathcal{C} \succ \mathcal{A}$).
- Charlies preferences are less clear ($\mathcal{C} :\ \mathcal{B} \sim \mathcal{A}$).

So we see that both Alice and Bob favour Charlie. Herein lays another King maker effect: whoever manages to 'win over' Charlie is likely to win the game; King will be who has the King maker, that is: the weakest player, on her side.

However our third assumption, that Charlie is indifferent whether Alice or Bob wins, may be weak. It is possible that Charlie can gain more (in points) from a coalition with Alice, since Alice is stronger and has thus more to 'offer'. This is a stabilising effect, which is more pronounced the more unequal the players are. Alice will have an interest in dividing Charlie and Bob, of course, and may thus spread her boons equally between them in a divide-and-conquer strategy.

Players can gain a significant advantage from this insight. Since I know that everyone will favour Charlie until briefly before the end, I should calculate that the pieces of Charlie will vanish from the board rather early. If I manage to position myself so that I take maximum advantage of this, I may be able to decide the game in my favour.

It is also clear that the point of time at which the actual strengths of the players become revealed (or obvious) plays a role. The stronger player will try to play stronger, but in an unnoticed way and even attempt to be taken for the weakest player.

From this it should be clear why three players should *always* play twice: once moving clockwise and once moving counter-clockwise. (In practice swapping seats it sometimes less confusing.) Since the duration of a three player is not too long this is usually what is done. It has two advantages: it levels out the unfairness between unequally strong players; and it gives every player equal interest in the score of any other player.

The game ends when one of the players has led three of her pieces out. Again the last round is played out, so all player have had the exact same number of movements. When you play twice with reversed order then you can simply write down the amount of pieces out as scores.

2.4. Zero sum scoring

The other way to deal with the issue of the eventual arbitrariness of preferences between players is to use zero sum scoring. This is the scoring method you should also use in tournaments to make results comparable.

We seek a mechanism to give Alice incentives to

1. not be indifferent to the scores of the others,

2. have a stable preference between Bob and Charlie,

3. be compensated for player's unequal abilities.

To make Alice not indifference it suffices if she is non-indifferent to either Bob or Charlie. Thus conditions (1) and (2) are met once we apply the same method as developed in the two-player setting: we take Alice's score and add how many pieces her 'chief enemy' has not moved out. As a result, the game will become zero-sum (or better: constant sum) with the constant sum of 9 points. Alice will not be indifferent, and have clear incentives for her preference of either Bob or Charlie.

But who should we make her 'chief enemy'? With other words: shall Alice get a point for each point that Bob doesn't make, or shall Alice get a point for each point that Charlie doesn't make?

Recall that Bob moves after her, which gives Alice a natural tendency to block him. This natural order of adversary we would like to stabilise; thus making Bob her 'chief enemy' makes sense. If every point that Bob doesn't make is a point for Alice, she will have a preference for Charlie.

Remember that Alice's result chiefly relies on the blocking by Charlie. If Charlie plays strong, she will likely score less. She should get compensated for that, which is Argument (3) in the list above, and she does.

Thus the rule is

- From a player's points subtract the points the player after her has made.

This means Alice *always* prefers Charlie over Bob.

Table 2.1.: Three-player zero sum scoring

From each player's points subtract the points of the player *after*.

	Alice $\mathcal{A}$	Bob $\mathcal{B}$	Charlie $\mathcal{C}$	$\sum$
Game 1				
points	3	2	1	6
amendment	$\mathcal{A}-\mathcal{B}$	$\mathcal{B}-\mathcal{C}$	$\mathcal{C}-\mathcal{A}$	
score	1	1	-2	**0**
Game 2				
points	3	2	2	7
amendment	$\mathcal{A}-\mathcal{B}$	$\mathcal{B}-\mathcal{C}$	$\mathcal{C}-\mathcal{A}$	
score	1	0	-1	**0**

In the example of Table 2.1 we see that in Game 1 Alice and Bob have made 3 and 2 points, but end up having a score of 1 both at the end of the procedure! But this is actually reasonable: Alice played after Charlie, who is obviously a much weaker player than her. Thus she had easier play than Bob who moved after her. Bob in comparison managed two out and keep Charlie down to just one point.

Ideally you would use this method and still change the order in which players move.

3. Four player Pentagame

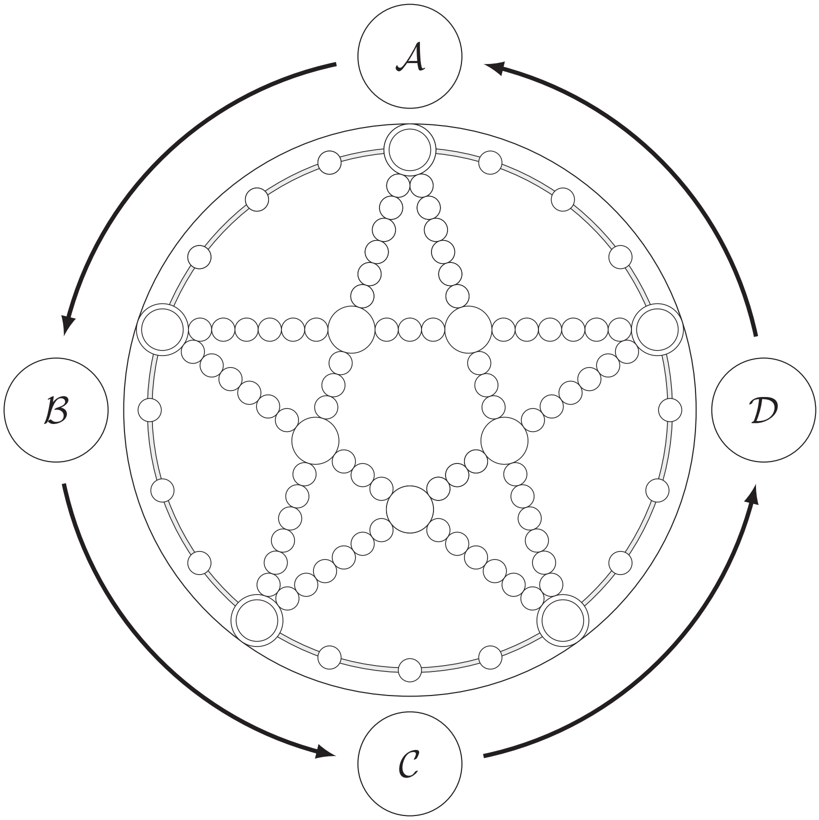


Figure 3.1.: Four players $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$

Four player settings are pretty demanding. Many options must be considered, so each player may end up thinking for quite a while before moving. In addition three other players move before a player moves again. We see that the duration of a game is more than exponential to the number of players involved. Consequently full four player games are lengthy with playing time easily exceeding 80 minutes.

In everyone-against-everyone quite a lot happens until a player is to move again; and much happens which is unexpected, which makes this a very entertaining setup and a very good parlour game well suitable for beginners.

In contrast, team play in a four player setting is highly challenging, even for advanced players, because every player has one partner and an opposition team to consider, where cooperative moves are possible. Team players experience less idle time than individual players—team play is extremely captivating.

Four players command together 20 pieces; that plus the 5 blocks makes 5 more pieces than there are lines on the board. This makes the board relatively blocked, so open lines are quite rare, pieces move relatively short distances and the game is comparatively lengthy.

There are still only five black blocks in the four player setting, while quite many grey blocks will enter play. One effect of this is that black blocks become a scarce resource. Another is that the game gets a little blocky until players remove grey blocks.

It may happen that a player moves out and gains the right to position a grey block, but all five of them are already on the board. In this case he simply re-positions one of the grey blocks on the board. This way the number of pieces on the board will slightly diminish (which is the reason for the distinction between black and grey blocks in the first place).

There are different ways to arrange a four-player game. They can play each against all, or they can form teams and play cooperatively in a team play setting. Both provide very interesting, though challenging game play.

3.1. Each against all

Four players each playing all is an entertaining and very exciting way to play; much unexpected happens, coalitions form and fall, and the win is always only by an edge.

To have four parties also implies that much happens before it is a player's turn again. Every strategy must take this high level of uncertainty into account. We have an almost chaotic situation, and psychology and politics are just an important part of the action. Often players call attention to the common rule that 'too much talking disqualifies', much disputed in the past. Much happens in the realm of partisanship, collusion and coalition, all of which of course are unstable within a framework of simple point scoring.

A single such game cannot be fair; the sitting order is of paramount importance. Play is not quite sharp, which is the Price of Anarchy in this *bellum omnium contra omnes.*

A serious drawback of playing each against all is that for fair play players would have to vary the order in which they play. But simple order reversal like in the three player setting is not enough to level their chances. Rather, four players need *six* permutations of the sitting order, which would make this a lengthy affair. Consequently, each against all is only common in social settings, but not in decent tournaments.

Players usually collude against whoever takes the lead. Thus often we have shifting 'three against one' settings. The politics involved can be quite interesting. Probably one of the most intriguing aspects of this type of game play is the challenge for the team of three to spot and use combinations against the leader.

Here is an example. This type of cooperation is only possible in four player mode. It takes advantage of the rule that a piece that gets moved out passively may only leave the board on its term (Example 12):

Example 12: Three players cooperate against one

1.	...	$\mathcal{A}$ ...
	$\leftrightarrows$	$\mathcal{B}$ moves a piece of $\mathcal{A}$ onto its goal!
	$\leftrightarrows$	$\mathcal{C}$ removes $\mathcal{A}'$s piece from its goal!
	$\rightarrow$	$\mathcal{D}$ blocks $\mathcal{A}$' piece from moving back!
2.	...	$\mathcal{A}$ cannot move out.

Finding such combinations without talking (too much) is a real challenge, and they are, when successful, quite exciting.

There are seven possible configurations of this play, all likely to appear: one true each against all, two two against two coalitions, and four one against three configurations, but all of these are unstable. Even using zero sum scoring does not solve this issue: if we again subtract the next player's points from any player's result, she nevertheless remains indifferent to the results of the other two. A four player each against all game always bears a fair degree of chaos.

3.2. Team play

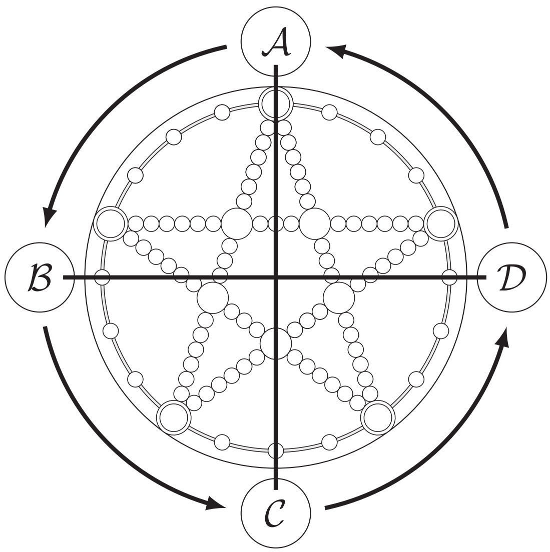


Figure 3.2.: $\mathcal{A}$ and $\mathcal{C}$ play against $\mathcal{B}$ and $\mathcal{D}$

This is well the most complex, and hence ultimate form (*Königsdisziplin*) of playing Pentagame: team play. Commonly partners face each other ($\mathcal{A}$ teams with $\mathcal{C}$ while $\mathcal{B}$ teams with $\mathcal{D}$ in Fig. 3.2. The winning team is usually the one that brings out any 5 of their pieces, but players can also agree to finish already after 2, 3 or 4 pieces of a party have reached their goal to shorten the otherwise relatively lengthy game.

We remember that the two most important move types for a player are *swap* $\leftrightarrow$ and *replace* $\rightarrow$. But now that players form teams, a new type of move enters play. A player can now also *cooperate* by swapping their piece with one from their partner. (We write '$\Leftrightarrow$', read 'coooperate') This move $\Leftrightarrow$ allows for a considerable gain in speed, which makes it a very strong move. And because $\Leftrightarrow$ is so strong, players have an incentive to make such moves possible.

Now we draw a tree that shows how in this setup every player has *three* choices per turn (rather than two when playing alone). Fig. 3.3 shows the higher decision complexity of such a game relative to the two-player game.

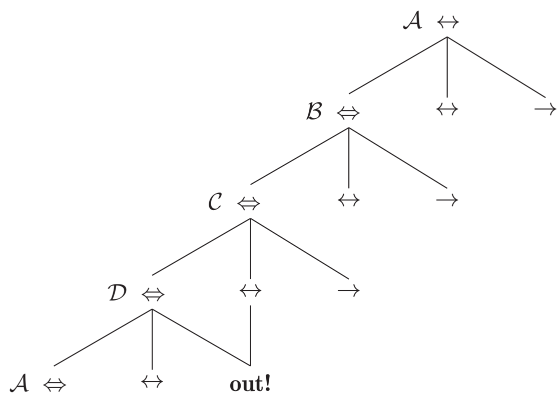


Figure 3.3.: Four player game tree game

It also shows a particular threat that only exists in this setting: to move out *with only two moves* rather than three as in the regular settings (Example 13):

Example 13: Two players cooperate against two

1.	$\leftrightarrow$	$\mathcal{A}$ moves two pieces on the ring
	...	$\mathcal{B}$ cannot stop both
	$\Leftrightarrow$	$\mathcal{C}$ cooperates and brings one piece of A even further!
	...	$\mathcal{D}$ doesn't see this and
2.	$\rightarrow +!$	$\mathcal{A}$ moves out! $(\mathcal{A}, \mathcal{C}) \succ (\mathcal{B}, \mathcal{D})$

We gain the following insights from Example 13:

1. At times, only cooperative play of both B and D can prevent A to move out.
2. Players must from the very start be aware of the strength of joint forces.
3. The minimal number of moves to reach a goal is reduced to two.
4. Cooperate $\Leftrightarrow$ yields considerable gain of speed.

A player reaching a goal in his second move is usually a strong wake up call for the other team. This demonstrates how much care team players must take.

Table 3.1.: Two-against-two player zero sum lookup table

points	vs.	zero sum score	vs.	constant sum score
5 : 5	$\rightarrow$	0 : 0	$\rightarrow$	5 : 5
5 : 4	$\rightarrow$	1 : −1	$\rightarrow$	6 : 4
5 : 3	$\rightarrow$	2 : −2	$\rightarrow$	7 : 3
5 : 2	$\rightarrow$	3 : −3	$\rightarrow$	8 : 2
5 : 1	$\rightarrow$	4 : −4	$\rightarrow$	9 : 1
5 : 0	$\rightarrow$	5 : −5	$\rightarrow$	10 : 0

Such team games can be made zero-sum games with the rule of the two-player game: a team will get their number of points plus the other team's missing number of points (Table 3.1).

4. Five player Pentagame

Five player games with the ordinary rules are possible but not necessarily recommended for three reasons: One, the board gets very full and very blocked, so it takes a long time. Two, there are many players that have to move before a player is to move again, so again it will take longer. Three, so much happens within one round that planning is not really possible.

But you can have a nice five-player game when you amend the rules a little. In deviation from the ordinary rules:

- Simply let each player have three pieces *of one colour.* So the one who sits near the white corner plays all white pieces, the one who sits near blue plays blue etc. which also makes the game easy to explain.
- A player needs three pieces out to win.
- All other rules remain the same.

One way to add more focus is to introduce zero sum scoring in the same way as before: from a player's points, subtract the points the player moving afterwards achieves.

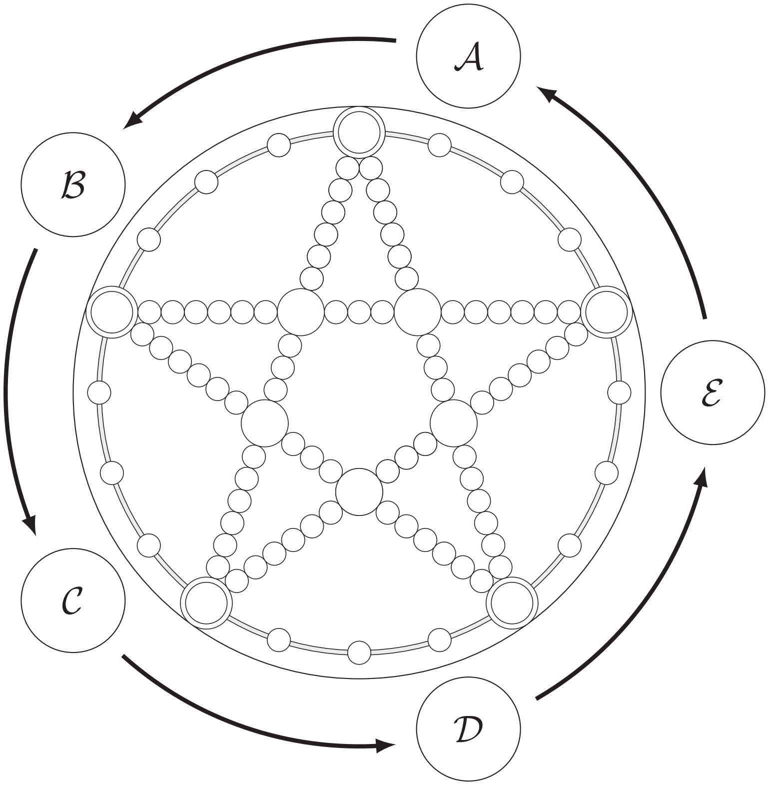


Figure 4.1.: Five players $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and $\mathcal{E}$

5. Material extensions

5.1. Simplicity

Some game are adorned by a plethora of rules. The charm of these games is operating in the ensuing jungle. Their drawback is the multitude of rules to be remembered, and the ensuing difficult optimisation process on the side of the player's. Since in such constructs rarely a clear strategy ensues, players are left with heuristics, which can be a lot of fun.

Simpler games are more likely to 'stand the test of time' because:

The easier the rules (read: the smaller the set of rules), the easier it is to find new players and 'propagate' the game.

For any game material exists a minimal set of working rules: the set of which no rule can be dropped without losing the qualities of clarity, complexity, decisiveness, drama; or a minimal set of rules that maximise these four qualities.

There may be other sets of rules which result in playable games, but they are likely larger.

Simple rules tend to give the largest number of options per player per ply, and this freedom is usually appreciated. This freedom per move results in a high Average Branching Factor, and this in turn creates the complexity required for a game to become a classic.

While counting of rules is not an entirely simple affair since it is sometimes difficult to clearly delineate one rule from another, it is nevertheless possible; in any description, Pentagame has no more than a dozen rules. We spare the reader to present a table comparing the quotient of a number of Complexity measures by rule count for a number of games. But it can safely be asserted that Pentagame is 'playing in the league' of the established classics.

Any minimal rule set can be extended with additional rules. Rule additions can be of three kinds:

a) Additions that *diminish the options* a player has; thus, the ABF is lowered by a factor; dice are an example because they limit the stops to which a player can move

b) Additions that give the player *more options;* thus, the ABF is increased by a factor; an example may be example cards that allow special movements

c) Additions that don't change the game but *add something unrelated.*

The logical consequence of adding diminishing rules type a) is that players can decide quicker, but might need more moves to finish a game. Thus the ABF decreases, while number of plies increases, and game play duration remains (almost) constant.

The consequence of adding enabling rules type b) is the opposite: players with more options will have to think harder, but the game will likely end after less moves. Thus we have a higher ABF, is countered by shorter game duration.

Finally, a plethora of type c) extensions are feasible; think of strip poker etc; but all this is beyond the scope of this book.

In all cases, players must remember and consider more rules. The minimum rule set usually prevails over all more complicated rule sets, because we all want maximum fun with a minimum of complication. With other words: to agree on a set of rules is a meta-game, and its equilibrium is the minimal set of rules.

The rules of Pentagame have been refined quite a while and can be considered a minimal, hence optimal, set of rules. Any extensions are thus what we call ‘house rules’: non-official, experimental, and most of the time superfluous.

5.2. A dice extension

Introduce colour dice and have players roll the die before their move; they may then only be allowed to move their figure of the colour shown by the die.

To make it more interesting you may consider these two versions of a move per player:

a) roll, then move
b) move, then roll

In version (b) each player rolls her die *after* having moved; her die remains in front of her, so that the other players can take into consideration with which figure she must move.—This can of course be played with a pair of dice per player...

5.3. A card extension

Using ordinary playing cards, deal each player a hand—it could be the same hand for all players. Then a player's turn is two-fold:

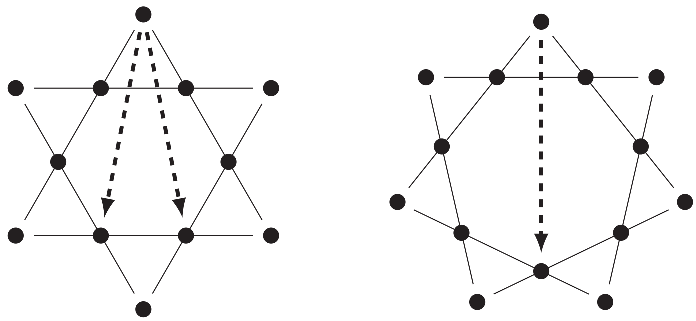
1. Play a move according to the ordinary rules.
2. Then, play one of your cards:
   a) Number cards have no effect at all
   b) Jack: slide a block along a path
   c) Queen: reposition a block
   d) King: add a block
   e) Ace: remove a block
3. Then it's the other player's move.

5.4. Different boards

Sometimes the question comes up whether it is possible to play the same rules on hexagon or a heptagon (Fig. 5.1). But any feasible board must offer goals opposite to starting nodes, which rules out six-pointed stars or indeed any polygons with an even number of vertices.

Uneven polygons of an order higher than five—like he heptagon—offer goals, but are unncessarily more complex. This also holds for all Platonic graphs. The pentagram is the smallest, simplest, and hence most feasible shape.

A three-dimensional version, where pieces start at one side of a sphere and proceed to the opposite pole, is theoretically possible but also unpractical. The five-pointed star is the most minimal board design, and hence the best and only.



(a) Hexagon (b) Heptagon

Figure 5.1.: Hexagon and Heptagon
(a) has no clear goals; (b) is unnecessarily complex

5.5. Conclusion: A minimal set

We have seen that due to its perfect symmetry Pentagame can be played with three and four players, and remarkably without any further amendment of the rules. With slight amendment it is even playable with five players.

We have also seen that it is suitable for tournaments, given a number of options to organise tournaments, computed the potential duration of games, matches, rounds etc. in a tournament, and extended this view to mixed tournaments.

Tournament practice has demonstrated a continuously rising learning curve.

We have discussed a number of rule variations and in passing proven that the existing rule set is a minimal set. Given our assumption that players prefer games the ‘higher’ the quotient from complexity ‘per rule’, we have shown that the existing rule set is optimum, and that Pentagame is likely to be a game ‘played by people’, rather than just being of interest for mathematicians.

Part VI.

Tournaments

1. Regular tournaments

1.1. Introduction

Pentagame never wears out and thus leads itself to repeated play, where players can become quite competitive. Tournaments are great fun. A good game possessing Thompson's quality of depth will allow for meaningful tournaments, as players can measure their wit with one another and increase their standing.

The actual depth of a game could be measured by the stability of any player ranking; give or take individual development, any given tournament result—which is a ranked list of players—should be stable to some degree. This would be the ultimate proof that Pentagame has depth in the sense of Thompson.[1]

It is relatively straightforward to compare two player games to two player games, and to compare three player games to three player games. We recall:

- Multiple two player games: have zero sum games (Section 4.3).
- Multiple three player games: play clockwise and counter-clockwise, have zero sum games. (Section 2.4).
- Multiple four player games: very complex, depends on the setting. (Chapter 3).

This is all you need to know—as long as you are dealing with just one table of players. But if you have multiple tables, you will want to shuffle the players by some means, identifying an overall winner. This is what we call a tournament. There are a number of tournament

[1] Thompson: *If a game has a large following, its depth can actually be measured by recording the results of games and determining how many distinct "levels" there are: if the players in class 1 all lose regularly to the players in class 2, who lose to players in class 3, etc., up to class n, then the value of n measures the depth of the game.*—But then n would always be the number of a game's players; so we need another measure. (JS)

modes to determine who should play whom next, with which we will deal now.

1.2. The First Five

Table 1.1.: The First Five

Andreas Grübel
Christoph Jantz
John Martineau
Gerhard Suchanek
Nathan Toups

Every tournament is a ritual. It is of paramount importance that anyone from the First Five present is addressed as Honourable Member of the First Five (Table 1.1). The Honourable Member of the First Five may present the prices to the winners.

The First Five own the Five Golden Hats; these I made from old Pentagame posters still depicting a coloured board and golden duckt tape.

1.3. The Pentagame Cup

The oldest price for a Pentagame tournament is of course the Pentagame Cup. This cup was presented by the inventor of the game in 2019, and it dedicated as perpetual trophy (Fig. 1.1. It shall be competed for yearly, with the holder obliged to organise the next tournament.

The Pentagame Cup is of steel, the cup is a half ellipsoid cup. The open end of the cup ends in five spikes crowned by round ball tips. The cup rests on three legs, somehow resembling a rocket. The sculpture is of unkwnown origin and was a flea market find in Berlin-Schöneberg in 2019. The wooden base was built by the donator.

Figure 1.1.: The Pentagame Cup (2020)

1.4. Player's pieces

Whichever type of tournament you choose, it is wise to assign each player a particular family of pieces, so that she can play with the same shapes in every game. Henceforth springs the tradition that players bring their own pieces. So far, we have seen and tested in this order:

- plastic squares, triangles
- plastic stars, moons
- glass beads, glued on black

- pawns with heads shaded black or silver
- pawns with bodys shaded black or grey
- rabbits, hedgehogs, cats, geese
- cars, planes, rockets
- plastic snakes, owls, and monkeys
- glass beads, large and coloured
- 3D-printed cubes, spheres, tetraeders and stars

Since virtually any shape can be used, it is a fun pasttime to create novel pieces.

1.5. Round-robin or league system

In the round-robin system, also called 'league' system, everyone plays everyone else once (or twice). It is easy to understand, popular, but lengthy and at times a little tedious. You need strongly committed players and regularly tight time tables. And of course such a system is hard to run with both two and three player tables. This system is appropriate when you have relatively few, but committed contenders of an even number.

1. Shuffle the participants,
2. Create a table with all possible combinations,
3. Have everyone play everyone once (or twice),
4. End when every pairing has been played out and the table is complete.

The round-robin or league system is great to play over a relatively long period of time. Table 1.2 is an example league table.

In practice two players should play twice: first a warm-up game, second the game that counts.

Pentagame Leage

Place: ____________________ Date: ______________

Zero-sum games!

	(real name)	$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\mathcal{G}$	$\mathcal{H}$	$\sum$	Rank
$\mathcal{A}$		$\times$									
$\mathcal{B}$			$\times$								
$\mathcal{C}$				$\times$							
$\mathcal{D}$					$\times$						
$\mathcal{E}$						$\times$					
$\mathcal{F}$							$\times$				
$\mathcal{G}$								$\times$			
$\mathcal{H}$									$\times$		

Table 1.2.: A league table

1.6. Knockout system

The knockout system eliminates everyone who loses. As a result players will really strive to play at the height of their ability in every single match. In comparison to the round-robin this is faster relative to the number of participants. It is unfair but prone to surprises, and it can thus be both frustrating and entertaining.

1. Shuffle players and assign first challengers randomly
2. Create a triangular matching table
3. Winners play winners, losers leave the tournament
4. The last remaining player wins.

The knockout system is great when you have a limited amount of time. The paired system as shown in Table 1.2 requires the number of players to be $n = 2^x$, so regularly a qualification tournament of a different type is needed.

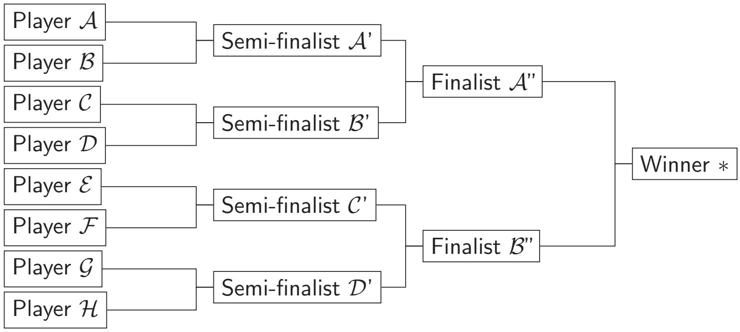


Figure 1.2.: Paired knockout tournament

A single table knockout tournament with up to five players is of course possible by starting with a multi-player setup, like four or five, and after every game have the player with the least score leaves the table, until you reach a final two player game.

1.7. Swiss system

This system is popular in all tournaments with very many participants, and—from a theoretical point of view—the 'best' system. It even allows players eventually to enter the tournament at a later stage or to drop out in the middle; it ends after a certain time limit has been reached.

The concept is that any player always gets paired with the next-high ranking player she has not yet competed with.

Usually (but not necessarily) you begin with some prior ranking, typically the result of a previous tournament, a pre-round or a rough estimate of player's abilities (effectively turning the Swiss system into a McMahon system).

The Swiss system offers the advantages that everyone generally encounters players of similar ability, that you need less games than in a round-robin and that the duration is not tied to the number of games. You may note that if you play a Swiss tournament for a sufficiently long time it morphs into a de-facto league tournament.

In a Pentagame tournament under the Swiss system the matches could be two or three player matches; the adjustment would then be as described in the section above.

The algorithm of the Swiss system can be summarised as follows:

1. Rank the players in descending order.
2. Working from top to bottom, pair every player with the next one in the list against whom she has not yet played.
3. Start the matches.
4. Record the results,
5. Adjust the ranking (i.e. total scores per player).
6. Repeat from #2 until reaching a time limit.

Table 1.3 may serve as reference. There are a number of methods to speed up and improve this system, which to cover would exceed the scope of this overview.

On the back side, such a system requires some central organisation.

1.8. Journey to the End of the Night

This system is fun when you include a Pentagame challenge in a race game. Parties arriving at the venue are required to play. The winner is allowed to leave the table, while the loser remains and has to play the next arrival until winning herself. Thus a win is required to proceed in the wider race. A harder version of this requires player to win *twice*.

Technically, this system is a truncated version of the Swiss system, because the Swiss tournament rule ‘every player plays the next opponent down the list she has not played yet’ holds.

Since every game involves two parties, the number of games to be played equals the number of players times the number of wins required.

Pentagame 2-player Swiss Tournament

Place: ______________ Date: __________

Zero-sum games!

Rank *as of last round*	Player *name or initial*	Total Score *from last round*	Playing *new opponent: next best no yet played*	Points *pieces out 0,...,3*	Score *zero sum score 0,...,6*	New *copy t rou*
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						

Table 1.3.: 2-Players Swiss Tournament Table

The only ranking produced in this manner will be number of games needed until leaving the table. Since pairing is relatively random, this system is relatively unfair.

1.9. Summary

If you have few but committed players, or want to stretch the tournament over a long period, use the league system.

If you want exciting action in shorter span of time, organise a knock-out tournament.

In all other cases and particularly when you have very many participants, use the Swiss system.

Next we shall investigate how to set up a tournament to include both two and three player setups—a mixed tournament. Such settings are of course more challenging.

2. Mixed Tournaments

2.1. Introduction

How do games *across* settings compare—with other words: how to compare two player games with three player games and four player games? How do, for example, three points made in a two-against-two setup compare to three points from a three player setting? This is what we are dealing with here. Leading questions will be: How much time do these games take? How many points does one on average score in each? And how many times must players play in each setting for fair play? Thus, how long must rounds be?

We begin this Section with some theory, which will allow us to identify a rather simple receipe, so please bear with us.

2.2. Comparing setups

Let us clarify what we are talking about.

A **game** is just what it is: some λ people playing once.

By virtue of the rules, every player has the same amount of movements in a game. Because of this and because there are only two reasonable opening movements, it does not matter much who begins; there is no large first mover advantage.

But the *sitting order* of the λ players makes a difference for three or more players, because it is a disadvantage to play after a very strong player, etc. So the first thing to do is to swap seats or change direction. Thus three players play *twice:* clockwise, and widdershins. Four and five players have more possible sitting combinations and would have to play many more times. We call this a **match** and a match consists of games with all possible *orders* of players.

The number of games in each match m grows clearly factorially and is $m = (\lambda - 1)!$

When you change the sitting order in the above fashion *and* have everyone start once, you play a **round.** The number of games per round r is $r = \lambda!$ which can be a rather large number. But alas—there is no much first-mover advantage, so we don't actually have to do rounds. We can stick with matches.

Multiples of matches we call a **competition.** A competition is thus the 'mini-tournament' you typically have on one table on the course of one evening.

If you then also have rivals change from table to table and find new challengers, you enter the realm of **tournaments.**

We will now look at the properties of games, matches and rounds for n players.

Games: The duration of a game T is (more than) quadratic to player number λ: more players mean more pieces to think about for each player; and more player to wait for until you move again.[1] Let's assume that the duration of a game is $T(\lambda) = 5 \cdot \lambda^2 \cdot \text{min}$. That means $T(2) = 20\,\text{min}$, $T(3) = 45\,\text{min}$, $T(4) = 80\,\text{min}$, which are somehow realistic values.[2]

Matches: We have shown above that the number of required games for fair play grows factorially with player number: all permutations of the sitting order must be tried. But we don't care for who starts. So the number of games per match is $m = (\lambda - 1)!$, which grows quickly.[3]

[1] If we call the time per game $T(\lambda)$ we have $T_g(\lambda) \in \mathcal{O}(\lambda^2)$.

[2] This implies that each player thinks about each piece about 1/2 minute, or has roughly 1 minute per move.

[3] Rounds would even require $r = n!$ games, and these would be lengthy; so the total duration of a round would be $T_r = \lambda!\lambda^2$. This is why this is so impractical.

Table 2.1.: Number of games and approximate duration of play

		2-player	3-player	4-player	
				4 vs. 4	2 vs. 2
Game	duration	20m	45m	1h20 m	
Match	games	1	2	6	2
change order	duration	20m	1h30m	8h	2h40m
Round	games	2	6	24	8
+everyone starts	duration	40m	4h30m	2d	16h

We collect the results in Table 2.1. Of particular interest for our quest is the row ‘match’, because we seek to compare matches of different types. To do so we will consider points *per time*.

For comparing the scores we must make all games zero-sum games as described above. That means that the two player always share 6 points and the three players share 9 points.

The table contains all information we need to compare games in other ways. Let us leave aside the long and rather different four-player setup and also stick with matches (ignore rounds).

2.3. Recipe

We have found out how the duration of the various setups relates to the number of players. From this we can draw the following recommendations for aggregating results from different type of games:

- Easily comparable are two and three player setups.
- Use zero-sum scoring, so that the points within every game sum up to zero.
- Seat players. A round is 90 minutes.
- Three players play one game clockwise and one counter-clockwise. They share $2 \cdot (3 \cdot 3) = 18$ points. Triple this score. The average per player is then 18 points. Use Table 2.2.

- Two players play 6 times on another table and share $6 \cdot 2 \cdot 3 = 36$ points. The average per player is also 18 points. Use Table 2.3.
- You can now simply add and compare scores and re-arrange players according to your tournament system. Use Table 2.4.

Pentagame Tournament - Three player match

Sheet No.: ______ Round No.: ______

Place: __________________ Date: ____________

Zero-sum games!

		Player 1	Player 2	Player 3
Game	Name:			
$\sim 45min$	Abbreviation:			
1^{st}	Pieces out			
clockwise	Score			
2^{nd}	Pieces out			
widdershins	Score			
RESULT	Total Score:			
	$(\times 3)$			
	Rank			

Table 2.2.: Three-player match table
Zero-sum games: Points you let go go to the player moving before you.
Copy total score ($\times 3$) into Tournament table

Pentagame Tournament - Two player match

Sheet No.: ______ Round No.: ______

Place: __________________ Date: ____________

Zero-sum games!

		Player 1	Player 2
Game	Name:		
$\sim 15min$	Abbreviation:		
1st	Pieces out		
	Score		
2nd	Pieces out		
	Score		
3rd	Pieces out		
	Score		
4th	Pieces out		
	Score		
5th	Pieces out		
	Score		
6th	Pieces out		
	Score		
RESULT	Total Score		
	Rank		

Table 2.3.: Two player match table
Zero-sum games: Points you let go go to your opponent.
Copy total score into Tournament table

Pentagame Mixed 2- and 3-player Tournament
Place: ____________ Date: __________
Zero-sum games!

		Pl. 1	Pl. 2	Pl. 3	Pl. 4	Pl. 5
	Name					
Round	Abbreviation					
$\sim 90min$	Initial rank					
1st	Playing against					
	Factor (1 or 3)					
	Score × factor					
	(New) total					
	New rank					
2nd	Playing against					
	Factor (1 or 3)					
	Score × factor					
	New total					
	New rank					
3rd	Playing against					
	Factor (1 or 3)					
	Score × factor					
	New total					
	New rank					
4th	Playing against					
	Factor (1 or 3)					
	Score × factor					
	New total					
	New rank					
5th	Playing against					
	Factor (1 or 3)					
	Score × factor					
RESULT	Total Score					
	Final Rank					

Table 2.4.: Mixed Tournament Table

Part VII.

Complexity

> Many combinatorial theorists [...] instead of studying particular games for which clever strategies can be demonstrated, they try to prove that certain classes of games are **hard** in the sense that any algorithm for playing all of them correctly must necessarily take a very large amount of computation.
>
> —Berlekamp, Conway, Guy [10]

1. Introduction

It is not so much playing but creating games that is a particularly human activity. Nightingales sing beautifully, but only humans compose, which is play according to rules.

> Nulla bestia talem habet cogitationem inveniendi ludum novum.
>
> *'No animal has the intellect to invent a new game.'*
>
> —Nicolaus Cusanus [41]

Only we humans *agree on games*, in that we come up with rules that draw limits to what is allowed [29]. Only humans teach games, or rules in general, one another. To agree on certain rules and to act within such a self-chosen framework *is* civilisation, and culture is its product [43].

A game is a competitive activity with defined rules.

Among the rules are usually spacial, temporal or material restrictions: run from here to there, trade on this floor, participate at this time; use these pawns, argue but fight not, and so forth.

Games can be classified in many ways. Film and theatre, sports, gambling, card games, fantasy games, computer games, even auctions and our behaviour within the social sphere can classify as games [13].

There are three approaches on games and play: 'theory of play', 'game theory' and 'social choice theory'.

Theory of play looks at the social function and psychology of games; we shall not be dealing with this [85].

Game theory is a branch of mathematics, concerned with the ideal moves.[1] It has applications in topics as diverse as theology [2], economics [35] and evolution [81]. This approach is what we will deal with further down in this book, analysing Pentagame in Part IV.

Social choice theory is concerned with how individuals agree on rules; it understands the negotiation of and agreement on an optimum rule set itself as a game. A result of these 'meta-games', where games compete among each other, is that certain games prevail and become classics, while others are rather short-lived.

Participants prefer a maximum amount of freedom and maximum chances to win. There is a quest for the best rules, which must thus be fair and efficient: a minimum of rules that create interesting games. Those games that stand that test of time become classics: where simple rules create complex play [90].

2. Complexity Measures

2.1. Overview

The four qualities a good abstract strategy game must have are *depth, clarity, drama* and *decisiveness*:

> **Depth** means that human beings are capable of playing at many different levels of expertise;
>
> **Clarity** means that an ordinary human being, without devoting his career to it, can form a jud[e]gment about what is the best move in a given situation;
>
> **Drama** [I]t should be possible for a player to recover from a weaker position and still win the game;
>
> **Decisiveness** [I]t should be possible ultimately for one player to achieve an advantage from which the other player cannot recover.
>
> —Thompson [90]

[1] Thus, the German term *Spiel* grasps the concept perfectly.

Only games that have these qualities stand the test of time, are 'games that survive' [5] [4]. We have already proven some of these qualities, namely clarity (by virtue of the rules) and decisiveness (in that it must always end). We shall now investigate depth (complexity) and drama.

There are ways to measure game complexity, resulting in a number of different complexity measures. The first attempt to define such measures was made by SHANNON in respect to Chess [**?**]. He introduced the idea of an 'average branching factor' and of 'game tree complexity'. A milestone in the literature is the much quoted PhD thesis of ALLIS [5], a pioneering attempt for generalisation and the comparison of different games.

A board game is by definition a game played on a board. It has a start position or state and an end state, and players need to reach this end state as fast as possible. Games can be classed depending on how much information is available and players can be classed how intelligently they play.

Such games or labyrinths can be more or less complex. This complexity *somehow* depends on the game's qualities, nameley:

1. the complexity of the graph (board), which consists of nodes and vertices,
2. the number of (available, different) pieces,
3. the rules.

How can we possibly find a measure for this complexity? A number of possible measures for a game's complexity (Table 2.1) are commonly used in the literature (e.g. [36] [5] [100]). They are all to some degree problematic in that there as convincing as they seem, no precise unique definitions for them exist, and many of them they can at best be estimated or empirically measured, where methods used vary from author to author. The values per game depend on certain assumptions, and often they depend on one another (Table 2.2).

State Space Complexity (SSC or $\mathfrak{S}$) is the number of states the game can take and thus a straightforward combinatorial measure depending

Table 2.1.: Common complexity measures

Name	Abbr.	Description
State Space Complexity	SSC	Possible positions
Average Branching Factor	ABF	Options per move
Game Tree Complexity	GTC	Size of game tree
Complexity Class	CC	'Hardness' for an algorithm

on nodes and pieces alone, but is independent of the number of moves. But not all states 'make sense', though the absolute maximum of legally possible positions is an upper bound. Depending on how me define 'make sense' we will be able to find lower measures. Each piece has a clear direction in which it strives and another that would be contradictory to rational play.

Average Branching Factor (ABF or F) is a far more problematic measure, since this is the assumption that since per move every player has only a certain number of options, this number of options per move could somehow be averaged.

This assumption does not seem reasonable to me since the number of available moves varies considerably over the course of the game; most notably, games tend to have very few possible opening moves and very few possible end moves with a maximum of choice appearing in the middle. Thus, the branching factor (function) should be based on and dependent of the SSC; we seek a method to achieve this in Section 2.3, but assume a constant ABF in the sections further down.

Game Tree Complexity (GTC) is typically a purely derived measure, which simply elates F to the power of the number of assumed typical moves m. It thus depends on the ABF.

GTC is problematic for two reasons. First, the game duration needs to be estimated and is typically just statistically known. Second, branches of the game tree will cross; the same position can occur by moves in a different temporal order. This would have to be considered for a proper comparable measure. Otherwise, GTC is just a fancy large number.

Finally, the Complexity Class (CC) measures what kind of computational time an efficient algorithm would take to solve a problem. Here, the question is how hard it is to solve Pentagame, which is at the time of writing an unsolved game.

Table 2.2.: Complexity measures dependencies

Abbr.	Basis
SSC	combinatorics
ABF	observation; *better: SSC*
GTC	ABF and duration
CC	?

To conclude: all these measures are somehow fancy numbers, but across games they are not defined in a hard sense. We will discuss them in a little more detail now. To do so, we set out with a very simple model of games, and again look at the one-player case before moving on to more complex settings.

2.2. Toy model: game $\mathfrak{r}$

Every game is a race game: the objective is always to solve a riddle in the most efficient way. At board games, this regularly is a matter of speed, where the two opponents hinder one another from reaching the goal first.

So if we describe a single player game as a race through a labyrinth, this labyrinth is static; in a two-player game, the labyrinth that one player faces changes with each move of the opponent.

You are invited to a simple solitary race game that we shall call $\mathfrak{r}$. It is 'played' on an 3×3 checkers board with 1 checkers piece. The piece may move from any square to any of the 6 adjacent squares.

You start by entering diagonally or straight at one side. Then you may move in any which direction. The game ends once you have reached the opposite side, where you remove the piece from the board again (Fig. 2.1).

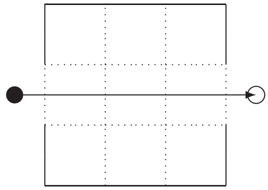
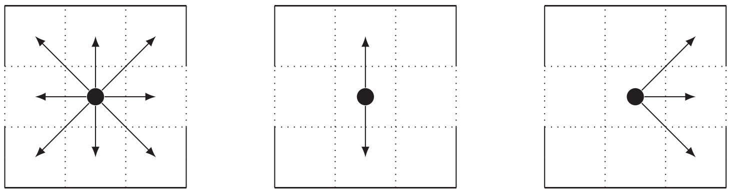


Figure 2.1.: Racing through an empty board (Game '$\mathfrak{r}$')



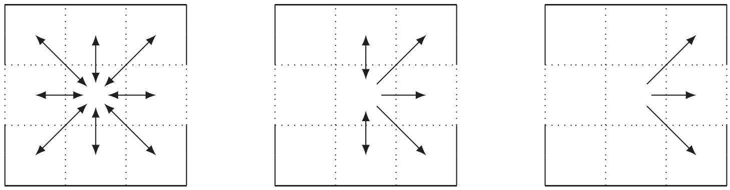
(a) 8 legal moves (b) 5 non-irrational (c) 3 rational

Figure 2.2.: Legal, non-negative and positive moves in '$\mathfrak{r}$'

There is obviously one initial state, followed by three possible states. Then there are many ways to proceed. But at the end there will be just 3 possible last states again, followed by the end state.

The absolute number of possible states this game can take depends on the number of pieces and stops. This is the State Space Complexity $\mathfrak{S}$ of the game. In our example it is easy to see that $\mathfrak{S}_{\mathfrak{r}} = 3 \cdot 3 + 2 = 11$. We observe that $\mathfrak{S}$ grows factorially with the number of stops and pieces z; $\mathfrak{S}(z) = \mathcal{O}(z!)$.

So it is straightforward to find the size of the State Space $\mathfrak{S}$. It is harder to find the duration of the game m and the Average Branching Factor F, since they depend on behavioural assumptions. How pieces behave depends on the rationality of the player. Remember that it is legal to move in any which direction. But the task is to be efficient in the pursue of the goal. Thus, under the assmuption that player play rationally, only moves will appear that actually bring the piece *forward* Fig. 2.3. Thus, we can now find an estimate for the Average Branchig Factor. Purely speaking, the ABF would be $F_{\mathfrak{r},sup} = 8$, since there are 6 legal moves. But we now see that there are actually only



(a) Bi-directed (b) Semi-directed (c) Directed

Figure 2.3.: Directions in '$\mathfrak{r}$'

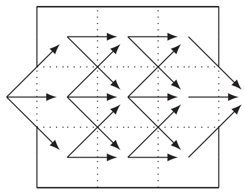


Figure 2.4.: The Directed State Space Graph for '$\mathfrak{r}$' is a DAG

$F_{\mathfrak{r},rational} = 3$ rational options. It comes to no surprise that the ABF *under the assumption of rational play* is a little less than half that of all legal moves (abstracting from the corner positions).

With this assumption we can draw the State Space of $\mathfrak{r}$ (Fig. 2.4). What we get is a Directed Acyclic Graph (DAG) that has one initial position and one end position. We see that in Fig. 2.4 all positions appear just once, but we also see that each position can be reached by a number of different paths.

We could also draw the Game Tree of our game $\mathfrak{r}$. This is nothing else but the State Space 'folded out'. The Game Tree Complexity of this game is the number of possible paths through the State Space. If there would not be the upper and lower border of our checkers board, it would be $GTC_{\mathfrak{r}=3^3=27}$.

So now the game $\mathfrak{r}$ is 'solved' in that we have found the values for the complexity measures (safe the Complexity Class, which is trivial).

Now we can expand the argument from a one-player case to the two-player case. Imagine another player enters from the right, and player take turns. Then the game becomes a little more challenging.

In this trivial case, the State Space would immediately be twice as long, since the game would took twice as many plies to end; each row of the State Space would be played by another player. It would also cease to resemble the board. In addition, depending on the opponent's move, some states may now be out of reach; so the State Space would become more complicated. And in really working games it would indeed become vastly more complex.

Nevertheless, this State Space would still only have one node per possible permutation; and it still would have a single starting point and a single end point, with vastly more branches in the middle. The shape of the State Space must be given by some function over the number of plies.

2.3. A general parabolic state space model

What we don't quite know is the exact shape of this function; but we can safely assume that it will have a peak in the middle, where entropy is maximum. Thus, we can assume the shape of a parabola: Fig. 2.5.

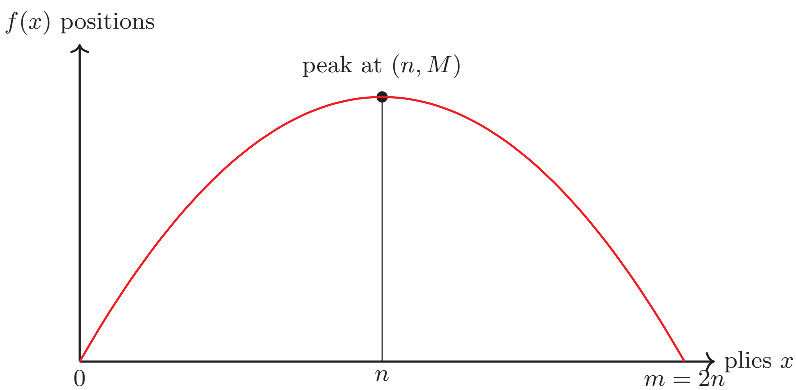


Figure 2.5.: Parabolic State Space model

What we know of this function is:

1. The integral, which is $\mathfrak{S}$;

2. The zeros at ply 0 and ply m,
3. A peak at ply n.

We also know that it is steep. It would of course be interesting to know or measure this steepness k.

Games typically have a more or less constant duration of moves. Actually, if players play perfect, then the game would always have the same length (and result); this does not mean that there cannot be more than one such perfect game. In our game $\mathfrak{r}$ with two players a number of possible games can be played, but the duration would always be 8 plies. For unsolved games this number is something we can only guess, or measure statistically.

We will also have to find the number of possible states at ply n; this is the number of positions with maximum entropy, and we call it M.

Since we know two zero points, 0 and m, we can write the parabola in the form

$$f(x) = k(x+0)(x-m) \tag{2.1}$$

where k is a stretch factor to be determined: the kurtosis. Using $m = 2n$ this can be written more conveniently as

$$f(x) = k(x^2 + 2n \cdot x) \tag{2.2}$$

For symmetry reasons we assume that the vertex of options M is reached at $n = m/2$ plies. We thus have

$$M = k(n+0)(n-2n) \tag{2.3}$$

This we can use to find a formula for the stretch factor a of the parabola:

$$k = \frac{M}{n \cdot (n-2n)} = -\frac{M}{n^2} \tag{2.4}$$

This yields

$$f(x) = -\frac{M}{n^2}\left(x^2 - 2nx\right) \tag{2.5}$$

Next we can profit from the fact that we know the total number of states $\mathfrak{S}$ by combinatorics alone. $\mathfrak{S}$ results from the number of stops z and pieces p (and the rules) alone:

$$\mathfrak{S} = \mathfrak{S}(z, p) \tag{2.6}$$

In our simple case of game $\mathfrak{r}$ we only have one piece and 11 stops, so $\mathfrak{S}_{\mathfrak{r}} = 11$, a very small number. But for any interesting real and typical game, it will be a number much larger by order of magnitude.

In perfectly complete games, all these stages appear somewhere in the state space (by definition), thus somewhere between ply 0 and ply m. Thus this number $\mathfrak{S}$ equals the surface under the parabola between 0 and m.

$$\mathfrak{S} = \int_0^{2n} -\frac{M}{n^2}\left(x^2 - 2nx\right) \mathrm{d}x = \frac{4Mn}{3} \tag{2.7}$$

Thus follows

$$M = \frac{3\mathfrak{S}}{4n} \tag{2.8}$$

which allows us to write Equation (2.5) as

$$f(x) = -\frac{3\mathfrak{S}}{4n^3}\left(x^2 - 2nx\right) \tag{2.9}$$

What we have now is a function that describes the *size* of the state space relative to the number of plies. Its derivatives are:

$$f'(x) = \frac{3\mathfrak{S}(n-x)}{2n^3} \tag{2.10}$$

$$f''(x) = -\frac{3\mathfrak{S}}{2n^3} \tag{2.11}$$

Thus we can calculate how many positions exist after x plies. In other words: When we have played x plies, how many different patterns can be on the board? Here we have a parabolic answer.

And now we can also see the stretch factor:

$$k = -\frac{3\mathfrak{S}}{4n^3} \tag{2.12}$$

Relative to $\mathfrak{S}$, which is (typically) large, n is (typically) very small; $\mathfrak{S} \gg n$ for typical games. Thus, in complete games our parabola is extremely 'steep'. Indeed, the steeper this parabola, the more rapid the game becomes complex, a quality we seek in games.

Since in reality we are dealing with discrete values (so that $x, (f) \in \mathbb{N}$) we will likely see a step function with some low values at the beginning and the end, and a large but more or less constant value for $f(x)$ in between.

The branching factor is $f'(x/2)$ and the *average* branching factor is

$$ABF = f'(n/4) = \frac{3\mathfrak{S}(n - (n/4))}{2n^3} = \frac{9\mathfrak{S}}{8n^2} \tag{2.13}$$

But S is typically very large, while n is typically small. If this theoretical value for the ABF is much larger than the observed number of options per move, the estimated value of the ABF, then *by far not all possible positions are actually achievable in a meaningful game.* This means the game is *incomplete,* and all meaningful games will be incomplete in this sense. This fits to observation: we can imagine countless positions that may never occur in meaningful games. In fact, if we scatter the material in any random oder onto the board, we are *most likely* to produce a position that does not appear in meaningful play.

The argument can be put on its head, calculating the (realistic) state $\mathfrak{s}$ space from an (observed) ABF:

$$\frac{8n^2}{9} \cdot ABF = \mathfrak{s} \tag{2.14}$$

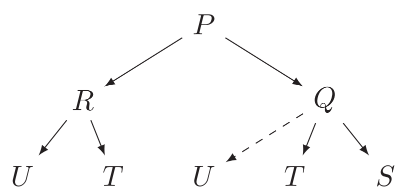
This is quite an interesting figure, because this allows to estimate the number of different positions *actually* available during *meaningful* game. Surprisingly, though, this number is typically low in order of magnitude.

3. Complexity of Pentagame

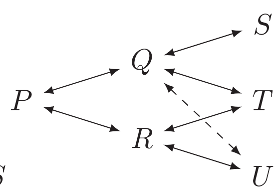
We will begin by looking at the *extensive form* of pentagame. This is a 'map' of all courses the game can take, and it is drawn as a tree. Each decision leads to a new branch in this tree. The leaves of the game tree are all possible games, the exact genesis of which can be read in the tree.

But in a game tree the same position or state can appear more than once. In other words, different points in this tree can represent the same position, but reached from different branches.

Thus, rather than mapping all possible games, it is possible to map all possible states (just once) and connect them, just like in a road map. Now every state can be reached from multiple other states. Such a diagram is called the *state space*. In the state space every state appears just once. Fig. 3.1 compares a game tree and a state space. From an initial state P two new states can be reached: R and Q, and so forth.



(a) Game tree



(b) State space

Figure 3.1.: Game tree vs. State space

In the extensive form or game tree lines never cross, and identical states can appear more than once (U, T). In the state space form lines do cross, but all states appear just once.

Note also that while a game tree grows in just one direction, the state space can (theoretically) be traversed in either direction. Indeed, it is the Ko-rule that makes all these edges 'one way'.

Game trees are usually depicted growing from top to bottom, while the state space is usually written from left to right; where the game tree has rows, the state space has colunns. The coloumns in which the states are ordered are ordered by the 'remonteness' of a position from the goal position.

We will now start with the tree form, or extensive form.

We will draw 'trees' as common in game theory, where they are called the 'extensive form' of a game (Fig. 3.2 and Fig. 3.3). We begin drawing at the top with noting Alice's initial move in the middle of the first row. We then proceed writing Bobs possible answers underneath; the result is a fork. The next row shows Alice's responses in further forks, and so forth. The numbers on each branch count variations of these options.

The swap on the ring or the replace on the pentagon are the only two sensible opening moves. Now we look at the *swap*.

When Alice swaps on the ring she must choose one piece to swap with. This must be one of her own pieces, anything else does not make sense.

The swap is popular, because then Alice moves two pieces at once (by convention pieces a and b), both of which still have many options, and does otherwise not commit to much. The board remains widely symmetrical. She has gained speed; but she has not gained much space, since after that opening move Bob still has all options he would have had she not moved at all.

Fig. 3.2 illustrates the options at hand: the first row is the initial swap, the second row Bob's answer, the third Alice's response. Thus the tree we sketch here shows the first three plies in a swap-game.

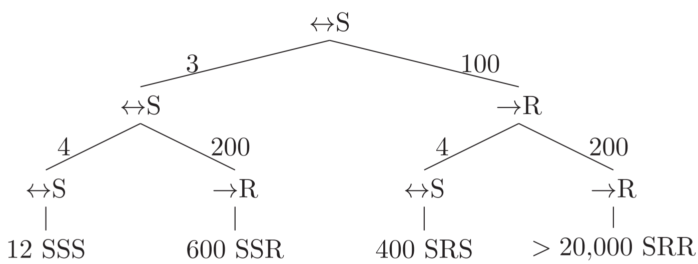


Figure 3.2.: The branch of 'swap games'

The second row shows Bobs two general options: to either to also *swap*, or to *replace* . If he swaps ↔S, he is on the left branch. He has three swaps to chose from: he can swap on an opposite, an adjacent, or the same arc as Alice did. If he plays *replace* →R, he moves into the pentagon. He has 10 different ways to do that and must then place the block on a line; there are 20 lines to choose from. That means he chooses one of $10 \cdot 20 = 200$ replace double moves. Since the board is still axially symmetrical half of these are equal, leaving 'only' 100 options.

The third row of Fig. 3.2 shows Alice's possible responses, where we again indicate the number of her options. Again, not all of her actions may be equally strong, though most of them are at worth considering. Now symmetry has vanished and there are 200 possible replace moves.

We can use this to give 'names' to the various swap openings: those that start with *swap-swap-swap* we call 'SSS' etc.

SSS There are 12 such openings, and they are very common. If you abstract from those where $\mathcal{B}$ swaps at the same arc as $\mathcal{A}$ there are only 6 such openings.

SSR Two swaps followed by a replace; there are 90 such openings;

SRS Swap-replace-swap, where $\mathcal{A}$ moves on the circle, but $\mathcal{B}$ invades the centre; there are 60 of these;

SRR Swap-replace-replace, of which there are more than 450 variants.

Altogether there are more than 21,012 different positions after just three movements in the branch of swap games.

What we see here is an exponential explosion of options in the first few plies.

Replace games are those where Alice starts with 'replace'. She then has to position the block on one of the 20 lines. Again it does not make any difference which colour she chooses.

Such replace openings are better than their reputation; Alice commits, Alice blocks a central node and thus directly two interesting options of Bob, and Alice has a block to determine the structure of the space. She gains space. The replace opening is a bold and honourable move.

As before we can draw a tree illustrating the first three plies of such openings (Fig. 3.3). Again the fist row is the initial move by Alice, the second row Bob's answer, the third Alice's response.

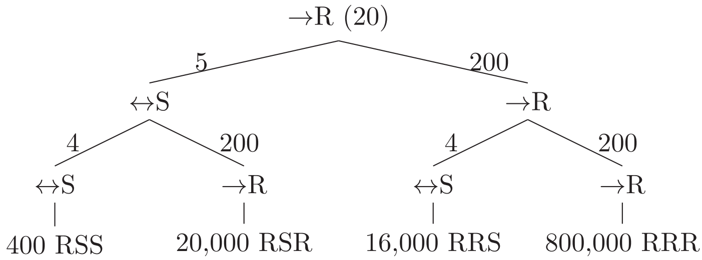


Figure 3.3.: The branch of 'replace games'

In comparison with the swap opening of which there actually is just one, there are 20 different replace openings, since Alice must one of 20 lines to place her block. Thus Fig. 3.3 begins with the factor 20 at the top.

Once Alice has played replace and set her block, the space is no longer symmetric. This means that the number of possible answers for Bob is effectively larger than in swap games. He has five options to swap ↔S. As for replace, he has 10 paths to play →R, then having him set a block on any of 20 lines; thus 200 options. The branch of replace games grows faster than of swap. There are 836,400 positions after just three plies.

What this shows us is that the large number of possible games makes it hard to learn opening theory in complete systematical way. We must resort to positional evaluation drawing on strategy and tactics (see Chapter 5); we can somehow 'prune' this tree and find out which of these options are actually strong than others.

While there are many options, some vanish quickly since every replacement closes one of the lines. Typically swapping options vanish first.

And each block disables somehow only one or two moves, the magnitude of the number of options is hardly affected. What →R does though is to limit not just the next, but all further moves—including moves by Alice, 'thinning out' the tree, adding structure. In comparison to 'swap'-games blocks get replaced one movement earlier, which results in less open lines.

Now that we have looked at the two branches of openings, we can compare Fig. 3.2 and Fig. 3.3. Recall that the numbers therein depend on our simplified model of the game that takes only nodes and lines into account. We can sum up the number of available positions in the lines of each of the figures (Table 3.1).

Table 3.1.: Development phase Branching Factor

ply	swap-branch	replace-branch
	No. of branches	
1	1	20
2	103	205
3	204	204
*4	*204	*204
⋮	⋮	⋮

We observe that after two moves, symmetry breaks; the number of options now need no longer be divided by two (for axial symmetry) nor by 5 (for radial symmetry). The branching factor stabilises. We see that both in the swap and in the replace branch there are 204 times as many options per move from move three on. That mean the game tree grows at the rate of

$$F(3) = 204 \tag{3.1}$$

at ply number 3.

Since this is independent of the history of the game, one can expect roughly the same *ever after*. This seems to be the number of options on a developed board, at least until the game enters another phase. We call it the Branching Factor of the Development phase, $F(3)$. The game enters this phase after one of $Q = (1 \cdot 103) + (20 \cdot 205) = 4203$ possible two-move-openings.

If the branching factor remains constant this is how many games there are playable in n plies:

$$p(n) = Q \cdot F(3)^n \tag{3.2}$$

If this tree continues to grow unrestrictedly in this manner, it will have a very large number of 'leaves'. Assume a game to take some 24 plies, then we get:

$$p(24) = 1.1 \cdot 10^{59} \tag{3.3}$$

If the branching factor is stable then this is the **number of playable games,** also called 'game tree complexity'. It is a number with 59 zeros. This number is larger than the number of atoms in a human, of stars in the galaxy and not that much lower than the number of atoms in the entire known universe.

This is certainly enough for more than a live time of play.

However we calculate this, the numbers are vast. Players cannot possibly foresee all possible games, so they must play tactically. Since strategy and tactics do not really seem to depend on the number of players, we will deal with that topic at once in Chapter 5.

Above we have viewed the game from its roots to its effects, mostly with the visual aide of tree diagrams. Such game trees show what stages the game can take and from where, given a certain beginning and given legal moves. Some of these may be better than others, but all finally lead to an end state, which can be thought of like a leaf of such a tree. We have also come to estimate that our tree has some billion such leaves; from the one source, the opening, the game can develop in that many different ways.

Now, we shall turn the perspective and begin from the outcomes. And we immediately recall that rather than a billion different outcomes, there are just a few: Alice wins, Bob wins, or they play a draw, for example. So somehow the game begins in the orderly fashion of the opening position, then develops on one of many million different paths (it diverges), and finally converges to an end result. Hence, the tree does not quite spread out; rather, it spreads out and (re-)converges, where different branches can re-unite towards the same outcome. It looks somehow like Fig. 3.4.

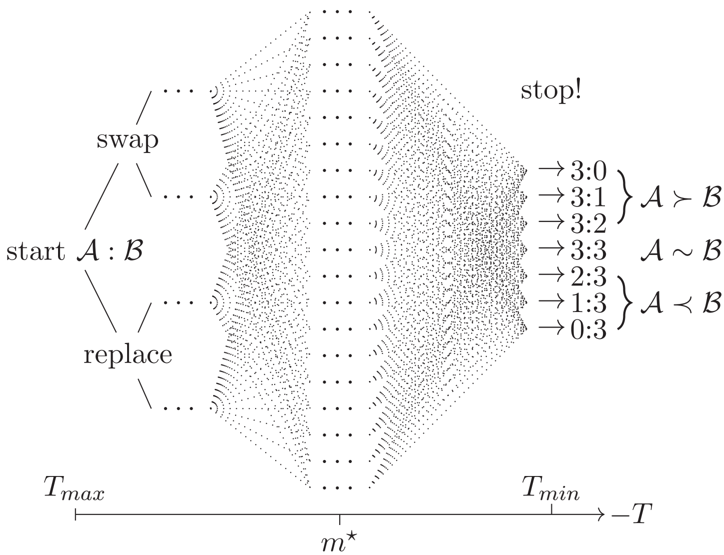


Figure 3.4.: State space (sketch)
Dotted lines sketch multitudes of nodes and paths.
The map need not be a complete graph.

In this figure, all states the game can reasonably take—positions—are vertices, linked by edges that show how the game can develop, just like in a flow chart. Such a map is conveniently called *state space*. The connection to the *game tree* is obvious: the game tree equals the state space 'spread out', so that it counts each game as a node rather than each position.

The State Space has a unique origin which is the starting position. It has an end defined by the 'sudden death'-rule. It has a theoretical end position which is also a unique position.

The middle we call the *peak*. All the positions on the peak will have maximum entropy between the ordered state of the beginning with maximal temperature and the end point of temperature zero.

The peak is where the state space reaches its largest extension and where entropy is maximum. There are nowhere more positions available than here.

The abscissa of Fig. 3.4 shows T for *Temperature* and t for time. The reasoning is the following:

The game develops according to certain rules. One of these is the imperative that pieces move closer to their goals until reaching them. A piece is called the 'hotter' the further it is from its goal. Thus the game starts at maximum 'temperature' and 'cools' until it ends. Notice that Pentagame is a *hot* game: it is always an advantage to move. It is also a game that first diverges, then converges.

Temperature can of course be measured in different ways; common measures include the number of moves a piece would need *ceteris paribus*, and the number of stops it is away from its goal. But however to measure it, it strictly falls with time. And time can be measured by number of moves (plies). This makes the graph a (weakly) directed graph.

The ordinate is a little less obvious. Here we see the number of possible states (after a given number of moves). This dimension can be interpreted as 'likelyhood'. A more 'likely' position—one more toward the abscissa—will at once have more possible 'ancestors' and more possible 'offspring' than a less likely position further away.

Every vertex on the graph could be associated a value, which is the weighted expected end result; this value is the position of the node from the abscissa.

The fact that this tree re-converges to certain outcomes, where it terminates, is what makes it a well-defined game: it is confined, and it ends; and it is of a complexity beyond 'control'. We will now investigate it a little further.

Our investigation of the State Space will rest on some fundamental values and assumptions.

Firstly, we count nodes and lines as the (most) *relevant* spaces n_{inf}.

The whole board has of course in fact n_{sup} spaces:

$$n_{inf} = 30 \tag{3.4}$$

$$n_{sup} = 100 \tag{3.5}$$

We will thus always have an upper bound that takes *all* stops and possibilities into account, and a lower bound that only looks at lines and nodes.

Secondly, there can be $\lambda = \{1, \ldots, 4\}$ players. We restrict our investigation for sake of simplicity to the two player case and set

$$\lambda = 2 \tag{3.6}$$

Thirdly, we need some value for the length of the State Space in time. Here we can build on our work from Section 2.6 and refer to experience. Playing time of course depends on the number of players, most likely in some exponential way; but we only deal with the two player case here. A typical two player takes this many plies:

$$M_{cut} = 24 \tag{3.7}$$

Now behold: this is when a game with the rule 'three out wins' ends. For reasons that will be clear soon we need a value for the duration such a game takes if played to the 'bitter end' of all pieces out. $M_{cut} = 24$ means that 6 pieces have needed 24 plies to reach their goals; 10 pieces would then need

$$^{*}M = \frac{10}{6} \cdot 24 = 40 \tag{3.8}$$

plies to reach their goals. In other words, we assume each pieces moves on average four times to reach a goal. For reasons that will become clear soon we define

$$^{*}m = \frac{^{*}M}{2} = 20 \tag{3.9}$$

We use the asterix (*) to remind us that these figures are best *ad hoc* guesses; upcoming results will be sensitive to this guess.

With this we can commence our analysis.

The number of physically possible positions of pieces on the board can be calculated with the multinominal coefficient. If a monkey were to place pieces at random on nodes and lines only, this is how many positions could appear.

The general case for λ players and 5 blocks is:

$$\mathfrak{S} = \frac{n!}{(n - 5\lambda - 5)! \cdot 5!} : 10 \tag{3.10}$$

The division by 10 is to take axial and radial symmetry into account. We insert $\lambda = 2$ and $n_{inf} = 30$ and $n_{sup} = 100$ from above.

$$\mathfrak{S}_{inf} = \frac{30!}{15! \cdot 5! \cdot (1!)^{10}} : 10 \approx 1.69 \cdot 10^{17} \tag{3.11}$$

By definition, the set of all these possible positions is the set of vertices in the state space.

The **state space complexity (SSC)** is then defined logarithmically:

$$SSC = \log_{10} \mathfrak{S} \tag{3.12}$$

The values we get are collected in Table 3.2. The actual or 'realistic' value is likely in the middle if you assume, for example, that every line consists of three *significant* different positions: beginning, middle, or end.

In any regard this number of available positions is very large. The number of atoms in a human body or the number of stars in the galaxy stands at about the same order of magnitude ($\approx \log_{10} x = 27$).

Table 3.3 lists the SSCs of a number of games given in the literature.

Table 3.2.: Pentagame State Space Complextiy ($\log_{10}$)

$SSC_{inf} =$	17
$SSC_{sup} =$	26

Table 3.3.: State Space Complexity (SSC) of games $\log_{10}$

Game	SSC	
Go	170	
Chess	47	
Othello	28	
Pentagame	26	upper bound
Pentagame	17	lower bound
Nine Men's Morris	10	

Now that we know the number of vertices in the State Space, we would like to know more about its shape. This echos our theory from Section 2.3.

Recall that at maximum temperature at the game's beginning there is just one position. At the end, when *all* pieces have left the board, there are negligible few positions, since only blocks are left. All other positions appear in between. They are spread out over the total time (in movements) it takes for the game to reach zero temperature, or every piece to leave the board.

The question on how exactly the positions are distributed over the course of the game is tricky. But in any case there is just one starting position and one end position, so there will be a peak somewhere in between.

The actual function that gives us number of possible positions after x plies is unkwown; what we know is that it has zeros at ply 0 and at ply m. The domain is $\{0 < t < 20\}$ but the range $\{0 < p < 10^{26}\}$, thus the curve is very steep.

It could be sinosoidal, or it could indeed be parabolic as discussed in Section 2.3 and sketched in Eq. (2.5).

Imagine someone blindfolded placing the pieces on random stops. The outcome will most likely be a 'middle position' and only very unlikely the beginning or an end position. This hints towards the shape of a probability function with a very high kurtosis; Fig. 3.5.

However, most literature on the complexity of games begins with

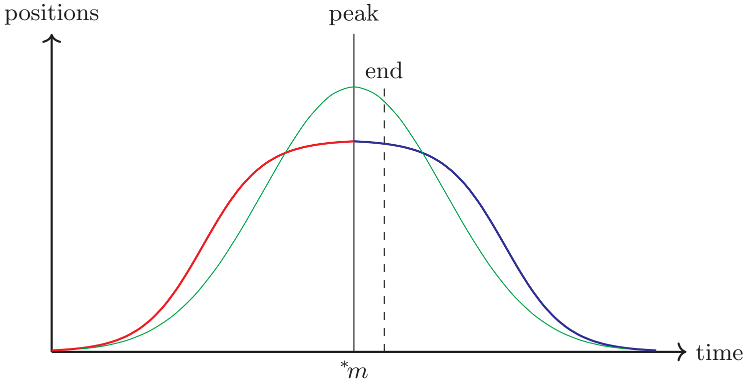


Figure 3.5.: Gaussian and sigmoid shaped

the assumption of a somehow constant Average Branching Factor. *If that is correct, the state space would have an exponential shape.*

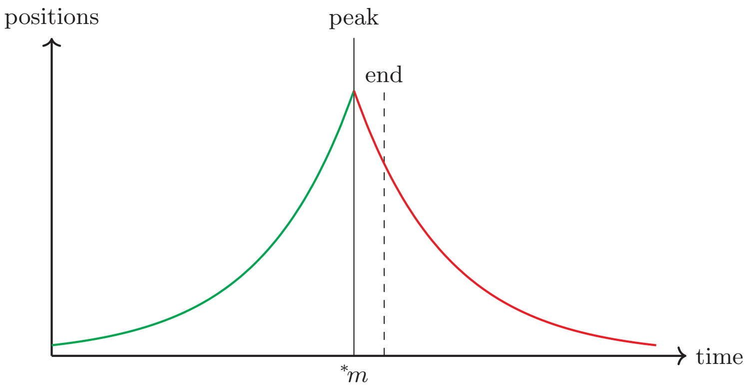


Figure 3.6.: Exponential

We begin with the assumption that every time a player moves she has approximately the same number of options *until the peak of the game*. After that peak and towards the end her options diminish in a mirrored fashion. This means we assume exponential growth up to the peak, as sketched in Fig. 3.6. The peak is the point where the number of possible positions reaches a maximum. We assume that to happen in the middle of the game. Consequently the peak is reached after

$$^{*}m = 20 \tag{3.13}$$

plies, shortly before the game ends due to the cutoff rule.

Now that we know the number of vertices and have a working hypothesis for the shape of the State Space, we can deduce a lower bound of the number of edges in it.

Every vertex (except the first) must be connected to *at least* one earlier vertex. From this we can get a lower bound of the number of edges; it corresponds to a tree that is not meshed, but where every vertex has exactly only one forerunner.

The area under the curve(s) in Fig. 3.6 must equal the absolute number of positions available.

$$\int_0^{2^{*}m} p(t)dt = \mathfrak{S} \tag{3.14}$$

Remember that half of all states must be to the left of the peak. Thanks to our assumption of exponential behaviour we are dealing with a geometic series. This allows straightforward calculation of the particular growth rate $F_\star$ that agrees with the total number of positions $\mathfrak{S}$. It is the lower bound for the Average Branching Factor of the game, and it simply computes as a partial sum:

$$\sum_{k=1}^{^{*}m} F_\star^k = \frac{\mathfrak{S}_{inf,sup}}{2} \tag{3.15}$$

The lower bounds for the **average branching factor of Pentagame** $b_\star$ under these assumptions are collected in Table 3.4.

Table 3.4.: Pentagame ABF lower bounds

$F_{\star,inf}$	≈ 7
$F_{\star,sup}$	≈ 20

That would be as if you can look at the board, see a position, and completely deduce how it came into being. Note that these values are independent of the number of options perceived by the player.

These lower bounds have an interesting interpretation: the actual number of real options per ply is not that great.

Since in reality some combinations of moves result in the same position there will be more edges in the State Space than this, as there are more options for the player to choose from that this. Thus the actual branching factor will be greater. How big is it, how much bigger than this lower limit, and what can this tell us?

Assume the game to be in one position. How many positions can be reached from there? Or generalised: how many possible positions follow after any given possibility? This is the quest for the branching factor.

If it is constant, the number of available positions grows exponentially. If it is not constant, what may it be on average? Game theorists like *Average* Branching Factors. Unfortunately the very concept of such an average implies a severe simplification, since the actual branching factor may change over the course of the game.

We have found $F(3) = 204$ for the first couple of plies; but now consider this: a piece at its origin has four ways to go alright, but later some of the paths available will leave it away from this goal.

Thus the large branching factor of the development phase will later diminish *at some rate*. A more precise aporoach would be to talk about a *branching function* rather than just a factor.

If we assume the ABF decline is constant, the Average Branching Factor is

$$^{\star}F = \frac{F(3)}{2} = 102 \tag{3.16}$$

Table 3.5 collects our results.

The first row in Table 3.5 shows the results from the analysis of options from Section 2.8. We have here a lower and an upper bound for the ABF; the actual ABF must be between these two extremes.

The second row is the result from **??** based on positions. The lower bound results from counting just nodes and lines. The upper bound takes all stops into account. Both of them are lower bounds in fact,

Table 3.5.: Branching Factor Estimates

	symb.	inf	mid	sup
based on options	F	6	$^{*}54$	3040
based on positions	$F_{\star}$	7		20
based on openings	$F(3)$			204
based on $F(3)$	$^{\star}F$		$^{*}102$	

since they show the value the ABF would have to have at least to allow for all positions. These are 'hard facts'. This proofs that $ABF \geq 20$.

The third row finally recalls the results from above, where the options in the third ply have been counted in respect to nodes only.

The fourth row lists the value we get under the assumption that the ABF starts at our measured value of $F(3)$ and then diminishes linearly to zero. Since it will on average be $^{\star}F$ we can use this Mid-game F_{mid} as best estimate for the overall ABF of Pentagame.

Table 3.6 lists the Average Branching Factors of some games. It seems the complexity of Pentagame is comparable to some of our 'harder' classics, Go and Chess.

Table 3.6.: Average Branching Factors (ABF) of games

Game	ABF	
Pentagame	3040	upper bound
Go	250	
Pentagame	102	best estimate
Chess	30	
Pentagame	20	lower bound
Othello	10	
Nine Men's Morris	10	

The ABF is the outdegree of the game tree of our model. This means that if a player who considers all future potential moves on the search for her best move should need about as many times as long to consider one more move further down the tree.

As a corollary we can compute the number of peak positions $\hat{p}$. This answers the question for the 'thickness' of the state space. Under the assumption of exponential behaviour there are

$$\hat{p} = F^{*m} \tag{3.17}$$

positions at the peak.

For the sake of comparability we define the Peak Size PS logarithmically:

$$PS_F = \log_{10} \hat{p}_F \tag{3.18}$$

Now we can plug in the lower bounds $F_\star$ and our best estimate for $F_\star$. The results are in Table 3.7.

Table 3.7.: Peak Size ($\log_{10}$)

PS_7	16
PS_{204}	46

This is the number of peak positions with maximum entropy. In these positions mobility of pieces should be maximum and the average temperature medium.

Note that the value of the ABFs depends crucially on the number of stops taking into account; for $F_\star$ we have only counted nodes and lines as reasonable stops.

Further note that the value of all ABFs depends crucially also on the average game length $^*M$ assumed.

Literature on game complexity likes comparing the Game Tree Complexity (GTC) of games. This is a misnomer, because it is a measure of magnitude, not of complexity. Anyway, the GTC is usually defined logarithmically as follows:

$$GTC_G = \log_{10}\left(ABF^M\right) \tag{3.19}$$

The GTC measures how many possible games can be played. We ca simply plug in the values we have found for the ABF and get a number. (We will look at the actual *complexity* further down.)

We can plug in our lower limit $b_\star = 20$, our best estimate $^\star F = 102$ and our upper bound $F_{sup} = 3034$. The results are collected in Table 3.8.

Table 3.8.: Game Tree Complexity ($\log_{10}$)

GTC_{inf}	52
GTC_{mid}	80
GTC_{sup}	139

These estimates are somehow 'Shannon numbers' for Pentagame. The middle value is in the same order of magnitude as the number of atoms in the universe. Whatever the actual value, it is clearly beyond countability, if not common comprehensibility.

Table 3.9.: Game Tree Complexity (GTC) of games ($\log_{10}$)

Game	GTC	
Go	360	
Pentagame	139	upper bound
Chess	123	
Pentagame	80	best estimate
Othello	58	
Pentagame	52	lower bound
Nine Men's Morris	50	

Table 3.9 lists the GTCs of a number of games for comparison as found in the literature. The comparison is of course pretty rough given the methodological differences in the determination of the ABFs, but a clear-cut comparison is beyond the scope of this work. Suffice

to remind that the GTC depends on the number of spaces technically available, the number of pieces, the number of (realistic?) options per piece, and the duration of play.

The branching factors will not remain constant over the course of the game. Once any piece has moved some distance towards its goal only about half of of options will be hot moves and the other half cold moves. One would thus expect the ABF of the game to *peak* at $F(3)$ and then diminish. This diminishment of options signals that the game has entered another phase, the Mid-game.

Sharp play 'prunes' the decision tree, and in decent play we will have much less options than in the random model above. Sharper play means less branching and thus less edges; but at the same time, sharper play leads to longer game duration, which may well offset this effect on GTC.

The pinnacle of play, absolutely perfect and omniscient play, should reduce the state space to a single line (as per ZERMELO'S theorem). Nevertheless, this would require full-search knowledge of the entire state space. This is not necessary. Once the player plays so sharp to actually consider only the mystical average $F_\star = 7$ consequences, each vertice will correspond exactly one edge—and play become 'perfect'.

It is not possible to perfectly play minimax without considering these $F_\star = 7$ average possibilities, but possible with considering these.

Another commonly used and more efficient search, optimally ordered α-β-search, will reach a complexity in the order of the square root of the GTC [49]. Given the size of the GTC this is of not much help though.

To speak of the 'complexity class of a game' is inaccurate, since it is not the game that has a complexity class, but (any) algorithm to solve it. Of special interest is the algorithm that solves the problem by complete calculation of the entire state space and / or game tree.

A machine, demon or human with complete knowledge of the state space can always decide which move is best simply by searching this library. Since the tree grows exponentially, this search will take time, and the time required per search is exponential to the length of the state space. We speak of polynomial time; and any problem that can be used in polynomial time is called a *problem in P*. Now since *all* problems that may occur in Pentagame can be reduced to P, Pentagame is P-complete.

To calculate any given further step means multiplication of the existing width of the tree by the ABF:

$$\mathcal{O}(c + c \cdot F + c \cdot F^2 + ... + c \cdot F^M) = \mathcal{O}(F^M) \qquad (3.20)$$

A more precise analysis would have to move on from our simplified models and model the branching functions more subtly. If the ABF is not constant, but the exponent in Eq. (3.20) itself growing with the duration or size of the game, then Pentagame is in EXP. But in any case, there clearly is exponential growth, albeit the exponent is the duration of play, which in Pentagame is relatively short.

Like all two-player combinatorial games, Pentagame falls into the class of PPAD, which is the class of all NP-hard problems for which there must be a solution [21]. That a solution exists for Pentagame is proven by ZERMELO's theorem, and that it is hard is demonstrated by the above.

The sizes of the state space as estimated above should make it tractable by brute force computing [36]. The question to be answered is of the type 'is htere a move for Alice so that for *every* move of Bob there is a move of Alice so that for *every* move of Bob there is a move of Alice...' [32]. The calculating power to solve this perfectly exceeds human strength, at least until the end of the game is near. The game is still 'in wonderland' [32]. As a result we have that Pentagame will remain challenging for human players.

4. Drama

A good game should also have drama: it should be possible for a player to recover from a weaker position and still win the game. Victory should not be achievable in a single successful blow; the suspense should continue through an extended campaign. Otherwise

> an early disadvantage makes the remainder of the game uninteresting: the doomed player rightly guesses that the puzzle he is trying to solve has no solution and that thinking about it is futile.
>
> —J. Mark Thompson

So far we have investigated the *vertices* of the state space independently from the choices players make. These choices result in *edges* that connect these vertices. Comparing the number of edges and the number of vertices measures meshedness, or redundancy.

If the number of edges equals the number of vertices, then paths never branch out. The more edges relative to vertices, the more paths branch out and intersect with one another, creating further possible paths.

We have a model for how many actual positions there are in general ($\mathfrak{S}$) and at the peak ($\hat{p}$); thus, we have investigated the positions or nodes within the state space. We now turn toward the edges that connect these positions. So rather than at positions, we now look at the paths or *games*: the absolute number of games ($\mathfrak{G}$) and the number of games at the peak ($\hat{g}$).

Now we have set down in the previous sections that rather than branching out further and further without any intersections, we know that the state space reaches its peak in the middle. So we assume the number of branches to behave like the number of options in **??**; exponential growth, a peak and exponential collapse. We can thus for sake of comparison calculate how many options or moves there should be at $^*m$, the peak. We define Peak Options PO logarithmically:

$$PO_k = \log_{10} F^{^*m} \tag{4.1}$$

Now we simply insert the values for the ABF we have found; the lower bound, the best estmate, the upper bound. The results are collected in Table 4.1.

This is not a number of *positions,* but of *moves.* It is an estimate for the number of edges at peak.

The next sensible thing to do is of course to compare the number of vertices at peak with the number of edges at peak. Table 4.2 does

Table 4.1.: Peak Options ($\log_{10}$)

PO_{20}	26
PO_{102}	40
PO_{3034}	69

just this; the first row counts just nodes and lines, the second row all stops.

We observe that this number of peak options is larger than the number of physically available positions. There are more moves leading to these positions than there are positions, and also more moves leading from them then there are positions.

Apply the pigeonhole principle: since there are more games than positions, some or many moves (or games, for that matter) must result in identical positions. For instance, Alice or Bob may make moves in reverse order, or a player manage to turn the odds.

Table 4.2.: Vertices and Edges at peak ($\log_{10}$)

PS		PO	
16	$\ll$	26	lower bound
		40	best estimate
30	$\ll$	69	upper bound

Now we can apply the pigeonhole principle: we see that the number of edges leading to a certain vertex is larger by several orders of magnitude. This means no less that there are legions of ways to reach a given middle position.

Notably the number of vertices is a ‘physical’ fact. The number of edges or paths in contrast crucially depends on how many options a player may actually consider worthwhile. A more irrational player will choose from more, a sharper player will choose from less options k. But in any case there are clearly more edges than vertices—*by* *order of magnitude.* Many paths must cross the same vertices; in other words, when player choose from $k > b$ options per move, then many games share the same position. This means that the game can morph from one game into another often.

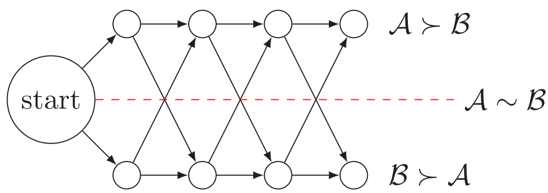


Figure 4.1.: Drama: more edges than vetices

Next imagine a horizontal line bisecting the state space in an upper and a lower half so that in all states above this line $\mathcal{A} \succ \mathcal{B}$ and in all states below this line $\mathcal{B} \succ \mathcal{A}$ (Fig. 4.1). Now as the state space opens up and re-converges, it likely contains paths that cross this line a number of times; and the higher the number of edges vs. vertices, the more often such crossings appear.

This shows that Pentagame has THOMPSON's quality of **drama:** who is in the lead may still change over the course of a game.

The fact that there is not only a binary outcome ($\mathcal{A} \succ \mathcal{B}$ vs. $\mathcal{B} \succ \mathcal{A}$) but that there are variable outcomes in score (3:0, 3:1 etc.) adds to the drama since there are more such decisive lines that can be crossed, and crossed multiple times within a game. Every such crossing can aptly be called a **crisis** or turning point in the game.

The degree of 'fuzziness' of play, or the 'trembling hand' of players using heuristics rather than perfect foresight, determines the richness of the state space, since this determines the number of options considered.

There will be more drama when players are more or less at equal footing. There lies some irony in the fact that looser play will likely lead to more drama and that looser play often goes with ignorance of both opportunities and catastrophes. Beginners often settle for pretty random moves.

Very often the thin line between winning and loosing is crossed without anyone noticing at once, opportunities for glory are missed and the end result determined more by chance and error than by skill—but such is drama.

5. Conclusion: Complexity, Drama and Decisiveness

We have been able to proof the following:

1. Pentagame is very complex, in the same ‘league’ as classics such as Go and Chess. It thus has **complexity.**

2. Pentagame can ‘swing’ from one state of who is winning to another often. It thus has **drama.**

3. Pentagame always ends; it thus has **decisiveness.**

We have also been able to give some examples of tactical moves and general advice on strategy, which helps in the face of the complexity proven.

These are Thompson’s qualities, among with clarity. Pentagame thus has what it takes to become a classic and stand the test of time.

★

Part VIII.

Conclusion

1. Summary

We can sum up the results from all prior parts and and conclude:

1. With Pentagame we have found a proper candidate for the missing pentagonal classic, q.e.f:
   a) We have created a pentagram shaped game board;
   b) We have found simple rules for complex and decisive play;
2. We have sought potential ancestors:
   a) We can explain its absence;
   b) We find it probable that such a game has existed before *Pente grammai* (or even *Petteia*);
3. We have analysed Pentagame;
   a) We have given advice on strategy;
   b) We have clarified the rules such as scoring etc.,
   c) We have proven that Pentagame has complexity, clarity, drama and decisiveness.

We thus conclude:

Pentagame is

- *probably* a resuscitated antique board game;
- *certainly* a game that has what it takes to be a classic.

q.e.d.

2. Outlook

The analysis of the game has shown some interesting properties, but it was a theoretical exercise. What is missing and what should be further challenges is the following:

1. A computer implementation of Pentagame to allow spatially independent play and data collection:
   a) A data structure to record moves, and states of the board,
   b) An interface to exchange such moves,
   c) A graphical representation.
2. The collection of data (through such a computer implementation) to allow statistical analyses, such as:
   a) The duration of play: average, variance etc.,
   b) The statistical quality of (opening) moves,
   c) The exact values of the game's qualities such as ABF, drama etc. over the course of play.
3. The development of artificial intelligence or heuristic methods to solve the game, with questions such as:
   a) The evaluation of positions and moves,
   b) The evaluation of (opponent's) strategies,
   c) The best move given a situation and an estimation of the opponent's actions.
4. Study of the psychology of players, such as
   a) The steepness of player's learning curves,
   b) Coalition behaviour in multi-player settings.

But most of all: *the game must be played!* For whenever we play it, we fathom a cornucopia of possibilities, combinations, and ideas. It is, after all, and ought to be, a *game.*

Annex

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