1. General

Weak form

$$(\mathbf{E}_t, \boldsymbol{\phi}) - (H, \nabla \times \boldsymbol{\phi}) + (\mathbf{J}, \boldsymbol{\phi}) = (\mathbf{g_1}, \boldsymbol{\phi}) \tag{1.1}$$

$$(H_t, \psi) + (\nabla \times \mathbf{E}, \psi) + (K, \psi) = (g_2, \psi) \tag{1.2}$$

$$(\mathbf{J}_t, \boldsymbol{\phi}) + \Gamma_e(\mathbf{J}, \boldsymbol{\phi}) - \omega_e^2(\mathbf{E}, \boldsymbol{\phi}) = (\mathbf{g_3}, \boldsymbol{\phi}) \tag{1.3}$$

$$(K_t, \psi) + \Gamma_m(K, \psi) - \omega_m^2(H, \psi) = (g_4, \psi) \tag{1.4}$$

Numerical scheme

For $\mathbf{J}$ and K:

$$\mathbf{J}^k + \mathbf{J}^{k-1} = \frac{4}{2+\tau\Gamma_e}\mathbf{J}^{k-1} + \frac{\tau\omega_e^2}{2+\tau\Gamma_e}(\mathbf{E}^k + \mathbf{E}^{k-1}) + \frac{2\tau}{2+\tau\Gamma_e}\mathbf{g_3}^{k-1/2}$$

$$K^k + K^{k-1} = \frac{4}{2+\tau\Gamma_m}K^{k-1} + \frac{\tau\omega_m^2}{2+\tau\Gamma_m}(H^k + H^{k-1}) + \frac{2\tau}{2+\tau\Gamma_m}g_4^{k-1/2}$$

Computational scheme:

$$\begin{aligned}\left(1 + \frac{\tau^2\omega_e^2}{4+2\tau\Gamma_e}\right)(\mathbf{E}^k, \boldsymbol{\phi}) - \frac{\tau}{2}(H^k, \nabla \times \boldsymbol{\phi}) &= \left(1 - \frac{\tau^2\omega_e^2}{4+2\tau\Gamma_e}\right)(\mathbf{E}^{k-1}, \boldsymbol{\phi}) + \frac{\tau}{2}(H^{k-1}, \nabla \times \boldsymbol{\phi}) \\ &\quad - \frac{2\tau}{2+\tau\Gamma_e}(\mathbf{J}^{k-1}, \boldsymbol{\phi}) - \frac{\tau^2}{2+\tau\Gamma_e}(\mathbf{g_3}^{k-1/2}, \boldsymbol{\phi}) + \tau(\mathbf{g_1}^{k-1/2}, \boldsymbol{\phi})\end{aligned} \tag{1.5}$$

$$\begin{aligned}\left(1 + \frac{\tau^2\omega_m^2}{4+2\tau\Gamma_m}\right)(H^k, \psi) + \frac{\tau}{2}(\nabla \times \mathbf{E}^k, \psi) &= \left(1 - \frac{\tau^2\omega_m^2}{4+2\tau\Gamma_m}\right)(H^{k-1}, \psi) - \frac{\tau}{2}(\nabla \times \mathbf{E}^{k-1}, \psi) \\ &\quad - \frac{2\tau}{2+\tau\Gamma_m}(K^{k-1}, \psi) - \frac{\tau^2}{2+\tau\Gamma_m}(g_4, \psi) + \tau(g_2^{k-1/2}, \psi)\end{aligned} \tag{1.6}$$

$$\mathbf{J}^k = \frac{2-\tau\Gamma_e}{2+\tau\Gamma_e}\mathbf{J}^{k-1} + \frac{\tau\omega_e^2}{2+\tau\Gamma_e}(\mathbf{E}^k + \mathbf{E}^{k-1}) + \frac{2\tau}{2+\tau\Gamma_e}\mathbf{g_3}^{k-1/2} \tag{1.7}$$

$$K^k = \frac{2-\tau\Gamma_m}{2+\tau\Gamma_m}K^{k-1} + \frac{\tau\omega_m^2}{2+\tau\Gamma_m}(H^k + H^{k-1}) + \frac{2\tau}{2+\tau\Gamma_m}g_4^{k-1/2} \tag{1.8}$$

2. Triangle Element

For a triangle K with vertices (x_i, y_i), $i = 1, 2, 3$ which are ordered conterclockwisely, let ϕ_{ij} be linear edge element basis function formed by vertices (x_i, y_i) and (x_j, y_j), then

$$\phi_{ij} = \lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i$$

particularly,

$$\phi_{i,i+1} = \frac{1}{2|K|}\begin{pmatrix} y_{i+2} - y \\ x - x_{i+2} \end{pmatrix}, \qquad i = \mod (i-1, 3) + 1$$

where

$$|K| = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

and basis functions in barycentric coordinate:

$$\lambda_1(x,y) = \frac{1}{2|K|}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad \lambda_2(x,y) = \frac{1}{2|K|}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix} \qquad \lambda_3(x,y) = \frac{1}{2|K|}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix}$$

Their gradients:

$$\nabla\lambda_1(x,y) = \frac{1}{2|K|}\begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix} \quad \nabla\lambda_2(x,y) = \frac{1}{2|K|}\begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix} \quad \nabla\lambda_3(x,y) = \frac{1}{2|K|}\begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix}$$

Let $\nabla\lambda_i = \frac{v_i}{2|K|}$.

For mass matrix:

Diagonal entries:

$$\begin{aligned}
\int_\triangle \phi_{12}^2 &= \int_\triangle (\lambda_1\nabla\lambda_2 - \lambda_2\nabla\lambda_1)^2 = \int_\triangle \lambda_1^2|\nabla\lambda_2|^2 - 2\lambda_1\lambda_2\left(\nabla\lambda_1\cdot\nabla\lambda_2\right) + \lambda_2^2|\nabla\lambda_1|^2 \\
&= \frac{|K|}{6}\left(|\nabla\lambda_2|^2 - \nabla\lambda_1\cdot\nabla\lambda_2 + |\nabla\lambda_1|^2\right) = \frac{1}{24|K|}(v_2^2 - v_1\cdot v_2 + v_1^2) \\
&= \frac{1}{24|K|}[(y_3-y_1)^2 - (y_3-y_1)(y_2-y_3) + (y_2-y_3)^2 + (x_1-x_3)^2 - (x_1-x_3)(x_3-x_2) + (x_3-x_2)^2]
\end{aligned}$$

$$\begin{aligned}
\int_\triangle \phi_{23}^2 &= \int_\triangle (\lambda_2\nabla\lambda_3 - \lambda_3\nabla\lambda_2)^2 = \int_\triangle \lambda_2^2|\nabla\lambda_3|^2 - 2\lambda_2\lambda_3\left(\nabla\lambda_2\cdot\nabla\lambda_3\right) + \lambda_3^2|\nabla\lambda_2|^2 \\
&= \frac{|K|}{6}\left(|\nabla\lambda_3|^2 - \nabla\lambda_2\cdot\nabla\lambda_3 + |\nabla\lambda_2|^2\right) = \frac{1}{24|K|}(v_3^2 - v_2\cdot v_3 + v_2^2) \\
&= \frac{1}{24|K|}[(y_1-y_2)^2 - (y_1-y_2)(y_3-y_1) + (y_3-y_1)^2 + (x_2-x_1)^2 - (x_2-x_1)(x_1-x_3) + (x_1-x_3)^2]
\end{aligned}$$

$$\begin{aligned}
\int_\triangle \phi_{31}^2 &= \int_\triangle (\lambda_3\nabla\lambda_1 - \lambda_1\nabla\lambda_3)^2 = \int_\triangle \lambda_3^2|\nabla\lambda_1|^2 - 2\lambda_3\lambda_1\left(\nabla\lambda_3\cdot\nabla\lambda_1\right) + \lambda_1^2|\nabla\lambda_3|^2 \\
&= \frac{|K|}{6}\left(|\nabla\lambda_3|^2 - \nabla\lambda_3\cdot\nabla\lambda_1 + |\nabla\lambda_1|^2\right) = \frac{1}{24|K|}(v_3^2 - v_3\cdot v_1 + v_1^2) \\
&= \frac{1}{24|K|}[(y_1-y_2)^2 - (y_1-y_2)(y_2-y_3) + (y_2-y_3)^2 + (x_2-x_1)^2 - (x_2-x_1)(x_3-x_2) + (x_3-x_2)^2]
\end{aligned}$$

Hence:

$$\int_\triangle \phi_{ij}^2 = \frac{1}{24|K|}(v_i^2 + v_j^2 - v_i\cdot v_j)$$

Upper entries:

$$
\begin{aligned}
\int_{\triangle} \phi_{12}\phi_{23} &= \int_{\triangle} (\lambda_1 \nabla\lambda_2 - \lambda_2\nabla\lambda_1)(\lambda_2\nabla\lambda_3 - \lambda_3\nabla\lambda_2) \\
&= \int_{\triangle} \lambda_1\lambda_2\nabla\lambda_2 \cdot \nabla\lambda_3 - \lambda_1\lambda_3|\nabla\lambda_2|^2 - \lambda_2^2\nabla\lambda_1 \cdot \nabla\lambda_3 + \lambda_2\lambda_3\nabla\lambda_1 \cdot \nabla\lambda_2 \\
&= \frac{|K|}{12}(\nabla\lambda_2 \cdot \nabla\lambda_3 - |\nabla\lambda_2|^2 - 2\nabla\lambda_1 \cdot \nabla\lambda_3 + \nabla\lambda_1 \cdot \nabla\lambda_2) \\
&= \frac{1}{48|K|}(v_2 \cdot v_3 - v_2^2 - 2v_1 \cdot v_3 + v_1 \cdot v_2)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\triangle} \phi_{12}\phi_{31} &= \int_{\triangle} (\lambda_1 \nabla\lambda_2 - \lambda_2\nabla\lambda_1)(\lambda_3\nabla\lambda_1 - \lambda_1\nabla\lambda_3) \\
&= \int_{\triangle} \lambda_1\lambda_3\nabla\lambda_1 \cdot \nabla\lambda_2 - \lambda_1^2\nabla\lambda_2 \cdot \nabla\lambda_3 - \lambda_2\lambda_3|\nabla\lambda_1|^2 + \lambda_1\lambda_2\nabla\lambda_1 \cdot \nabla\lambda_3 \\
&= \frac{|K|}{12}(\nabla\lambda_1 \cdot \nabla\lambda_2 - 2\nabla\lambda_2 \cdot \nabla\lambda_3 - |\nabla\lambda_1|^2 + \nabla\lambda_1 \cdot \nabla\lambda_3) \\
&= \frac{1}{48|K|}(v_1 \cdot v_2 - 2v_2 \cdot v_3 - v_1^2 + v_1 \cdot v_3)
\end{aligned}
$$

$$
\begin{aligned}
\int_{\triangle} \phi_{23}\phi_{31} &= \int_{\triangle} (\lambda_2 \nabla\lambda_3 - \lambda_3\nabla\lambda_2)(\lambda_3\nabla\lambda_1 - \lambda_1\nabla\lambda_3) \\
&= \int_{\triangle} \lambda_2\lambda_3\nabla\lambda_1 \cdot \nabla\lambda_3 - \lambda_3^2\nabla\lambda_1 \cdot \nabla\lambda_2 - \lambda_1\lambda_2|\nabla\lambda_3|^2 + \lambda_1\lambda_3\nabla\lambda_2 \cdot \nabla\lambda_3 \\
&= \frac{|K|}{12}(\nabla\lambda_1 \cdot \nabla\lambda_3 - 2\nabla\lambda_1 \cdot \nabla\lambda_2 - |\nabla\lambda_3|^2 + \nabla\lambda_2 \cdot \nabla\lambda_3) \\
&= \frac{1}{48|K|}(v_1 \cdot v_3 - 2v_1 \cdot v_2 - v_3^2 + v_2 \cdot v_3)
\end{aligned}
$$

Hence:

$$\int_{\triangle} \phi_{ij}\phi_{jk} = \frac{1}{48|K|}(v_i \cdot v_j + v_j \cdot v_k - v_j^2 - 2v_i \cdot v_k)$$

For stiffness matrix:

$$\nabla \times \phi_{ij} = 2\nabla\lambda_i \times \nabla\lambda_j$$

Hence

$$\int_{\triangle} \nabla \times \phi_{ij} = 2|K|\nabla\lambda_i \times \nabla\lambda_j = \frac{1}{2|K|} v_i \times v_j$$

Remark 1. *For 2-D vector $\alpha = (a, b)$ and $\beta = (c, d)$, the curl is a scalar (actually a 3-D vector along the orthogonal direction), given by*

$$\alpha \times \beta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

In the program, the local mass matrix $matM$ is defined by

$$matM = \begin{bmatrix} \frac{1}{24} & \frac{1}{48} & \frac{1}{48} \\ \frac{1}{48} & \frac{1}{24} & \frac{1}{48} \\ \frac{1}{48} & \frac{1}{48} & \frac{1}{24} \end{bmatrix}$$

and stiffness matrix $matBM$

$$matBM = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$