LabNum

Lista 3 - ~~2018~~ 2019

Ex 1 (Cap. 14 Ex 2)

Let $f(x)$ be a given function that can be evaluated at points $x_0 \pm jh$, $j = 0, 1, 2 \ldots$ for any fixed value of h, $0 < h \ll 1$.

(a) Find a second order formula (i.e. truncation error $O(h^2)$) approximating the third derivative $f'''(x_0)$. Give the formula, as well as an expression for the truncation error, i.e. not just its order.

Vamos considerar as expansões de Taylor de $f(x_0+h)$, $f(x_0-h)$, $f(x_0+2h)$, $f(x_0-2h)$

$$(\mathrm{I})\quad f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{IV}(x_0) + \frac{h^5}{120}f^{V}(x_0) + O(h^6)$$

$$(\mathrm{II})\quad f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + \frac{h^4}{24}f^{IV}(x_0) - \frac{h^5}{120}f^{V}(x_0) + O(h^6)$$

$$(\mathrm{III}):\quad f(x_0+2h) = f(x_0) + 2hf'(x_0) + \frac{4}{2}h^2f''(x_0) + \frac{8}{6}h^3f'''(x_0) + \frac{16h^4f^{IV}(x_0)}{24} + \frac{32}{120}h^5f^{V}(x_0) + O(h^6)$$

$$(\mathrm{IV}):\quad f(x_0-2h) = f(x_0) - 2hf'(x_0) + \frac{4}{2}h^2f''(x_0) - \frac{8}{6}h^3f'''(x_0) + \frac{16}{24}h^4f^{IV}(x_0) - \frac{32}{120}h^5f^{V}(x_0) + O(h^6)$$

Manipulando as expressões

$$(\mathrm{I}-\mathrm{II})\quad f(x_0+h) - f(x_0-h) = 2h f'(x_0) + \frac{2}{6}h^3f'''(x_0) + \frac{2}{120}h^5f^{V}(x_0) + O(h^6) \Rightarrow (A)$$

$$\mathrm{III}-\mathrm{IV})\quad f(x_0+2h) - f(x_0-2h) = 4h f'(x_0) + \frac{16}{6}h^3f'''(x_0) + \frac{64}{120}h^5f^{V}(x_0) + O(h^6) \Rightarrow (B)$$

Finalmente, vamos anular f' da expressão

$$2A - B)\quad 2\big(f(x_0+h) - f(x_0-h)\big) - \big(f(x_0+2h) - f(x_0-2h)\big) = \frac{-12}{6}h^3f'''(x_0) - \frac{60}{120}h^5f^{V}(x_0) + O(h^6)$$

Isolando $f'''(x_0)$

$$f'''(x_0) = \frac{2f(x_0+h) - 2f(x_0-h) + f(x_0-2h) - f(x_0+2h)}{-2h^3} + \frac{1}{2}\frac{h^5}{(-2h^3)}f^{V}(x_0) + O(h^3) \Rightarrow$$

$$f'''(x_0) = \frac{2f(x_0-h) - 2f(x_0+h) + f(x_0+2h) - f(x_0-2h)}{2h^3} - \frac{h^2}{4}f^{V}(x_0) + O(h^3)$$

Então obtemos a aproximação

$$\boxed{f'''(x_0) = \frac{2f(x_0-h) - 2f(x_0+h) + f(x_0+2h) - f(x_0-2h)}{2h^3}}$$

Com um erro $\approx -\frac{h^2}{4} f^{v}(\xi) + O(h^3) \approx O(h^2)$ conforme especificado

Vamos usar a extrapolasão de Richardson para obter uma expressão para esse erro

Usando a aproximaxão para h e para $2h$

$$(\mathrm{I}):\quad f'''(x_0) = \frac{2f(x_0-h) - 2f(x_0+h) + f(x_0+2h) - f(x_0-2h)}{2h^3} - \frac{h^2}{4} f^{v}(x_0) + O(h^3)$$

$$(\mathrm{II}):\quad f'''(x_0) = \frac{2f(x_0-2h) - 2f(x_0+2h) + f(x_0+4h) - f(x_0-4h)}{16h^3} - \frac{8h^2}{4} f^{v}(x_0) + O(h^3)$$

$$(\mathrm{I}-\mathrm{II}):\quad 0 = \frac{16(f(x_0-h) - f(x_0+h)) + 6(f(x_0+2h) - f(x_0-2h)) - f(x_0+4h) + f(x_0-4h)}{16h^3} + \frac{7h^2}{4} f^{v}(x_0) + O(h^3)$$

Isolando $-\frac{h^2}{4} f^{v}(x_0) = \text{Erro}$

$$\boxed{-\frac{h^2}{4} f^{v}(x_0) \cong \frac{16\left[f(x_0-h) - f(x_0+h)\right] + 6\left[f(x_0+2h) - f(x_0-2h)\right] - f(x_0+4h) + f(x_0-4h)}{112\, h^3}}$$