Robot Dynamics I

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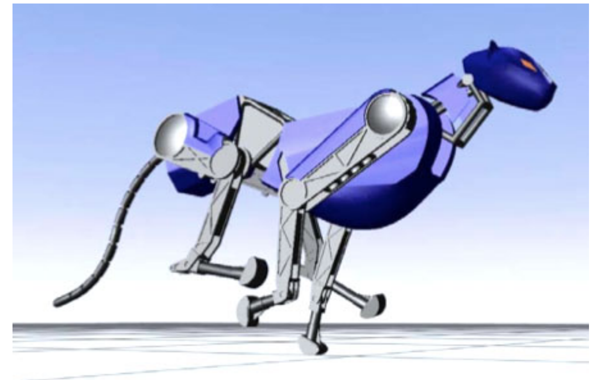
What you would learn

- Newton-Euler formulation to obtain manipulator’s equations of motion.
- Physical interpretation of the manipulator’s equations of motion

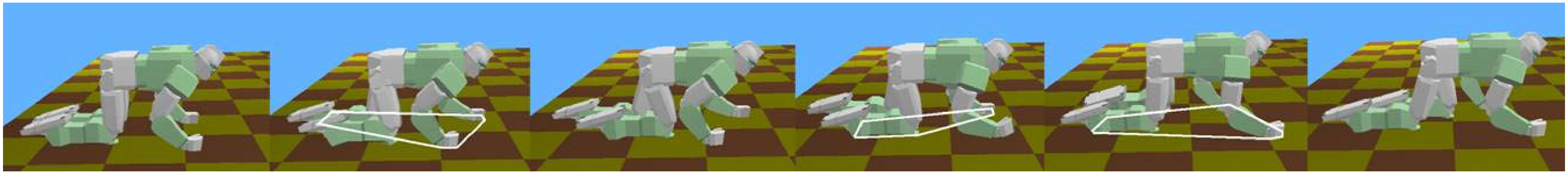
Introduction



- **Dynamics** (Wikipedia): a branch of physics (specifically classical mechanics) concerned with the study of forces and torques and their effect on motion, as opposed to kinematics, which studies the motion of objects without reference to its causes.
- Manipulator dynamics:
  - The way in which motion of the manipulator arises from torques applied by the actuators or from external forces applied to the manipulator

Introduction

- Usage of robot dynamics:
  - Controller design: Model-based controller typically perform better than non model-based ones
  - Simulation study: Help to check the perform of a robot and test the control strategies before working on the physical robots



Introduction

- Typically, robot manipulator is modeled as a rigid multi-body system
- Two main problems associated with robot dynamics:

(For control of manipulator)

Inverse dynamics

joints' motions
$(q_1(t), \dots, q_n(t)$
$\dot{q}_1(t), \dots, \dot{q}_n(t)$
$\ddot{q}_1(t), \dots, \ddot{q}_n(t))$

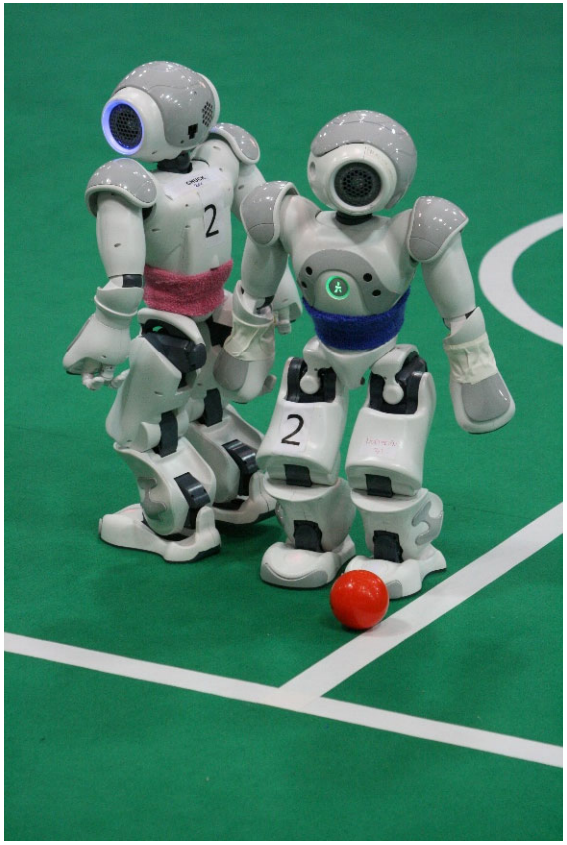
joints' torques
$(\tau_1(t), \dots, \tau_n(t))$

Forward (direct) dynamics

(Mainly used for simulation)

Robot Dynamics

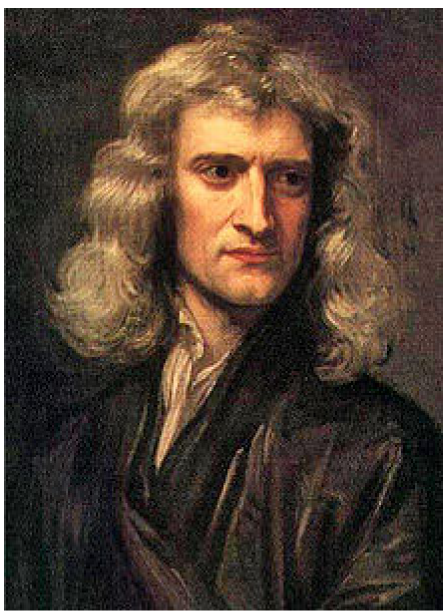
- Dynamics of multi-body systems
  - Formulations:
    - Newton-Euler
      - Based on Newton's Second Law of Motion and Euler's equation of motion
    - Lagrangian
      - Based on work and energy



Newton-Euler Formulation of Equations of Motion

- Dynamic equations of a rigid body represented by two sets of equations:
  - 1) Translational motion of mass centroid (or centre of mass, C)

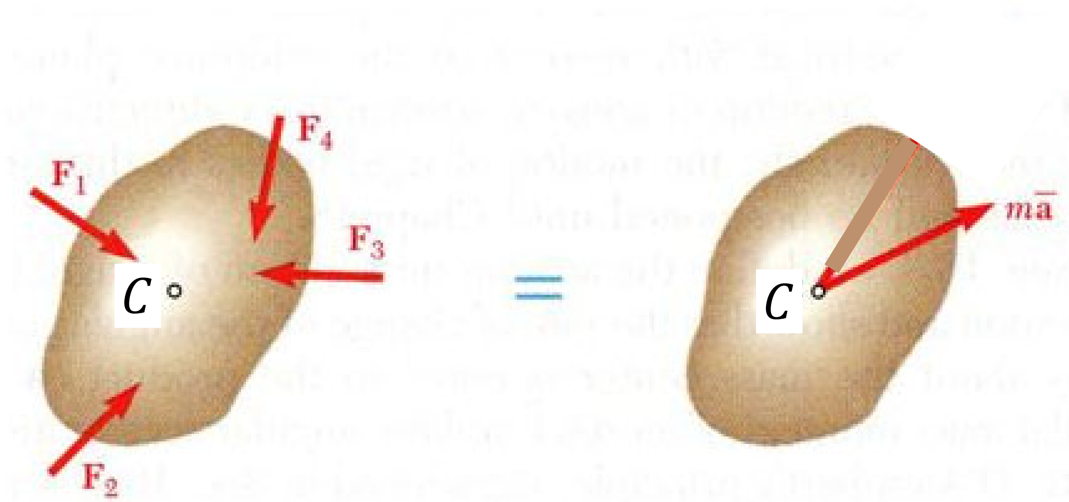
Newton's equation of motion



Sir Isaac Newton (1642 – 1727)

$$\sum_j \mathbf{F}_j = m \cdot \mathbf{a}$$

$\mathbf{F}_j$: force j acting on body
m : mass of body
a : acceleration of body's mass centroid



Newton-Euler Formulation of Equations of Motion

- Dynamic equations of a rigid body represented by two sets of equations:
  - 2) Rotational motion about centroid

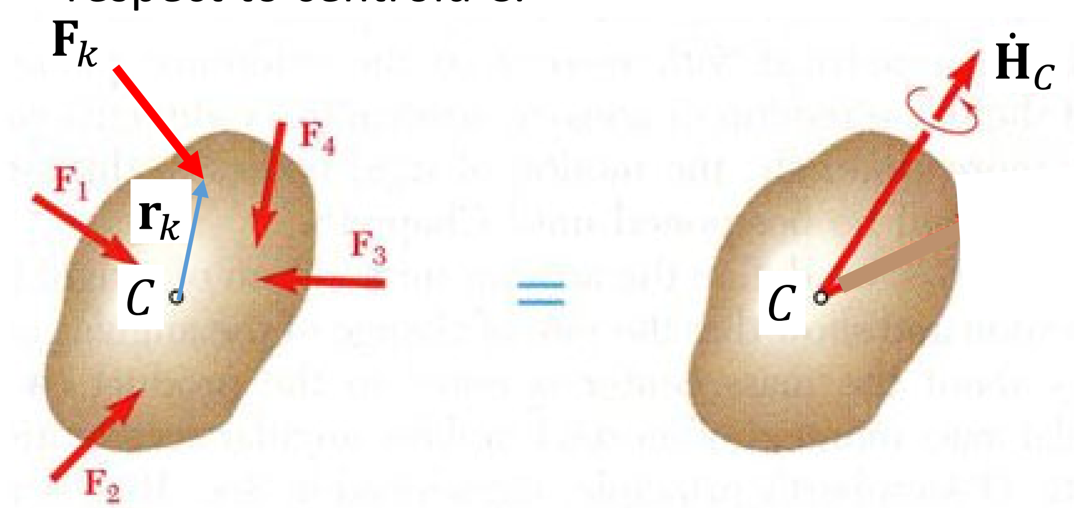
Euler's equation of motion

Leonhard Euler (1707 – 1783)

$$\sum_k \mathbf{T}_k = \frac{d\mathbf{H}_\mathbf{C}}{dt}$$

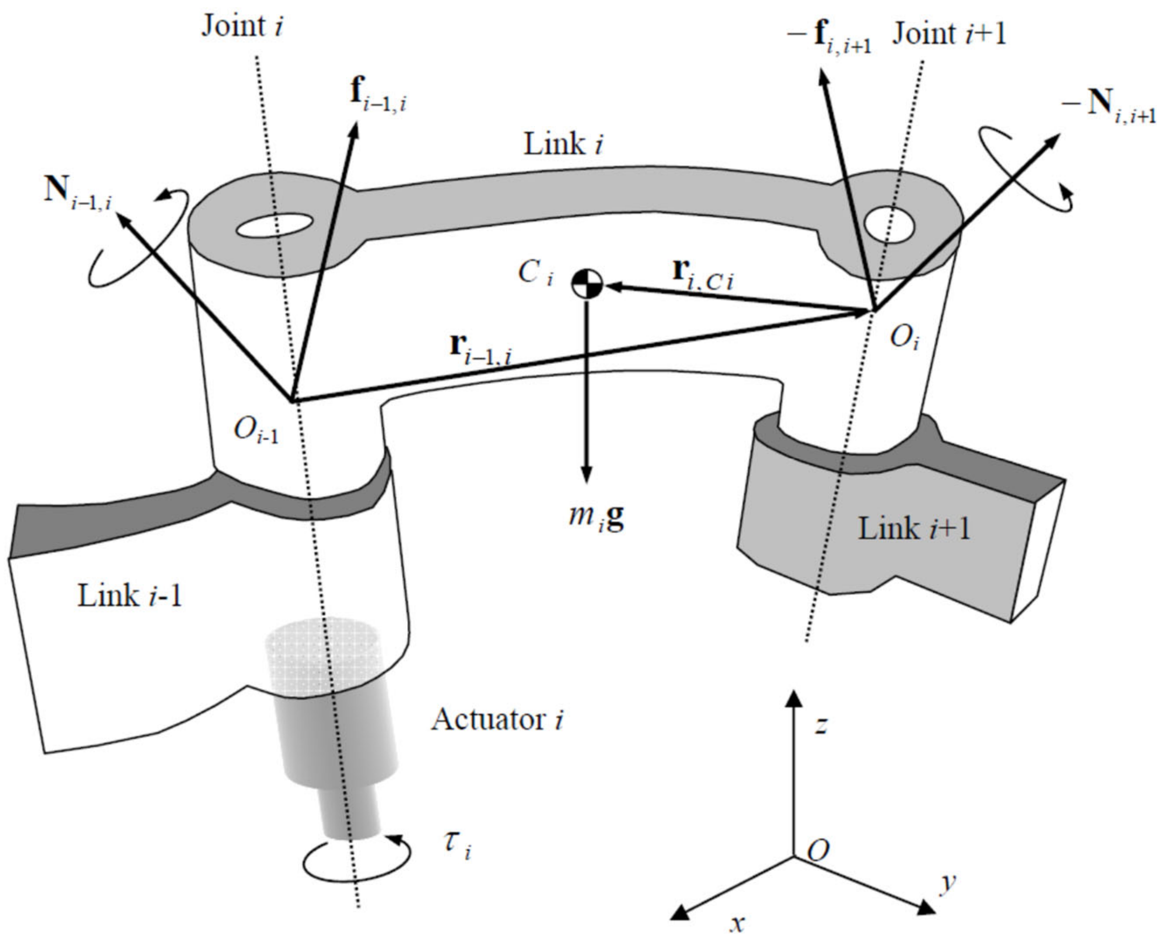
$\mathbf{T}_k$: moment k acting on body
$\mathbf{H}_C$: angular momentum of body about its mass centroid C

If $\mathbf{T}_k$ is due to force $\mathbf{F}_\mathrm{k}$ acting on the body, $\mathbf{T}_k = \mathbf{r}_k \times \mathbf{F}_k$, where $\mathbf{r}_k$ is a position vector of the contact point with respect to centroid C.



Newton-Euler Formulation of Equations of Motion

- Considering free body diagram of an individual link (i):



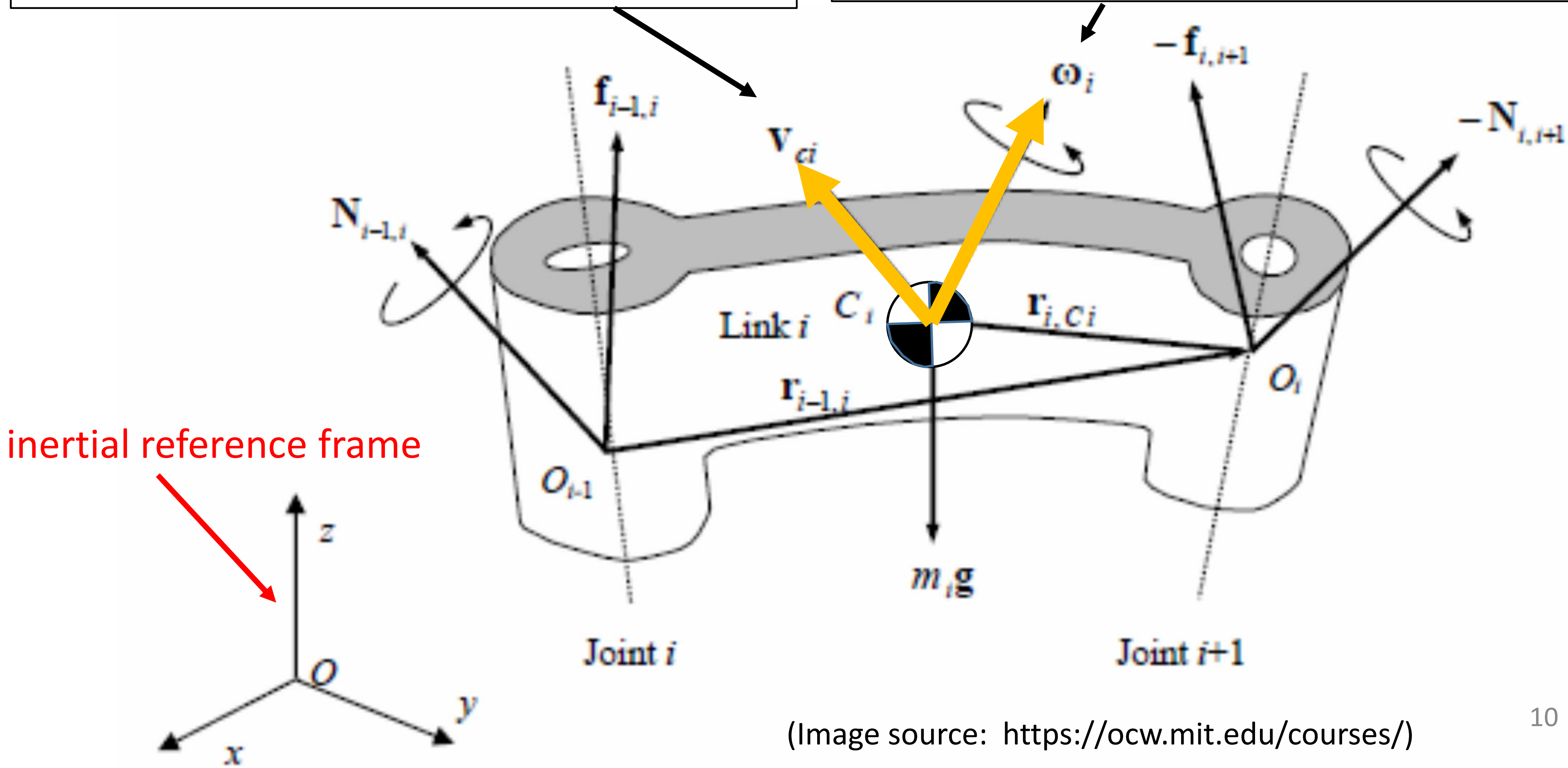
(Image source: https://ocw.mit.edu/courses/)

Newton-Euler Formulation of Equations of Motion

- Considering free body diagram of an individual link (i):

$\boldsymbol{v}_{ci}$: linear velocity of centroid C_i of link i with reference to inertial reference frame *O-xyz*

$\boldsymbol{\omega}_i$: angular velocity of link i with reference to inertial reference frame *O-xyz*



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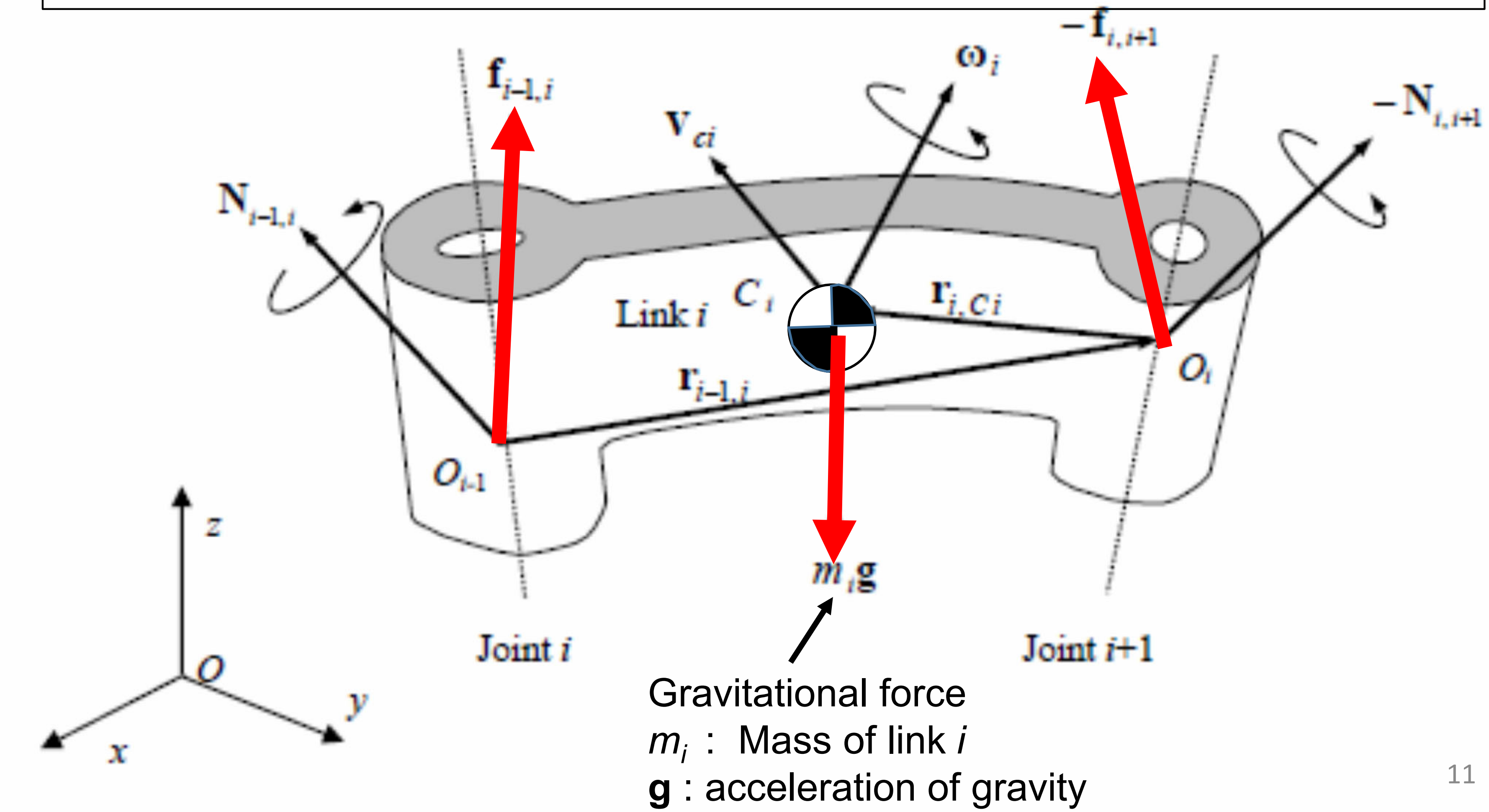
Newton-Euler Formulation of Equations of Motion

- Considering free body diagram of an individual link (i):

$\mathbf{f}_{i\text{-}1,i}$ and $-\mathbf{f}_{i,i+1}$ are coupling forces applied to link i by links i-1 and i+1, respectively

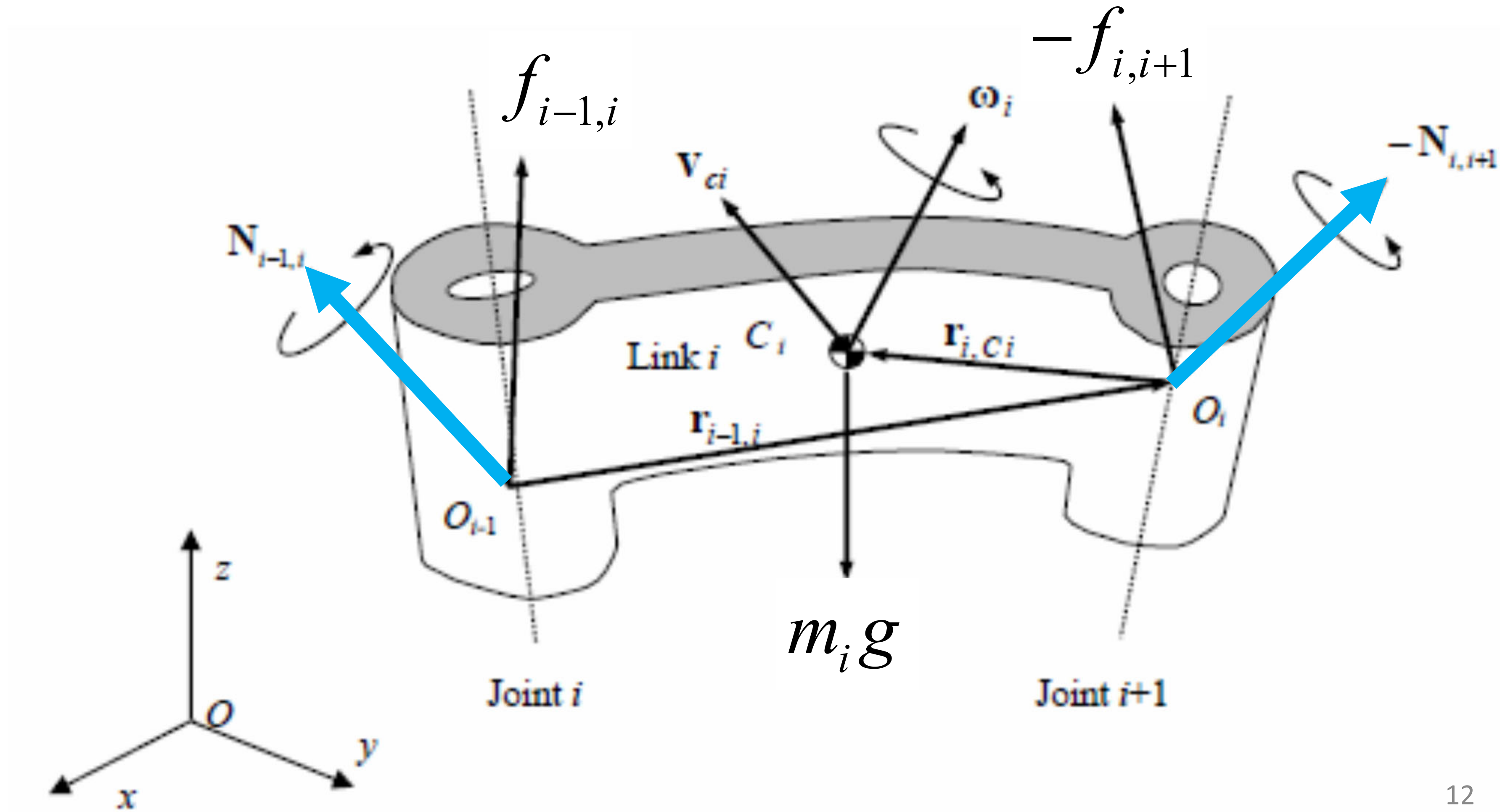


Gravitational force
m_i : Mass of link i
$\mathbf{g}$: acceleration of gravity

Newton-Euler Formulation of Equations of Motion

- Considering free body diagram of an individual link (i):

$\mathbf{N}_{i-1,i}$ and **$-\mathbf{N}_{i,i+1}$** are coupling moment applied to link i by links i-1 and i+1, respectively

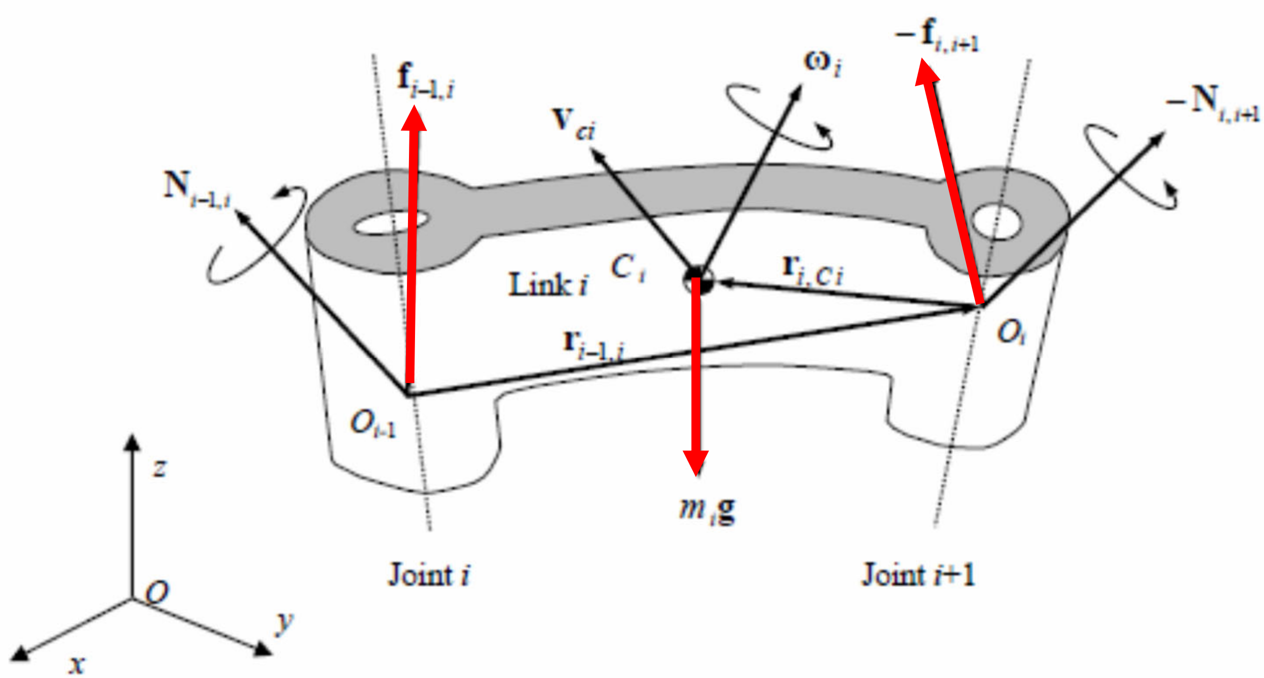


Newton-Euler Formulation of Equations of Motion

- 1) Translational motion of centroid (or centre of mass)

Newton's equation of motion based on D'Alembert's principle (summation of actual forces acting on the link and inertial force* = 0),

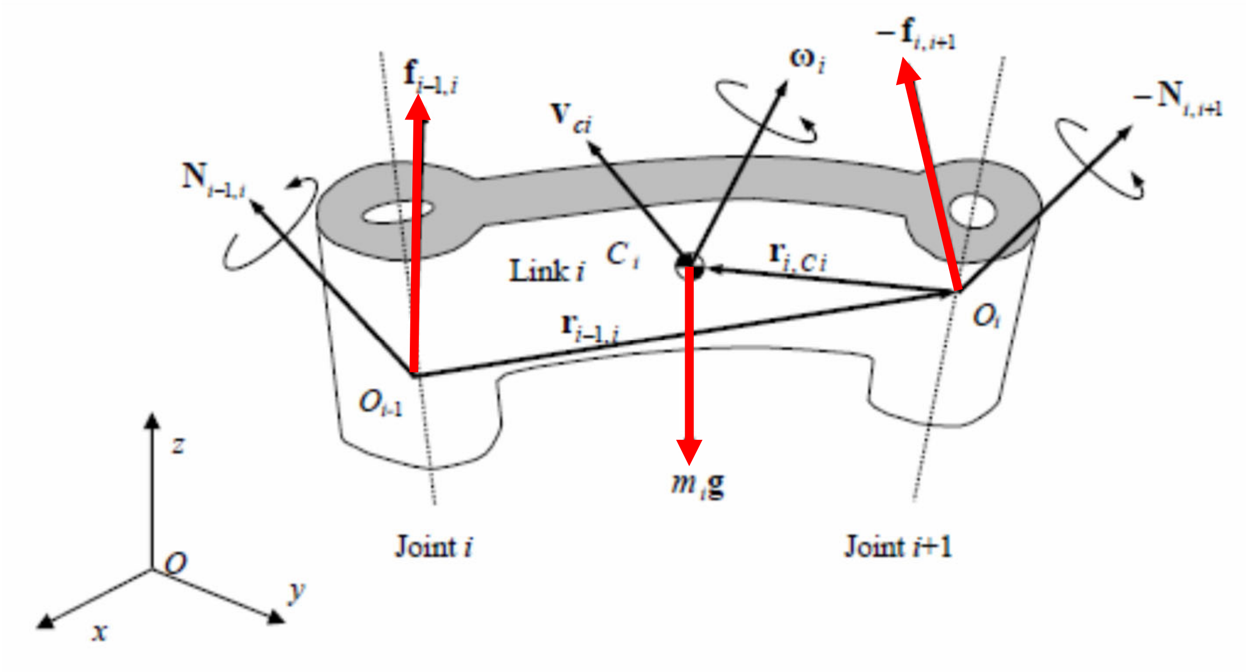
$$\sum_j \mathbf{F}_j \underbrace{- m \cdot \mathbf{a}}_{\text{inertial force*}} = 0$$



Newton-Euler Formulation of Equations of Motion

- Hence, equation of motion for translational motion of centroid is as follows:

$$\overbrace{\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i\mathbf{g}}^{\sum_j \boldsymbol{F}_j} \overbrace{- m_i\dot{\mathbf{v}}_{ci}}^{-m\boldsymbol{a}} = \mathbf{0}, \quad i = 1,\cdots,n \qquad (1)$$

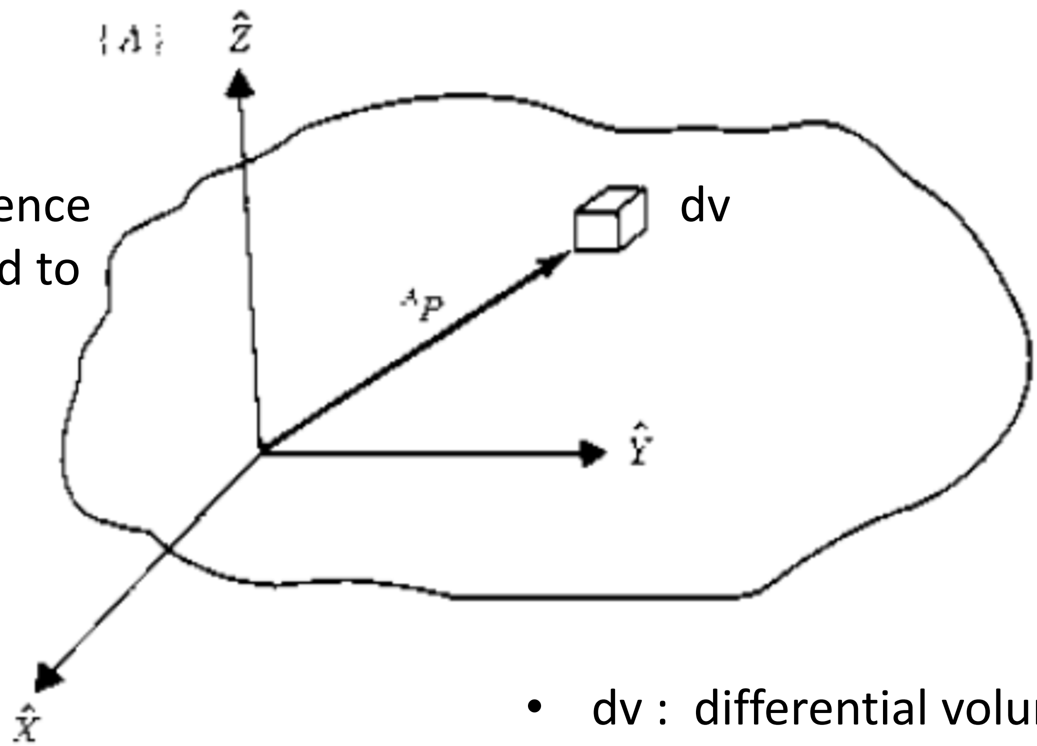


*Inertial force is given by $-m_i\dot{\mathbf{v}}_{\mathbf{ci}}$ where $\dot{\mathbf{v}}_{\mathbf{ci}}$ is time derivative of $\boldsymbol{v}_{ci}$.

Newton-Euler Formulation of Equations of Motion

- Inertia tensor (inertia matrix): set of quantities that give information about distribution of mass of a rigid body relative to a reference frame

Considering reference frame {A} attached to rigid body



- dv : differential volume element with density ρ
- $^{A}P = [x\ y\ z]^T$: position vector which locates differential volume element relative to frame {A}

Newton-Euler Formulation of Equations of Motion

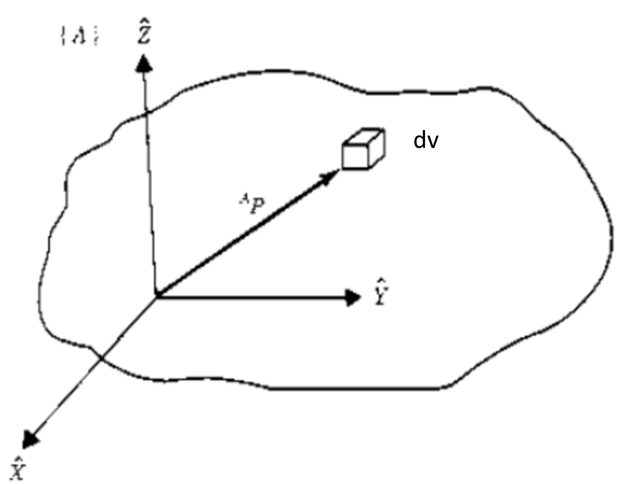
- Inertia tensor (inertia matrix):

$$
{}^{A}\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

3x3 symmetric matrix

(Leading superscript A: frame of reference of inertia tensor)

where (2)



$$I_{xx} = \int_V \left(y^2 + z^2\right)\rho dv \qquad I_{xy} = \int_V xy\rho dv$$

$$I_{yy} = \int_V \left(x^2 + z^2\right)\rho dv \qquad I_{xz} = \int_V xz\rho dv$$

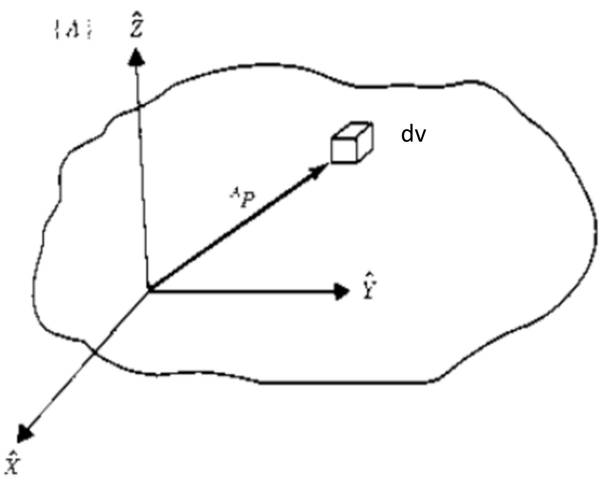
$$I_{zz} = \int_V \left(x^2 + y^2\right)\rho dv \qquad I_{yz} = \int_V yz\rho dv$$

Each volume integral is taken over entire rigid body

Newton-Euler Formulation of Equations of Motion

- Inertia tensor (inertia matrix):

Mass moments of inertia	Mass products of inertia
$I_{xx} = \int_V (y^2 + z^2)\rho dv$	$I_{xy} = \int_V xy\rho dv$
$I_{yy} = \int_V (x^2 + z^2)\rho dv$	$I_{xz} = \int_V xz\rho dv$
$I_{zz} = \int_V (x^2 + y^2)\rho dv$	$I_{yz} = \int_V yz\rho dv$



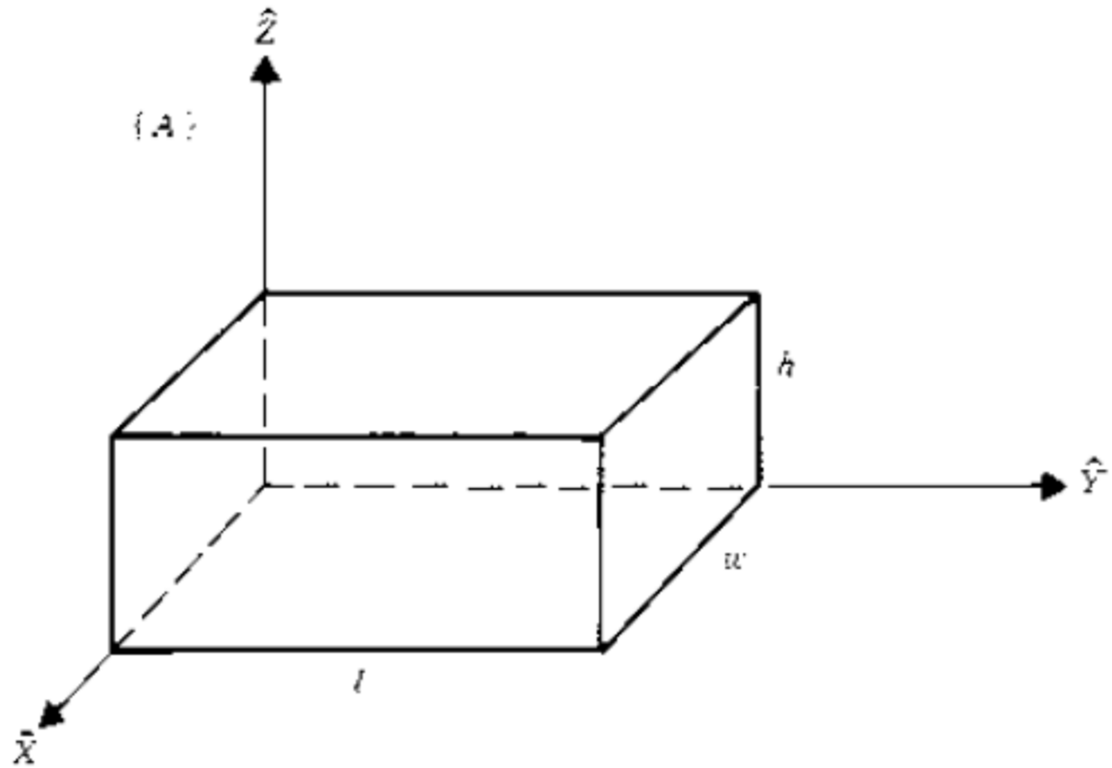
Newton-Euler Formulation of Equations of Motion

- Remarks:
  - If axes of reference frame are aligned to **principal axes**:
    - Mass products of inertia = 0
    - Mass moments of inertia = Principal moments of inertia

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Newton-Euler Formulation of Equations of Motion

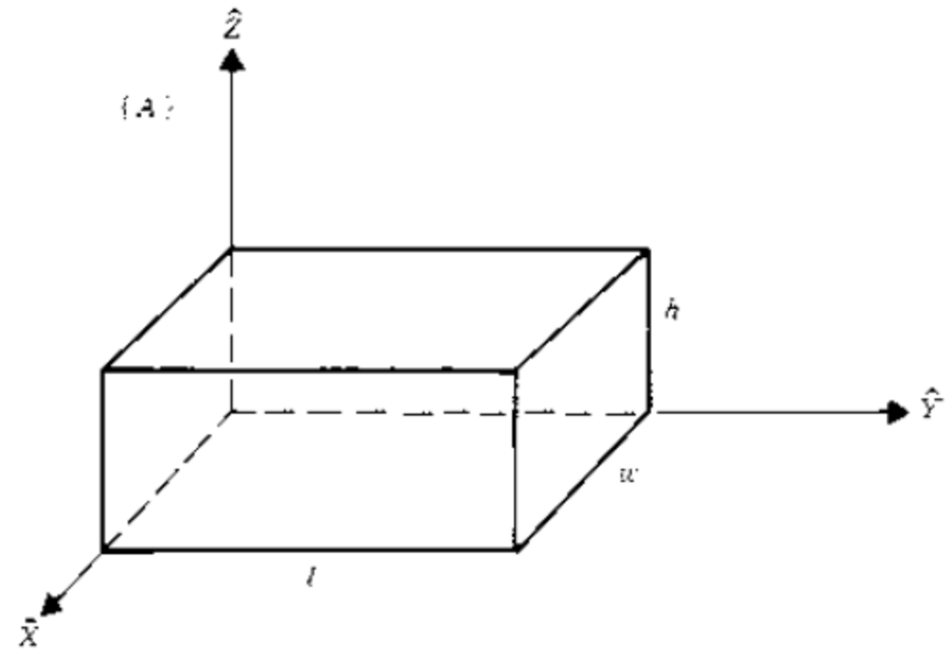
- Example 1 (Inertia tensor): Find inertia tensor for rectangular body of uniform density ρ with respect to coordinate system shown:



Newton-Euler Formulation of Equations of Motion

- Example 1 (cont.):

Mass moments of inertia:



$$
\begin{aligned}
I_{xx} &= \int_0^h \int_0^l \int_0^w \left(y^2 + z^2\right) \rho\, dx\, dy\, dz \\
&= \int_0^h \int_0^l \left(y^2 + z^2\right) w \rho\, dy\, dz \\
&= \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w \rho\, dz \\
&= \left(\frac{hl^3 w}{3} + \frac{h^3 l w}{3}\right)\rho \\
&= \frac{m}{3}\left(l^2 + h^2\right)
\end{aligned}
$$

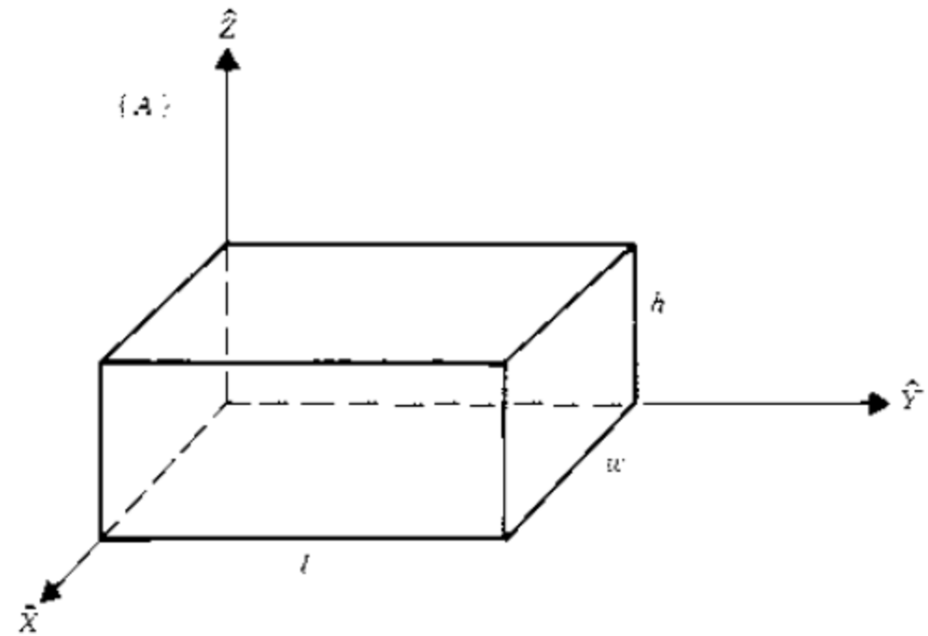
Similarly:

$$I_{yy} = \frac{m}{3}\left(w^2 + h^2\right)$$

$$I_{zz} = \frac{m}{3}\left(l^2 + w^2\right)$$

Newton-Euler Formulation of Equations of Motion

- Example 1 (inertia tensor):



Mass products of inertia:

$$I_{xy} = \int_0^h \int_0^l \int_0^w xy\rho dx dy dz$$

$$= \int_0^h \int_0^l \frac{w^2}{2} y\rho dy dz$$

$$= \int_0^h \frac{w^2 l^2}{4} \rho dz$$

$$= \frac{m}{4} wl$$

Similarly:

$$I_{xz} = \frac{m}{4} hw$$

$$I_{yz} = \frac{m}{4} hl$$

Hence, inertia tensor:

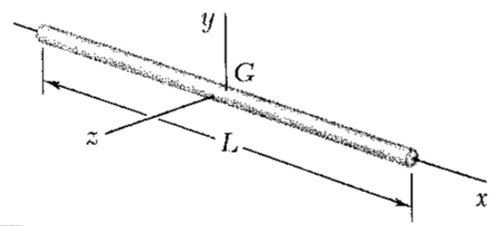
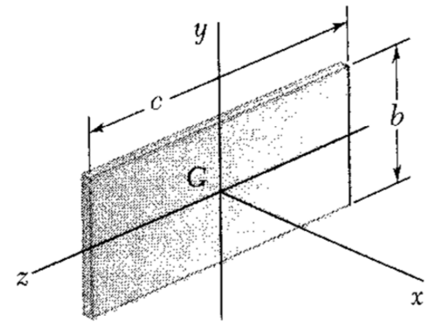
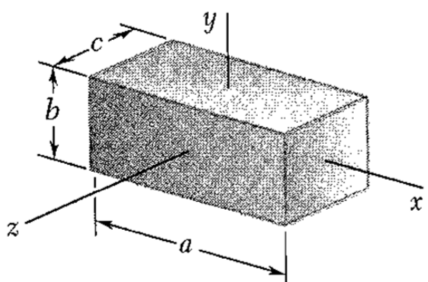
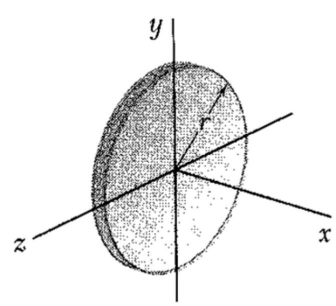
$$^{A}\mathbf{I} = \begin{bmatrix} \frac{m}{3}\left(l^2 + h^2\right) & -\frac{m}{4} wl & -\frac{m}{4} hw \\ -\frac{m}{4} wl & \frac{m}{3}\left(w^2 + h^2\right) & -\frac{m}{4} hl \\ -\frac{m}{4} hw & -\frac{m}{4} hl & \frac{m}{3}\left(l^2 + w^2\right) \end{bmatrix}$$

Newton-Euler Formulation of Equations of Motion

- Remarks:
  - Mass moments of inertia always positive.
  - Eigenvalues of inertia tensor are principal moments of inertia. Associated eigenvectors are principal axes.

Newton-Euler Formulation of Equations of Motion

- Mass moments of inertia of common geometric shapes

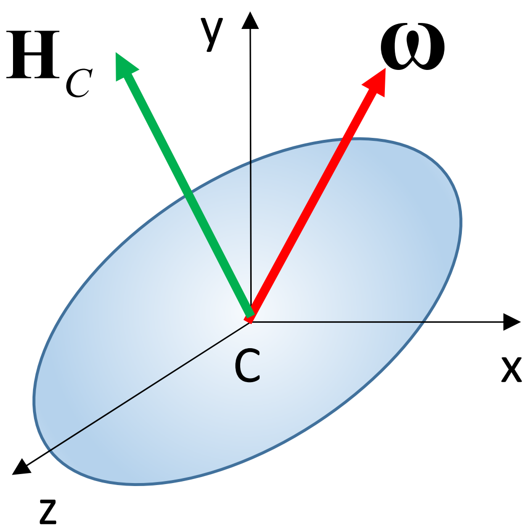
Slender rod		$I_y = I_z = \frac{1}{12} mL^2$
Thin rectangular plate		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} mc^2$ $I_z = \frac{1}{12} mb^2$
Rectangular prism		$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m(c^2 + a^2)$ $I_z = \frac{1}{12} m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2} mr^2$ $I_y = I_z = \frac{1}{4} mr^2$

I_x is same as I_{xx} of previous slides. Similarly for I_y and I_z.

Newton-Euler Formulation of Equations of Motion

- Angular momentum of a body about mass centroid C:

$$\boxed{\mathbf{H}_C = {}^{C}\mathbf{I} \cdot \boldsymbol{\omega}} \in R^3 \text{ (i.e.(3x1) vector)}$$



$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Angular velocity vector (3x1)

$${}^{C}\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Centroidal inertia tensor of link *i* (3x3)

Leading superscript C of **I** will be omitted in subsequent slides

Newton-Euler Formulation of Equations of Motion

- Time derivative of angular momentum of body:

Cross product

$$\frac{d\mathbf{H}_C}{dt} = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \left(\mathbf{I}\boldsymbol{\omega}\right)$$

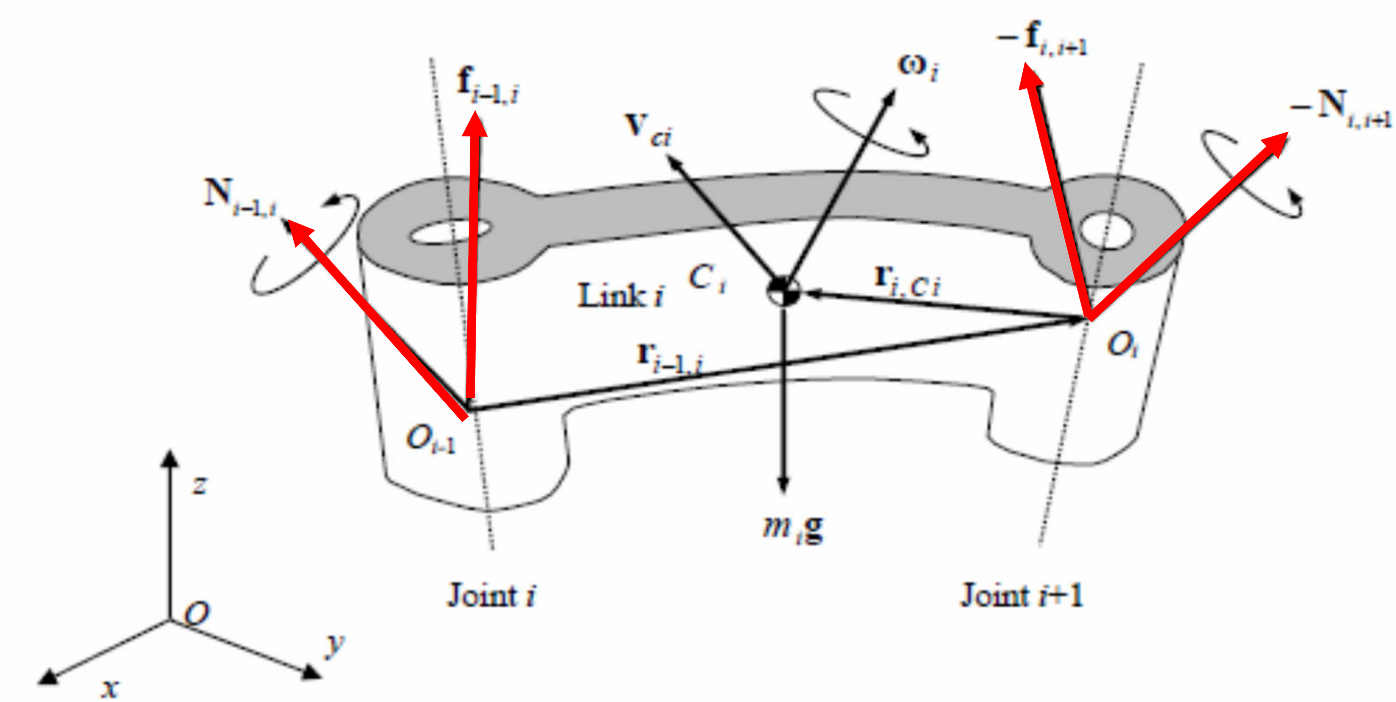
ω : angular velocity of body (3x1 vector)
$\dot{\omega}$: angular acceleration of body (3x1 vector)
I : centroidal inertia tensor of body (3x3 matrix)
(with reference to frame attached to body)

Newton-Euler Formulation of Equations of Motion

- 2) Rotational motion about the centroid

Euler's equation of motion

$$\sum_k \mathbf{T}_k \underbrace{- \frac{d\mathbf{H}_\mathbf{C}}{dt}}_{\text{inertial torque}} = 0$$

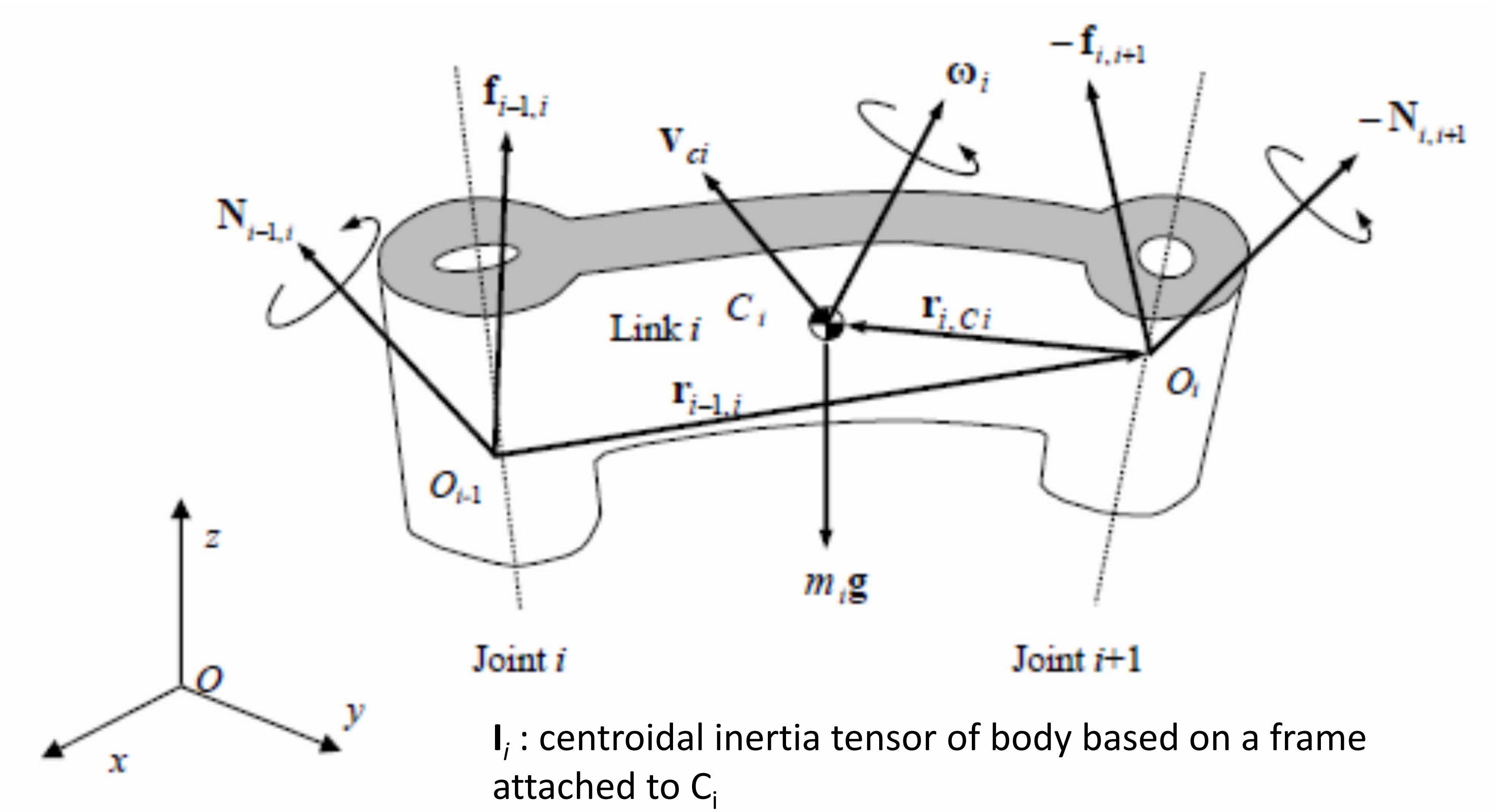


Newton-Euler Formulation of Equations of Motion

- Euler's equation (Balance of moments about C_i):

$$\overbrace{\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - \left(\mathbf{r}_{i-1,i} + \mathbf{r}_{i,Ci}\right) \times \mathbf{f}_{i-1,i} + \left(-\mathbf{r}_{i,Ci}\right) \times \left(-\mathbf{f}_{i,i+1}\right)}^{\sum_k \boldsymbol{T}_k} \overbrace{- \mathbf{I}_i \dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times (\mathbf{I}_i \boldsymbol{\omega}_i)}^{-\frac{d\boldsymbol{H}_C}{dt}} = \mathbf{0}, \qquad (3)$$

$$i = 1, \cdots, n$$



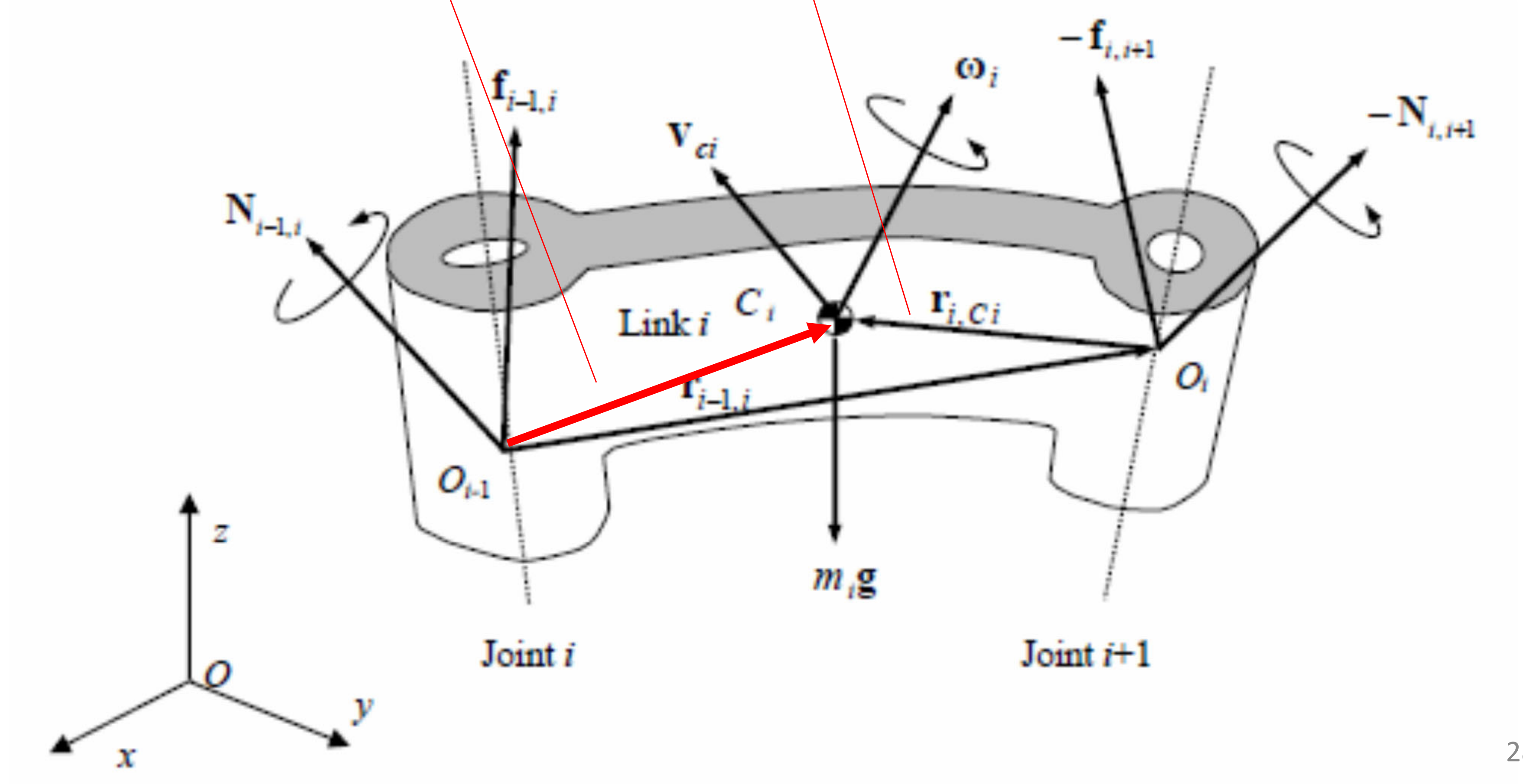
$\mathbf{I}_i$: centroidal inertia tensor of body based on a frame attached to C_i

Newton-Euler Formulation of Equations of Motion

- Euler's equation (Balance of moments about C_i):

$$\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - \underline{\left(\mathbf{r}_{i-1,i} + \mathbf{r}_{i,Ci}\right)} \times \mathbf{f}_{i-1,i} + \left(-\underline{\mathbf{r}_{i,Ci}}\right) \times \left(-\mathbf{f}_{i,i+1}\right) - \mathbf{I}_i \dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times \left(\mathbf{I}_i \boldsymbol{\omega}_i\right) = \mathbf{0}, \quad (3)$$

$$i = 1, \cdots, n$$



Newton-Euler Formulation of Equations of Motion

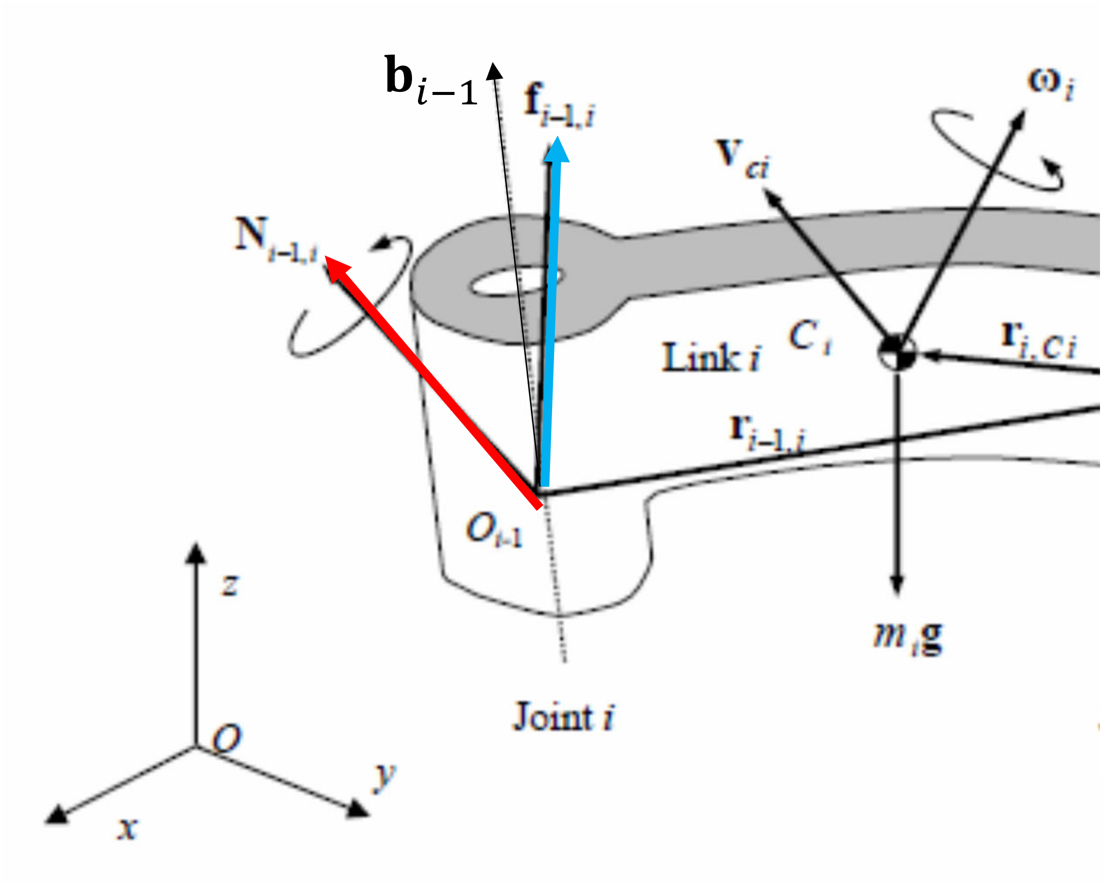
- Eqs (1) and (3) govern dynamic behavior of individual link. Complete set of equations for whole manipulator arm is obtained by evaluating both equations for all links, *i* = 1,…,*n*.

$$\boxed{\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i \mathbf{g} - m_i \dot{\mathbf{v}}_{ci} = \mathbf{0}} \tag{1}$$

$$\boxed{\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - \left(\mathbf{r}_{i-1,i} + \mathbf{r}_{i,Ci}\right) \times \mathbf{f}_{i-1,i} + \left(-\mathbf{r}_{i,Ci}\right) \times \left(-\mathbf{f}_{i,i+1}\right) - \mathbf{I}_i \dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times \left(\mathbf{I}_i \boldsymbol{\omega}_i\right) = \mathbf{0},} \tag{3}$$

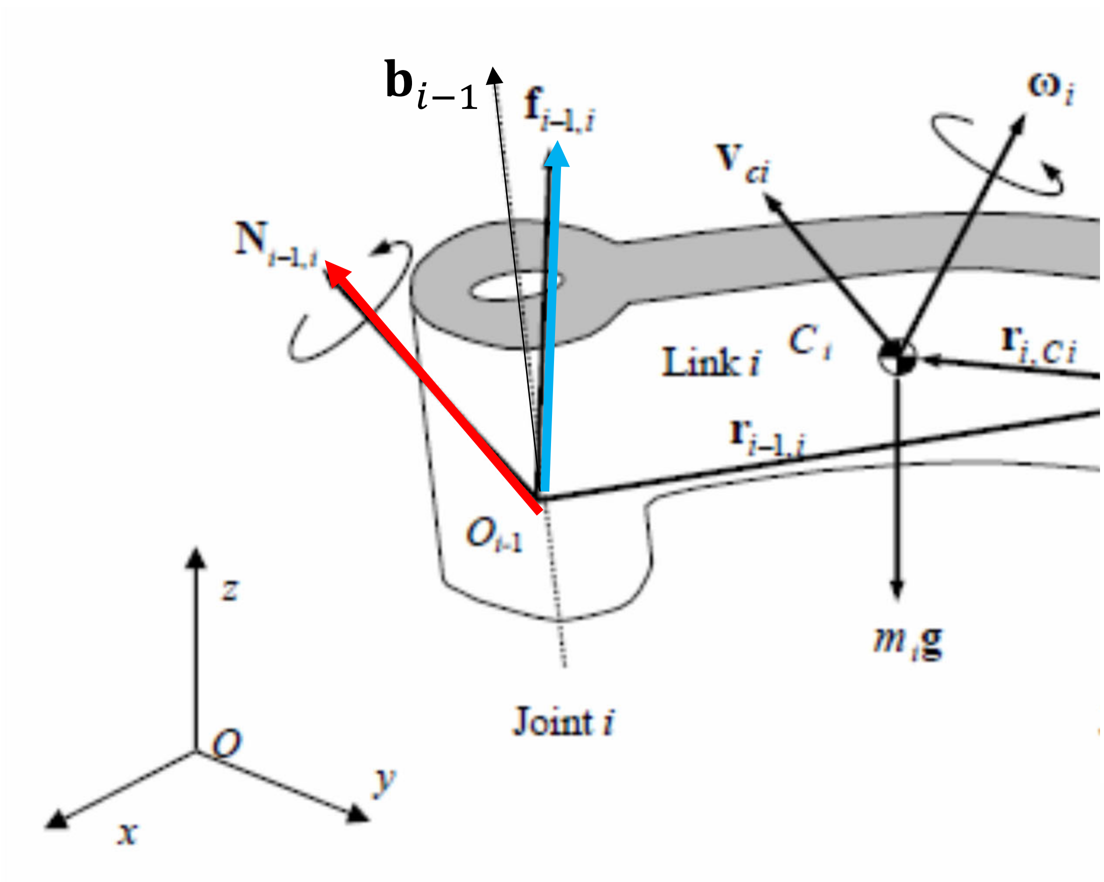
Newton-Euler Formulation of Equations of Motion

- Coupling forces $\mathbf{f}_{i-1,i}$ and moments $\mathbf{N}_{i-1,i}$ include workless constraint forces (internal) and joint forces/torques τ_i



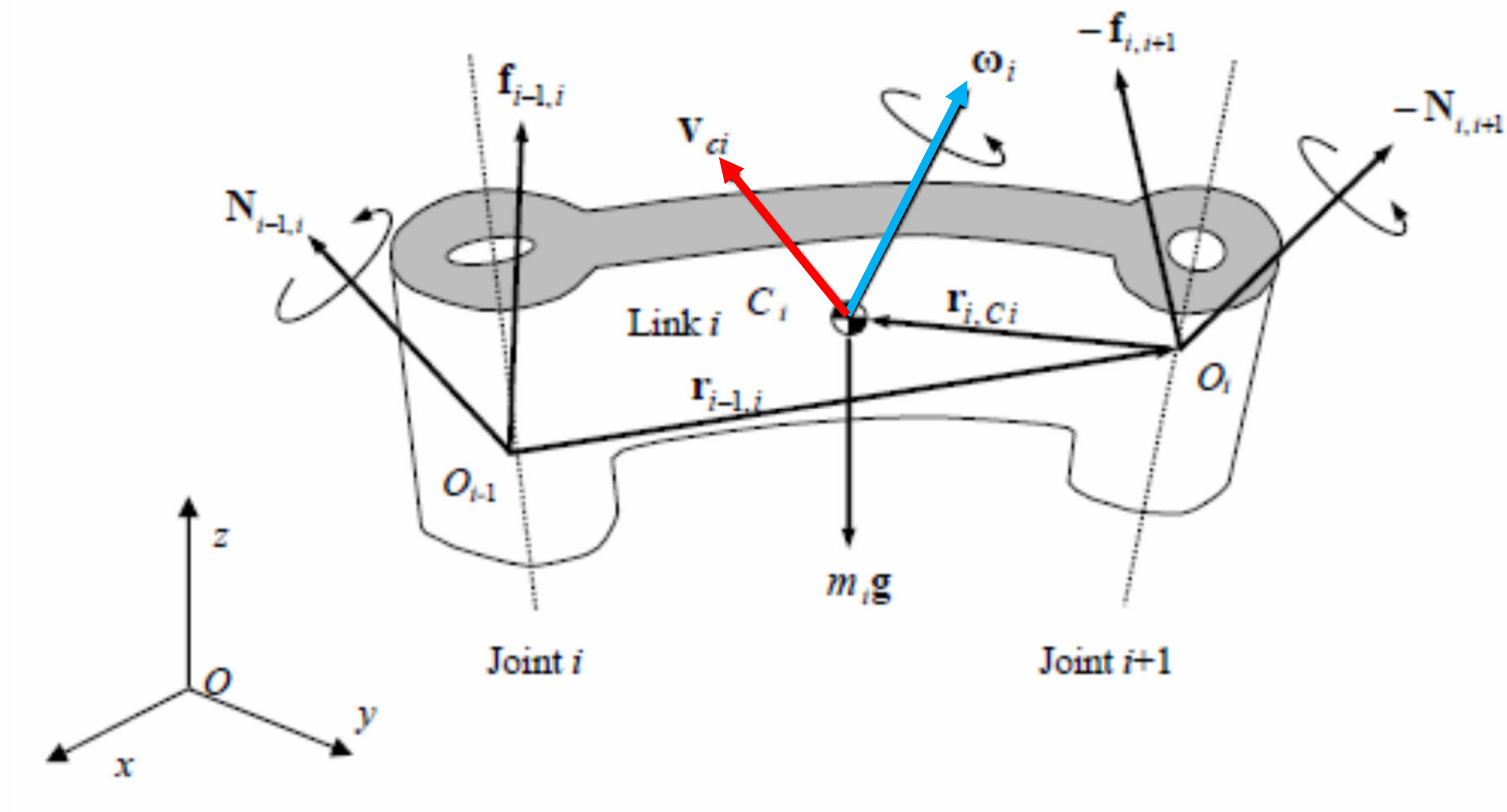
Newton-Euler Formulation of Equations of Motion

- For prismatic joint, $\tau_i = \mathbf{b}_{i-1}^T\,\mathbf{f}_{i-1,i}$
- For revolute joint, $\tau_i = \mathbf{b}_{i-1}^T\,\mathbf{N}_{i-1,i}$



Newton-Euler Formulation of Equations of Motion

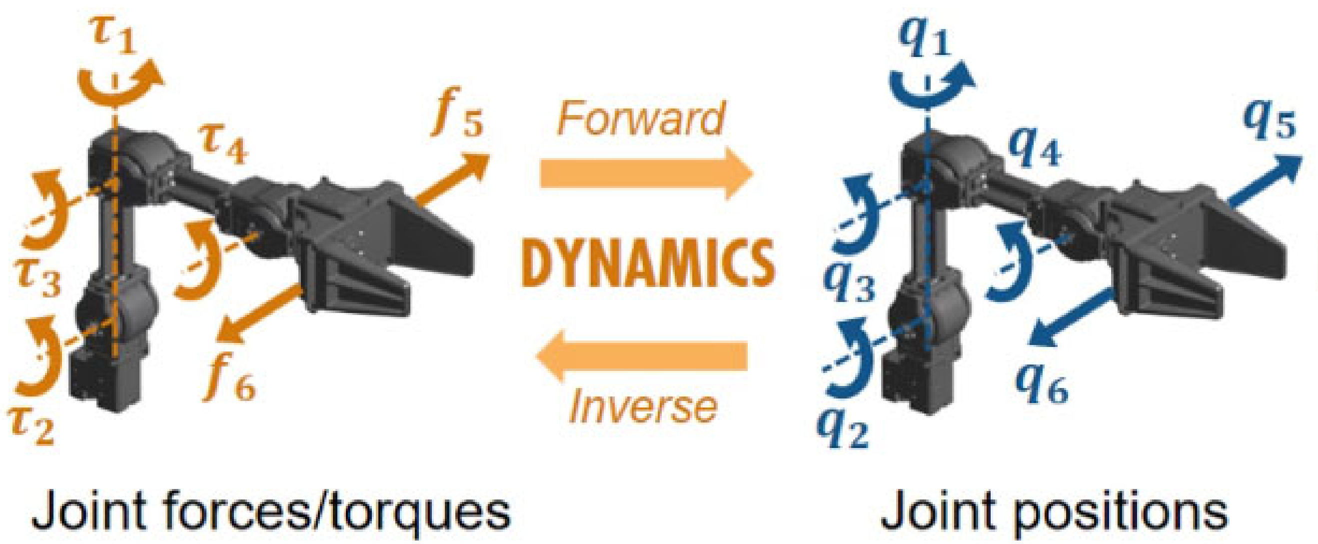
- Individual centroid position, velocity, acceleration variables not suitable output variables (not independent and subject to geometric constraints of the arm)



Newton-Euler Formulation of Equations of Motion:

- Desired to have dynamic equations consist of input-output terms such as:
  - Joint forces/torques
  - Joint displacement variables

    (which are complete and independent generalized coordinates that locate whole robot mechanism)



(Image Source: fr.mathworks.com)

Newton-Euler Formulation of Equations of Motion:

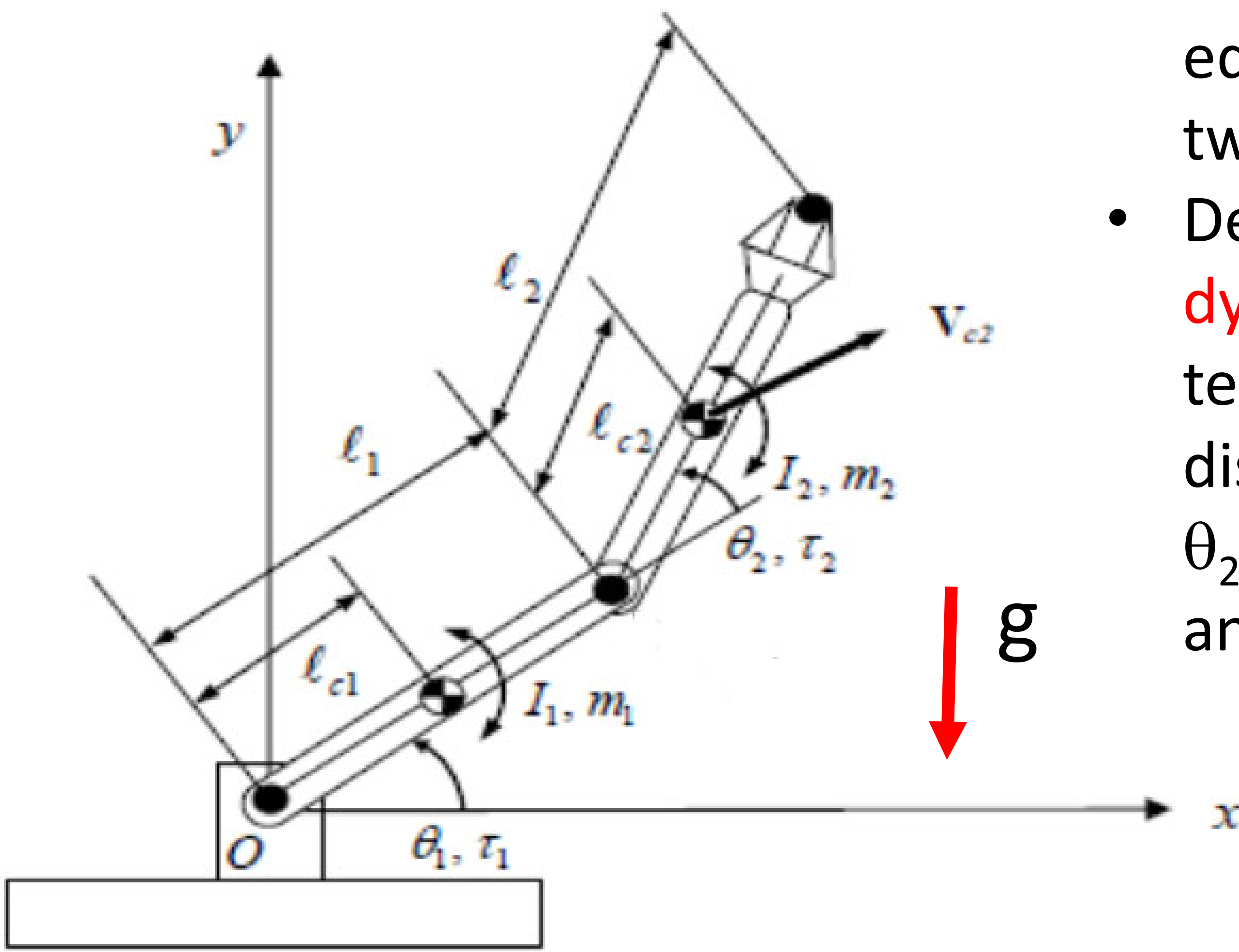
- Desired to have dynamic equations consist of input-output terms such as:
  - Joint forces/torques
  - Joint displacement variables

    (which are complete and independent generalized coordinates that locate whole robot mechanism)

Closed-form dynamic equations**:** explicit input-output form in terms of joint displacement vector **q** and joint torque vector $\boldsymbol{\tau}$

Example 2

- Two degree-of-freedom (dof) planar manipulator

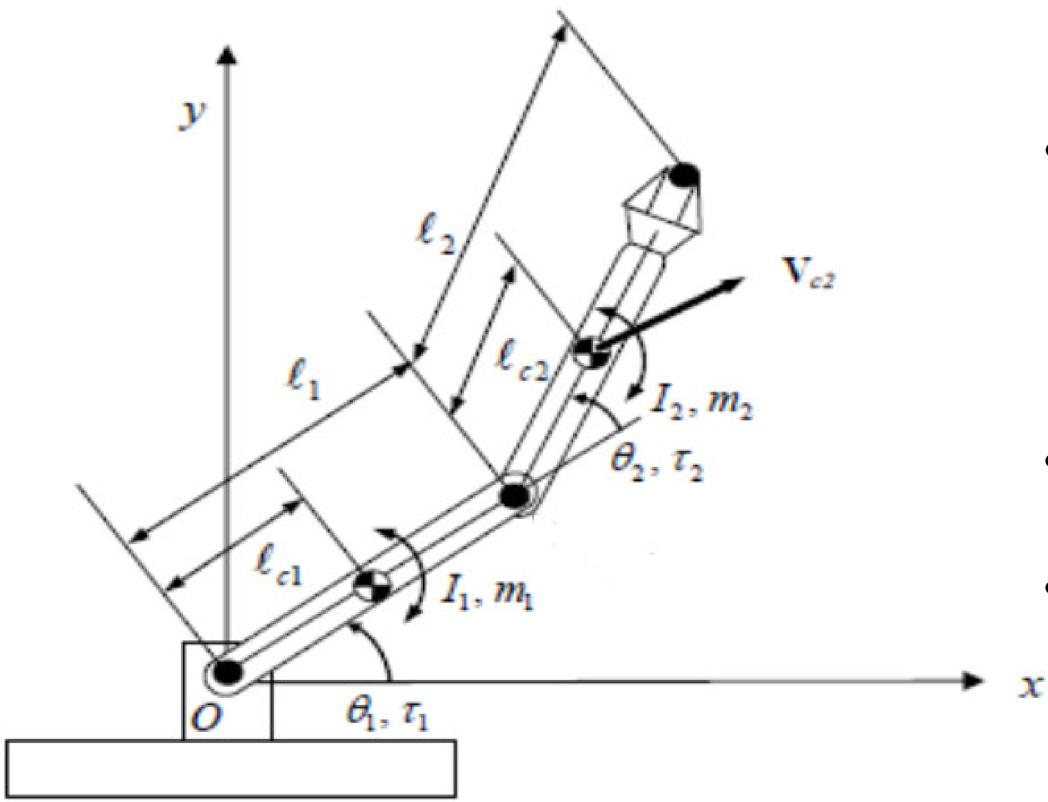


(Image source: https://ocw.mit.edu/courses/)

- Obtain Newton-Euler equations of motion for two individual links
- Derive closed-form dynamic equations in terms of joint displacements θ_1 and θ_2, and joint torques τ_1 and τ_2

Example 2

- First obtain Newton-Euler equations of motion for the two individual links:



- Assume centroid of link *i* is located on centre line passing through adjacent joints at a distance l_{ci} from joint *i*
- For 2D problem, centroidal inertia tensor is reduced to a scalar moment of inertia about centroid (denoted by I_i)

- $\mathbf{v}_{ci}$: Velocity of centroid of each link (vector in xy plane)
- ω_i : Angular velocity (scalar)

Example 2

From earlier slide:

$$\mathbf{f}_{i-1,i} - \mathbf{f}_{i,i+1} + m_i\mathbf{g} - m_i\dot{\mathbf{v}}_{ci} = \mathbf{0}$$

$$\mathbf{N}_{i-1,i} - \mathbf{N}_{i,i+1} - \left(\mathbf{r}_{i-1,i} + \mathbf{r}_{i,Ci}\right)\times\mathbf{f}_{i-1,i} + \left(-\mathbf{r}_{i,Ci}\right)\times\left(-\mathbf{f}_{i,i+1}\right) - \mathbf{I}_i\dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i\times\left(\mathbf{I}_i\boldsymbol{\omega}_i\right) = \mathbf{0},$$

From Eqs (1) and (3):

- Link 1 (i = 1):

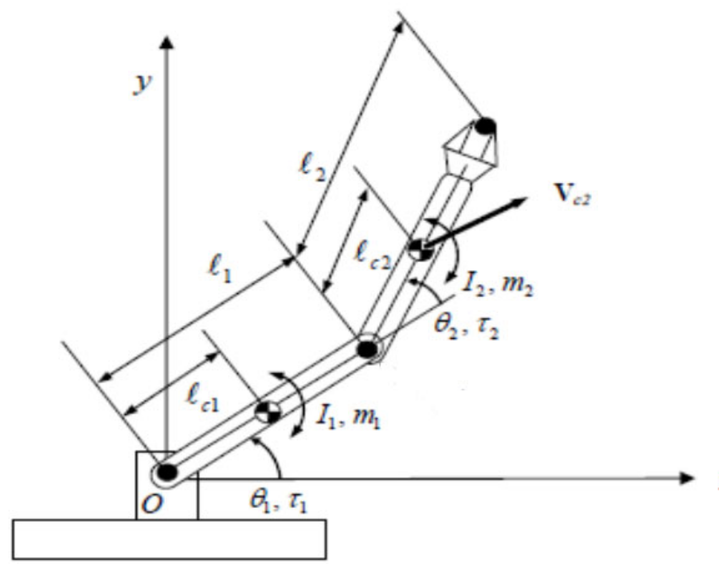
$$\mathbf{f}_{0,1} - \mathbf{f}_{1,2} + m_1\mathbf{g} - m_1\dot{\mathbf{v}}_{c1} = \mathbf{0} \qquad (4a)$$

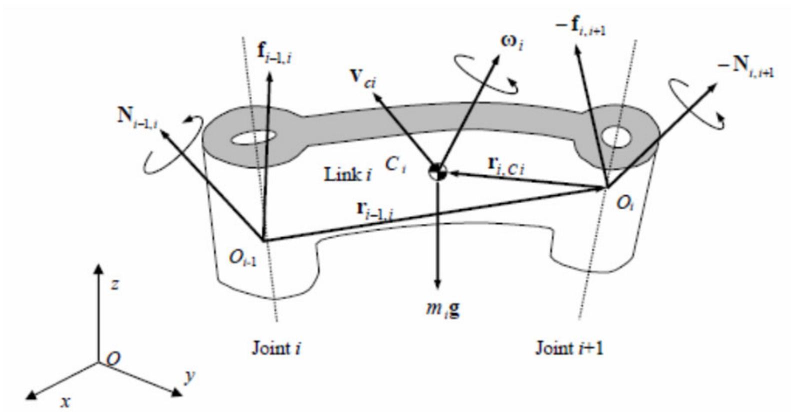
$$N_{0,1} - N_{1,2} + \mathbf{r}_{1,c1}\times\mathbf{f}_{1,2} - \mathbf{r}_{0,c1}\times\mathbf{f}_{0,1} - I_1\dot{\omega}_1 = 0 \qquad (4b)$$

- Link 2 (i = 2):

$$\mathbf{f}_{1,2} + m_2\mathbf{g} - m_2\dot{\mathbf{v}}_{c2} = \mathbf{0} \qquad (5a)$$

$$N_{1,2} - \mathbf{r}_{1,c2}\times\mathbf{f}_{1,2} - I_2\dot{\omega}_2 = 0 \qquad (5b)$$

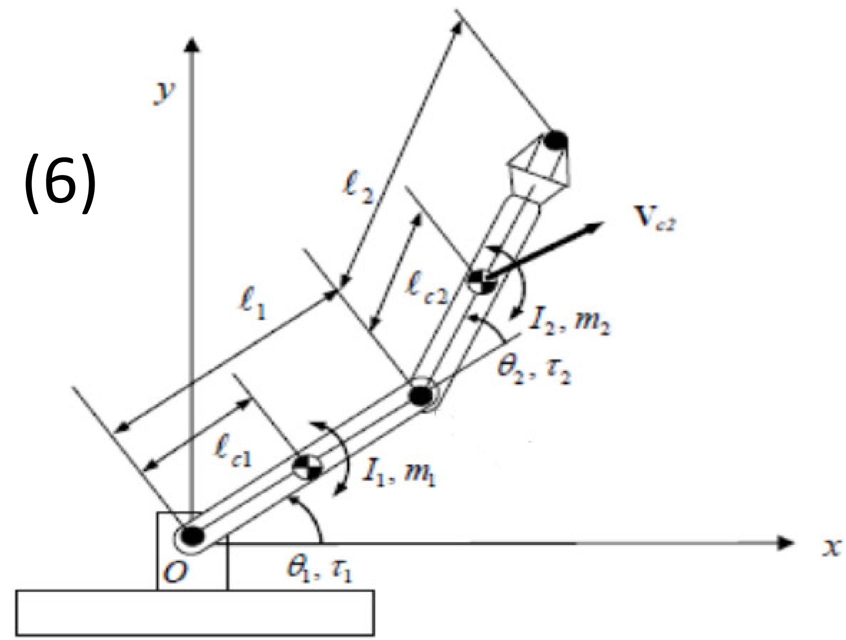




Example 2

- To obtain closed-form dynamic equations → To explicitly involve joint torques in dynamic equations
- For this planar manipulator, two joint torques simply equal to respective coupling moments:

$$N_{i-1,i} = \tau_i , \quad i = 1,2 \qquad (6)$$



Example 2

From earlier slides:

$$\mathbf{f}_{1,2} = m_2\dot{\mathbf{v}}_{c2} - m_2\mathbf{g} \quad \text{(5a)}$$

$$N_{i-1,i} = \tau_i, \quad i = 1,2 \quad \text{(6)}$$

- Starting from Eq. (5b), substituting Eq (6) and eliminating $\mathbf{f}_{1,2}$ using Eq. (5a):

$$N_{1,2} - \mathbf{r}_{1,c2} \times \mathbf{f}_{1,2} - I_2\dot{\omega}_2 = 0 \quad \text{(5b)}$$

$$\tau_2 - \mathbf{r}_{1,c2} \times \left(m_2\dot{\mathbf{v}}_{c2} - m_2\mathbf{g}\right) - I_2\dot{\omega}_2 = 0 \quad \text{(Sub. 6 and 5a)}$$

$$\boxed{\tau_2 - \mathbf{r}_{1,c2} \times m_2\dot{\mathbf{v}}_{c2} + \mathbf{r}_{1,c2} \times m_2\mathbf{g} - I_2\dot{\omega}_2 = 0} \quad \text{(7)}$$

$N_{1,2}$ *and* $\boldsymbol{f}_{1,2}$ eliminated!

Example 2

From earlier slide: $\mathbf{f}_{0,1} - \mathbf{f}_{1,2} + m_1\mathbf{g} - m_1\dot{\mathbf{v}}_{c1} = \mathbf{0}$ (4a)

$\mathbf{f}_{1,2} = m_2\dot{\mathbf{v}}_{c2} - m_2\mathbf{g}$ (5a)

$N_{i-1,i} = \tau_i, \quad i = 1,2$ (6)

- Similarly for Eq. (4b), substituting Eq (6), eliminating $\mathbf{f}_{0,1}$ using Eq. (4a) and eliminating $\mathbf{f}_{1,2}$ using Eq. (5a):

$$N_{0,1} - N_{1,2} + \mathbf{r}_{1,c1} \times \mathbf{f}_{1,2} - \mathbf{r}_{0,c1} \times \mathbf{f}_{0,1} - I_1\dot{\omega}_1 = 0 \qquad \text{(4b)}$$

$$\tau_1 - \tau_2 + \mathbf{r}_{1,c1} \times \mathbf{f}_{1,2} - \mathbf{r}_{0,c1} \times \left(\mathbf{f}_{1,2} - m_1\mathbf{g} + m_1\dot{\mathbf{v}}_{c1}\right) - I_1\dot{\omega}_1 = 0 \qquad \text{(Sub. 4a and 6)}$$

$$\tau_1 - \tau_2 - \mathbf{r}_{0,1} \times \mathbf{f}_{1,2} + \mathbf{r}_{0,c1} \times m_1\mathbf{g} - \mathbf{r}_{0,c1} \times m_1\dot{\mathbf{v}}_{c1} - I_1\dot{\omega}_1 = 0$$

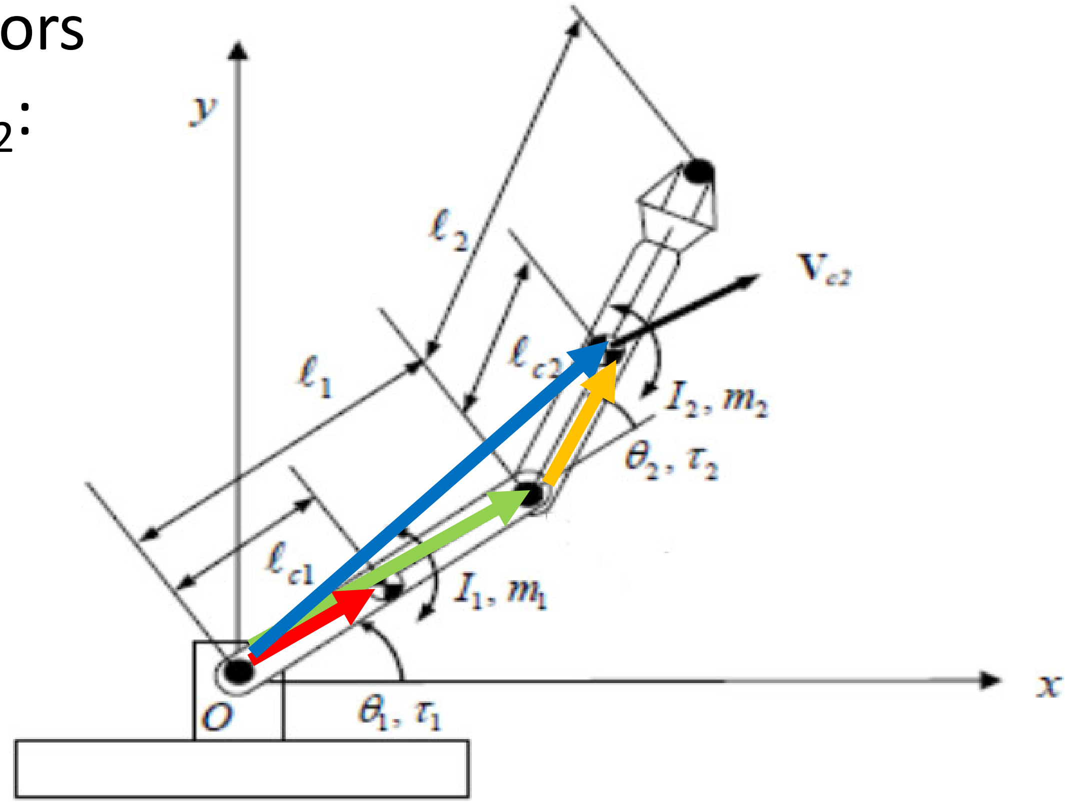
$$\tau_1 - \tau_2 - \mathbf{r}_{0,1} \times \left(m_2\dot{\mathbf{v}}_{c2} - m_2\mathbf{g}\right) + \mathbf{r}_{0,c1} \times m_1\mathbf{g} - \mathbf{r}_{0,c1} \times m_1\dot{\mathbf{v}}_{c1} - I_1\dot{\omega}_1 = 0 \qquad \text{(Sub. 5a)}$$

$$\boxed{\tau_1 - \tau_2 - \mathbf{r}_{0,c1} \times m_1\dot{\mathbf{v}}_{c1} - \mathbf{r}_{0,1} \times m_2\dot{\mathbf{v}}_{c2} + \mathbf{r}_{0,c1} \times m_1\mathbf{g} + \mathbf{r}_{0,1} \times m_2\mathbf{g} - I_1\dot{\omega}_1 = 0} \qquad (8)$$

$N_{0,1}, N_{1,2}, \boldsymbol{f}_{0,1}$ *and* $\boldsymbol{f}_{1,2}$ eliminated!

Example 2

Next, express key position vectors using joint displacements θ_1, θ_2:



$$\mathbf{r}_{0,c1} = \begin{pmatrix} l_{c1}\,\mathrm{c}_1 \\ l_{c1}\,\mathrm{s}_1 \end{pmatrix} \qquad \text{(9a)}$$

$$\mathbf{r}_{0,1} = \begin{pmatrix} l_1\,\mathrm{c}_1 \\ l_1\,\mathrm{s}_1 \end{pmatrix} \qquad \text{(9b)}$$

$$\mathbf{r}_{1,c2} = \begin{pmatrix} l_{c2}\,\mathrm{c}_{12} \\ l_{c2}\,\mathrm{s}_{12} \end{pmatrix} \qquad \text{(9c)}$$

$$\mathbf{r}_{0,c2} = \mathbf{r}_{0,1} + \mathbf{r}_{1,c2} = \begin{pmatrix} l_1\,\mathrm{c}_1 \\ l_1\,\mathrm{s}_1 \end{pmatrix} + \begin{pmatrix} l_{c2}\,\mathrm{c}_{12} \\ l_{c2}\,\mathrm{s}_{12} \end{pmatrix} = \begin{pmatrix} l_1\,\mathrm{c}_1 + l_{c2}\,\mathrm{c}_{12} \\ l_1\,\mathrm{s}_1 + l_{c2}\,\mathrm{s}_{12} \end{pmatrix} \qquad \text{(9d)}$$

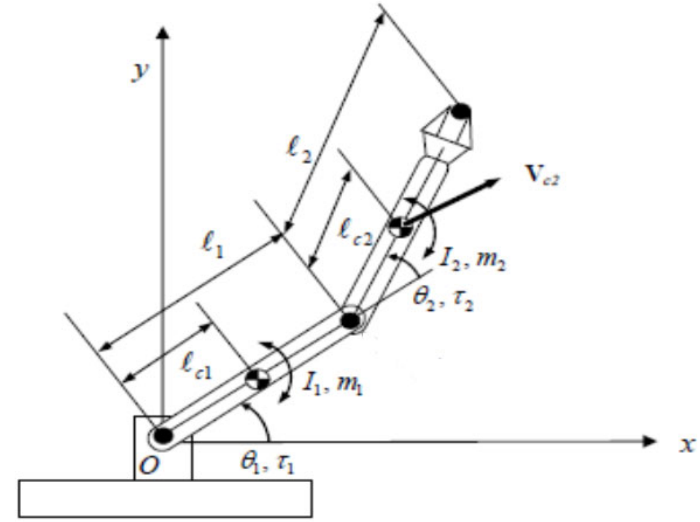
Notation: $\mathrm{c}_1 = \cos(\theta_1); \mathrm{s}_2 = sin(\theta_2);\ \mathrm{c}_{12} = \cos(\theta_1 + \theta_2); \mathrm{s}_{12} = sin(\theta_1 + \theta_2)$

Example 2

- $\mathbf{v}_{ci}$, $\dot{\boldsymbol{v}}_{ci}$, ω_i and $\dot{\omega}_i$ expressed in terms of joint variables:

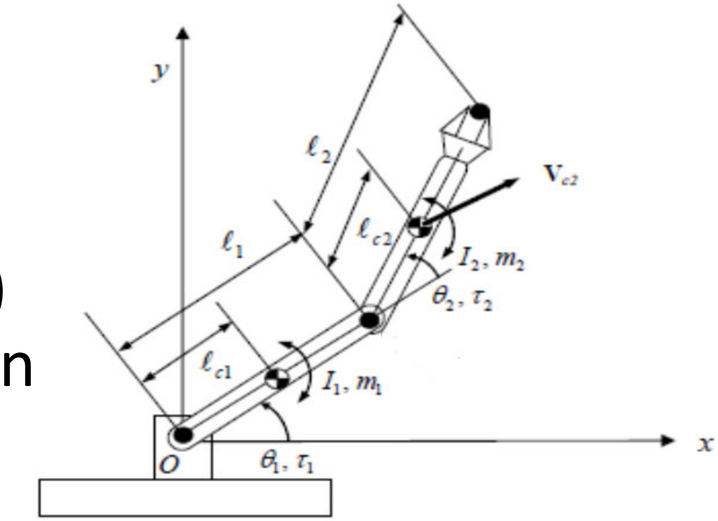
$$\left.\begin{aligned}\omega_1 &= \dot{\theta}_1, \quad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\omega}_1 &= \ddot{\theta}_1, \quad \dot{\omega}_2 = \ddot{\theta}_1 + \ddot{\theta}_2\end{aligned}\right\} \text{(10a)}$$

$$\left.\begin{aligned}\mathbf{v}_{c1} &= \dot{\mathbf{r}}_{0,c1} = \begin{pmatrix} -l_{c1}\dot{\theta}_1\,\mathrm{s}_1 \\ l_{c1}\dot{\theta}_1\,\mathrm{c}_1 \end{pmatrix} \\ \dot{\mathbf{v}}_{c1} &= \begin{pmatrix} -l_{c1}\ddot{\theta}_1\,\mathrm{s}_1 - l_{c1}\dot{\theta}_1^2\,\mathrm{c}_1 \\ l_{c1}\ddot{\theta}_1\,\mathrm{c}_1 - l_{c1}\dot{\theta}_1^2\,\mathrm{s}_1 \end{pmatrix}\end{aligned}\right\} \text{(10b)}$$



$$\left.\begin{aligned}\mathbf{v}_{c2} &= \dot{\mathbf{r}}_{0,c2} = \begin{pmatrix} -\{l_1\,\mathrm{s}_1 + l_{c2}\,\mathrm{s}_{12}\}\,\dot{\theta}_1 - l_{c2}\dot{\theta}_2\,\mathrm{s}_{12} \\ \{l_1\,\mathrm{c}_1 + l_{c2}\,\mathrm{c}_{12}\}\,\dot{\theta}_1 + l_{c2}\dot{\theta}_2\,\mathrm{c}_{12} \end{pmatrix} \\ \dot{\mathbf{v}}_{c2} &= \begin{pmatrix} -\{l_1\ddot{\theta}_1\,\mathrm{s}_1 + l_{c2}\ddot{\theta}_1\,\mathrm{s}_{12}\} - \{l_1\dot{\theta}_1^2\,\mathrm{c}_1 + l_{c2}\left(\dot{\theta}_1^2 + \dot{\theta}_2\dot{\theta}_1\right)c_{12}\} - l_{c2}\ddot{\theta}_2\,\mathrm{s}_{12} - l_{c2}\left(\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^{\,2}\right)c_{12} \\ \{l_1\ddot{\theta}_1\,\mathrm{c}_1 + l_{c2}\ddot{\theta}_1 c_{12}\} + \{-l_1\dot{\theta}_1^2\,\mathrm{s}_1 - l_{c2}\left(\dot{\theta}_1^2 + \dot{\theta}_2\dot{\theta}_1\right)\mathrm{s}_{12}\} + l_{c2}\ddot{\theta}_2 c_{12} - l_{c2}\left(\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^{\,2}\right)\mathrm{s}_{12} \end{pmatrix} \\ &= \begin{pmatrix} -l_1\ddot{\theta}_1\,\mathrm{s}_1 - l_{c2}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\mathrm{s}_{12} - l_1\dot{\theta}_1^2\,\mathrm{c}_1 - l_{c2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^{\,2}\right)c_{12} - 2l_{c2}\dot{\theta}_1\dot{\theta}_2 c_{12} \\ l_1\ddot{\theta}_1\,\mathrm{c}_1 + l_{c2}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)c_{12} - l_1\dot{\theta}_1^2\,\mathrm{s}_1 - l_{c2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^{\,2}\right)s_{12} - 2l_{c2}\dot{\theta}_1\dot{\theta}_2\,\mathrm{s}_{12} \end{pmatrix}\end{aligned}\right\} \text{(10c)}$$

Example 2



- Substituting Eqs. (9) and (10) into Eqs. (7) and (8), closed-form dynamic equations in terms of θ_1 and θ_2 is obtained:

$$\tau_1 = H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1 \qquad \text{(11a)}$$

$$\tau_2 = H_{21}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 + G_2 \qquad \text{(11b)}$$

where

$$H_{11} = m_1 l_{c1}^2 + I_1 + m_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos\theta_2\right) + I_2 \qquad \text{(12a)}$$

$$H_{22} = m_2 l_{c2}^2 + I_2 \qquad \text{(12b)}$$

$$H_{12} = H_{21} = m_2\left(l_{c2}^2 + l_1 l_{c2}\cos\theta_2\right) + I_2 \qquad \text{(12c)}$$

$$h = m_2 l_1 l_{c2}\sin\theta_2 \qquad \text{(12d)}$$

$$G_1 = m_1 l_{c1} g\cos\theta_1 + m_2 g\left\{l_{c2}\cos\left(\theta_1 + \theta_2\right) + l_1\cos\theta_1\right\} \qquad \text{(12e)}$$

$$G_2 = m_2 g l_{c2}\cos\left(\theta_1 + \theta_2\right) \qquad \text{(12f)}$$

Scalar g represents the acceleration of gravity along the negative y-axis

- More generally, closed-form dynamic equations of n-degree-of-freedom manipulator (assume frictionless joints and no interaction force/moments):

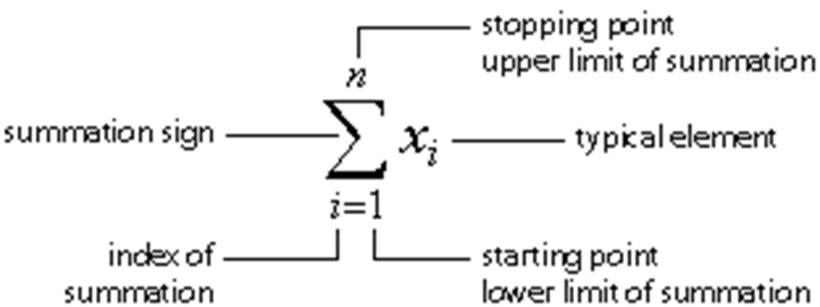
$$\boxed{\tau_i = \sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} h_{ijk}\dot{q}_j\dot{q}_k + G_i, \quad i = 1, \cdots, n} \qquad (13)$$

$$\underbrace{H_{ij},\ h_{ijk} \text{ and } G_i}_{}$$

Functions of joint displacements

$$q_1, q_2, \ldots, q_n$$

Summation notation:



Double summation notation:

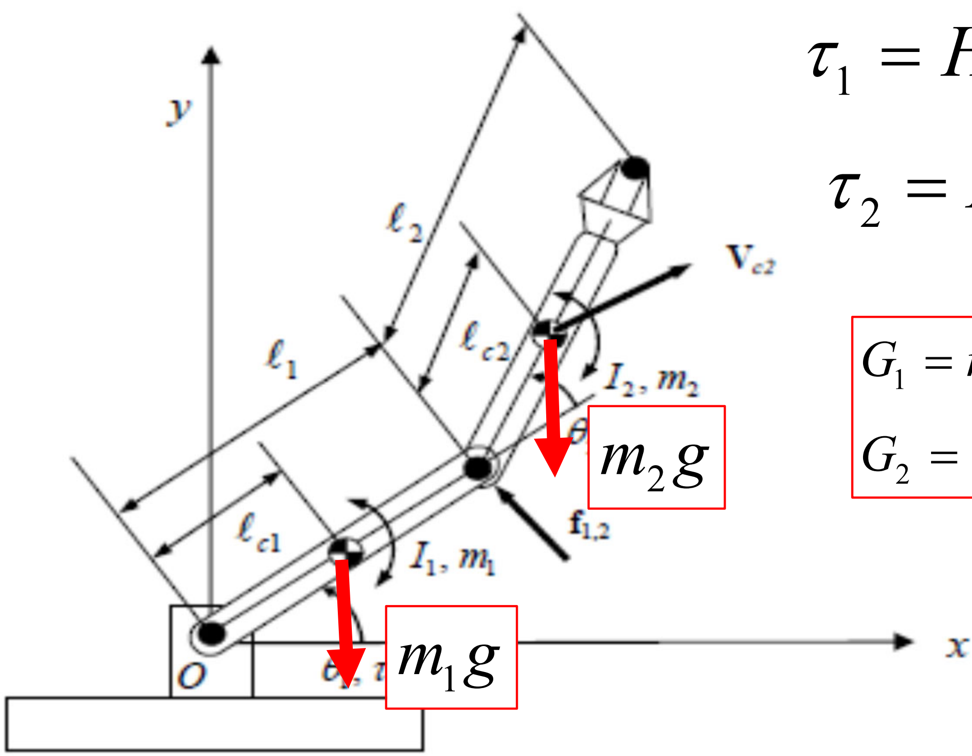
The double summation operator is used to sum up twice for the same variable:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij} = \sum_{i=1}^{n}(x_{i1} + x_{i2} + \ldots + x_{im})$$

$$= (x_{11}+x_{21}+\ldots+x_{n1})+(x_{12}+x_{22}+\ldots+x_{n2})+\ldots+(x_{1m}+x_{2m}+\ldots+x_{nm})$$

Physical Interpretation of the Dynamic Equations

- G_i : effect of gravity.
  - G_1 is resulting moment about joint axis 1 due to gravitational forces acting on masses m_1 and m_2.
  - G_2 is resulting moment about joint axis 2 due to gravitational force acting on mass m_2.
  - Both dependent upon arm configuration

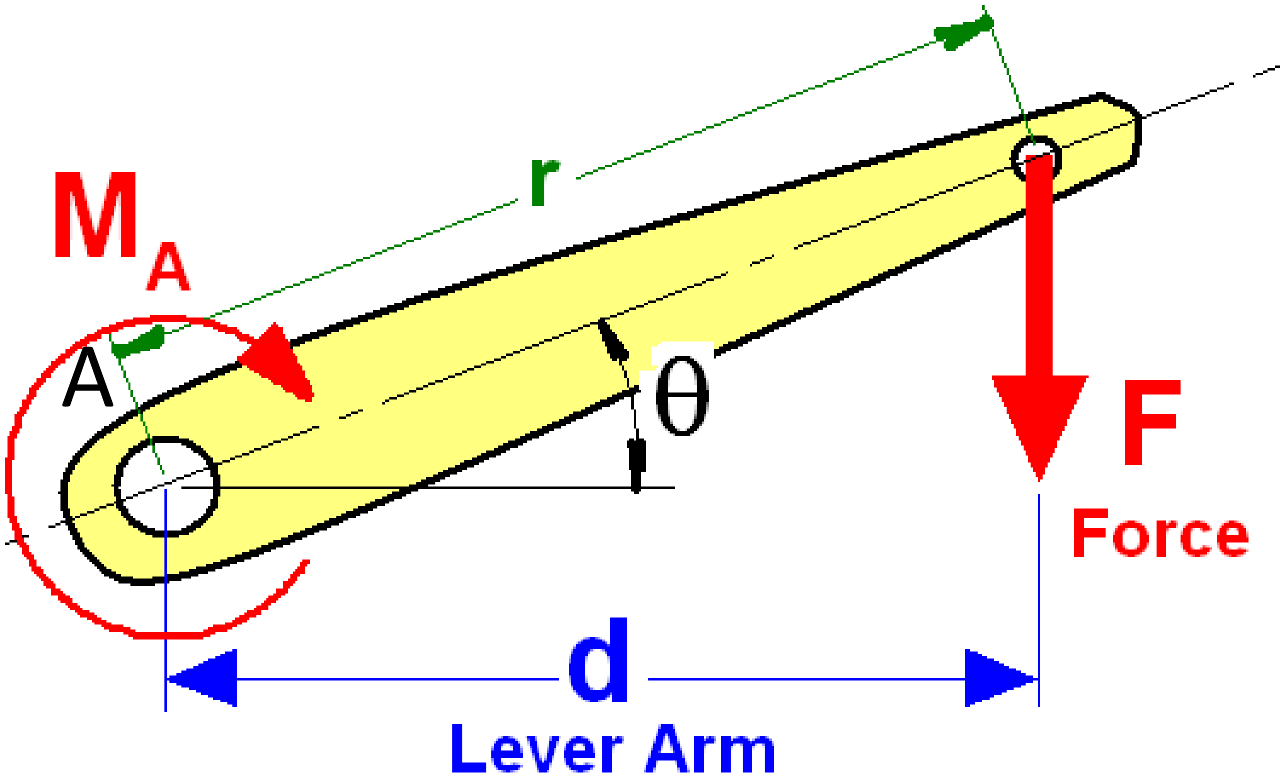


$$\tau_1 = H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + \boxed{G_1}$$

$$\tau_2 = H_{12}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 + \boxed{G_2}$$

$$G_1 = m_1 l_{c1} g\cos\theta_1 + m_2 g\left\{l_{c2}\cos\left(\theta_1+\theta_2\right) + l_1\cos\theta_1\right\}$$

$$G_2 = m_2 g l_{c2}\cos\left(\theta_1+\theta_2\right)$$



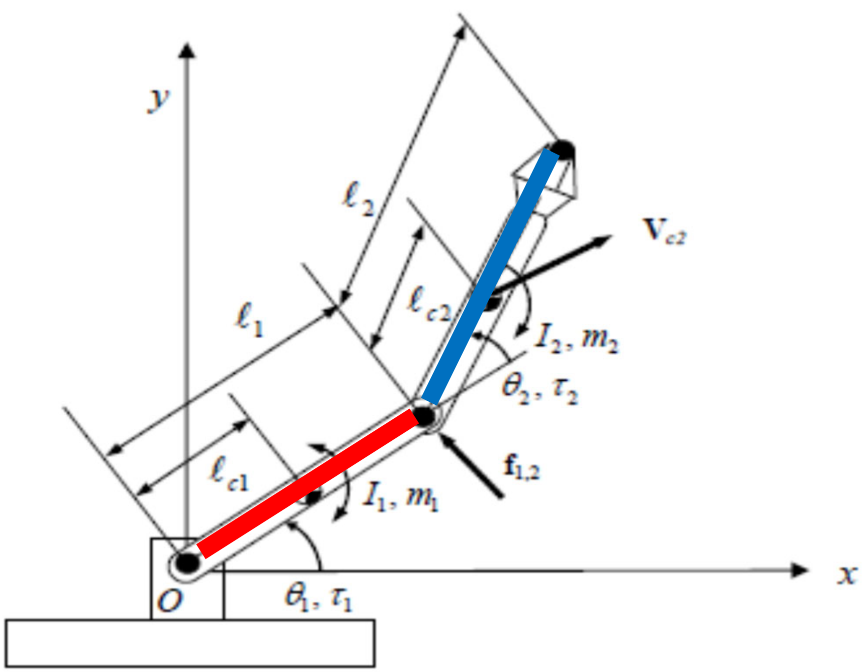
Moment of F about A, $M_A = F\,d = Fr\cos\theta$

+

Physical Interpretation of the Dynamic Equations

- $H_{11}\ddot{\theta}_1$
  - Consider $\dot{\theta}_1 = \dot{\theta}_2 = \ddot{\theta}_2 = 0$,

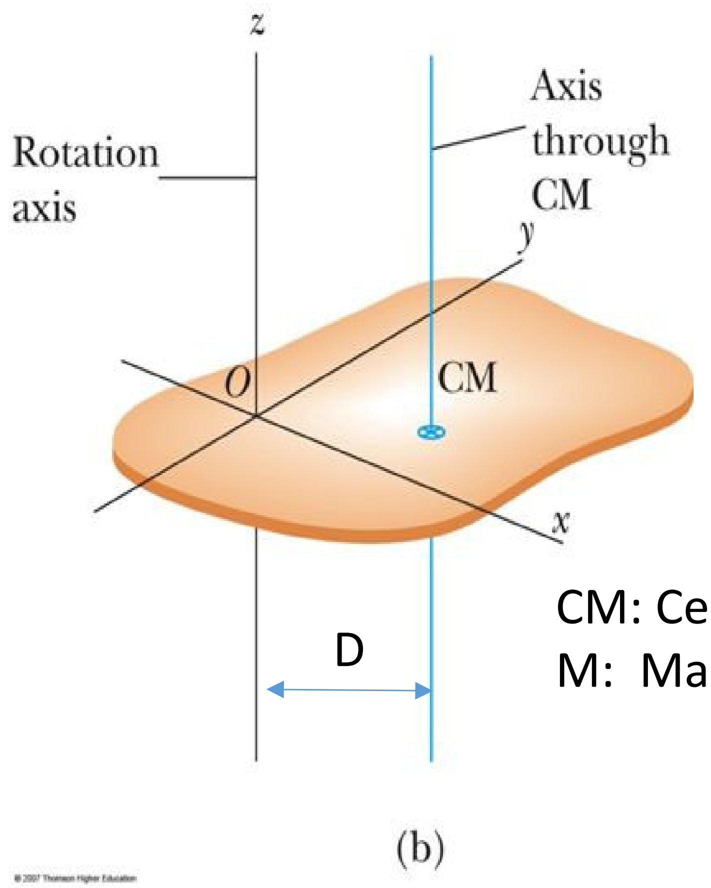
    H_{11} accounts for total moment of inertia of both links seen by first joint when joint 2 is immobilized.



$$H_{11} = \boxed{m_1 l_{c1}^2 + I_1} + \boxed{m_2\left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}\cos\theta_2\right) + I_2}$$

Parallel-Axis Theorem

- The theorem states $I = I_{CM} + MD^{2}$
  - I is about any axis parallel to the axis through the centre of mass of the object
  - I_{CM} is about the axis through the centre of mass
  - D is the distance from the centre of mass axis to the arbitrary axis



CM: Centre of Mass
M: Mass

Physical Interpretation of the Dynamic Equations

- $H_{11}\ddot{\theta}_1$ (cont.)
  - First two terms of H_{11} : $\boxed{m_1 l_{c1}^2 + I_1}$

Moment of inertia of link 1 with respect to joint 1

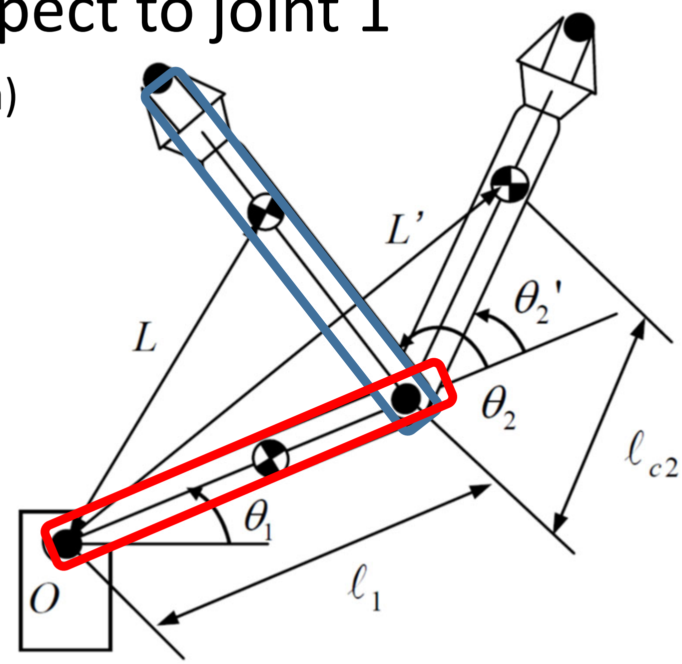
(Applying parallel-axis theorem of moment of inertia)

- Other terms : $\boxed{m_2\left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}\cos\theta_2\right) + I_2}$

Moment of inertia of link 2 with respect to joint 1

(Applying parallel-axis theorem of moment of inertia)

$$
\begin{aligned}
& m_2 L^2 + I_2 \\
&= m_2\left(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}\cos\theta_2\right) + I_2
\end{aligned}
$$



Physical Interpretation of the Dynamic Equations

- $H_{12}\ddot{\theta}_2$: Effect of joint 2 acceleration on the first joint (dynamic coupling)

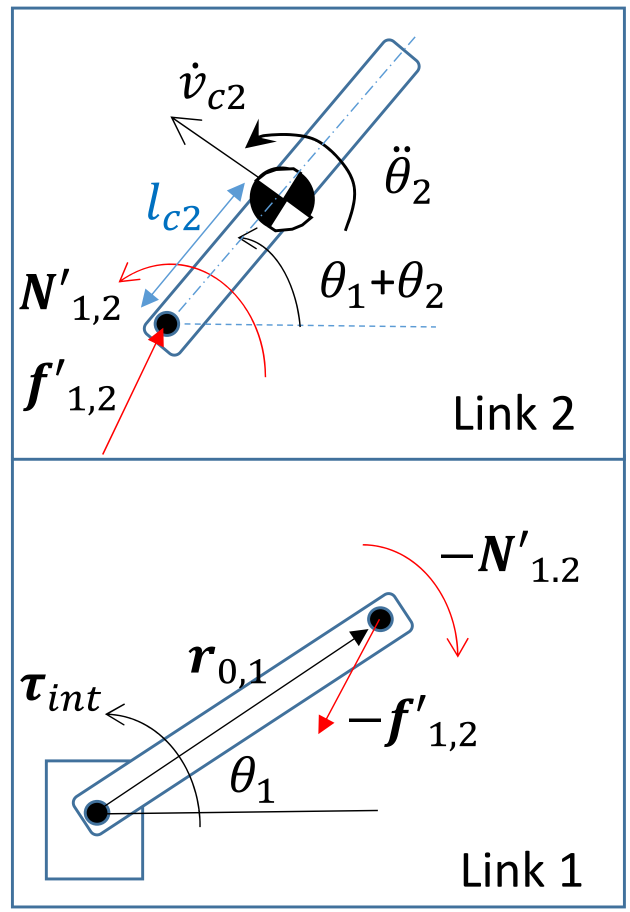
  Consider $\dot{\theta}_1 = \dot{\theta}_2 = \ddot{\theta}_1 = 0$:

  - When joint 2 is accelerated from rest while joint 1 is fixed, internal reaction force and moment: $\mathbf{f}'_{1,2}$, $N'_{1,2}$ at joint 2 will be present.

$$\mathbf{f}'_{1,2} = m_2\dot{\mathbf{v}}_{c2}$$

where $\quad \dot{\mathbf{v}}_{c2} = l_{c2}\begin{pmatrix} -s_{12} \\ c_{12} \\ 0 \end{pmatrix}\ddot{\theta}_2$

$$N'_{1,2} = (I_2 + m_2 l_{c2}^2)\ddot{\theta}_2$$

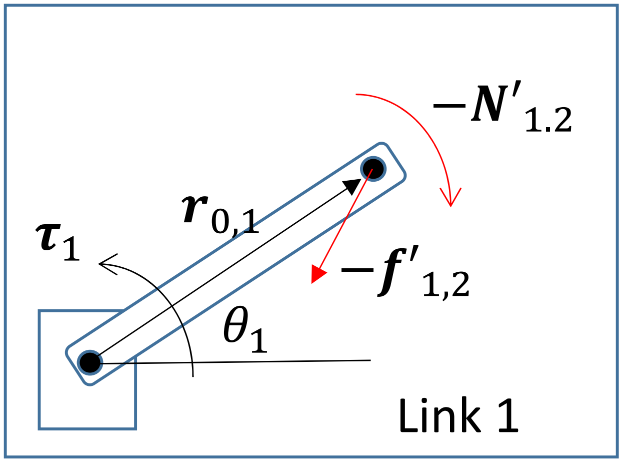


Physical Interpretation of the Dynamic Equations

- $H_{12}\ddot{\theta}_2$: Effect of joint 2 acceleration on the first joint (dynamic coupling)

  - Coupling force $\mathbf{f'}_{1,2}$ and moment $N'_{1,2}$ cause an equivalent torque τ_{int} about first joint axis:

$$
\begin{aligned}
\tau_{\text{int}} &= -N'_{1,2} + \mathbf{r}_{0,1} \times \left(-\mathbf{f}'_{1,2}\right) \\
&= -(I_2 + m_2 l_{c2}^2)\ddot{\theta}_2 - l_1 \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} \times \left(m_2 l_{c2} \begin{pmatrix} -s_{12} \\ c_{12} \\ 0 \end{pmatrix} \ddot{\theta}_2 \right) \\
&= -\{I_2 + m_2(l_{c2}^2 + l_1 l_{c2} \cos\theta_2)\}\ddot{\theta}_2 \quad (15) \\
&= -H_{12}\ddot{\theta}_2
\end{aligned}
$$



$$
\mathbf{r}_{0,1} = \begin{pmatrix} c_1 \\ s_1 \\ 0 \end{pmatrix} l_1
$$

Trigonometric Identities

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Reduction Formulas:

$$\sin(-\theta) = -\sin\theta \qquad \sin\theta = -\sin(\theta - \pi)$$
$$\cos(-\theta) = \cos\theta \qquad \cos\theta = -\cos(\theta - \pi)$$
$$\tan(-\theta) = -\tan\theta \qquad \tan\theta = \tan(\theta - \pi)$$

Sum or Difference of Two Angles:

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Half–Angle Formulas:

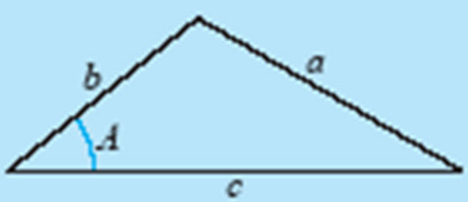
$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$
$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Double–Angle Formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Reciprocal Identities:

$$\csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Cross product

The cross product of two **vectors** in 3-space is

$$\vec{u} \times \vec{v} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix} \qquad \vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

An easy way to remember this is to write it in the form

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

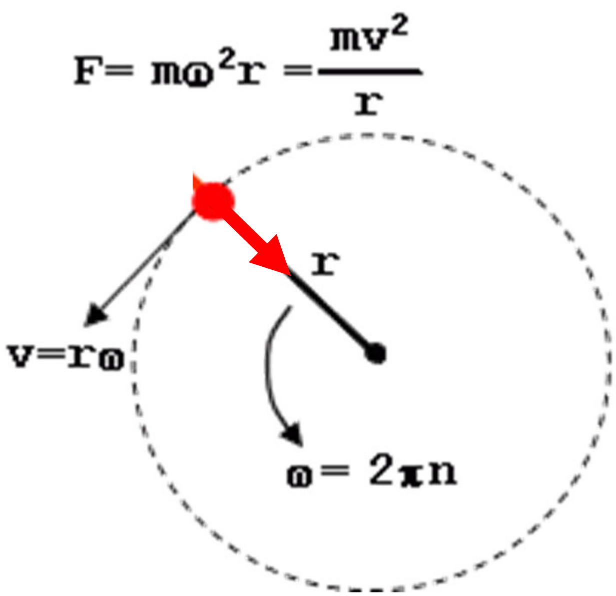
where $\vec{e_x} = (1,0,0)^T$, $\vec{e_y} = (0,1,0)^T$, and $\vec{e_x} = (0,0,1)^T$.

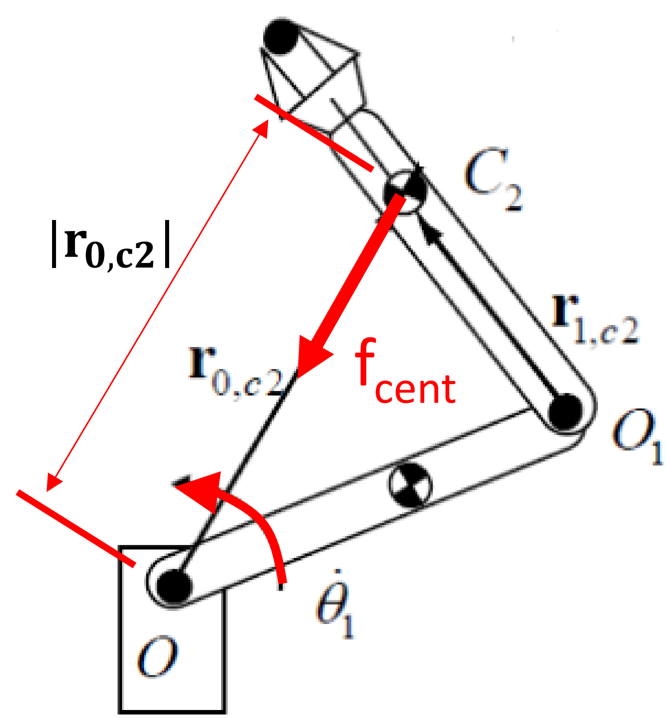
Physical Interpretation of the Dynamic Equations

- $h\dot{\theta}_1^2$ (effect on joint 2 torque, τ_2)
  - Set $\dot{\theta}_2 = \ddot{\theta}_1 = \ddot{\theta}_2 = 0$

  Magnitude of centripetal force acting upon mass centroid of link 2 due to $\dot{\theta}_1$:

$$|\mathbf{f}_{cent}| = m_2 \dot{\theta}_1^2 |\mathbf{r}_{0,c2}| \qquad (16)$$

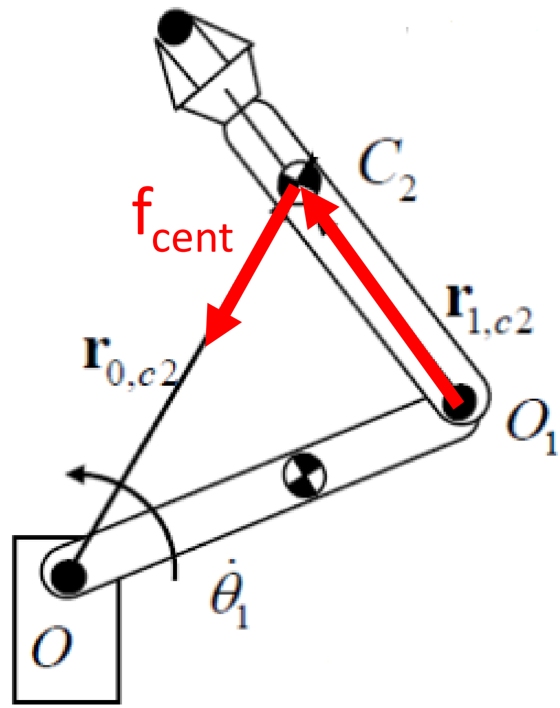




Physical Interpretation of the Dynamic Equations

- $h\dot{\theta}_1^2$ (cont)
  - $\mathbf{f}_{\text{cent}}$ is effected by equivalent torque τ_{cent} at joint 2:

$$\boxed{\tau_{cent} = \mathbf{r}_{1,c2} \times \mathbf{f}_{cent} = m_2 l_1 l_{c2} \dot{\theta}_1^2 \sin\theta_2} \quad \text{(17a)}$$



$$\mathbf{r}_{1,c2} = \begin{pmatrix} l_{c2}\mathrm{c}_{12} \\ l_{c2}\mathrm{s}_{12} \\ 0 \end{pmatrix} \qquad \mathbf{r}_{0,c2} = \begin{pmatrix} l_1\mathrm{c}_1 + l_{c2}\mathrm{c}_{12} \\ l_1\mathrm{s}_1 + l_{c2}\mathrm{s}_{12} \\ 0 \end{pmatrix}$$

$$\mathbf{f}_{cent} = |\mathbf{f}_{cent}| \frac{-r_{0,c2}}{|r_{0,c2}|} = -m_2\dot{\theta}_1^2 |r_{0,c2}| \frac{1}{|r_{0,c2}|} \begin{pmatrix} l_1\mathrm{c}_1 + l_{c2}\mathrm{c}_{12} \\ l_1\mathrm{s}_1 + l_{c2}\mathrm{s}_{12} \\ 0 \end{pmatrix}$$

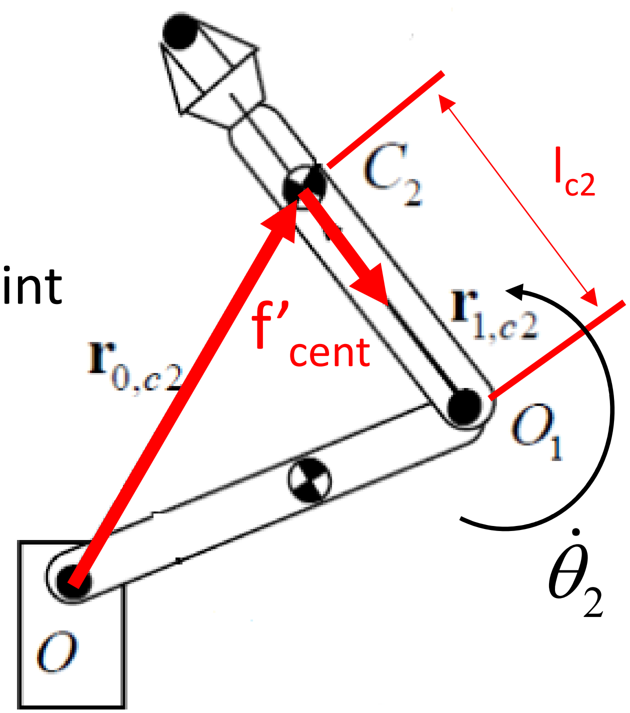
$$= -m_2\dot{\theta}_1^2 \begin{pmatrix} l_1\mathrm{c}_1 + l_{c2}\mathrm{c}_{12} \\ l_1\mathrm{s}_1 + l_{c2}\mathrm{s}_{12} \\ 0 \end{pmatrix}$$

Physical Interpretation of the Dynamic Equations

- $-h\dot{\theta}_2^2$ (effect on joint 1 torque, τ_1)
  - Set $\dot{\theta}_1 = \ddot{\theta}_1 = \ddot{\theta}_2 = 0$
  - Magnitude of centripetal force acting upon link 2 due to $\dot{\theta}_2$:

$$\left|\mathbf{f}'_{cent}\right| = m_2 l_{c2} \dot{\theta}_2^2$$

  - $\mathbf{f'}_{\text{cent}}$ is effected by an equivalent torque τ'_{cent} at joint 1:



$$\boxed{\tau'_{cent} = \mathbf{r}_{0,c2} \times \mathbf{f}'_{cent} = -m_2 l_1 l_{c2} \dot{\theta}_2^2 \sin\theta_2} \quad \text{(17b)}$$

$$\mathbf{r}_{0,c2} = \begin{pmatrix} l_1 c_1 + l_{c2} c_{12} \\ l_1 s_1 + l_{c2} s_{12} \\ 0 \end{pmatrix} \qquad \mathbf{r}_{1,c2} = l_{c2} \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix}$$

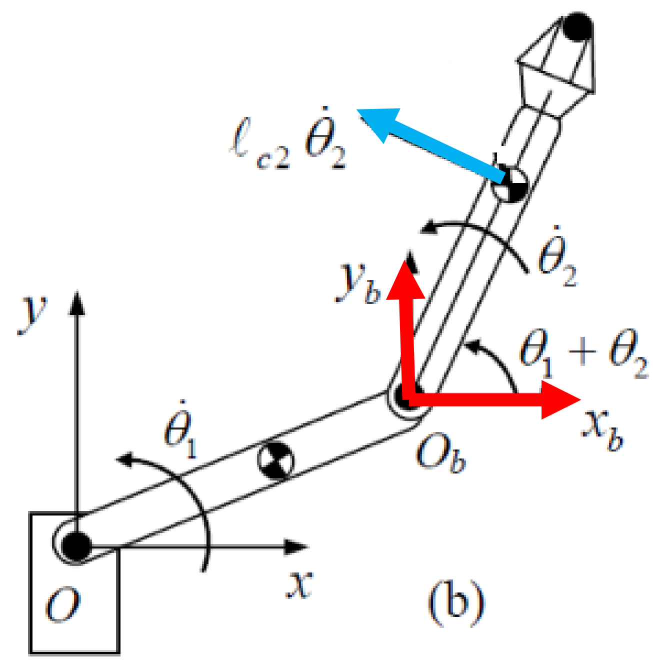
$$\mathbf{f'}_{cent} = |\mathbf{f'}_{cent}| \frac{-r_{1,c2}}{|r_{1,c2}|} = -m_2 l_{c2} \dot{\theta}_2^2 \begin{pmatrix} c_{12} \\ s_{12} \\ 0 \end{pmatrix}$$

Physical Interpretation of the Dynamic Equations

- $-2h\dot{\theta}_1\dot{\theta}_2$ (effect on joint 1 torque, τ_1)
  - Let O_b-$x_b y_b$ be coordinate frame attached to tip of link *1* (parallel to base coordinate frame at the instant)
  - Mass centroid of link *2* moves at velocity of

$$\boldsymbol{v_b} = l_{c2}\dot{\theta}_2 \begin{bmatrix} -s_{12} \\ c_{12} \\ 0 \end{bmatrix}$$ relative to O_b-$x_b y_b$

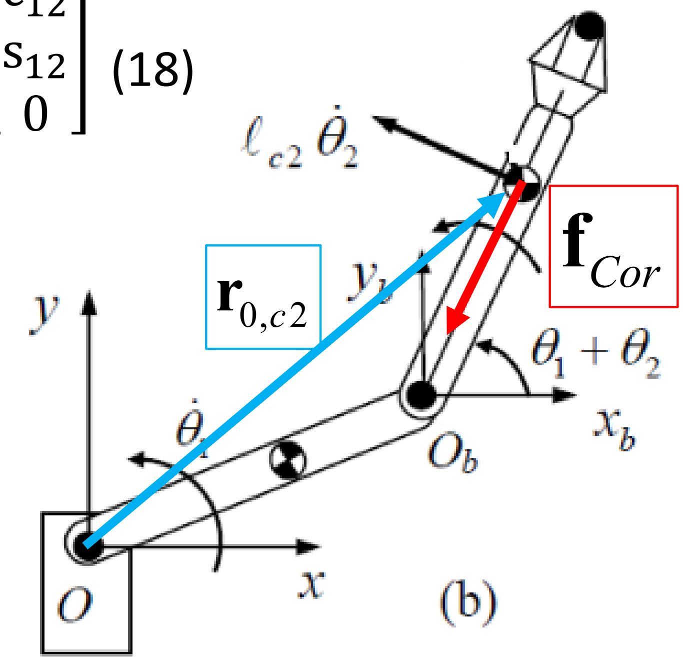
When mass particle *m* moves at velocity of $\mathbf{v}_b$ relative to moving coordinate frame rotating at angular velocity *ω*, mass particle has so-called *Coriolis force* given by $2m(\boldsymbol{\omega} \times \mathbf{v}_b)$.



Physical Interpretation of the Dynamic Equations

- $-2h\dot{\theta}_1\dot{\theta}_2$ (cont):
  - Let $\mathbf{f}_{Cor}$ be force acting on link 2 due to Coriolis effect:

$$\mathbf{f}_{Cor} = 2m_2\left(\underbrace{\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}}_{\boldsymbol{\omega}_1} \times \underbrace{\left(l_{c2}\dot{\theta}_2\begin{bmatrix}-s_{12}\\c_{12}\\0\end{bmatrix}\right)}_{\mathbf{v}_b}\right) = -2m_2 l_{c2}\dot{\theta}_1\dot{\theta}_2\begin{bmatrix}c_{12}\\s_{12}\\0\end{bmatrix} \quad (18)$$



  - Equivalent torque at joint 1 due to Coriolis force is:

$$\tau_{Cor} = \mathbf{r}_{0,c2} \times \mathbf{f}_{Cor} = -2m_2 l_{c2}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2) \quad (19)$$

$$\mathbf{r}_{0,c2} = \begin{pmatrix} l_1 c_1 + l_{c2}c_{12}\\ l_1 s_1 + l_{c2}s_{12}\\ 0\end{pmatrix}$$

Remark: Since Coriolis force given by Eq. (18) acts through joint 2, it does not create a moment about second joint.

Physical Interpretation of the Dynamic Equations

Summary:

- Dynamic equations of a robot arm are characterized by:
  - Configuration-dependent inertia
  - Gravity torques, and
  - Interaction torques caused by
    - Accelerations of the other joints and
    - Existence of Centripetal (Centrifugal) and Coriolis effects
  - For example: 2DOF planar manipulator

$$\tau_1 = H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1$$

$$\tau_2 = H_{12}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + h\dot{\theta}_1^2 + G_2$$

Physical Interpretation of the Dynamic Equations

- Dynamic equations can also be expressed in **matrix-vector form**:

$$\mathbf{H}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}\left(\mathbf{q}, \dot{\mathbf{q}}\right) + \mathbf{G}(\mathbf{q}) = \boldsymbol{\tau}$$

For 2-DOF planar manipulator example:

$$\mathbf{H}(\boldsymbol{q}) = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix}$$ – Manipulator inertia matrix

$$\mathbf{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} -h\dot{\theta}_2^2 - 2h\dot{\theta}_1\dot{\theta}_2 \\ h\dot{\theta}_1^2 \end{bmatrix}$$ – Coriolis/centripetal(centrifugal)

$$\mathbf{G}(\boldsymbol{q}) = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$ – Gravity

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \qquad \boldsymbol{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \quad \ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

What have you learned

- Newton-Euler formulation of manipulator dynamics
- Physical interpretation of manipulator dynamic equations