600.363/463 Solution to homework 6
Posted: 10/22/03
Due: Wednesday, 10/29/03

22.5-6

Let $C_1, C_2, \cdots, C_k (k \geq 1)$ be the strongly connected components of G. Let $C_l = \{v_1, v_2, \cdots, v_i\}$. Now in G', add one edge between $\{v_{j \bmod i}, v_{j+1 \bmod i}\}, j = 1, 2, \cdots i$ from $v_{j \bmod i}$ to $v_{j+1 \bmod i}$. This creates a directed cycle between the i vertices in the strongly connected component C_l. In addition to the above, add one each from some vertex u in a strongly connected component C_i to a vertex v in another strongly connected component C_j in the direction from u to v iff any vertex in C_i is connected to a vertex in C_j. So condition (a) of the question is met here as all the vertices in the directed cycle form a strongly connected component. Condition (b) is easily met by the construction of the component graph. For (c), observe that the directed cycle is a strongly connected graph with least number of edges. So each of the strongly connected components in G' have the least number of edges between vertices in the same component. For meeting condition (b), there must be at least one edge between strongly connected components that are connected and in our construction we add only one edge.

To explicitly construct G', first obtain the strongly connected components of G and add the edges as specfied in the above. The algorithm takes $O(V + E)$ time as the time to obtain strongly connected components in $O(V + E)$ and for the construction we need only $O(V + E)$ time again.

22.5-7

Algorithm:

1. Run STRONG-CONNECTED-COMPONENTS(G).
2. Take each strong connected component as a virtual vertex and create a new virtual graph G'.
3. Run TOPOLOGICAL-SORT(G').

4. Check if for any consevutive vertices (V_i, V_{i+1}) in a topological sort of G', there is an ede (V_i, V_{i+1}) in graph G'. if so, the original graph is semiconnected. Otherwise, it isn't.

Proof:
It is easy to show that G' is a DAG. Consider consecutive vertices V_i and V_{i+1} in G'. If there is no edge from V_i to V_{i+1}, we also conclude that there is no path from V_{i+1} to V_i since V_i finished after V_{i+1}. From the definition of G', we conclude that, there is no path from any vertices in G who is represented as V_i in G' to those represented as V_{i+1}. Thus G is not semi-connected.
If there is an edcge between anyu consecutive vertices, we claim that there is an edge between any two vertices. Therefore, G is semi-connected.
Running-time:
$T = T_{SCC} + T_{TopoSort} = O(E + V)$.