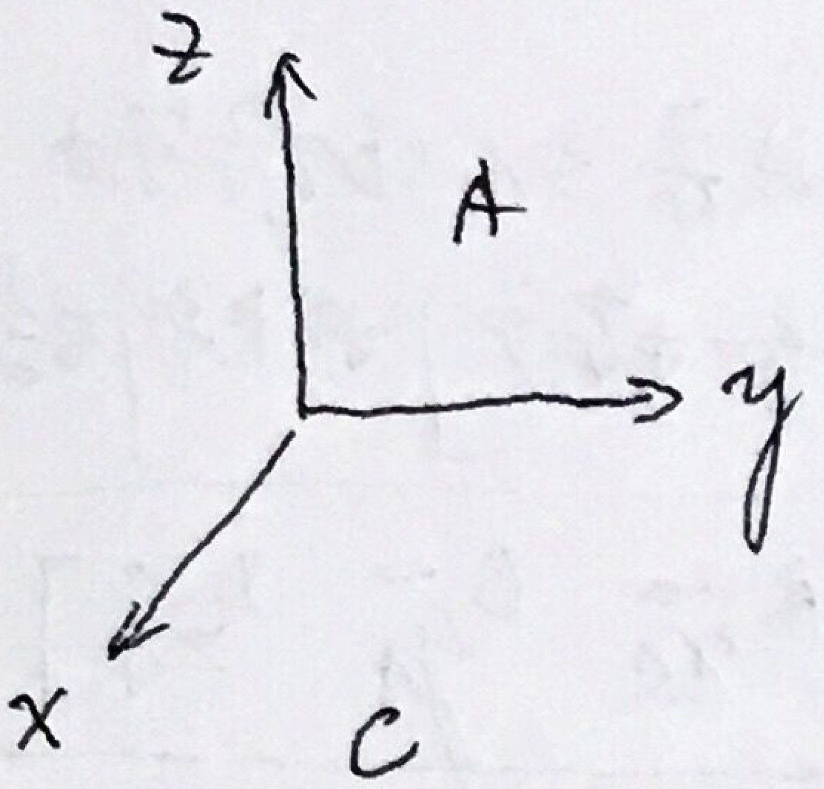
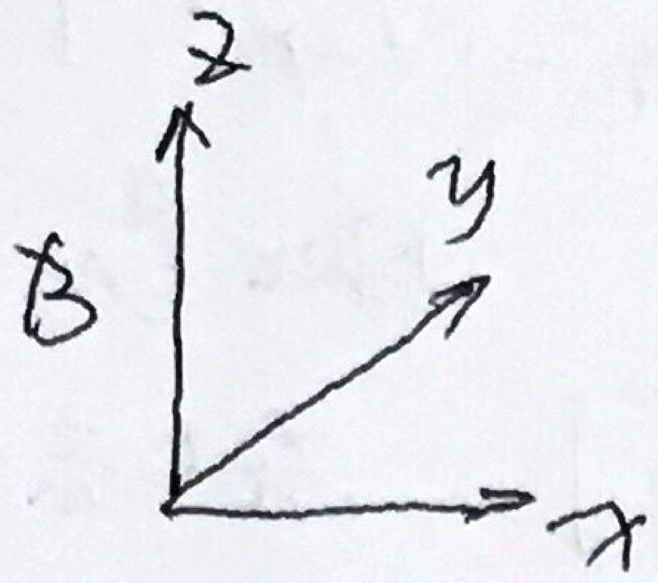
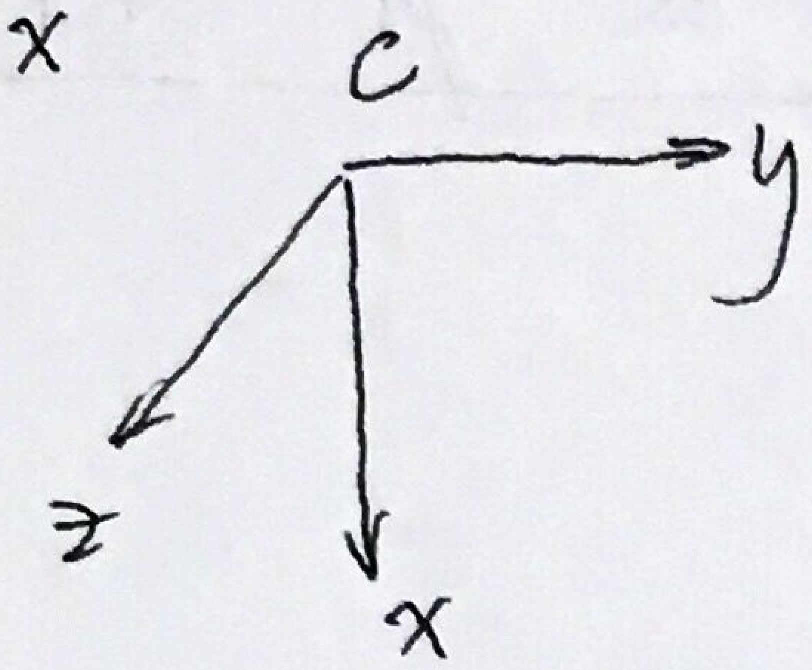
四元数与坐标转换基本运算：

给定坐标系A、B、C如下，B由A绕其z轴转90°得到，C由A绕其y轴旋转90°得到.







① 四元数的构造.

B由A绕 ${}^A\vec{v}$ 旋转 θ_A 得到，则定义

$$\boxed{{}^A_Bq = \left(\cos\frac{\theta_A}{2}, -\sin\frac{\theta_A}{2}\,{}^A\vec{v}\right)}$$

表示A相对于B或B到A的旋转.

$${}^A_Bq = \left(\cos\frac{90°}{2}, -\sin\frac{90°}{2}\,{}^A z_A\right) = \left(\frac{\sqrt{2}}{2}, 0, 0, -\frac{\sqrt{2}}{2}\right) \quad \{{}^A z_A, 90°\}$$

$${}^A_Cq = \left(\cos\frac{90°}{2}, -\sin\frac{90°}{2}\,{}^A y_A\right) = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, 0\right) \quad \{{}^A y_A, 90°\}$$

② 坐标转换. $\boxed{{}^B\vec{v} = {}^A_Bq \otimes {}^A\vec{v} \otimes ({}^A_Bq)^*}$ 简记为 ${}^B\vec{v} = {}^A_Bq \otimes {}^A\vec{v}$.

取 ${}^A\vec{v} = {}^A\vec{x}_A$，则A系x轴在B系表示 ${}^B\vec{x}_A$：

$${}^B\vec{x}_A = {}^A_Bq \otimes {}^A\vec{x}_A \otimes ({}^A_Bq)^* = \left(\frac{\sqrt{2}}{2}, 0, 0, -\frac{\sqrt{2}}{2}\right) \otimes (0,1,0,0) \otimes \left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right) = (0,0,-1,0)$$

$= (0,-1,0)^T = -{}^B\vec{y}_B$，即为B系y轴负方向■

同理，${}^C\vec{x}_A = {}^A_Cq \otimes {}^A\vec{x}_A \otimes ({}^A_Cq)^* = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, 0\right) \otimes (0,1,0,0) \otimes \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 0\right) = (0,0,0,1)$

$= (0,0,1)^T = {}^C\vec{z}_C$，即为C系z轴方向■

考虑B与C之间的关系，根据四元数乘法.

$$\boxed{{}^B_Cq = {}^A_Cq \otimes {}^B_Aq} = {}^A_Cq \otimes ({}^A_Bq)^{-1}$$

$$= \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, 0\right) \otimes \left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right).$$

则B系x轴在C系表示为

$${}^C\vec{x}_B = {}^B_Cq \otimes {}^B\vec{x}_B \otimes ({}^B_Cq)^*$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \otimes (0,1,0,0) \otimes \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$$

$= (0,1,0)^T = {}^C\vec{y}_C$，即为C系y轴方向■

四元数乘法.

$a = [a_1, a_2, a_3, a_4]$

$b = [b_1, b_2, b_3, b_4]$

$$a \otimes b = \begin{bmatrix} a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 \\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 \\ a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 \\ a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 \end{bmatrix}^T$$

③ 四元数与旋转矩阵的转换. 已知 $^A_Bq = [q_1, q_2, q_3, q_4]$.

$$^A_BM = \begin{bmatrix} 2q_1^2-1+2q_2^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & 2q_1^2-1+2q_3^2 & 2(q_3q_4-q_1q_2) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_1q_2) & 2q_1^2-1+2q_4^2 \end{bmatrix}.$$

对于系A, B, C.

$$^A_BM = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [{}^B\vec{x}_A \vdots {}^B\vec{y}_A \vdots {}^B\vec{z}_A].$$

$$^A_CM = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [{}^C\vec{x}_A \vdots {}^C\vec{y}_A \vdots {}^C\vec{z}_A].$$

因此 ^A_BM 即为A系3个坐标轴在B系的表示按列排列结

$$\boxed{^A_BM = [{}^B\vec{x}_A \vdots {}^B\vec{y}_A \vdots {}^B\vec{z}_A]}$$

④ 用旋转矩阵进行坐标转换.

$$\boxed{^B\vec{x}_A = {}^A_BM \cdot {}^A\vec{x}_A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -{}^B\vec{y}_B$$

$$^C\vec{x}_A = {}^A_CM \cdot {}^A\vec{x}_A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = {}^C\vec{z}_C.$$

$$\boxed{^B_CM = {}^A_CM \cdot {}^B_AM} = {}^A_CM \cdot {}^A_BM^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = [{}^C\vec{x}_B \vdots {}^C\vec{y}_B \vdots {}^C\vec{z}_B]$$

⑤ 四元数、旋转矩阵与欧拉角的转换.

基本单位旋转矩阵

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

若A系绕其自身z轴转γ角度,再绕新的y、x轴分别转β、α,则表示为A按照zyx顺序转(γ, β, α)得到B.

$$\boxed{^B_AM = R_z(\gamma)R_y(\beta)R_x(\alpha)}$$

上面定义的B、C中,B系按照zyx顺序转(-90°, -90°, 0)得到C系,则

$$^C_BM = R_z(-90^\circ)R_y(90^\circ).$$

$$^B_CM = [R_z(-90^\circ) \cdot R_y(90^\circ)]^T = R_y(90^\circ)^T \cdot R_z(-90^\circ)^T = \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{^A_CM} \cdot \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{^B_AM} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$