Numerical Methods

1 Worksheet 1

Exercise 1.1. Write a function that takes a ID number as input and returns the corresponding ID letter as output.

Exercise 1.2. Write a function that takes natural numbers a_0, a_1 y N as inputs and returns a vector consisting of the first N elements of the sequence defined as

$$a_n = a_{n-1} + a_{n-2}$$

2 Worksheet 2

Exercise 2.1. Write a program that, given a decimal number, returns a vector of 0s and 1s representing its binary equivalent.

Exercise 2.2. Write a program that, given a binary number represented by a vector of 0s and 1s, returns the corresponding decimal number.

Exercise 2.3. Determine the machine epsilon. To do so, calculate $1+x$ with $x = 2^{-i}$ for $i = 1, 2, \ldots$ while $1 + x > 1$. Compare with the MATLAB command *eps*.

Exercise 2.4. Given the function $f(x) = \operatorname{sen}(x)$, we can use the incremental quotient

$$\frac{\operatorname{sen}(x+h) - \operatorname{sen}(x)}{h}$$

at $x = 1$ to make an estimate of

$$0,540302305868140 = \cos(1) = f'(1) = \lim_{h \to 0} \frac{\operatorname{sen}(1+h) - \operatorname{sen}(1)}{h}$$

Write a function that takes a natural number N and returns a matrix satisfying:
1. the first column is the vector $\left[10^{-1}, 10^{-2}, 10^{-3}, \cdots, 10^{-N}\right]'$;
2. the second column is the estimated value;
3. the third column is the absolute error made;
4. the third column is the relative error made;

Verify that there is a loss of precision due to cancellation.

Exercise 2.5. The exact roots of the quadratic equation

$$x^2 - \left(64 + 10^{-15}\right) x + 64 \times 10^{-15} = 0$$

are $x1 = 64$ y $x2 = 10^{-15}$. Calculate its roots, verifying that the result obtained for the smaller one does not agree with the exact one in any significant figure.

Exercise 2.6. Check the results from the example we studied in class about the calculation of $(1/7)^{100}$ using the sequence

$$a_{n+2} = \frac{22}{7} a_{n+1} - \frac{3}{7} a_n \quad n \geq 2$$

By seeking solutions of the form $a_n = \lambda^n$ the following expression for the general term of the sequence can be obtained:

$$a_n = C_1 \left(\frac{1}{7}\right)^n + C_2 3^n$$

Therefore, taking $a_0 = 1$ y $a_1 = 1/7$, we have that $C_1 = 1$ y $C_2 = 0$, arriving at

$$a_n = \left(\frac{1}{7}\right)^n$$

3 Worksheet 3

Exercise 3.1 Write a program that calculates the 1-norm, infinity norm, and Fröbenius norm of a given matrix. Verify the results obtained with the MATLAB command *norm*.

Exercise 3.2. Write a specific program to calculate the product of an upper triangular matrix (resp. lower) by a vector, and the product of two upper triangular matrices (resp. lower).

Exercise 3.3. (PALU factorization) Write a function that takes a square invertible matrix A of n-th order as input, and returns matrices P, L and U from the $PA = LU$ factorization.

4 Worksheet 4

Say we want to calculate a numerical estimate for the solution of the following problem:

$$\begin{cases} y'' - y = 0 & x \in (0,1) \\ y(0) = 1 \\ y(1) = e \end{cases}$$

which we know has solution $y = e^x$. We follow these steps:

1.Take an uniform partition of size h of the interval $(0,1)$. That is, given a number N, consider the vector $(0, h, 2h, 3h, \cdots, Nh, 1)$. Note that the node vector has N interior nodes, therefore $h = 1/(N+1)$.

2. Estimate the second derivative,

$$y''(x_i) \simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

where, $x_i = ih$ y $y_i \simeq y(x_i)$ is the estimate we want to calculate.

3. Substitute the approximation into the differential equation at the point x_i, thus arriving at the following linear system of equations:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - y_i = 0 \quad i = 1, \cdots, N$$

Note that y_0 and y_{N+1} are known values; here $y_0 = 1$ and $y_{N+1} = \exp(1)$. Since h is a small number, and dividing by small numbers can cause stability problems, consider the linear system

$$y_{i+1} - 2y_i + y_{i-1} - h^2 y_i = 0 \quad i = 1, \cdots, N$$

4.This system can be written in matrix form as $AY_h = b$, where

a) A is a tridiagonal matrix given by

$$A = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix} - h^2 I$$

with I the identity matrix.

b) b is a vector given by

$$b = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ -e \end{pmatrix}$$

5. Thus, we arrive at

$$Y_h = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N-1} \\ y_N \end{pmatrix} = A \backslash b$$

6. Using the boundary conditions, we get that the numerical of $y(x)$ at $(0, h, 2h, \cdots, Nh, 1)$ is given by the vector $Y_h = [1; Y_h; \exp(1)]$.

Exercise 4.1. Write a program that solves the previous problem and draws:
i) In the same graphic, the real solution (in red) and the estimated solution (in green).
ii) In a different graphic, the error made.

Exercise 4.2. Repeat the previous exercised for this boundary problem:

$$\begin{cases} y'' + y' - 5y = x \quad x \in (0,2) \\ y(0) = 3 \\ y(2) = e^4 - 2 \end{cases}$$

Which we know has solution $y = e^{2x} + 2 - 2x$. This time, approximate the derivatives by

$$y'(x_i) \simeq \frac{y_{i+1} - y_{i-1}}{2h}, \quad y''(x_i) \simeq \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$