Computing the noncomputable

Tien D Kieu[*]
Centre for Atom Optics and Ultrafast Spectroscopy,
Swinburne University of Technology, Hawthorn 3122, Australia

February 1, 2008

Abstract

We explore in the framework of Quantum Computation the notion of computability, which holds a central position in Mathematics and Theoretical Computer Science. A quantum algorithm that exploits the quantum adiabatic processes is considered for the Hilbert's tenth problem, which is equivalent to the Turing halting problem and known to be mathematically noncomputable. Generalised quantum algorithms are also considered for some other mathematical noncomputables in the same and of different noncomputability classes. The key element of all these algorithms is the measurability of both the values of physical observables and of the quantum-mechanical probability distributions for these values. It is argued that computability, and thus the limits of Mathematics, ought to be determined not solely by Mathematics itself but also by physical principles.

[*]email: kieu@swin.edu.au

Contents

Wir müssen wissen
Wir werden wissen
David Hilbert

1 Introduction

This article is a brief introduction to an effort to compute the otherwise mathematical noncomputable. The new ingredients here for this effort are those supplied by physical principles, the quantum principles in particular, and situated outside the domain of Mathematics where the limits of computability are set.

In the next Section we will summarise the important concept of Turing machines and their relationship to the mathematical recursive functions. They will set the scene for the notion of computability as delimited by the Church-Turing hypothesis. We then review the noncomputable results for the Turing halting problem and the equivalent Hilbert's tenth problem, which is directly accessible for some quantum-mechanical exploration in a later Section. We also introduce some other noncomputable problems and the Chaitin's Ω numbers before moving on to a discussion of Quantum Mechanics and Quantum Computation in Secs. 3 and 4.

In Sec. 3, we emphasise the concept of coherent states and the quantum adiabatic theorem in order to introduce the model of Quantum Computation by adiabatic processes in Sec. 4. With this computation model we then propose a quantum algorithm for Hilbert's tenth problem in Sec. 5 (where the illustrating examples are done in collaboration with Andrew Rawlinson), and discuss some of its finer points in the following Sec. 6. To explore the limits of this quantum algorithm we also consider some generalisation for other noncomputables in Sec. 7.

Before concluding with some remarks in the final Section, we present a reformulation of Hilbert's tenth problem, as inspired by our quantum algorithm, in terms of an infinitely coupled set of differential equations in Sec. 8.

2 Effective Computability and Noncomputability

2.1 Turing machines

The concept of computation is at the heart of Mathematics and hard sciences but it was only made precise by Alan Turing relatively recently, at the beginning of the twentieth century [20]. With the introduction of theoretical and mathematically well-defined machines, Turing was able to capture the essence of computation processes and algorithms. There are a few other models of computation [26] but they all, except possibly the quantum model to be considered later on, can be shown to be equivalent to that given by Turing machines.

Turing machines are equipped with an infinite one-dimensional tape over which a head can move in discrete single steps backward and forward. What symbol is already on the tape directly beneath the head and what state the head is in will specify the new symbol

(including the blank symbol) to be written, if at all, over the old one and the new state of the head and the direction in which the head moves next. There needs to be only a finite number of such symbols, similar to the finiteness of the number of letters in the English alphabet. The number of available states for the head is also finite. Everything about the machine is actually finite despite the appearance of the infinite tape – of which only a finite, but unbounded, portion will ever be used. The infiniteness requirement for the tape is only there to enable the accommodation of unbounded but finite sequences of "words" and "sentences" which can be meaningfully made out from the symbol alphabets.

Initially, a finite portion of an otherwise blank tape is prepared with some suitably encoded input for the Turing machine. Each Turing machine can do only one specific task as completely defined by the transition function T:

$$\langle \text{new symbol, new state, next movement} \rangle \;=\; T(\text{present symbol, present state}). \quad (1)$$

Once the task is done, the machine goes into a special halt state and stops with the output written out on the tape ready to be inspected. Transition functions are but representations for computer programs implementing algorithms in general; so we will use interchangeably the names Turing machines and programs.

Such deceptively simple machines are so powerful that they can in fact capture the notion of computation by algorithms as we intuitively know it. Turing went further to introduce his universal machines, which are not restricted to do just a single computational task but can be "reprogrammed" to do many different ones. The key conceptual leap that enables Turing universal machines is the recognition that the descriptions of Turing machines, given by the transition functions, are not at all different to and can be encoded in the same way as the machines' very own inputs. The input to a Turing machine and the description of the Turing machine itself can be put together to form a new, single input to yet another machine, a Turing universal machine. This latter machine only needs to emulate the encoded machine, which can be any well-formed Turing machine, and thus acquires the status of being universal. Universal machines are not unique and one can be encoded into the form of an input for another as well.

The ability for a machine to act on its own kind enables us to investigate the capability and limitations of Turing machines using no other instruments other than the machines themselves. It is precisely this self-referential property that Gödel exploited to embed statements *about* arithmetics in statements *of* arithmetics in his famous Incompleteness Theorem [30]. The embedding in the Theorem is the same as the encoding of Turing machines into input forms acceptable for universal machines and is achieved by converting the finite description of a Turing machine into a unique non-negative integer which can then be expressed in binary or decimal or any other convenient notation. The conversion is possible as we are only dealing here with machines having a finite number of states, a finite number of symbols in its alphabet, and only a finite number of movements for their heads. Multi-tape Turing machines have been introduced and can simplify certain computation; so can non-deterministic Turing machines whose next computation step at a given instant can be chosen, with certain probabilities, among a finite number of possibilities. Nevertheless,

these generalisations cannot compute what a traditional Turing machine cannot do: all of the Turing machines have the same power in terms of computability [26].

Viewed from this perspective, (universal) Turing machines are just functions from non-negative integers (encoding the input) to non-negative integers (encoding the output). But simple arguments can show that the class of functions realised by Turing machines cannot be the same as the whole class of functions from the set of natural numbers to the same set. On the one hand, the number of Turing machines is only countably infinite because each machine can be mapped into a unique integer. On the other hand, the whole class of functions from natural numbers to natural numbers can be shown by the Cantor diagonal arguments [38] to be uncountably infinite (in fact, this class of functions has the same cardinality as the set of reals). A pictorial and heuristic way to visualise this fact is where each mapping from natural numbers to themselves is represented by a real-valued angle between the two semi-infinite half-lines, one of which contains the domain of the function, the other the range. There are as "many" functions as there are angles between the two lines.

The set of functions that are captured by and identified with Turing machines has been identified as that of so-called partial recursive functions and is briefly reviewed in the next Section. From here on, we also call Turing machines interchangeably with programs and partial recursive functions.

2.2 Recursive functions

We begin with the set of *basic primitive recursive functions*, where all the variables are in the set of non-negative integer $\mathcal{N}$,

1. The successor function: $s(x) = x + 1$;
2. The zero function: $z(x) = 0$;
3. The projections functions: $p_i^{(n)}(x_1, \cdots, x_n) = x_i$, $1 \leq i \leq n$.

These are simple and intuitive functions.

From this basic set, other *primitive recursive functions* can be built using a finite number of the following operations

1. Composition: $f(x_1, \cdots, x_k) = h(g_1(x_1, \cdots, x_k), \cdots, g_n(x_1, \cdots, x_k))$; where h is an n-variable function and the n functions g_i are k-variable.
2. Primitive recursion: Let g and h be functions with n and $n+2$ variables, respectively. The $n+1$-variable function f can then be defined as

$$\begin{aligned} f(x_1, \cdots, x_n, 0) &= g(x_1, \cdots, x_n); \\ f(x_1, \cdots, x_n, y+1) &= h(x_1, \cdots, x_n, y, f(x_1, \cdots, x_n, y)). \end{aligned} \tag{2}$$

It is easy to see that the primitive recursive functions so constructed are *total* in the sense that they are defined for *all* the non-negative integers.

The set of primitive recursive functions is quite large but still inadequate and not large enough to encompass all the functions that are admissible and computable by the Turing machines. For this, we have to introduce yet another necessary operation, in addition to the two above, the unbounded μ-minimisation which is defined as:

$$\begin{aligned} f(x_1, \cdots, x_n) &= \text{the least } y \text{ such that } p(x_1, \cdots, x_n, y) = 0; \\ &\equiv \mu y[p(x_1, \cdots, x_n, y)]; \end{aligned} \tag{3}$$

where $p(x_1, \cdots, x_n, y)$ is a total function for all the non-negative integral values of the variables. The general, mechanical method to find y that is applicable to all total functions p is to step y through 0, 1, 2, $\cdots$ until $p = 0$ is satisfied, if this can indeed be satisfied. (For a particular function p one might be able to find y in a different and quicker way, but the mechanical method is applicable in general without any privy knowledge about characteristics of p, as long as it is total.)

All these operations acting, in a finite number of steps, on the basic functions define the set of *partial recursive functions* which contains the set of primitive recursive functions as a proper subset. The name *partial recursive* signifies the fact that there is no guarantee that a (least) y can always be found to satisfy the condition $p(x_1, \cdots, x_n, y) = 0$ for given $x_1, \cdots, x_n$ in the (unbounded) μ-minimisation. If such a y cannot be found, f in (3) is undefined at that point. The problem to establish whether f is defined or not at a general point $(x_1, \cdots, x_n)$ in its arguments, i.e. whether such a y can be found, is a decision problem which could have a binary answer, yes or no. However, this particular problem is a mathematically undecidable problem, in the sense that there exists no general algorithm that can always yield the required answer for any partial recursive function.

We define the set of *μ-recursive functions* being the subset of partial recursive functions where there always exists a decision procedure for the μ-minimisation operation, if needed, that is, there always exists a value for y. This set is bigger than the set of primitive recursive functions. And it can be proved that a function is μ-recursive if and only if there exists a Turing machine which can compute it and then halts, i.e., if and only if it is Turing computable.

The Church-Turing thesis stipulates that all the functions that are *effectively computable* are in fact Turing computable, and vice versa, thus restricting the intuitive and informal notion of computability to the well-defined mechanical operations of Turing machines. This identification is a thesis and not a theorem, not even a conjecture, because it can never be proven right in the mathematical sense. Nevertheless, it can be shown to be wrong if a counter-example can be found in which the computation steps are clearly and acceptably identified. Since it is testable, the thesis does not have the status of an axiom in Mathematics either.

We dispute this thesis by showing in a later Section that there exist computable functions, computable by executing well-defined quantum mechanical procedures in a finite manner, that are not Turing computable.

The evaluation of partial recursive functions, even though well defined (through the three operations of composition, primitive recursion and μ-minimisation upon basic functions and their resultants), may require an infinite number of steps – in contrast to the μ-recursive functions whose evaluation requires only a finite number of steps. That is, partial recursive functions can be implemented on a Turing machine with well-defined execution steps but the machine could never halt in some cases.

One example of non-halting is the simplest program to find a number that is not the sum of four square numbers [35]. All that the program can do is to step through the natural numbers one-by-one and test each of these numbers by direct substitution of square numbers in increasing order of magnitudes. When a number can be found which is not the sum of four squares, the program prints out that number and halts. But this is a false hope. Constructed in this way, such a program never halts because there exists Lagrange's Theorem which confirms that all numbers can in fact be expressed as the sum of four squares!

Another example can be seen through the famous but unproven Goldbach's conjecture that every even number greater than 2 can be written as a sum of two primes. The simplest program would be one which steps through the even numbers one by one, and for each of these even numbers tests for all prime numbers (less than the even numbers). If an even number is found which is not a sum of two primes, the program prints out that number and halts. In other words, the program halts if and only if the Goldbach's conjecture is false. So far, no counter-example to Goldbach's conjecture has been found this way. However, we cannot rely on such a program simply because it may never halt.

There are many other important conjectures and problems in Mathematics which can be resolved if we somehow have a way to tell whether simple Turing machines/programs, directly constructed similar to those above, will halt or not. Thus is the all-important Turing halting problem: given a Turing machine, the question is whether there is a general algorithm which is able to determine if the machine would halt with a specified input. As is well-known in the framework of classical computability, this halting problem is as undecidable as the question whether a partial recursive function is defined at a given argument.

2.3 The Turing halting problem

The halting problem for Turing machines is a manifestation of undecidability: a Turing computation is equivalent to the evaluation of a partial recursive function, which is only defined for a subset of the integers. As this domain is classically undecidable, one cannot always tell in advance whether the Turing machine will halt (that is, whether the input is in the domain of the partial recursive function) or not (when the input is not in the domain).

A version of the proof of the unsolvability of the halting problem based on the Cantor diagonal argument goes as follows. The proof is by contradiction with the assumption of the existence of a computable halting function $h(p, i)$ which has two integer arguments - p

is the Gödel encoded integer number for the algorithm and i is its (encoded) integer input:

$$h(p,i) \;=\; \begin{cases} 0 & \text{if } p \text{ halts on input } i \\ 1 & \text{if } p \text{ does not} \end{cases} \tag{4}$$

One can then construct a program $r(n)$ having one integer argument n in such a way that it calls the function $h(n,n)$ as a subroutine and

$$\begin{cases} r(n) \text{ halts if } h(n,n) = 1 \\ r(n) \text{ loops infinitely (i.e., never stops) otherwise.} \end{cases}$$

The application of the halting function h on the program r and input n results in

$$h(r,n) \;=\; \begin{cases} 0 & \text{if } h(n,n) = 1 \\ 1 & \text{if } h(n,n) = 0 \end{cases} \tag{5}$$

A contradiction is clearly manifest once we put $n = r$ in the last equation above.

The construction of such a program r is transparently possible, unless the existence of a computable h is wrongly assumed. Thus the contradiction discounts the assumption that there is a classically algorithmic way to determine whether any arbitrarily given program with arbitrary input will halt or not.

However, this contradiction argument might be avoided if we distinguish and separate the two classes of quantum and classical algorithms. A *quantum* function $qh(p,i)$, similar to eq. (4), can conceivably exist to determine whether any classical program p will halt on any classical input i or not. The contradiction in eq. (5) would be avoided if the quantum halting qh cannot take as an argument the modified program r, which is now of *quantum* character because it now has quantum qh as a subroutine. This will be the case if qh can only accept integers while quantum algorithms, with proper definitions, cannot in general be themselves encoded as integers. In fact, the no-cloning theorem [45] of quantum mechanics does restrict the type of operations available to quantum algorithms.

In essence, the way we will break the self-referential reasoning here by the differentiation between quantum and classical algorithms is similar to the way Bertrand Russell resolved the set theory paradox (to do with "The set of all sets which are not members of themselves") by the introduction of classes as distinct from sets. (For other lines of arguments, see [41, 40, 8] for example.)

To investigate the decidability of the Turing halting problem in the framework of quantum computability, we will need to isolate the point which causes the classical undecidability. With this aim we turn to the Diophantine equations in the next Section.

2.4 Relation to Diophantine equations and Hilbert's tenth problem

Identities between polynomials with integer coefficients in several unknowns over the natural numbers have been studied for some time in mathematics under the name of Diophantine equations. At the turn of the last century, David Hilbert listed, as challenges for

the new century, 23 important problems among which the problem number ten is the only decision problem and could be rephrased as:

> *Given any polynomial equation with any number of unknowns and with integer coefficients (that is, any Diophantine equation): To devise a universal process according to which it can be determined by a finite number of operations whether the equation has integer solutions.*

There are many important and interesting mathematical conjectures which can be proved or disproved depending on whether some corresponding Diophantine equations have an integer solution or not. These are, for instances, Goldbach's conjecture which we already mentioned, the Riemann hypothesis about the positions in the complex plane of the zeroes of the zeta function, the four-colour conjecture for planar maps, Fermat's last theorem, etc ... Thus, we can appreciate the importance of having a general method for all Diophantine equations, instead of considering each of them individually on its own merits.

However, there are only few special cases of Diophantine equations that are solvable. These include linear (first-degree) equations in the unknowns, of which the existence and absence of solutions can be inferred from the Euclid's algorithm. Also solvable are second-degree equations with only two unknowns, that is, in quadratic forms. But Hilbert asked for a *single* general and finite decision procedure that is applicable for any Diophantine equations. Little was it known that this problem is ultimately connected to the seemingly unrelated notions of computation and Turing machines which were to be introduced some 40 years later. See [11] for a general introduction to Diophantine equations and Hilbert tenth problem.

Eventually, the Hilbert's tenth problem was finally shown to be undecidable in 1970 through a crucial step by Matiyasevich [28, 9]: The Hilbert's tenth problem could be solved if and only if the Turing halting problem could also be solved. The two are simply equivalent. Consequently, as we have a proof in the last Section that the Turing's is not solvable, the Hilbert's tenth problem is noncomputable/undecidable in the most general sense if one accepts, as almost everyone does, the Church-Turing thesis of computability. One would thus have to be content with the fact that individual Diophantine equations need to be considered separately, and may or may not be solved with a different approach anew each time.

For a precise discussion and the history of the negative result, see [28, 9]; for a semi-popular account, see [5]. We briefly sketch below some key arguments leading to the undecidable result.

It is recognised that Turing machines as devices mapping inputs to outputs are equivalent to partial recursive functions, whose domain are restricted to proper subsets of $\mathcal{N}$. Without loss of generality, we can always restrict the computation of Turing machines to that of numeric partial recursive functions – because for non-numeric algorithms there always exist mappings into the computation of some corresponding partial recursive functions. The question whether a Turing machine would halt or not upon some particular input is now equivalent to the question whether the corresponding argument value is in the domain of the corresponding function or not. When a machine cannot halt with a given

input is when the partial recursive function corresponding to that machine is not defined at that particular argument.

Now, there is a universal representation, known as the Kleene normal form, for any partial recursive function f

$$f(x_1, \cdots, x_n) \;\; = \;\; \psi(\mu y[\tau(g, x_1, \cdots, x_n, y)]), \tag{6}$$

where ψ and τ are *fixed* primitive recursive functions, independent of the particular function f on the left hand side. The information about f on the right hand side of (6) is encoded in the first argument of the function τ as the Gödel number g of f. (We recall that the class of partial recursive functions is countably infinite and each function can be systematically given a unique identification number according to some Gödel numbering scheme.) It is the μ-minimisation operation in (6) that could potentially turn a total primitive recursive function, which is defined everywhere, into a partial recursive function. Thus, given $(x_1, \cdots, x_n)$ if there exists a (least) y that for some integer g solves $\tau(g, x_1, \cdots, x_n, y) = 0$ then the Turing machine, which corresponds to g, would halt upon the input corresponding to $(x_1, \cdots, x_n)$. If no such y exists, the Turing machine in question would not halt.

Gödel, as quoted in [36], has shown that the μ-minimising operation (3) can always be represented as some arithmetic statement with a set of identities between multi-variate polynomials over the integers, together with a finite number of the existential quantifiers, $\exists$, and bounded versions of the universal quantifiers, $\forall$. (The boundedness comes from the requirement of some *least* number that satisfies a μ-minimisation as defined in (3).) Note that quantifiers hold a crucial role in arithmetic. Restricted arithmetics with only the addition operation or the multiplication operation, but not both, are in fact complete: any of their statements can be decided to be provable or not within the relevant frameworks. On the other hand, we have the famous Gödel's Incompleteness Theorem for 'ordinary' arithmetic which has both addition and multiplication operations. Normally we would think that multiplication is just a compact way to express long and repeated additions, and might thus be puzzled over the difference in completeness. But it is the quantifiers that make such a huge difference.

Arithmetic statements for solutions of Diophantine equations (i.e. the Hilbert's tenth problem) can only involve existential quantifiers, but the elimination of bounded universal quantifiers in the μ-minimisation necessitates the appearance of exponentiation of variables. A famous example of variable exponentiation is the equation of Fermat's last theorem:

$$(x+1)^{(u+3)} + (y+1)^{(u+3)} + (z+1)^{(u+3)} \;\; = \;\; 0, \tag{7}$$

which has no non-negative integer solutions for the unknowns x, y, z and u.

After various attempts by many people, it was finally shown that exponentiation is indeed Diophantine – that is, variable exponents can be eliminated to result in only polynomial Diophantine equations. This proves the equivalence between the two decision problems and implies that the Hilbert's tenth is not computable in the Turing scheme of computation.

There are many other noncomputable problems, some are also equivalent to the Turing halting problem, and others belong to altogether different non-computability classes.

2.5 Some other noncomputables

Similarly based on the Turing machines, many other problems can be shown to be equivalent to the Turing halting problem, and thus noncomputable. We can name in this class [26] the tiling problem in a plane, Post's correspondence problem, Thue's word problem, Wang's domino problem, etc ... For example, the tiling problem asks for a decision procedure to see if any given finite number of sets of tiles (but each set has an infinite number of the same tiles available) can tile the first quadrant of the plane or not, with a specified tile at the origin at the lower left corner and some adjacency rules for the tiling.

One could also ask questions related to and generalising the Hilbert's tenth problem such as [9]: Is there a single algorithm for testing whether the number of solutions for any Diophantine equation is finite, or is infinite, or equal to one, or is even?

Besides the above equivalent to the Turing halting problem, there are yet others which belong to different classes in the non-computability hierarchy. The most famous of all must be the Gödel's Incompleteness Theorem. It establishes that any finitely axiomatic, consistent mathematical system sufficiently complex to embrace Arithmetic must be incomplete – that is, there exist some statements whose truth cannot be confirmed or denied from within the system, resulting in undecidability. (Interestingly, it is precisely the statement about consistency of the system that is neither provable nor deniable.) We can see that this undecidability is more than that of Turing machines, whose applicable mathematical statements can only have bounded universal quantifiers as opposed to arbitrarily unbounded universal quantifiers of Arithmetic at large.

Chaitin, approaching from the perspectives of Algorithmic Information Theory [6, 7], has shown that there exist many unprovable statements in Arithmetic simply because they have irreducible algorithmic contents, measurable in bits, that are more than the complexity, also measurable in bits, of the finite set of axioms and inference rules of the system. Infinitely irreducible algorithmic complexity is randomness which exists even in pure mathematics. And more frustratingly, we can never prove randomness since we could only ever deal with finite axiomatic complexity. Chaitin, to illustrate the point, has introduced the number Ω as the halting probability for a random program, with some random input, being emulated by a particular Turing machine. Ω is an average measure over all programs run on the universal Turing machine. This number has many interesting properties, and has been generalised to a quantum version [41], but we only mention here the linkage between this number and polynomial Diophantine equations. When expressed in binary, the value of the k-th bit of Ω is respectively 0 or 1 depending on whether some Diophantine equation, corresponding to the Turing machine in consideration,

$$C(k, N, x_1, \cdots, x_K) \;\; = \;\; 0, \tag{8}$$

for a given $k > 0$, has solutions in non-negative integers $(x_1, ..., x_K)$ for finitely or infinitely many values of the parameter $N > 0$.

From this Diophantine representation we can see that the noncomputability of Ω is "more" than the noncomputability of the Turing halting problem. Even if, somehow, we have an algorithm for the latter to decide the equations in (8) for each N respectively, we

would still need to apply the algorithm for an infinite number of times just to get a single digit for Ω! An infinite number of times cannot be normally performed in a finite time, hence the different class of noncomputability from that of the Turing halting problem. We will come back to the Ω number with a hypothetical algorithm in Quantum Field Theory later.

We end our brief review of relevant mathematical concepts here with a discussion of computable numbers. If we had restricted such numbers to only those that can be output by some Turing machine (which then halts), then we would have had to effectively restrict ourselves to integers and to treat irrational numbers, like $\sqrt{2}$ and π, as noncomputable. This is clearly undesirable. A better and more practical definition of computable numbers are those which we can approximate to an arbitrary degree of accuracy with (integer/rational) outputs of some Turing machines. This definition of computability of a number when it can be evaluated to any degree of accuracy is sufficient to interpret the number and to establish its relationship with other numbers. And that all we need in computing any number: its place in relation to other numbers.

Note that, in this sense, the Chaitin's Ω is not computable because we cannot estimate this number to an arbitrarily high accuracy, although it is bounded from above by unity (being the maximum probability) and can be approximated only from below by some very slowly converging process [6] whose convergence rate is indeterminable.

3 Quantum Mechanics

3.1 The postulates of measurement and associated problems

Quantum Physics, including Quantum Mechanics and Quantum Field Theory, is the most successful theory that we have in Science for the description and prediction of phenomena in Nature. And so far there is not a single discrepancy between the theory and experiments.

According to Quantum Mechanics, pure states of a physical system can capture the most that can be said about the system and are associated with vectors, unique up to phases, in some abstract linear vector Hilbert space. When the system, particularly when it is a subsystem of a bigger entity, cannot be described by a pure state but is in a mixed state, the language of density matrices would be necessary for its description. Acting on the Hilbert space are linear operators, of which hermitean and unitary operators are of particular interest.

In the Schrödinger picture where the time dependency is explicitly carried by the states, the time evolution of the system is governed by the Schrödinger equation, in which the hermitean Hamiltonian operators play a unique role. In general, each physical observable is associated with a hermitean operator; the Hamiltonian operator, for instance, is associated with the system's energy. The real-valued eigenvalues (which can be continuous or discrete) of these hermitean operators restrict the obtainable values under observation. Each time when the associated observable is measured, only one single value, among the eigenvalues given, is obtained. Repetitions of the measurement under identical conditions could yield

different measured values each time. And the probability of getting a particular eigenvalue in a measurement is given by the square of the absolute value of the inner product between the corresponding eigenvector and the state describing the system at that instant. If the system is in a mixed state described by a density matrix, the probability is then given by the trace of the product between the density matrix and the corresponding projector associated with the eigenvector in question.

After a measurement, the state of the system is a pure state whose representing vector is the same as the eigenvector, up to a phase, corresponding to the eigenvalue obtained. Note that measurement thus is a non-unitary and irreversible operation in general, unless the system is already in the observable eigenstate. Different observables can be measured simultaneously, with the same degree of statistical accuracy, only when the associated hermitean operators commute with each other.

But already seeded in the summary of quantum mechanical postulates above is a fundamental problem of inconsistency. The act of measurement, on the one hand, is a process unfolded itself in time. On the other hand, why should it not be governed by the unitary Schrödinger time-evolution operator?

Even when we extend the system to a larger system consisting of the considered system and the measuring apparatus, we still face the same Quantum Measurement Problem, under a slightly different guise. In setting up a measurement we effectively establish some correlation between the state of the measured system and the pointers of the measuring apparatus. This correlation is one-to-one for the eigenstates of the operator representing the measured observable: if the system is in the state $|e_i\rangle$, $i = 1, 2$, then the pointer, initially in a neutral state $|A_0\rangle$, is subsequently in the state $|A_i\rangle$ respectively,

$$|e_i\rangle|A_0\rangle \rightarrow |e_i\rangle|A_i\rangle. \tag{9}$$

Now, according to the quantum mechanical principles of superposition and linearity, it follows that if the system is in a superposition then so is the pointer,

$$(\alpha|e_1\rangle + \beta|e_2\rangle)\,|A_0\rangle \rightarrow \alpha|e_1\rangle|A_1\rangle + \beta|e_2\rangle|A_2\rangle. \tag{10}$$

The problem is that we have never been able to observe the classical pointer in an entangled state as in the right hand side of the last expression. Instead, we either get the pointer position A_1 with a probability proportional to $|\alpha|^2$ or A_2 with a probability proportional to $|\beta|^2$.

To resolve this problem one could postulate, as in the Copenhagen interpretation, *some* undefined separation between the quantum and the classical worlds, or modify the theory to have elements of non-linearity admitted. (This second situation is unlike mathematics in the way that the postulates/axioms of a physical theory can themselves also be subjects of investigation.) But if we believe in Quantum Mechanics as the universal and fundamental theory then we have a difficult problem at hand. Various resolution attempts ranging from the many-world interpretation to decoherence have not been deemed successful.

We shall leave the measurement problem here, and only wish to emphasise that the power of all quantum algorithms in quantum computation relies crucially on such mysterious measurement processes.

Also for later use, we now introduce the concept of *measurable* quantities [17]. Analogous to the concept of computable numbers discussed in Sec. 2.5, a number w is deemed measurable if there exists a finite set of instructions for performing an experiment such that a technician, given an abundance of unprepared raw materials and an allowed error ϵ, is able to obtain a rational number within ϵ of w. The technician is analogous to the computer, the instructions analogous to the computer program, the "abundance of unprepared raw materials" analogous to the infinite Turing tape, initially blank.

In particular, not only the (stochastic) outcomes of an observable are obviously measurable and of interest but so are the probability distributions for these outcomes. The probabilities can be obtained within any given accuracy by increasing the number of measurement repetitions. Later, we make full use of this crucial fact that *the quantum mechanical probabilities are and measurable in principle.*

3.2 Peculiarities of Quantum Mechanics

One important, but least understood, property of Quantum Mechanics is the randomness in the outcome of a quantum measurement. Even if we prepare the initial quantum states to be *exactly* the same in principle, we can still have different and random outcomes in subsequent measurements. Such randomness is a fact of life in the quantum reality of our universe.

To reflect that intrinsic and inevitable randomness, the best that Quantum Mechanics, as a physical theory of nature, can do is to list, given the initial conditions, the possible values for measured quantities and the probability distributions for those values. Both the values and the probability distributions are computable in the sense discussed at the end of Sec. 2.5 for Hilbert spaces with finite dimensions. With countably infinite dimensions, Quantum Mechanics can be *noncomputable* unless we can control the accuracy of calculated values by suitable truncation of the infinite dimensional space. (The mathematical noncomputability of the Hilbert's tenth problem is reflected through this infinite dimensions of the Fock space in our formulation of the problem. But there is a way out as discussed below.)

On the other hand, not only the values registered in the measurement of some measurable but also the associated probability distributions are measurable in the sense that they can be obtained to any desirable accuracy by the act of physical measurements. Normally, the values for measurable are quantised so they can be obtained exactly; the probability distributions are real numbers but can be obtained to any given accuracy by repeating the measurements again and again (each time from the same initial quantum state) until the desired statistics can be reached. That is how the computable numbers from Quantum Mechanics can be judged against the measurable numbers obtained from physical experiments. Thus far, there is no evidence of any discrepancy between theory and experiments.

Randomness is, by mathematical definition, incompressible and irreducible. In Algorithmic Information Theory, Chaitin [6] defines randomness by program-size complexity: a binary string is considered random when the size of the shortest program that generates that string is not "smaller", as measured in bits, than the size of the string itself. We refer

the readers to the original literature for more technically precise definitions for the cases of finite and infinite strings.

Paradoxically, the quantum reality of Nature somehow allows us to *compress* the *infinitely incompressible* randomness into the *apparently finite* act of preparing the same quantum state over and over again for subsequent measurements! This quantum mechanically implied infinity seems to be both needed for and consistent with the finitely measured, see [25] and references therein for further discussion.

3.3 Coherent states

One of the simplest and most widely applicable problems in Quantum Mechanics is that of the (one-dimensional) Simple Harmonic Oscillator (SHO) with the Hamiltonian

$$H_{\rm SHO} \;=\; (P^2 + X^2)/2, \tag{11}$$

which can also be expressed as

$$H_{\rm SHO} \;=\; a^\dagger a + \frac{1}{2}. \tag{12}$$

The operators $a^\dagger$, a are linearly related to the position and momentum operators, which satisfy the commutation relation $[P, X] = i$,

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}(a + a^\dagger), \\ P &= \frac{i}{\sqrt{2}}(a - a^\dagger). \end{aligned} \tag{13}$$

The operators $a^\dagger$, a thus satisfy different commutation relations

$$\begin{aligned} &[a, a^\dagger] = 1, \\ &[a, a] = [a^\dagger, a^\dagger] = 0. \end{aligned} \tag{14}$$

The spectrum of the number operator $N = a^\dagger a$ that appears in (12) is discrete and spans over the natural numbers. Its eigenstates are termed the number states $|n\rangle$,

$$N|n\rangle \;=\; n|n\rangle; \quad n = 0, 1, 2, \cdots \tag{15}$$

These eigenstates also constitute an orthonormal basis for a Fock space, a special type of Hilbert space, and can be constructed by the operators $a^\dagger$ acting on the special "vacuum" state $|0\rangle$, the lowest-eigenvalue state,

$$|n\rangle \;=\; \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle, \tag{16}$$

from which follow the recursive relations

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a|n\rangle &= \sqrt{n}|n-1\rangle. \end{aligned} \tag{17}$$

These relations lead us to the names *creation* and *annihilation* operators respectively for $a^\dagger$ and a.

The number state $|n\rangle$ can be realised in Quantum Optics as one having a definite number of n photons, all at the same frequency. But these number states are not the states of travelling optical modes generated by idealised lasers [13], which have an indefinite number of photons. For the description of these modes, we need the coherent states [44], $|\alpha\rangle$, which are the eigenstates of a and are labeled by the complex number α,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{18}$$

With the relation to the number states,

$$\begin{aligned} |\alpha\rangle &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \\ &= \mathrm{e}^{-\frac{|\alpha|^2}{2}} \mathrm{e}^{\alpha a^\dagger}|0\rangle, \end{aligned} \tag{19}$$

the coherent states are not orthogonal but can still be used for spanning the Hilbert space. They have some unique and nice properties, one of which is that they are the states that optimise the amplitude-phase Heisenberg uncertainty relation. The other fact, which we will exploit later, is that they are also the ground states, i.e. the eigenstates having the lowest "energy" eigenvalues, for the family of semi-definite Hamiltonians

$$H_\alpha = (a^\dagger - \alpha^*)(a - \alpha). \tag{20}$$

With the simple substitution $a_\alpha = a - \alpha$ we are back to a family of α-labeled SHO Hamiltonians (11), except for the additive constant, all of which have the same spectrum over the natural numbers but with different sets of eigenvectors $|n_\alpha\rangle$.

3.4 The quantum adiabatic theorem

The dynamical evolution of a quantum system is governed by the Hamiltonian through the Schrödinger equation. If the system is closed then the Hamiltonian is time independent. If the system is subject to external influences, whose dynamics are not of direct concern to the investigation, then the Hamiltonian is time dependent; and the modification in the quantum state of the system critically depends on the time T during which the change of the Hamiltonian takes place. This dependency is particularly simplified when the rate of change of the external fields is very fast compared to some intrinsic time scale, whence we can apply the sudden approximation, or is very slow, whence we can appeal to the quantum adiabatic theorem.

The sudden approximation says that if the time change T is sufficiently fast relative to the inverse of the average Hamiltonian during that time, $\Delta\bar{H}$,

$$T \ll \hbar/\Delta\bar{H}, \tag{21}$$

then the dynamical state of the system remains essentially unmodified.

On the other hand, in the case of an infinitely slow, or adiabatic passage, if the system is initially in an eigenstate of the Hamiltonian at the initial time it will, under certain conditions, pass into the eigenstate of the Hamiltonian at the final time, that derives from it by continuity [29]. This is the content of the adiabatic theorem, provided the following conditions are satisfied throughout the relevant time interval:

- The instantaneous eigenvalues remain distinct;
- The first and second derivatives of the instantaneous eigenvectors with respect to time are well-defined and piece-wise continuous.

This important theorem has been exploited for a model of quantum computation involving the ground state and to be discussed below. In practice, the evolution time T is finite but the more it satisfies the condition

$$T \gg \frac{\| \Delta H \|}{g^2}, \tag{22}$$

then the higher the probability that the system remains in the instantaneous eigenstate. In the above,

$$\| \Delta H \| \equiv \max_{0\leq t\leq T} |\langle e(t)|H(T) - H(0)|g(t)\rangle| , \tag{23}$$

and

$$g \equiv \min_{0\leq t\leq T} (E_e(t) - E_g(t)) , \tag{24}$$

where $|g(t)\rangle$ and $|e(t)\rangle$ are respectively the instantaneous ground state and first excited state with instantaneous eigenvalues $E_g(t)$, $E_e(t)$. The critical scale is this gap g.

4 Quantum Computation

The underlying laws of all physical phenomena in Nature, to the best of our knowledge, are those given by quantum physics. However, the best present day computers, as they are of classical nature, cannot even in principle simulate quantum systems efficiently. Feynman pointed out in 1982 that [16], see also [2], only quantum mechanical systems may be able to simulate other quantum systems more efficiently. Furthermore, according to Moore's law, the exponential rate of miniaturisation of micro-electronic semiconductor devices will soon, if not already, take us to the sub-micron and nano dimensions and beyond. On

this scale, quantum physics will become more and more relevant in the design and production of computer components. Heat dissipation in irreversible computation will be yet another problem at these dimensions. Even though reversible classical computation can be implemented in principle, quantum computation, being almost reversible except the final read-out by measurement, could automatically minimise this heating problem.

Lastly, the notion of effective computability, as delimited by the Church-Turing thesis "Every function which would be naturally regarded as computable can be computed by the universal Turing machine", begs the question whether it could be extended with quantum principles. Initial efforts seem to confirm that quantum computability is no more than classical and mathematical computability [3]. However, more recent indications may prove otherwise [22, 8]. We will come back to this computability notion in a Section below.

All of the above have inevitably led to the recent convergence of quantum physics, mathematics, and computing and information processing.

4.1 "Standard" Model of Quantum Computation

According to the "standard" model of quantum computation (see [33] for instance), which is a direct generalisation of classical digital computing, the fundamental unit of a quantum computer is the quantum bit, shortened as *qubit*, which is the generalisation of a binary bit. Physical implementation of a qubit could be any (measurable) two-state system: the up and down values of a quantum spin, or the two polarisation states of a photon, etc ... But unlike the binary bit, a qubit can be in a superposition state of its two states/values. Upon measurement the superposition is destroyed, revealing one of the two classical values of a qubit. The two states of a qubit, denoted by $|0\rangle$ and $|1\rangle$, should be unambiguously distinguishable by measurement and thus be orthogonal to each other.

There are three stages of operation for a quantum computer, corresponding to the input, the processing, and finally the output. The input preparation stage can be and has been carried out in laboratories for certain well-known systems. So can the output stage in which the output is read out by an act of measurement – even though quantum measurement is not that well understood, as already alluded to in Sec. (3.1). The speed of state preparation and measurement, which should be carried out in such a way as not to perturb other subsystems/qubits not being directly measured, is crucial for quantum computation. The information processing stage is the most difficult to be implemented. In principle, it is governed by unitary evolution of a set of qubits well isolated from the surroundings to avoid as much of the decoherence effects of the environment as possible. The discovery of error correcting codes for quantum computation was a pleasant surprise. Without this possibility, realisation of quantum computation would have been unthinkable as computers are inevitably and constantly subject to errors induced by either the internal interactions or the environment or both.

The power of a quantum computer firstly lies in the massive parallelism resulting directly from the superposition possibility of the quantum states. If each qubit is a superposition of two states then the measurement of such a superimposed N-qubit system could in general access 2^N distinguished states simultaneously. However, such quantum parallelism

is not that useful because of the probabilistic nature of the measured outcomes. (After all, many classical wave systems, like water waves, can also have superposition but cannot provide a better computation model.) The second and most important power of quantum computation is thought to have its root in *quantum entanglement* [4], which has no counterpart in the classical world (even though it might be expensively simulated by classical means). Quantum entanglement provides the extra dimensions in information storage and processing that distinguishes the quantum from the classical. It is the entanglement that allows us to *control* the massive quantum parallelism through selective interference of different computational paths to extract the information desired.

These characteristics have been exploited to reduce the computational complexity of some problems. So far there are only few quantum algorithms discovered [39, 19]; most notable is Shor's factorisation algorithm which employs Quantum Fourier Transformation. QFT is the only known quantum algorithm that could offer an exponential increase in computational speed, due to the interference of different computation paths (as Fourier Transformation is intimately linked to interference) and to quantum entanglement. More quantum algorithms are urgently needed.

The approach above with qubits and unitary gates of so-called quantum networks has been accepted as the standard model for quantum computation. It has been argued [3] that the computability obtainable in this model is not better but is the same as classical computability.

However, it is not the only model available.

4.2 Quantum Adiabatic Computation

Among the alternative models for quantum computation is the recent proposal [15] to employ quantum adiabatic processes for computation. The idea is to encode the solution of some problem to be solved into the ground state, $|g\rangle$, of some suitable Hamiltonian, H_P. But as it is easier to implement the Hamiltonian than to obtain the ground state, we should start the computation in yet a different and readily obtainable initial ground state, $|g_I\rangle$, of some initial Hamiltonian, H_I, then deform this Hamiltonian in a time T into the Hamiltonian whose ground state is the desired one, through a time-dependent process,

$$\mathcal{H}\left(\frac{t}{T}\right) \;=\; \left(1-\frac{t}{T}\right)H_I + \frac{t}{T}H_P. \tag{25}$$

The adiabatic theorem of Quantum Mechanics, Sec. 3.4, stipulates that if the deformation time is sufficiently slow compared to some intrinsic time scale, the initial state will evolve into the desired ground state with high probability – the longer the time, the higher the probability.

5 Quantum algorithm for the Hilbert's tenth problem

5.1 General approach

It suffices to consider non-negative solutions, if any, of a Diophantine equation. Let us consider a particular example

$$(x+1)^3 + (y+1)^3 - (z+1)^3 + cxyz = 0, \qquad c \in Z, \tag{26}$$

with unknowns x, y, and z. To find out whether this equation has any non-negative integer solution by quantum algorithms, it requires the realisation of a Fock space. Upon this Hilbert space, we construct the Hamiltonian corresponding to (26)

$$H_P \;=\; \Big((a_x^\dagger a_x + 1)^3 + (a_y^\dagger a_y + 1)^3 - (a_z^\dagger a_z + 1)^3 + c(a_x^\dagger a_x)(a_y^\dagger a_y)(a_z^\dagger a_z)\Big)^2,$$

which has a spectrum bounded from below – semidefinite, in fact.

Note that the operators $N_j = a_j^\dagger a_j$ have only non-negative integer eigenvalues n_j, and that $[N_j, H_P] = 0 = [N_i, N_j]$ so these observables are simultaneously measurable. The ground state $|g\rangle$ of the Hamiltonian so constructed has the properties

$$\begin{aligned} N_j|g\rangle &= n_j|g\rangle, \\ H_P|g\rangle &= \Big((n_x+1)^3 + (n_y+1)^3 - (n_z+1)^3 + cn_xn_yn_z\Big)^2 |g\rangle \equiv E_g|g\rangle, \end{aligned}$$

for some (n_x, n_y, n_z).

Thus, a projective measurement of the energy E_g of the ground state $|g\rangle$ will yield the answer for the decision problem: The Diophantine equation has at least one integer solution if and only if $E_g = 0$, and has not otherwise. (If $c = 0$ in our example, we know that $E_g > 0$ from Fermat's last theorem.)

If there is one unique solution then the projective measurements of the observables corresponding to the operators N_j will reveal the values of various unknowns. If there are many solutions, finitely or infinitely as in the case of the Pythagoras theorem, $x^2+y^2-z^2 = 0$, the ground state $|g\rangle$ will be a linear superposition of states of the form $|n_x\rangle|n_y\rangle|n_z\rangle$, where (n_x, n_y, n_z) are the solutions. In such a situation, the measurement may not yield all the solutions. However, finding all the solutions is not the aim of a decision procedure for this kind of problem.

Notwithstanding this, measurements of N_j of the ground state would always yield some values (n_x, n_y, n_z) and a straightforward substitution would confirm if the equation has a solution or not. Thus the measurement on the ground state either of the energy (with respect to the hermitean operator H_P, provided the zero point can be calibrated) or of the number operators will be sufficient to give the result for the decision problem.

The quantum algorithm with the ground-state oracle is thus clear:

1. Given a Diophantine equation with K unknown x's

$$D(x_1, \cdots, x_K) \;=\; 0, \tag{27}$$

we need to simulate on some appropriate Fock space the quantum Hamiltonian

$$H_P = \left(D(a_1^\dagger a_1, \cdots, a_K^\dagger a_K)\right)^2. \tag{28}$$

2. If the ground state could be obtained with high probability and unambiguously verified, measurements of appropriate observables would provide the answer for our decision problem.

The key ingredients are the availability of a countably infinite number of Fock states, the ability to construct/simulate a suitable Hamiltonian and to obtain and verify its ground state. As a counterpart of the semi-infinite tape of a Turing machine, the Fock space is employed here instead of the qubits of the more well-known model of quantum computation. Its advantage over the infinitely many qubits which would otherwise be required is obvious.

One way to construct any suitable Hamiltonian so desired is through the technique of ref. [27]. We consider the hermitean operators, where j is the index for the unknowns of the Diophantine equation,

$$\begin{aligned} X_j &= \frac{1}{\sqrt{2}}(a_j + a_j^\dagger), \\ P_j &= \frac{i}{\sqrt{2}}(a_j - a_j^\dagger), \\ [P_j, X_k] &= i\delta_{jk}. \end{aligned} \tag{29}$$

Together with the availability of the fundamental Hamiltonians

$$X_j, P_j, (X_j^2 + P_j^2), \pm(X_k P_j + P_j X_k), \text{ and } (X_j^2 + P_j^2)^2 \tag{30}$$

one could construct the unitary time evolutions corresponding to Hamiltonians of arbitrary hermitean polynomials in $\{X_j, P_j\}$, and hence in $\{a_j^\dagger a_j\}$, to an arbitrary degree of accuracy. These fundamental Hamiltonians correspond to, for examples, translations, phase shifts, squeezers, beam splitters and Kerr nonlinearity [27].

With the polynomial Hamiltonian constructed, we need to obtain its ground state. Any approach that allows us to access the ground state will suffice. One way is perhaps to use that of quantum annealing or cooling [21]. Another way is to employ the quantum computation method of evolution with time-dependent Hamiltonians.

5.2 Time-dependent Hamiltonian approach

Of the theory of Quantum Mechanics we cannot in general compute in infinite dimensional Hilbert spaces the various observables, including the probability distributions for these. The reason is that we can only employ some truncated versions of the Hilbert spaces but may not know, from mathematical arguments solely, the relationship between the truncation and the accuracy of those calculated from the theory is. On the other hand, we do have some access to the presumably infinite physical world; but the access is quite limited,

through the act of measurement only. Our approach is to combine the two, and exploit the measurability in the physical world to compute the mathematically noncomputable. Below is an algorithm based on the exploitation of both the presumably infinite physical world and the theory of Quantum Mechanics calculated in a finite manner on Turing machines. The algorithm presented may not be the most efficient; there could be many other variations making better use of the same philosophy.

In the physical world, it is in general easier to implement some Hamiltonian than to obtain its ground state. We thus should start with yet a different and readily obtainable initial ground state, $|g_I\rangle$, of some initial Hamiltonian, H_I, then deform this Hamiltonian in a time T into the Hamiltonian whose ground state is the desired one, through a time-dependent process represented by the interpolating Hamiltonian $\mathcal{H}(t/T)$.

One could start, for example, with the Hamiltonian H_I, for some $\alpha' s$,

$$H_I = \sum_{i=1}^{K} (a_i^\dagger - \alpha_i^*)(a_i - \alpha_i), \tag{31}$$

which admits the readily achievable coherent state $|g_I\rangle = |\alpha_1 \cdots \alpha_K\rangle$ as the ground state. Then, one forms the slowly varying Hamiltonian $\mathcal{H}(t/T)$ in (25), which interpolates in the time interval $t \in [0, T]$ between the initial H_I and H_P.

- *Step 0:* Choose an evolution time T, a probability p which can be made arbitrarily closed to unity, and an accuracy $0 < \epsilon < 1$ which can be made arbitrarily small.

- *Step 1 (on the physical apparatus):* Perform the *physical* quantum time-dependent process which is governed by the time-dependent Hamiltonian $\mathcal{H}(t/T)$ and terminates after a time T. Then, by projective measurement (either of the observable H_p or the number operators $\{N_1, \ldots, N_K\}$) we obtain some state of the form $|\cdots n_i \cdots\rangle$, $i = 1, \cdots, K$.

- *Step 2 (on the physical apparatus):* Repeat the physical process in *Step 1* a number of times, $L(\epsilon, p)$, to build up a histogram of measurement frequencies (for all the states obtained by measurement) until we get a probability distribution $P(T; \epsilon)$ at the time T with an accuracy ϵ for all the measured states. The convergence of this repetition process is ensured by the Weak Law of Large Numbers in probability theory. (An overestimate of the number of repetitions is $L \geq 1/(\epsilon^2(1-p))$.) Note the lowest energy state so obtained, $|\vec{n}_c\rangle$, as the candidate ground state.

Any normalised (or normalisable) state in general has a finite support, that is, in its expansion in any complete basis of the infinite space the magnitudes of the coefficients tend to zero as their indices tend to infinity. We are interested in the ground state but in general we cannot mathematically (on (Turing) classical computer) determine the size of the error due to truncation of the basis. But we have the physical world available out there helping us to estimate the size of the truncation error by direct comparison with the measured probability distributions of the last *Steps*. Once the truncation size can be thus obtained,

we can control and estimate, with some probability (confidence) p, the fluctuations of the numerically obtained ground state to confirm whether the ground state energy is zero or not. Explicitly,

- *Step 3 (on the classical computer):* Choose a truncated basis of M vectors made up of $|\alpha_1 \cdots \alpha_K\rangle$ and its excited states by successive applications of the displaced creation operators $b_i^\dagger \equiv (a_i^\dagger - \alpha_i^*)$ on the initial state.

- *Step 4 (on the classical computer):* Solve the Schrödinger equation in this basis for $\psi(T)$, with the initial state $\psi(0) = |\alpha_1 \cdots \alpha_K\rangle$, to derive a probability distribution $P_{\rm est}(T; M)$ (through $|\langle\psi(T)| \cdots n_i \cdots\rangle|^2$) which is similar to that of *Step 2* and which depends on the total number M of vectors in the truncated basis.

- *Step 5 (on the classical computer):* If the two probability distributions are not uniformly within the desired accuracy, that is, $|P_{\rm est}(T; M) - P(T; \epsilon)| > \epsilon$, we enlarge the truncated basis by increasing the size M and go back to the *Step 4* above.

- *Step 6 (on the classical computer):* If the two probability distributions are uniformly within the desired accuracy, that is, $|P_{\rm est}(T; M) - P(T; \epsilon)| < \epsilon$, then use this truncated basis to diagonalise H_P to yield, within an accuracy which can be determined from ϵ, the approximated ground state $|g'\rangle$ and its energy $E_{g'}$.

- *Step 7 (on the classical computer):* We can now estimate in this truncated basis the gap between the groundstate and the first excited state. From this gap, we can make use of the quantum adiabatic theorem and choose a time T such that the probability to have the system mostly in the ground state

$$\Big||\langle g'|\psi(T)\rangle|^2 - 1\Big| < \epsilon.$$

  We then go back to *Step 1* with this choice of T, which is to amplify and thus confirm the candidate ground state as the real ground state.

$E_{g'}$, the energy of the numerically obtained ground state, is approximately the same as the H_p-energy E_c associated with the candidate ground state $|\vec{n_c}\rangle$ obtained in measurement,

$$E_{g'} = E_c + \delta$$

Note that E_c can be assumed to be non-zero; otherwise, if it is zero then we will have found a solution for the Diophantine equation and can stop the algorithm then. The fluctuation δ in $E_{g'}$ due to the truncation can also be numerically estimated as a function of ϵ – it can be easily shown that the fluctuation $\delta(\epsilon)$ tends to zero as ϵ goes to zero. Now, to confirm that $E_{g'}$ is in fact non-zero we only need to enforce the condition

$$|\delta(\epsilon)| \;<\; E_c, \tag{32}$$

by choosing ϵ in a self-consistent manner. From this we can conclude (with probability p) that $E_{g'} > 0$ and thus the Diophantine equation has no root.

To illustrate that the statistics, influenced by the spectral flow, are sufficient for the identification purpose, let us assume that the lowest-energy state $|\vec{n_c}\rangle$ is the one obtained physically $t = T$ but is not the true ground state. After choosing a truncated basis, the computation on a classical computer for some evolution time, which is a parameter of both the numerical computation and the physical process, and will have two distinct scenarios:

1. Either the numerical computation clearly shows that $|\vec{n_c}\rangle$ is not even the ground state in this truncated basis. That is, we numerically find another state $|g'\rangle$ which has lower energy (wrt H_P) than that of $|\vec{n_c}\rangle$.

2. Or the numerical seems to be consistent that this state $|\vec{n_c}\rangle$ is the "ground state" in our arbitrarily truncated basis.

In the first case, we can immediately eliminate $|\vec{n_c}\rangle$ for it is not even the "ground state" in a truncated space, let alone in the infinite space. In the second case, we repeat both the numerical computation and the physical process but with longer evolution times, chosen in *Step 7* above. The numerically so-computed probability for obtaining the state $|g'\rangle$, even though initially agrees with (within some accuracy), will have to eventually depart from the physically obtained probability. This is a consequence of the quantum adiabatic theorem. In the case of our classically simulated but truncated world, the state $|g'\rangle$ *is* the ground state and thus the obtaining probability will, as asserted by the theorem, increase with the times evolved (for evolution times greater than some scale). But if the state $|\vec{n_c}\rangle$ is *not* the true ground state in the infinite world as we have assumed on the other hand, its measured probability must decrease with the times evolved (as the probability for the true ground state would increase for evolution times greater than some scale).

We only discuss here the changing rate of probabilities as functions of evolved time, but note that this rate is only part of the much richer body of information obtainable from the statistics. The full body of information should and could, if needed, be exploited further for the identification of the true ground state.

The proof of convergence and of veracity this algorithm is to be presented elsewhere.

In summary, to solve the Hilbert's tenth problem we need *both* the physical adiabatic processes to obtain a candidate state *and* the numerical Quantum Mechanics to verify this is the ground state through the usual statistical predictions from the Schrödinger equation with a few low-lying energy states of $\mathcal{H}(t/T)$. This way, we can overcome the problem of where to truncate the infinite basis for a numerical study of Quantum Mechanics, and reconcile with the Cantor diagonal arguments which state that the problem could not be solved entirely in the framework of classical computation.

The key factor in the ground state verification is *the probability distributions*, which are measurable in practice, as mentioned in Sec 3.1. However, in using the probability distributions as the verifying criteria, we have to assume that Quantum Mechanics is able to describe Nature correctly to the precision required. Note also that we have here a peculiar situation in which the computational complexity, that is, the computational time, might not be known exactly *before* carrying out the quantum computation – although it can be estimated approximately.

5.3 Simple illustrations (*in collaboration with Andrew Rawlinson)*

In recapitulation, we need to be able to obtain and identify the ground state of some quantum Hamiltonian which is of infinite dimensions. On the one hand, we cannot implement the infinity of dimensions of the Hilbert space in the framework of Turing computation. On the other hand, in the physical, finite-time process governed by the time-dependent Hamiltonian $\mathcal{H}(t/T)$ we will always obtain some state at $t = T$; but how to identify the end states? We have argued that we need the measurability of the probability distributions. These distributions are the end products of the *dynamical process*, that is, of the time evolution through which the measured values are obtained. Quantum mechanically, the probability distributions can be deduced from the amplitudes which are the inner products of the state of the system at that time (the time of measurement) with various eigenstates of the observable (the Hamiltonian operator H_P in our situation here).

Let us take the example of the simple linear equation

$$x - 6 \;\; = \;\; 0. \tag{33}$$

The spectral flows of the corresponding $\mathcal{H}(t/T)$ are given in Figures 1a, 1b and 1c for various truncations of the infinite dimensions. The spectral flows in the Figures are clearly different. This difference implies different measurement probabilities and statistics when the ground state of H_I is the starting state for the evolution process. On the classical computer, one would have to be content with a truncated version of the infinite-dimensional Fock space. The whole issue of mathematical (non)computability is where to truncate! A comparison between the computed statistics (computable to any desired accuracy with ever expanding truncated bases) and the measured statistics (measurable to any desired accuracy by repeating the experiments a necessary number of times) will enable us to single out the real ground state of H_P.

In this extremely simple example, Figures 1b and 1c show the ground state $|n_x\rangle = |6\rangle$ with early truncation of the basis. Further addition of higher number states to the basis does not, as expected, alter the general characteristics of the spectral flow of these low-lying states but only helps sharpen the numerical values for the various statistics.

Let us take another example of the Pythagoras equation

$$(x+1)^2 + (y+1)^2 - (z+1)^2 \;\; = \;\; 0, \tag{34}$$

of which the corresponding spectral flows are presented in Figures 2a, 2b and 2c for various truncations of the infinite dimensions. The Figure 2c shows the two solutions $(2, 3, 4)$ and $(3, 2, 4)$ of (34) in a sufficiently large truncated basis. In general, the information where to sufficiently truncate the basis will have to be provided by the physical process with appropriately found T, not by mathematical arguments alone. Note that we do not discuss here the complicated issue of degeneracy in the ground state of H_P.

6 Discussion of finer points

6.1 Difference from classical algorithms

We do not look for the zeroes of the polynomial, $D(x_1, \cdots, x_K)$, which may not exist, but instead search for the absolute minimum of its square which always exists,

$$0 \leq \min \left(D(x_1, \cdots, x_K)\right)^2 \leq \left(D(0, \cdots, 0)\right)^2, \tag{35}$$

and is finite because $\lim_{x\to\infty} \left(D(x_1, \cdots, x_K)\right)^2$ diverges.

While it is equally hard to find either the zeroes or the absolute minimum in classical computation, we have converted the problem to the realisation of the ground state of a quantum Hamiltonian and there is no known quantum principle against such an act. In fact, there is no known physical principles against it. Let us consider the three laws of thermodynamics concerning energy conservation, entropy of closed systems and the unattainability of absolute zero temperature. The energy involved in our algorithm is finite, being the ground state energy of some Hamiltonian. The entropy increase which ultimately connects to decoherence effects is a technical problem for all quantum computation in general.

It may appear that even the quantum process can only explore a finite domain in a finite time and is thus no better than a classical machine in terms of computability. But there is a crucial difference:

- In a classical search, even if the global minimum is encountered, it cannot generally be proved that it is the global minimum (unless it is a zero of the Diophantine equation). Armed only with mathematical logic, we would still have to compare it with all other numbers from the infinite domain yet to come, but we obviously can never complete this comparison in finite time – thus, mathematical noncomputability.

- In the quantum case, the global minimum is encoded in the ground state. Then, by energetic tagging, the global minimum can be found in finite time and confirmed, if it is the ground state that is obtained at the end of the computation. And the ground state may be identified and/or verified by physical principles. These principles are over and above the mathematics which govern the logic of a classical machine and help differentiate the quantum from the classical. Quantum mechanics could "explore" an infinite domain, but only in the sense that it can select, among an infinite number of states, one single state (or a subspace in case of degeneracy) to be identified as the ground state of some given Hamiltonian (which is bounded from below). This "sorting" can be done because of physical principles which are not available to mathematical computability. Note that infinite dimensions are common in Quantum Mechanics, for example it is well known that no finite-dimensional matrices can satisfy the commutator
$$[x, p] = i\hbar.$$

6.2 Difference from the standard model of Quantum Computation

Our proposal is in contrast to the claim in [3] that quantum Turing machines compute exactly the same class of functions, albeit perhaps more efficiently, which can be computed by classical Turing machines. We can only offer here some speculations about this apparent discrepancy. The quantum Turing machine approach is a direct generalisation of that of the classical Turing machines but with qubits and some universal set of one-qubit and two-qubit unitary gates to build up, step by step, dimensionally larger, but still dimensionally finite unitary operations. This universal set is chosen on its ability to evaluate any desirable classical logic function. Our approach, on the other hand, is from the start based on infinite-dimension Hamiltonians acting on some Fock space and also based on the special properties and unique status of their ground states. The unitary operations are then followed as the Schrödinger time evolutions. Even at the Hamiltonian level higher orders of the operators a and $a^\dagger$, i.e. not just two-body but many-body interactions in a sense, are already present. This proliferation, which is even more pronounced at the level of the time-evolution operators, together with the infinite dimensionality and the unique energetic status of the vacuum could be the reasons behind the ability to compute, in a finite number of steps, what the dimensionally finite unitary operators of the standard quantum Turing computation cannot do in a finite number of steps. Note that it was the general Hamiltonian computation that was discussed by Benioff and Feynman [2, 16] in the conception days of quantum computation.

Indeed, Nielsen [32] has also found no logical contradiction in applying the most general quantum mechanical principles to the computation of the classical noncomputable, unless certain Hermitean operators cannot somehow be realised as observables or certain unitary processes cannot somehow be admitted as quantum dynamics. And up to now we do not have any evidence nor any principles that prohibit these kinds of observables and dynamics.

6.3 Possible pitfalls

Our quantum algorithm is based on the key ingredients of:

- The exactitude, to the level required, of the theory of Quantum Mechanics in describing and predicting physical processes;
- Our ability to physically implement certain Hamiltonians having infinite numbers of energy levels;
- Our ability to physically obtain and verify some appropriate ground state.

If any of these is not achievable or approximable with controllable accuracy, the quantum algorithm simply fails and further modifications may or may not work.

For example, we could implement the algorithm with Quantum Optical apparatuses, in which a beam of quantum light is the physical system on which final measurements

are performed and the number of photons is the quantity measured. The Hamiltonians could then be physically simulated by various components of mirrors, beam splitters, Kerr-nonlinear media (with appropriate efficiency), etc ... We should differentiate the relative concepts of energy involved in this case; a final beam state having one single photon, say, could correspond to a *higher* energy eigenstate of H_P than that of a state having more photons! Only in the final act of measuring photon numbers, the more-photon state would transfer more energy in the measuring device than the less-photon state.

A fundamental problem [12] is that the Hamiltonians which we need to be simulated in the optical apparatuses are only *effective Hamiltonians* in that their descriptions are only valid for certain range of number of photons. When there are too many photons, a mirror, for example, may respond in a different way from when only a few photons impinge on it, or the mirror simply melts. That is, other more fundamental processes/Hamiltonians different than the desirable effective Hamiltonians would take over beyond certain limit in photon numbers. Thus it seems that we cannot have available an unbounded number of levels for the quantum algorithm.

This situation is not unlike that of the required unboundedness of the Turing tape. In practice, we can only have a finite Turing tape/memory/register; and when the register is overflowed we would need to extend it.

Similar to this, we would have to be content with a finite range of applicability for our simulated Hamiltonians. But we should also know the limitation of this applicability range and be able to tell when in a quantum computation an overflow has occurred – that is, when the range of validity is breached. We could then use new materials with extended range of (photon number) applicability or modify the Diophantine equations until the appropriate ground state is verifiably obtained. A simple way to reduce the number of photons involved would be a shifting the unknowns in the Diophantine equations by some integer amounts, $x_i \to x_i + n_i$, as in the example of (7).

This way of patching results in some approximation for our algorithm. The important thing is that the approximation is *controllable* because, within some given accuracy, only a finite number of low-lying energy levels of H_P can influence the statistics for obtaining the ground state . In other words, the probability distribution, necessary for the verification of the ground state, is still measurable to within any given accuracy with appropriate increase in the physically available number of low-lying levels.

Controllability of the environmental effects is also another crucial requirement for the implementation of our quantum algorithm. The coupling with the environment causes some fluctuations in the energy levels. Ideally, the size of these fluctuations should be controlled and reduced to a degree that is smaller than the size of the smallest energy gap such as not to cause transitions out of the adiabatic process. This may present a difficulty [34] for our algorithm, but one difficulty of technical nature rather than of principle. Even though these fluctuations in the energy levels are ever present and cannot be reduced to zero, there is no physical principle, and hence no physical reason, why their size cannot be reduced to a size smaller than some required scale. The situation is similar, and closely related, to the fact that even though we cannot attain the temperature of absolute zero, there is nothing in principle preventing us from approaching as closely as desired to the absolute zero.

However, we may not need to physically reduce the fluctuations in the energy levels to that degree. They may only need to be understood and incorporated well into our numerical estimation for a realistic estimate of the transition probabilities.

Only physical experiments actually performed, nevertheless, could definitely and finally settle the implementation issues discussed in this Section or elsewhere.

7 Implications for other decision problems and Gödel's Incompleteness Theorem

Generalised noncomputability and undecidability set the boundary for computation carried out by mechanical (including quantum mechanical) processes, and in doing so help us to understand much better what can be so computed. With this in mind, we mention here the considerations [23] about some modified version of the Hilbert's tenth problem and about the computation of Chaitin's Ω number of Sec. 2.5. These problems are all inter-related through questions about existence of solutions of Diophantine equations.

As pointed out in Sec. 2.5, we can also ask different questions [9] whether some Diophantine equation has a finite number of non-negative integer solutions (including the case in which it has no solution) or an infinite number, or whether the number of solutions is even, etc ... In general, we cannot tell the degree of degeneracy of the ground state; but with some modifications, the quantum algorithm above for the Hilbert's tenth problem can be generalised to tackle this new class of questions. The possibility of such a generalisation confirms the mathematical equivalence between the Hilbert's tenth problem and this new class.

Notwithstanding this, the situation is different with the evaluation of Chaitin's Ω number. We now have to appeal to the hypothetical ability to physically construct Hamiltonians involving a countably infinite number of distinct pairs of creation and annihilation operators. (Recall that for the Hilbert's tenth problem and its equivalence, we only need as many pairs as the number of unknowns in a Diophantine equation.) In other words, we would have to possess the ability to create or simulate Hamiltonians in Quantum Field Theory. That is, if we stick to the example of quantum computation with Quantum Optics, we would have to have individual control over infinitely many different optical modes, each with a distinct frequency. Clearly, the situation is worse than before and is even more hypothetical.

The difficulty we encounter for the Ω number is nothing but another manifestation of the different classes in the hierarchy of classical noncomputability. In Sec. 2.5 we pointed out that, on the one hand, the evaluation of this number is as classically noncomputable as the completeness of Arithmetic and that, on the other hand, Gödel's Incompleteness is "more" noncomputable than the Hilbert's tenth problem.

Our decidability study so far only deals with the property of being Diophantine, which does not cover the property of being arithmetic in general (which could involve unbounded universal quantifiers). As such, our consideration has no direct consequences on Gödel's In-

completeness theorem. However, it is conceivable that Gödel's theorem may lose its restrictive power once the concept of mathematical proof is suitably generalised with quantum principles.

8 Back to Mathematics

While the proposal is about some quantum processes to be implemented physically, it illustrates the surprisingly important rôle of Physics in the study of computability. This is an unusual state of affairs when Physics, which has its roots in the physical world out there, could perhaps help in setting the limits of Mathematics.

Inspired by this connection between the two, we present next a reformulation of the Hilbert's tenth problem. The reformulation is made possible since physical theories in general, and Quantum Mechanics in particular, have enjoyed the support and rigour of mathematical languages. We wish to stress here that, in spite of the inspiration, the connection is established entirely in the domain of Mathematics; we need not appeal to some real physical processes as we do with the proposed quantum algorithms above.

Mathematically, all we need to do is to sort out the instantaneous ground state $|g\rangle$ among the infinitely many eigenvectors of $\mathcal{H}(S)$ in (25); but this is a hard task. The trick we will use [24], as inspired by quantum adiabatic processes, is to tag the state $|g\rangle$ by some other known state $|g_I\rangle$ which is the ground state of some other operator H_I and can be smoothly connected to $|g\rangle$ through some continuous parameter $s \in [0,1]$. To that end, we consider the interpolating operator (25), rewritten as

$$\begin{aligned} \mathcal{H}(s) &= H_I + f(s)(H_P - H_I), \\ &\equiv H_I + f(s)W, \end{aligned} \tag{36}$$

which has an eigenproblem at each instant s,

$$[\mathcal{H}(s) - E_q(s)]|E_q(s)\rangle = 0, \qquad q = 0, 1, \cdots \tag{37}$$

with the subscript ordering of the sizes of the eigenvalues, $|E_0(1)\rangle \equiv |g\rangle$, and $f(s)$ not necessarily linear but some continuous and monotonically increasing function in $[0,1]$

$$f(0) = 0; \qquad f(1) = 1. \tag{38}$$

Clearly, $E_0(0) = E_I$ and $E_0(1) = E_g$. It turns out that for the linkage $E_0(s)$ to connect a ground state to another ground state we require that

$$[H_P, H_I] \neq 0. \tag{39}$$

The details are presented in [24] to arrive at

$$\frac{d}{ds}|E_q\rangle = f'(s)\sum_{l\neq q}^{\infty} \frac{\langle E_l|W|E_q\rangle}{E_q - E_l}|E_l\rangle, \tag{40}$$

$$\frac{d}{ds}E_q(s) = f'(s)\langle E_q(s)|W|E_q(s)\rangle. \tag{41}$$

Equations (40) and (41) form the set of infinitely coupled differential equations providing the tagging linkage as desired.

Analytical and numerical methods can now be employed to investigate the unknown ground state of H_P from the constructively known spectrum of H_I as the initial conditions. In this reformulation, the Diophantine equation in consideration has at least one integer solution if and only if

$$\lim_{s\to 1} E_0(s) \;=\; 0. \tag{42}$$

The limiting process is necessary here since H_P, i.e. $\mathcal{H}(1)$, will have a degenerate spectrum because of certain symmetry (H_P commutes with $a_i^\dagger a_i$).

The equations above are infinitely coupled and cannot be solved explicitly in general. But we are only interested in certain information about the ground state. And since the influence on the ground state by states having larger and larger indices diminishes more and more thanks to the denominators in (40) (once no degeneracy is assured), this information may be derived, numerically or otherwise, with some truncation to a finite number of states involved. This may not work for all the Diophantine equations.

While the ground-state outcome for our differential equations might or might not be computable, it should be noted that there are instances [1, 37] where very simple differential equations, such as the wave equations, could have noncomputable outcomes because of ill-posed initial conditions.

With care we can slightly modify the derivation for (40,41) to come up with similar equations even when there is some degeneracy in $[0, 1]$. But for the condition (42) to be the indicator for the existence of solutions of the Diophantine equation, simple topological consideration only requires *that the initial ground state* $|E_0(0)\rangle = |g_I\rangle$ *is not degenerate and that this state does not cross with any other state in the open interval* $s \in (0, 1)$. With the freedom of choice for H_I satisfying (39), we should be able to eliminate any symmetry in the open interval $s \in (0, 1)$ for $\mathcal{H}(s)$ in order to have a stronger condition of a totally avoided crossing.

9 Concluding remarks

We have discussed and emphasised the issue of computability in principle, not that of computational complexity. This attempt of broadening the concept of effective computability, taking into account the quantum mechanical principles, has been argued to be able in principle to decide some of the classical undecidables, the Hilbert's tenth problem and thus the Turing halting problem in this instance. If the quantum algorithm is realisable, and we do not have any evidence of fundamental nature to the contrary, the Church-Turing thesis for effective computability should be modified accordingly. But first, in need of further investigations are the effects of errors in the implementation of Hamiltonians, and of decoherence, and of measurement on the final outcome of our algorithm.

On the other hand, if for any reasons the algorithm is not implementable *in principle* then it would be an example of information being limited by physics, rather than by logical

arguments alone. That is, there might be some fundamental physical principles, not those of practicality, which prohibit the implementation. Or, there might not be enough physical resources (ultimately limited by the total energy and the lifetime of the universe) to satisfy the execution of the quantum algorithm. (In this case it is likely that the Turing program under consideration, even if it eventually halts upon some input, would take a running time longer than the lifetime of the universe.) In either case, the whole exercise is still very interesting as the unsolvability of those problems and the limit of mathematics itself are also dictated by physical principles and resources.

That some generalisation of the notion of computation could help solve the previous undecidability/noncomputability has been recognised before in mathematics. But quantum physics has not been recognised as the missing ingredient until very recently. Our quantum algorithm presented here could in fact be regarded as an infinite search through the integers in a finite amount of time, the type of search required to solve the Turing halting problem. Apart from ours, there also exist in the literature some other efforts where physical principles are explored [8, 14, 32, 40] for some possible extension of the notion of computability.

It should be emphasised again here that not only the values of some observables are measured but so are the probability distributions of these values. These measurable probability distributions are also to be compared against those obtained from the theory of Quantum Mechanics. This helps identify the ground state, and thus the answer for the decision problem, and is the pivotal element of our algorithm to separate quantum computability from Turing computability. In doing so, we have to assume that Nature is describable by Quantum Mechanics correctly at least to the precision required. If this is not the case, testing a mathematically solvable Diophantine equation might yield some evidence for the failure of Quantum Mechanics as a theory of Nature.

Also implied in the discusions above is the fact that Quantum Mechanics *is*, as a theory, non-computable. This fact is not so widely recognised.

To understand quantum computability and its limits, we have also considered, with mixed results, some problems generalised from the Hilbert's tenth and the evaluation of Chaitin's Ω number. Nevertheless, our study is an illustration of "Information is physical".

Inspired by Quantum Mechanics, we have also reformulated the question of solution existence of a Diophantine equation into the question of certain properties contained in an infinitely coupled set of differential equations. In words, we encode the answer of the former question into the smallest eigenvalue and corresponding eigenvector of a hermitean operator bounded from below. And to find these eigen-properties we next deform the operator continuously to another operator whose spectrum is known. Once the deformation is also expressible in the form of a set of coupled differential equations, we could now start from the constructive knowns as a handle to study the desired unknowns.

Note that the reformulation is entirely based on mathematics. If a general mathematical method could be found to extract the required information from the differential equations for any given Diophantine equation then one would have the solution to the Hilbert's tenth problem itself. This may be unlikely but not be as contradictory as it seems –because the unsolvability of the Hilbert's tenth problem is only established in the framework of

integer arithmetic and in Turing computability, not necessarily in Mathematics in general. Tarski [42] has shown that the question about the existence of *real* solutions of polynomials over the reals is, in fact, *decidable*.

It seems appropriate to end here with a quotation from the man whose famous Incompleteness result has often been misquoted as spelling the end for computability/provability in Arithmetic. In Gödel's own words [18]:

> "... On the other hand, on the basis of what has been proved so far, it remains possible that there may exist (and even be empirically discoverable) a theorem-proving machine which in fact *is* equivalent to mathematical intuition, but cannot be *proved* to be so, nor even be proved to yield only *correct* theorems of finitary number theory."

Perhaps, quantum computation *is* that possibility? (In the affirmative case, the notion of proof may also need to be reconsidered, see [25] and references therein.)

Acknowledgements

I am indebted to Alan Head and Peter Hannaford for discussions, comments and suggestions. I would also like to thank Cristian Calude, Jack Copeland, Bryan Dalton, John Markham, Michael Nielsen, Boris Pavlov, Andrew Rawlinson, Falk Scharnberg and Khai Vu for discussions; Gregory Chaitin, Peter Drummond, Gabor Etesi, Yuri Matiyasevich, Ray Sawyer, Karl Svozil, Boris Tsirelson and Ray Volkas for email correspondence. Comments by anonymous referees are also gratefully acknowledged. I wish to acknowledge a Visiting Scientist position and the hospitality extended to me during my stay at the CTP at MIT, and also a visit at the IAS at Princeton. Discussions with Stephen Adler, Enrico Deotto, Edward Farhi, Jeffrey Goldstone, and Sam Gutmann have helped me sharpening up the arguments presented herein.

References

[1] Barrow, J.D., 2000, *Complexity*, **5**, 28-34.

[2] Benioff, P., 1980, The computer as a physical system. *J. Stat. Phys.*, **22**, 563-591.

[3] Bernstein, E., and Vazirani, U., 1997, Quantum complexity theory. *SIAM J. Comput.*, **26**, 1411.

[4] Bell, J.S., 1987, *Speakable and unspeakable in quantum mechanics*, (Cambridge University Press).

[5] Casti, J.L., 2001, *Mathematical Mountaintops* (OUP).

[6] Chaitin, G.J., 1992, *Algorithmic Information Theory* (Cambridge University Press).

[7] Calude, C.S., and Chaitin, G.J., 1999, Randomness everywhere, *Nature*, 400.

[8] Calude, C.S., and Pavlov, B., 2001, Coins, Quantum Measurements and Turing's Barrier, Archive `quant-ph/0112087`.

[9] Davis, M., 1982, *Computability and Unsolvability* (Dover).

[10] Davis, M., 2000, *Engines of Logic* (Norton).

[11] Davis, M., and Hersh, R., (November 1973) *Sci. Am.*, 84-91.

[12] Drummond, P., 2002, private communication.

[13] Enk, S.J. van, and Fuchs, C.A., 2001, Archive `quant-ph/0111157`.

[14] Etesi, G., and Németi, I., 2001, Non-Turing computations via Malament-Hogarth space-times, Archive `gr-qc/0104023`.

[15] Farhi, E., Goldstone, J., Gutmann, S., and Sipser, M., 2000, Quantum computation by adiabatic evolution, Archive `quant-ph/0001106`.

[16] Feynman, R.P., 1982, Simulating physics with computers. *Int. J. Theor. Phys.*, **21**, 467.

[17] Geroch, R., and Hartle, J.B., 1986, *J. Found. Phys.*, **16**, 533-50.

[18] Gödel, K., 1951, statement delivered to the American Mathematical Society.

[19] Grover, L. K., 1997, Quantum mechanics helps in searching for a needle in a haystack, *Phys. Rev. Lett.*, **79**, 325-328.

[20] Hodges, A.P., 1983, *Alan Turing: the enigma*, (Burnett Books and Hutchinson).

[21] Kadowaki, T., and Nishimori, H., 1998, Quantum annealing in the transverse Ising model, *Phys. Rev. E*, **58**, 5355.

[22] Kieu, T.D., 2001, Quantum algorithm for the Hilbert's tenth problem, Archive `quant-ph/0110136`.

[23] Kieu, T.D., 2001, Gödel's Indompleteness, Chaitin's Ω and Quantum Physics, Archive `quant-ph/0111062`.

[24] Kieu, T.D., 2001, A reformulation of the Hilbert's tenth problem through Quantum Mechanics, Archive `quant-ph/0111063`.

[25] Kieu, T.D., 2002, Quantum principles and mathematical computability, Archive `quant-ph/0205093`.

[26] Lewis, H.R., and Papadimitriou, C.H., 1981, *Elements of the Theory of Computation* (Prentice Hall).

[27] Lloyd, S., and Braunstein, S. L., 1998, Quantum computation over continuous variables, Archive `quant-ph/9810082`.

[28] Matiyasevich, Y. V., 1993, *Hilbert's Tenth Problem*, (MIT Press).

[29] Messiah, A., 1976, *Quantum Mechanics*, Vol. II, (North Holland).

[30] Nagel, E., and Newman, J.R., 1958, *Gödel's Proof* (New York University Press).

[31] Neumann, J. von, 1983, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press).

[32] Nielsen, M.A., 1997, Computable functions, quantum measurements, and quantum dynamics, *Phys. Rev. Lett.*, **79**, 2915-8.

[33] Nielsen, M.A. & Chuang, I.L., 2000, *Quantum Computation and Quantum Information* (Cambridge University Press).

[34] Nielsen, M.A., 2002, private communication.

[35] Penrose, R., 1997, *The Large, the Small and the Human Mind* (CUP).

[36] Peter, R., 1967, *Recursive Functions* (Academic Press).

[37] Pour-Ell, M. B., and Richards, I., 1982, *Int. J. Theor. Phys.*, **21**, 553.

[38] Rogers, H. Jr., 1987, *Theory of Recursive Functions and Effective Computability* (MIT Press).

[39] Shor, P.W., 1997, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, *SIAM J. Comput.*, **26**, 1484-1509.

[40] Stannett, M., 2001, Hypercomputation is Experimentally Irrefutable. Tech. Report CS-01-04 Dept of Computer Science, Sheffield University.

[41] Svozil, K., 1998, The Church-Turing Thesis as a Guiding Principle for Physics, *Unconventional Models of Computation*, ed. by Cristian S. Calude, John Casti and Michael J. Dinneen (Springer, Singapore), pp. 371-385.

[42] Tarski, A., 1951, *A Decision Method for Elementary Algebra and Geometry* (University of California Press).

[43] Uspensky, V.A., 1987, *Gödel's Incompleteness Theorem* (Mir).

[44] Walls, D.F., and Milburn, G.J., 1995, *Quantum Optics* (Springer).

[45] Wooters, W.K., and Zurek, W.H., 1982, A single quantum cannot be cloned, *Nature*, **299**, 802-803.

Tien D Kieu came to Australia as a Vietnamese refugee in 1980 and obtained a B Sc (Hons) from the University of Queensland in 1984 and a Ph D from the University of Edinburgh in 1988. He then held postdoctoral positions at the Universities of Edinburgh and Oxford, including a Junior Research Fellowship at Linacre College. In 1991, he took up an Australian Postdoctoral Fellowship and then a Queen Elizabeth II Fellowship at the University of Melbourne. At the beginning of 2001, after three years at CSIRO in the position of Senior Research Scientist and Project Leader, he joined the Swinburne University of Technology as a Professorial Fellow. He is currently interested in Quantum Computation and Quantum Measurement.

Figures 1: The spectral flow of $\mathcal{H}(s)$, with $\alpha = 0.5$ and with $s \equiv t/T$ is the scaled evolution time, corresponding to the linear equation (33) in the truncated bases with 5, 9, and 25 states respectively in Figures 1a, 1b and 1c. Figures 1b and 1c contain the unique solution $|n_x\rangle = |6\rangle$. Enlargement of the truncated basis from here on would not change the characteristics of the spectral flow of these low- lying states, but only approximates better the measurement probabilities as functions of evolution time.

Figures 2: The spectral flow of $\mathcal{H}(s)$, with $\alpha = 0.5$ and with $s \equiv t/T$ is the scaled evolution time, corresponding to the Pythagoras equation (34) in the truncated bases with 84, 220, 1771 states respectively in Figures 2a, 2b, and 2c. Figure 2c contains two solutions of the Pythagoras equation (34), $|n_x, n_y, n_z\rangle = |2, 3, 4\rangle$ and $|3, 2, 4\rangle$. Further enlargement of the truncated basis will increase the degeneracy of the ground state.