600.363/463 Final Exam
8:20-9:50 Monday 12/8/03

1 Set-covering problem (20 points)

A *dominating set* in a graph $G = (V, E)$ is a subset $D \subseteq V$ of vertices, such that for every vertex $v \in V$, either $v \in D$ or there is a neighbor u of v that belongs to D. A *minimum* dominating set is a dominating set with minimum size.

Argue that there is an approximation algorithm for finding a minimum dominating set with approximating factor $O(\log |V|)$.

Solution:

This problem can be reduced to the minimum set-covering problem. Given the graph $G = (V, E)$, let $X = V$, and for each vertex v, define $S_v = \{u : (u, v) \in E\}$ containing all neighbors of v in the graph. Bu definition, $\cup_v S_v = X$. Hence this is a "'legal"' instance of set-covering problem.

Suppose we run the set-cover approximation algorithm on the above instance, and let $C = \{S_{v_1}, S_{v_2}, ...S_{v_k}\}$ be the output of the algorithm. Since $\cup_{i=1}^{k} S_{v_i} = X = V$, we have by definition that $\{v_1, v_2, ...v_k\}$ is a dominating set in G. It remains to argue that the size of this dominating set is $O(\log n)$ times optimal. By what we have shown in class, the set cover C has size $O(\log n)$ times the size of the minimum set cover in $(V, \{S_v\})$. But it is easy to verify that the size of the minimum set cover in $(V, \{S_v\})$ is at most the size of the minimum dominating set in G: This is true since every dominating set in G corresponds to a set cover of the same size in $(V, \{S_v\})$. In particular this is true for the minimum-size dominating set, and so the minimum set cover cannot be larger. (It actually has the same size, but we don't need this for the argument.)

2. Online Steiner Tree (20 points)

Consider on-line arrivals of terminal $x_1, x_2, ...x_n$. To generate the Steiner tree, the terminal x_i is connected to terminal x_j for $j < i$. Suppose instead of connecting terminal x_i to the closest terminal, the pair (x_i, x_j) is approximately closest to a factor of c.

Compare the steiner tree generated in this way with the optimal Steiner tree, and show how the approximation ratio deteriorates with c. Provide a detailed proof for your answer.

3. (20 points)

We may use MST to approximate the single-source shortest path problem. For a given source, the distance from the source s to any other node v is approximated as the distance from s to v via the MST.

a. Prove that MST approximates shortest paths from a given source within a ratio of n. Show an example for which this bound is almost tight.

b. Prove that shortest paths from a given source approximates MST within a ratio of n. Show an example for which this bound is almost tight.

Solution:

a. Let's denote the length of the shortest path from s to vertex v as d_v, the distance betwwen them via MST is: d_v^{MST}. Suppose there are k edges between s and v in the MST. We argue that each of these k edges is shorter than or equal to d_v. If not, we can connect v with s, and delete the edge that is longer than d_v. In this way we will have a spanning tree that is smaller than MST. That's impossible. So any of these k edges must be shorter than d_v. Besides, there are at most $|V|-1$ edges between s and any vertex v on the MST. This means: $d_v^{MST} \leq (|V|-1) \cdot d_v$. So the approximation ratio is $(|V|-1)$.

An example is: a graph like a wheel. There are m vertices $v_1, v_2, ...v_m$ on the wheel. Each vertices on the wheel is connected to a vertex s on the centroid. Each vertice on the wheel is connected with its two neighbors. The distances between neighboring vertices are all equal to $c \leq 1$. Besides, the center s is connected to all the vertices on the wheel and the distance is 1. In this graph a MST is $s--v_1--v_2--...--v_m$. The shortest path from s to v_m on the MST is: $1+(m-1)c$. If $c=1$, $d_m^{MST} = m$, while the shortst path d_m is 1. Here the $|V| = 1+m$.

b. The shortest paths althogether has $n-1$ edges. Each edge is less than the total length of the MST. This must be ture, since if not, we can always improve the shortest path by going from the source via the MST to the other vertex. So: the length of the shortest path tree is less than or equal to $(n-1) \cdot lengthofMST$, which means the approximation is no more than $n-1$.

The example is similar as in a. Now the distance between neighboring vertices on the wheel is 1, and the distance from the center to each vertex on the wheel is c. c is bigger than 1. This graph has a similar MST as in a. So the total length of the MST is: $c+(m-1)$. If we choose the center as the source, the shortest path tree will include all the edges from s to $v_i, i = 1, ...m$. The length of this tree is: mc. So the approximation ratio is: $\frac{mc}{c+(m-1)} = m \cdot \frac{1}{1+(m-1)/c}$ So if c is $c \to \infty$, then the ratio is close to $m = n-1$.

4. Preflow problem (40 points)

Consider the push-flow method with a floating source, which means the height of the source s is not fixed to $|V|$. Instead the height is updated during the process, starting from 1. But it cannot go above a fixed parameter M.

Besides, we consider the case of a simple graph that the capacity of every edge is equal to 1. Upon initialization, the height of s is set as 1, and it pushes flows to all its neighbors. Then the excess of s is set as 0 instead of a negative number as in the original algorithm.

Below, you are requested to find out for which ranges of M certain statements become true. For some statements there might not exist a range of M that satisfies the conditions, for some statements all values of M work, and for some statements M must be above certain value, e.g,, V or V^2 or 2^V. For full credit, indicate the range of M for which statement is true and show counter example when M is not in this range.

a. the algorithm computing max flow.

b. source s never has an augmenting path to receiver t

c. source s never has an augmenting path to receiver t while $height(s) = M$

Consider a non-source vertex v with excess and height h. Below, you are requested to find out for which ranges of h certain statements become true. For some statements there might not exist a range of h that satisfies the conditions, for some statements all values of h

work, and for some statements h must be above certain value that may depend on M, which you should view as a parameter here. For full credit, indicate the range of h for which statement is true and show counter example when h is not in this range.

d. v has an augmenting path to receiver t

e. v has *no* augmenting path to receiver t

f. v has an augmenting path to source s

g. v has *no* augmenting path to source s

Solution:

a. $M \geq |V|$

b. No such M.

c. $M \geq |V|$

d. No such h.

e. $h > M$.

f. For any edge.

g. No.