Mathematical Systems, Direct Proofs and Counter Examples

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Discrete Mathematics 2

1 Introduction

In this paper we will explore and expound through ,mathematical proofing, several fundamental theorems and inequalities in Discrete Math. These proving challenges will task us to revisit and refine our basic understanding of Algebra, Number Theory and most of all writing proofs.

This paper requires the knowledge and skill of writing cohesive mathematical proofs by establishing a domain, hypothesis and a conclusion. Relying heavily on establishing statements that are supported by mathematical laws, theorems and lemmas. Using these fundamental concepts, we are able to structure and justify our arguments or counter arguments Our goal is to prove, disprove our hypothesis and possible create a better and more logically sound counter argument

2 Euler's Number Proof

Task: Prove that e, Euler's number is not a rational number

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \notin \mathbb{Q}$$

Figure 1.0

Recalling the definition of a rational number, denoted as $\mathbb{Q}$,

$$\frac{a}{b} = \mathbb{Q}$$

Figure 1.1

Let us create a hypothesis, assume that euler's number is a rational number.

hypothesis:

$$e \in \mathbb{Q}$$

Suppose that the summation from k = 0 to ∞ can be expressed in a more simpler form

$$\sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} ... + \frac{1}{k!} = \frac{a}{b}$$

Figure 1.2

$$e = \frac{a}{b}$$

Figure 1.2

According to our hypothesis where $e \in \mathbb{Q}$, the reciprocal of eulers number should also be rational

$$e = \frac{a}{b} \quad = \quad e^x = \frac{b}{a}$$

s.t. x = -1

Figure 1.3

$$e^{-1} = \frac{1}{e} = \frac{b}{a} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

s.t. $a, b \in \mathbb{Z}$

Figure 1.4

Using a, let us group the summation of the infinite series into two terms. The first term will be the summation from 0 to a and the second term will be the summation from a to infinity.

$$e^{-1} = \frac{b}{a} = \sum_{k=0}^{a} \frac{(-1)^k}{k!} + \sum_{k=a+1}^{\infty} \frac{(-1)^k}{k!}$$

Figure 1.5

$$\frac{b}{a} - \sum_{k=0}^{a} \frac{(-1)^k}{k!} = \sum_{k=a+1}^{\infty} \frac{(-1)^k}{k!}$$

Figure 1.6

Multiply each term with

$$(-1)^{(a+1)}a!$$

$$\text{term 1:}$$

$$(-1)^{(a+1)}b(a-1)!$$

this term is an integer since all constants, variables and integers. any operation between integers that integers excluding division and root, will always result an integer

$$\text{term 2:}$$

$$(-1)^{(a+1)}(\frac{a!}{2!} \cdots + \frac{(-1)^a a!}{a})$$

in this series, every denominator is always less than the numerator, it will always result an integer since the denominator is a factor of the numerator

$$\forall a, b \in \mathbb{Z}, a > b$$

$$\frac{a!}{b!} = c, s.tc \in \mathbb{Z}$$

since term 1 and term 2 is a subtraction between two integers, we can conclude that the left side of the equation is an integer

$$\mathbb{Z} = \sum_{k=a+1}^{\infty} \frac{(-1)^k}{k!}$$

Let us expound on the term on the right side of the equation

$$= \frac{1}{a+1} - \frac{1}{(a+1)(a+2)} + \ldots$$

in this expression, an upper and lower bound is defined. The first term is the upper bound. The succeeding terms are the lower bound such that it is lower than the first term as k reaches infinity, the summation of the succeding terms will approach zero. Therefore this this expression will result in a number that is irrational

$$\frac{1}{a+1} - \frac{1}{(a+1)(a+2)} + \ldots \notin \mathbb{Z}$$

3 Minskowski's Inequality for Sums

Task: Prove Minskow's Inequality for Sums

$$[\sum_{k=1}^{n} |a_k + b_k|^p]^{\frac{1}{p}} \leq [\sum_{k=1}^{n} |a_k|^p]^{\frac{1}{p}} + [\sum_{k=1}^{n} |b_k|^p]^{\frac{1}{p}}$$

$$\forall a, b, p \in \mathbb{R}$$

$$p > 1, (a_k, b_k) > 0$$

Let us directly prove Minskowski's Inequality for sums by the triangle Inequality theorem

$$\text{Triangle Inequality Theorem}$$

$$|a + b| \leq |a| + |b|$$

Let us first raise terms on both sides to P

$$|a + b|^p \leq |a|^p + |b|^p$$

Do summation notation for all k from 1 to n

$$\sum_{k=1}^{n} |a_k + b_k|^p \leq \sum_{k=1}^{n} |a_k|^p + \sum_{k=1}^{n} |b_k|^p$$

raise to 1 over p by property of absolute value

$$[\sum_{k=1}^{n} |a_k + b_k|^p]^{\frac{1}{p}} \leq [\sum_{k=1}^{n} |a_k|^p]^{\frac{1}{p}} + [\sum_{k=1}^{n} |b_k|^p]^{\frac{1}{p}}$$

$$|x + y| \leq |x| + |y|, \forall (x, y) \in \mathbb{R}$$

4 Triangle Inequality Theorem

Task: Prove that...

$$|x + y| \leq |x| + |y|, \forall(x, y) \in \mathbb{R}$$

hypothesis

$$(|x| + |y|)^2 \geq |a + b|^2$$

$$(x + y)^2 = x^2 + 2xy + y^2 \leq |x|^2 + 2|x||y| + |y|^2$$

By the known properties of absolute value where $x \leq |x|$

$$x^2 + 2xy + y^2 = (x + y)^2 = |a + b|^2$$

Therefore...

$$\sqrt{(|a| + |b|)^2} \geq \sqrt{|a + b|^2}$$

$$|a| + |b| \geq |a + b|$$

5 Sedrakayan's Lemma

Sedrakayan's Lemma, denoted as...

$$\frac{(\sum_{i=1}^{n} u_i)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=i}^{n} \frac{(u_i)^2}{v_i}$$

Let us directly prove...

hypothesis: Sedrakayan's Lemma is a Direct Consequence of
Cauchy-Schwarz Inequality

Recall Cauchy-Schwarz Inequality...

$$< a, b > |^2 \ \leq < a, a >< b, b >$$

$$\forall u_i, v_i \in \mathbb{R}^+$$

the square of the dot product of two vectors is less than or equal to the product of the dot product of each vector with itself.

Let $a_i = u_i, b_i = \sqrt{v_i}$

$$| < a, b > |^2 = (\sum_{i=1}^{n} a_i b_i)^2 = (\sum_{i=1}^{n} u_i \sqrt{v_i})^2$$

$$< a, a >< b, b >= (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2) = (\sum_{i=1}^{n} u_i^2)(\sum_{i=1}^{n} v_i)$$

Therefore...

$$(\sum_{i=1}^{n} u_i \sqrt{v_i})^2 \leq (\sum_{i=1}^{n} u_i^2)(\sum_{i=1}^{n} v_i)$$

divide both sides by $\sum_{i=1}^{n} v_i$

$$\frac{(\sum_{i=1}^{n} u_i)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}$$

quod erat demonstratum

6 Sedrakayan's Lemma 2

Now that we have directly proven Sedrakayan's Lemma through Cauchy-Schwarz Inequality, let us try to directly prove if the Inequality holds of u_i and v_i are square roots of an even integer

There are two ways of tackling this thought experiment, one where we are dealing with Perfect Square roots of an even integer and one where we do not. Both ways require some supplemental proofs.

Part 1: Assuming that we are dealing with Perfect Squares

let us expound on square roots of an even integer. This statement may not always be true as for example the square root of 8 is 2.83, which is not even and it is not an integer.

Recall the definition of an even number...

$$\forall n \in \mathbb{Z}\text{is EVEN} \iff \exists k \in \mathbb{Z}, n = 2k$$

Recall the definition of a perfect square...

$$\forall n \in \mathbb{Z}\text{is a PERFECT SQUARE} \iff \exists k \in \mathbb{Z}$$
$$k^2 = n, k = \sqrt{n}$$
$$s.t.n \in \mathbb{Z}^+$$

Let us prove that the square root of an even perfect square is an even number

let $n = m^2$
Unique prime factorization of m

$$m = \prod p_i^{n_i}$$

prime factorization for n

$$n = \prod p_i^{2n_i}$$

n^2 is for sure to be even. The prime factorization reveals that one if its prime numbers must be 2 the primes that factor n^2 are the same as those that factor m, therefore m must be even therefor the square root of an even perfect square is even

let us assume that both are perfect square roots of an even integer $u_i = 2a_i$ and $v_i = 2b_i$ $a_i, b_i \in \mathbb{Z}$

$$\frac{(\sum_{i=1}^{n} 2a_i)^2}{\sum_{i=1}^{n} 2b_i} \leq \sum_{i=1}^{n} \frac{(2a_i)^2}{2b_i}$$

$$\text{simplify...}$$

$$\frac{4(\sum_{i=1}^{n} a_i)^2}{2\sum_{i=1}^{n} b_i} \leq \sum_{i=1}^{n} \frac{4(a_i)^2}{2b_i}$$

$$\text{simplify further}\ldots$$

$$\frac{2(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} b_i} \leq 2[\sum_{i=1}^{n} \frac{(a_i)^2}{b_i}]$$

$$\text{multiply both sides by } \tfrac{1}{2}$$

$$\frac{(\sum_{i=1}^{n} a_i)^2}{\sum_{i=1}^{n} b_i} \leq \sum_{i=i}^{n} \frac{(a_i)^2}{b_i}$$

Therefore Sedrakayan's Lemma holds true if u_i and v_i are square roots of an even perfect square

Part 2: Assuming that we are not dealing with Perfect Squares

recalling the Cauchy-Schwarz inequality for Real Numbers... s

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$$

let $a_i = u_i$ and $b_i = \sqrt{v_i}$

$$(\sum_{i=1}^{n} u_i \sqrt{v_i})^2 \leq (\sum_{i=1}^{n} u_i^2)(\sum_{i=1}^{n} \sqrt{v_i}^2)$$

since we are dealing with the square root of even integers, substitute the values $u_i = \sqrt{2a_i}$, $\sqrt{v_i} = \sqrt{2b_i}$

$$(\sum_{i=1}^{n} \sqrt{2a_i}\sqrt{2b_i})^2 \leq (\sum_{i=1}^{n}[\sqrt{2a_i}]^2)(\sum_{i=1}^{n}[\sqrt{2b_i}]^2)$$

$$4\sum_{i=1}^{n} a_i b_i \leq 4\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$$

divide both sides by... $4\sum_{i=1}^{n} b_i$

$$\frac{4\sum_{i=1}^{n} a_i b_i \leq 4\sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i}{4\sum_{i=1}^{n} b_i}$$

$$4(\frac{1}{4})[\sum_{i=1}^{n} a_i b_i (\frac{1}{[\sum_{i=1}^{n} b_i]})] \leq 4(\frac{1}{4})\sum_{i=1}^{n} a_i (\frac{1}{[\sum_{i=1}^{n} b_i]})\sum_{i=1}^{n} b_i (\frac{1}{[\sum_{i=1}^{n} b_i]})$$

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \leq \sum_{i=1}^{n} \frac{a_i}{b_i}$$

Remember that $u_i = \sqrt{2a_i}, v_i = \sqrt{2b_i}$
modify the equality to get a and b

$$u_i = \sqrt{2a_i}$$

$$a = (\frac{u_i^2}{2})$$

$$\sqrt{v_i} = \sqrt{2b_i}$$

$$b = (\frac{\sqrt{v_i}^2}{2})$$

Substitue and square the numerator to get Sedrakayan's Lemma

$$\frac{\sum_{i=1}^{n} \frac{u_i^2}{2}}{\sum_{i=1}^{n} \frac{\sqrt{v_i}^2}{2}} \leq \sum_{i=1}^{n} \frac{\frac{u_i^2}{2}}{\frac{\sqrt{v_i}^2}{2}}$$

$$\frac{(\sum_{i=1}^{n} u_i)^2}{\sum_{i=1}^{n} v_i} \leq \sum_{i=1}^{n} \frac{u_i^2}{v_i}$$

Therefore Sedrakayan's Lemma holds if u_i and v_i are square roots of an even integer

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