

MAC0325 Combinatorial Optimization

Midterm

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Exercise 1

Let P^* be an optimal solution for (1.1), i.e., an (r, s) -path in D of minimum cost. Let $k^* := c(P^*)$ be the optimal value of (1.1).

Define the function

$$R : [k^*] \rightarrow \mathcal{P}(V)$$

$$i \mapsto R_i := \{v \in V : \text{there is a } (r, v)\text{-path } P \text{ in } D \text{ such that } c(P) < i\}$$

Informally, $v \in R_i$ means that the cost between r and v is smaller than i .

Proposition 1. *Let $i \in [k^*]$ and let $v \in R_i$. Then $v \in R_j$ for any $i \leq j \leq k^*$*

Proof. Since $v \in R_i$ there is a (r, v) -path P with $c(P) < i$. But then, if $j \geq i$, $c(P) < j$, thus $v \in R_j$ \square

Proposition 2. *Let $j, t \in \mathbb{N}$ with $j, t \geq 1$, let $a \in A$ such that $a \in \delta^{\text{out}}(R_t)$ and let $(u, v) := \varphi(a)$. Then*

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) = j \implies a \in \delta^{\text{out}}(R_i) \text{ for each } i \in \{t - j + 1, \dots, t\}$$

Proof. Since $a \in \mathbb{1}_{\delta^{\text{out}}(R_t)}$ we have that $u \in R_t$ and $v \notin R_t$. And $v \notin R_i$ for any $i \in [t]$, because if v was in any R_i then it would be in R_t by Proposition 1.

Take the set $S := \{\delta^{\text{out}}(R_{t-j+2}), \dots, \delta^{\text{out}}(R_t)\}$, we have that $|S| = j - 1$, so if a was only in the sets of S , the sum of incidence vectors would be at most $j - 1$.

Thus there is at least one R_k with $k \leq t - j + 1$ such that $a \in \delta^{\text{out}}(R_k)$, but then $u \in R_k$, and by Proposition 1, for every $i \geq t - j + 1 \geq k$, $u \in R_i$. But we know that $v \notin R_i$, so $a \in \delta^{\text{out}}(R_i)$. And this proves our proposition. \square

Lemma 1. *k^* is a feasible solution of (1.2).*

Proof. Since all arcs have natural costs, we have that

$$(1.3) \quad k^* \in \mathbb{N}$$

Moreover, take the (r, r) -path $\langle r \rangle$ with cost $0 < i$, so that $r \in R_i$. And suppose that $s \in R_i$ then there would be a (r, s) -path P' with cost $c(P') < i \leq k^* = c(P^*)$ which contradicts the optimality of P^* , thus $s \notin R_i$. Indeed,

$$(1.4) \quad r \in R_i \text{ and } s \notin R_i \text{ for each } i \in [k^*]$$

Now, let $t \in \mathbb{N}$. We will prove by induction on t that

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

The base case $t = 0$ holds because there is no $i \in [0]$, then

$$\sum_{i \in [0]} \mathbb{1}_{\delta^{\text{out}}(R_i)} = 0 \leq c$$

(Induction Hypothesis) Suppose that

$$\sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

Now, suppose, by the sake of contradiction, that there is at least one arc $a \in A$, with $(u, v) := \varphi(a)$, such that

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) > c(a)$$

We have

$$\begin{aligned} 0 &\leq c(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \text{ (by induction hypothesis)} \\ &< \sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \\ &= \mathbb{1}_{\delta^{\text{out}}(R_t)}(a) \\ &\leq 1 \end{aligned}$$

Summing up, we got

$$0 < \mathbb{1}_{\delta^{\text{out}}(R_t)}(a) \leq 1 \implies a \in \delta^{\text{out}}(R_t)$$

And

$$\begin{aligned} 0 &\leq c(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) < 1 \implies \\ c(a) &= \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \implies \\ c(a) + 1 &= \sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \end{aligned}$$

Finally, by Proposition 2, taking $j = c(a)+1$ we have that $a \in \delta^{\text{out}}(R_{t-(c(a)+1)+1})$. Which means that $u \in R_{t-c(a)}$, so there is a (r, u) -path P of cost $c(P) < t - c(a)$

So we can create the (r, v) -path $P' := P \cdot \langle u, a, v \rangle$ of cost

$$c(P') = c(P) + c(a) < t - c(a) + c(a) = t$$

This shows that $v \in R_t$, a contradiction because $a \in \delta^{\text{out}}(R_t)$.
Hence, there can not be any arc $a \in A$ with

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) > c(a)$$

So,

$$(1.5) \quad \sum_{i \in [k^*]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

Hence, by (1.3), (1.4) and (1.5), k^* is a feasible solution of problem (1.2). \square

Theorem 1. k^* is the optimal value of (1.2).

Proof. Let $k' \in \mathbb{N}$ be the optimal value of (6.2). Let $R' : [k'] \rightarrow \mathcal{P}(V)$ be a function that satisfies the constraints of (6.2). In particular

$$(1.6) \quad \sum_{i \in [k']} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

Since $r \in R'_i$ and $s \notin R'_i$ for each $i \in [k']$, then any (r, s) -path P in D has one arc traversed by R_i , i.e, there is $a \in A(P)$ such that $a \in \delta^{\text{out}}(R_i)$ (Exercise 3.1 from lectures). This implies that

$$\begin{aligned} k^* = c(P^*) &= \sum_{a \in A(P^*)} c(a) \\ &\geq \sum_{a \in A(P^*)} \sum_{i \in [k']} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) && \text{(by (1.6))} \\ &\geq \sum_{a \in A(P^*)} \sum_{i \in [k'], a \in \delta^{\text{out}}(R_i)} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \\ &\geq k' && \text{(by (exercise 3.1))} \end{aligned}$$

So $k' \leq k^*$ and we know by Lemma 1 that k^* is a feasible value of (6.2). Hence k^* is the optimal value of (6.2). \square

Exercise 2

(i)

Proposition 3. *Let $S \subseteq V$ be a stable set. Then $V \setminus S$ is a vertex cover of G .*

Proof. Suppose, by the sake of contradiction that $V \setminus S$ is not a vertex cover, i.e., there is an edge $e \in E$ such that $e \cap (V \setminus S) = \emptyset$. Then we can use some set manipulation and get (in the right there is the manipulation used)

$$\begin{aligned}
 e \cap (V \setminus S) = \emptyset &\implies \\
 (e \cap V) \setminus (e \cap S) = \emptyset &\implies (A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)) \\
 e \setminus (e \cap S) = \emptyset &\implies \\
 (e \setminus e) \cup (e \setminus S) = \emptyset &\implies (A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)) \\
 e \setminus S = \emptyset &\implies \\
 e \subseteq S &\implies \\
 e \in E[S] &
 \end{aligned}$$

But this last statement contradicts the fact that S is stable. Thus $V \setminus S$ is a vertex cover. \square

Proposition 4. *Let $K \subseteq V$ be a vertex cover of G . Then $S := V \setminus K$ is a stable set.*

Proof. The proof is the other direction of the proof we just made. Suppose, by the sake of contradiction that S is not a stable set, i.e., there is an edge $e \in E[S]$. Then

$$\begin{aligned}
 e \in E[S] &\implies \\
 e \subseteq S &\implies \\
 e \setminus S = \emptyset &\implies \\
 (e \setminus e) \cup (e \setminus S) = \emptyset &\implies \\
 e \setminus (e \cap S) = \emptyset &\implies (A \setminus B) \cup (A \setminus C) = (A \setminus (B \cap C)) \\
 (e \cap V) \setminus (e \cap S) = \emptyset &\implies (e = e \cap V) \\
 e \cap (V \setminus S) = \emptyset &\implies ((A \cap B) \setminus (A \cap C)) = A \cap (B \setminus C) \\
 e \cap (V \setminus (V \setminus K)) = \emptyset &\implies \\
 e \cap (V \cap K) = \emptyset &\implies (A \setminus (A \setminus B) = A \cap B) \\
 e \cap K = \emptyset &
 \end{aligned}$$

But this last statement contradicts the fact that K is a vertex cover. Thus S is a stable set. \square

Theorem 2. $\alpha(G) + \tau(G) = |V|$

Proof. Let K^* be a vertex cover of size $\tau(G)$. Let $S^* := V \setminus K^*$. By Proposition 4, S^* is a stable set and its size is $|S^*| = |V| - \tau(G)$.

Let S' be a maximum stable set, i.e., $|S'| = \alpha(G)$. Now let $K' := V \setminus S'$. By Proposition 3, K' is a vertex cover.

Since K^* is minimum

$$\begin{aligned} |K'| &\geq |K^*| \implies \\ |K'| - |V| &\geq |K^*| - |V| \implies \\ |V| - |K'| &\leq |V| - |K^*| \implies \\ |S'| &\leq |S^*| \end{aligned}$$

But S' is the maximum stable set, so $|S^*| = \alpha(G)$, then $\alpha(G) = |V| - \tau(G) \implies \alpha(G) + \tau(G) = |V|$. \square

(ii)

Since G is a bipartite graph, by Konig's matching theorem (8.8 from lectures) we have that $\nu(G) = \tau(G)$.

We claim that $\rho(G) = \alpha(G)$.

Let $S \subseteq V$ be a stable set of size $\alpha(G)$ and let $F \subseteq E$ be an edge cover of size $\rho(G)$ (we know it exists because there is no isolated vertex).

For each vertex $v \in S$ there is one edge $f \in F$ such that $v \in f$ but every distinct vertex require one distinct edge in the cover, because S is stable, so there is no edge that covers two vertices in S . Thus $|F| \geq |S|$.

Now suppose there are two edges $e, f \in F$ that covers the same vertex in S , say $e := \{s, k\}, f := \{s, l\}$, for $s \in S$ and $k, l \notin S$

By Lemma 1 we know that $V \setminus S$ is a minimum vertex cover, but then we could substitute $\{k, l\}$ by just $\{s\}$ and obtain a smaller vertex cover, which is a contradiction.

Thus there is one and only one edge in F for each vertex in S . And then $|F| = |S|$, i.e., $\rho(G) = \alpha(G)$ and then $\nu(G) + \rho(G) = \tau(G) + \alpha(G) = |V|$, by item (i).

(iii)

(I ran out of time)

Exercise 3

Necessity:

Let $\{M_1, M_2, \dots, M_k\}$ be k disjoint perfect matchings in G . Then each vertex $v \in V$ is saturated by each one of the M_i matchings by one distinct edge. It means that there are at least k edges that saturates v , i.e.,

$$(3.1) \quad |\delta(\{v\})| \geq k \text{ for each } v \in V$$

For any $R \subseteq V$ we have that the number of edges of $\delta(R)$ is equal to the number of edges that leaves (or we could say enter) any vertex, except for the edges that joins two vertices in S , i.e.,

$$(3.2) \quad |\delta(R)| = \sum_{v \in S} |\delta(\{v\})| - |E[S]| \text{ for each } S \subseteq V$$

From this, taking $R = V$ we have that

$$\begin{aligned} 0 = |\delta(V)| &= \sum_{v \in V} |\delta(\{v\})| - |E[V]| \\ &= \sum_{v \in V} |\delta(\{v\})| - |E| \\ &\geq \sum_{v \in V} k - |E| && \text{(by 3.1)} \\ &= |V|k - |E| \\ &\geq |U|k - |E| \\ &= tk - |E| \end{aligned}$$

Thus

$$(3.3) \quad |E| \geq tk$$

Finally, let $P \subseteq U$ and $Q \subseteq W$, the number of edges between P and Q is $E[P \cup Q]$ because there is no edge between two vertices of P nor between two vertices of Q , so every edge in $E[P \cup Q]$ needs to be between them.

Now, summing up everything, we got

$$\begin{aligned} E[P \cup Q] &= \sum_{v \in P \cup W} |\delta(\{v\})| - |\delta(P \cup W)| && \text{(by 3.2)} \\ &\geq \sum_{v \in P \cup W} |\delta(\{v\})| - |E| \\ &\geq \sum_{v \in P \cup W} |\delta(\{v\})| - tk && \text{(by 3.3)} \\ &= \sum_{v \in P} |\delta(\{v\})| + \sum_{v \in Q} |\delta(\{v\})| - tk \quad \text{(because } P \text{ and } Q \text{ are disjoint)} \\ &\geq \sum_{v \in P} k + \sum_{v \in Q} k - tk && \text{(by 3.1)} \\ &= |P|k + |Q|k - tk \\ &= k(|P| + |Q| - t) \end{aligned}$$

This concludes the proof for necessity.

Sufficiency:

We will prove, by induction on k that if there are at least $k(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$, then G has k perfect matchings.

The base case $k = 0$ holds, because every graph has 0 perfect matchings (if it has more, than it still has (at least) 0).

Now suppose that if G is a graph in which there are at least $(k - 1)(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$, then G has $k - 1$ perfect matchings. (Induction Hypothesis)

We can add edges to G , so that there are at least $k(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$.

By the induction hypothesis, we know that there are at least $k - 1$ perfect matchings, we just need to find another one. But we claim that these edges that were added saturates all vertices, and so there is a new perfect matching, disjoint of the others.

Let $u \in U$, take $P = \{u\} \subseteq U$ and $Q = W$. Then we have that the number of edges between P and W is at least $k(|P| + |Q| - t) = k(1 + t - t) = k$.

Thus for any vertex $u \in U$ there are at least k edges between u and W . By an analogous argument we can show that there are at least k edges from any vertex $w \in W$ to the set U .

So every vertex in U and W is an end point of at least one edge that is not covered by any matching, thus these edges contains a new perfect matching.

Hence G has k disjoint perfect matchings.

This ends the proof for sufficiency.