Computational Science: Modeling and Simulation

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Chapters 6 and 13

DYNAMICAL SYSTEMS, CHAOS & FRACTALS


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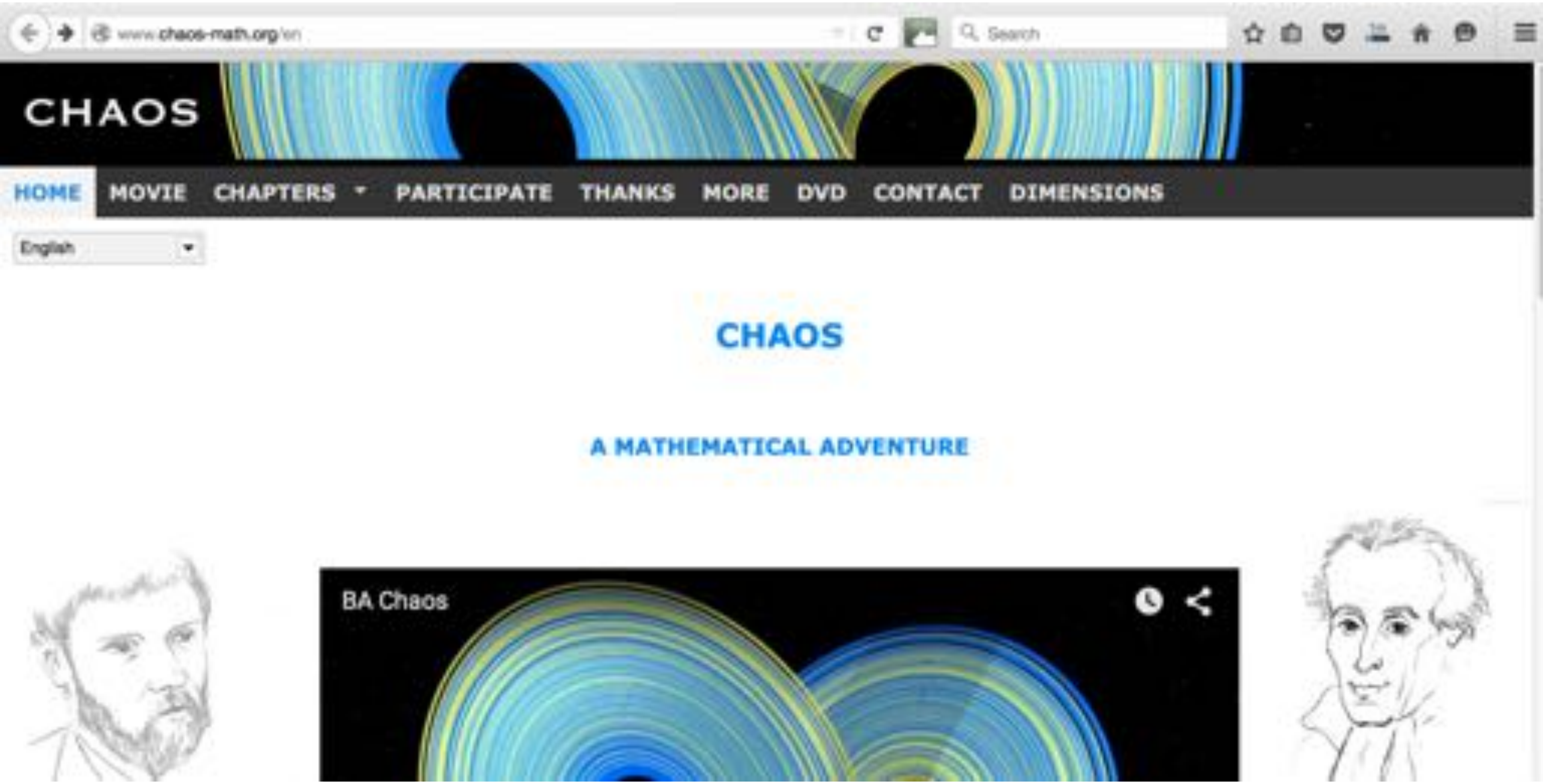
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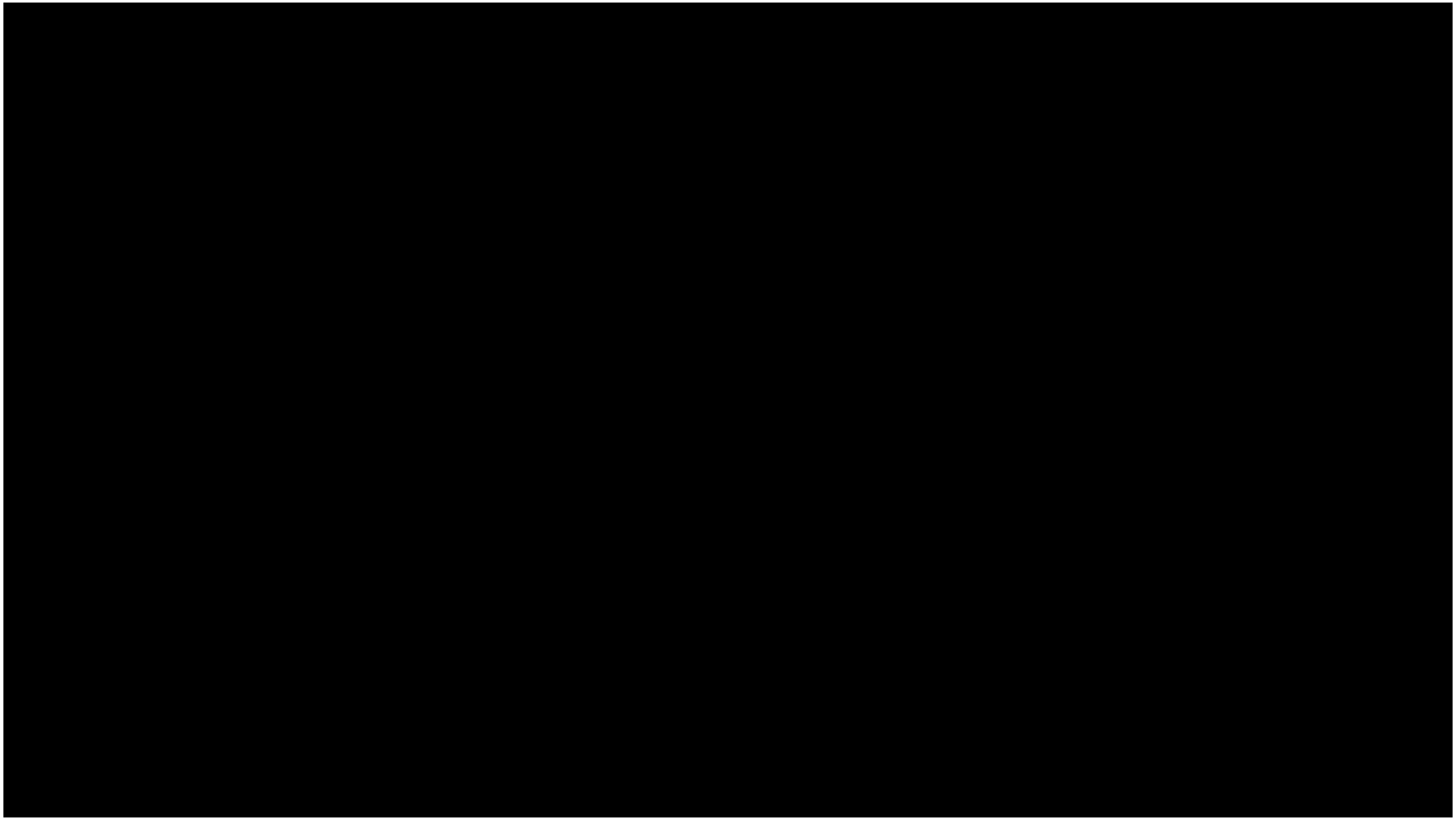


Chapter 6

The Chaotic Motion of Dynamical Systems

We study simple nonlinear deterministic models that exhibit chaotic behavior. We will find that the use of the computer to do numerical experiments will help us gain insight into the nature of chaos.

6.1 Introduction

Most natural phenomena are intrinsically nonlinear. Weather patterns and the turbulent motion of fluids are everyday examples. Although we have explored some of the properties of nonlinear systems in Chapter 4, it is easier to introduce some of the important concepts in the context of ecology. Our first goal will be to motivate and analyze the one-dimensional difference equation

$$x_{n+1} = 4rx_n(1 - x_n) \tag{6.1}$$

Linear models

Population evolution models

$$P_{n+1} = aP_n,$$

EXERCÍCIO:

Create a Jupyter Notebook to implement the model above.
Your program should plot the trajectories.
Do they converge? Do they diverge? How is the dependency on P0, a, n?

Nonlinear models

Population evolution models

From linear to nonlinear models

$$P_{n+1} = aP_n,$$

$$P_{n+1} = P_n(a - bP_n).$$

$$P_n = (a/b)x_n$$

Population ration wrt standard

$$x_{n+1} = ax_n(1 - x_n).$$

$$r = a/4$$

$$x_{n+1} = f(x_n) = 4rx_n(1 - x_n).$$

One dimensional map

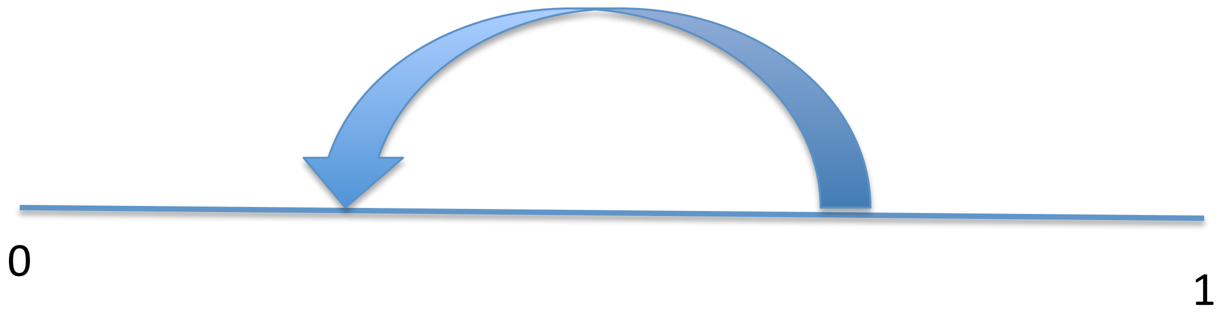
Modify your program to implement this new model

Nonlinear models

Population evolution models

One dimensional map

$$x_{n+1} = f(x_n) = 4rx_n(1 - x_n).$$



The sequence of values $x_0, x_1, x_2, \ldots$ is called the *trajectory*. To check your understanding, suppose that the initial value of x_0 or *seed* is $x_0 = 0.5$ and $r = 0.2$. Do a calculation to show that the trajectory is $x_1 = 0.2, x_2 = 0.128, x_3 = 0.089293, \ldots$ The first thirty iterations of (6.5) are shown for two values of r in Figure 6.1.

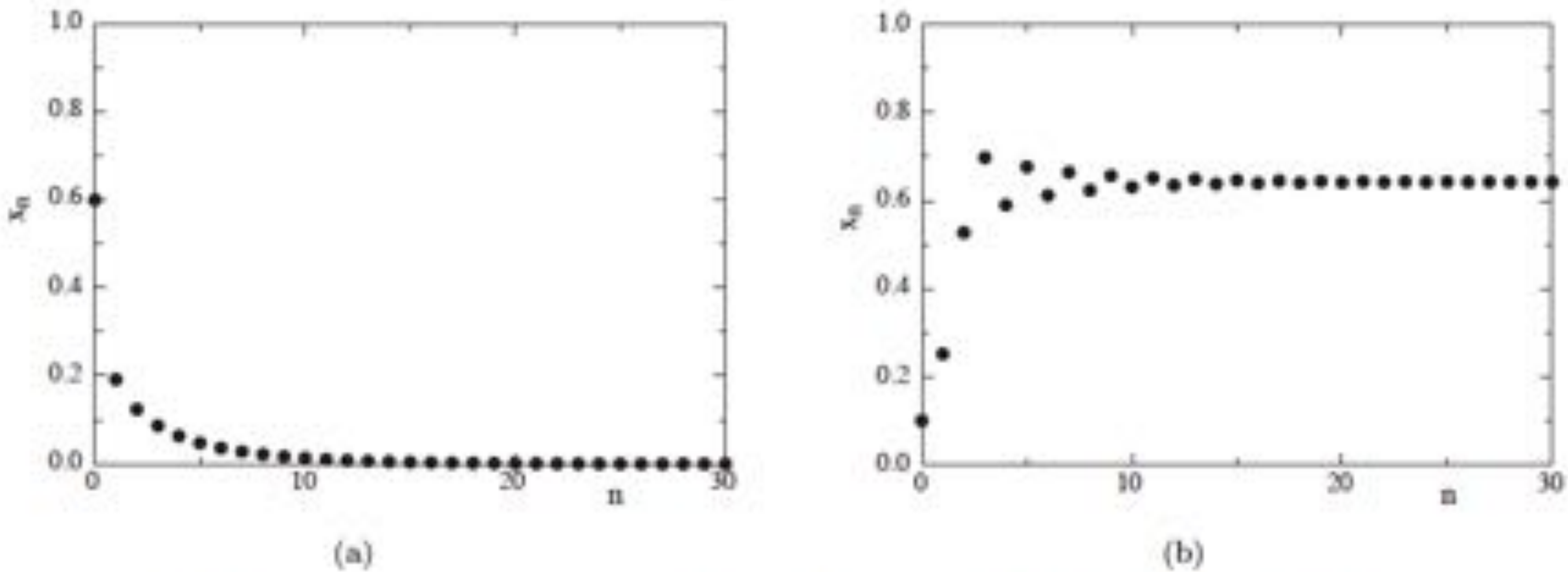


Figure 6.1: (a) The trajectory of x for $r = 0.2$ and $x_0 = 0.6$. The stable fixed point is at $x = 0$. (b) The trajectory for $r = 0.7$ and $x_0 = 0.1$. Note the initial transient behavior.

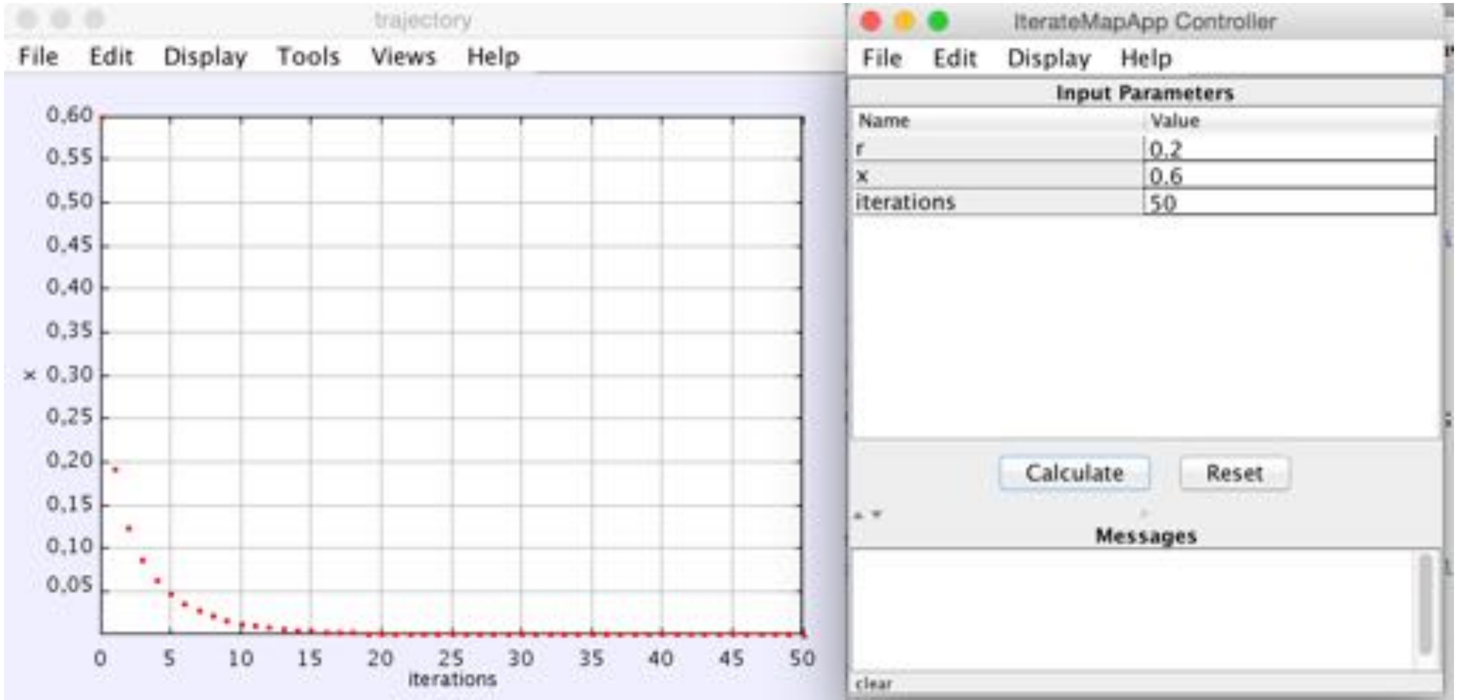
Listing 6.1: The **IterateMapApp** class iterates the logistic map and plots the resulting trajectory.

```
package org.opensourcephysics.sip.ch06;
import org.opensourcephysics.frames.*;
import org.opensourcephysics.controls.*;

public class IterateMapApp extends AbstractCalculation {
   int datasetIndex = 0;
   PlotFrame plotFrame = new PlotFrame("iterations", "x", "trajectory");

   public IterateMapApp() {
      plotFrame.setAutoclear(false); // keep data between calls to calculate
   }

   public void reset() {
      control.setValue("r", 0.2);
      control.setValue("x", 0.6);
      control.setValue("iterations", 50);
      datasetIndex = 0;
   }
```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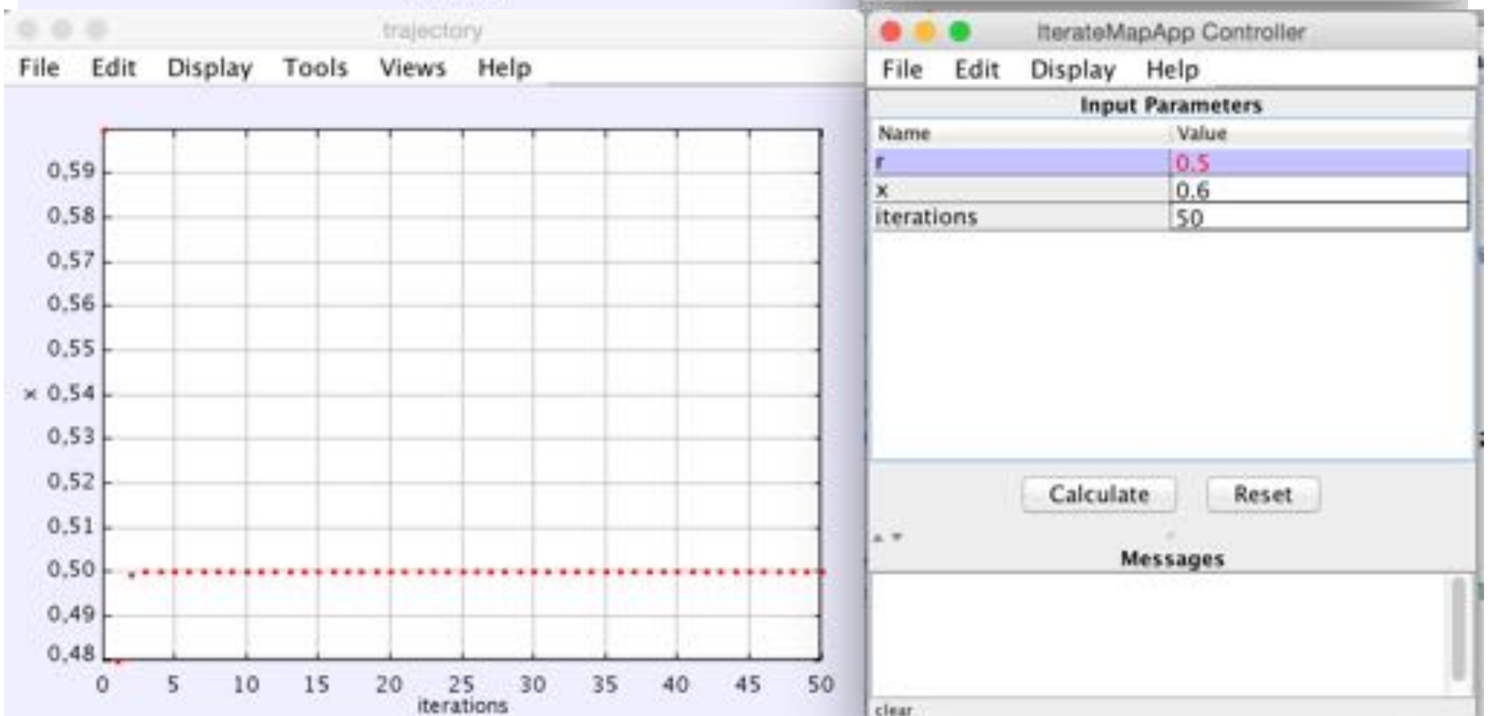
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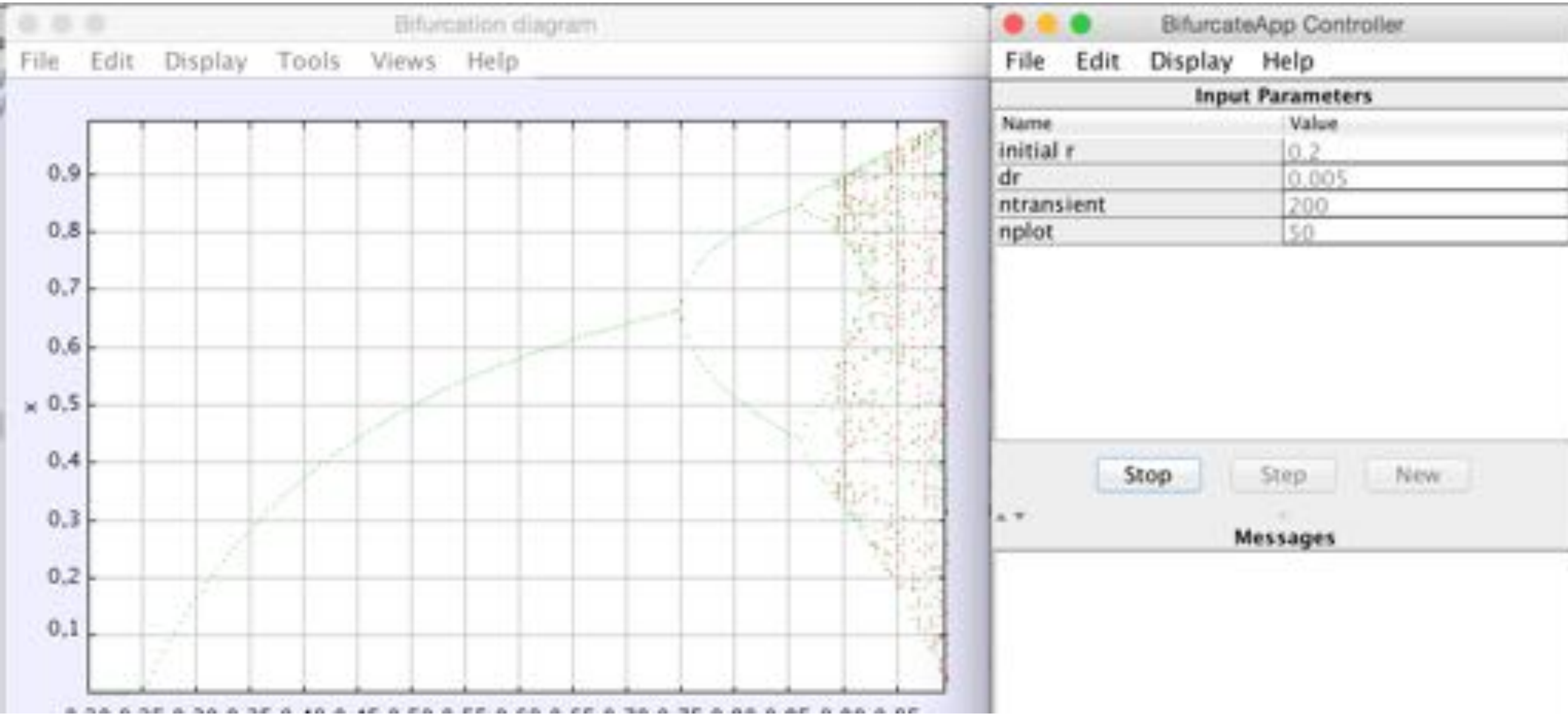
Problem 6.1. The trajectory of the logistic map

a. Explore the dynamical behavior of the logistic map in (6.5) with $r = 0.24$ for different values of x_0. Show numerically that $x = 0$ is a *stable fixed point* for this value of r. That is, the iterated values of x converge to $x = 0$ independently of the value of x_0. If x represents the population of insects, describe the qualitative behavior of the population.

b. Explore the dynamical behavior of (6.5) for $r = 0.26, 0.5, 0.74$, and 0.748. A fixed point is *unstable* if for almost all values of x_0 near the fixed point, the trajectories diverge from it. Verify that $x = 0$ is an unstable fixed point for $r > 0.25$. Show that for the suggested values of r, the iterated values of x do not change after an initial *transient*, that is, the long time dynamical behavior is *period* 1. In Appendix 6A we show that for $r < 3/4$ and for x_0 in the interval $0 < x_0 < 1$, the trajectories approach the *stable attractor* at $x = 1 - 1/4r$. The set of initial points that iterate to the attractor is called the *basin* of the attractor. For the logistic map, the interval $0 < x < 1$ is the basin of attraction of the attractor $x = 1 - 1/4r$.

c. Explore the dynamical properties of (6.5) for $r = 0.752, 0.76, 0.8$, and 0.862. For $r = 0.752$ and 0.862 approximately 1000 iterations are necessary to obtain convergent results. Show that if r is greater than 0.75, x oscillates between two values after an initial transient behavior. That is, instead of a stable cycle of period 1 corresponding to one fixed point, the system has a stable cycle of period 2. The value of r at which the single fixed point x^* splits or *bifurcates* into two values $x_1{}^*$ and $x_2{}^*$ is $r = b_1 = 3/4$. The pair of x values, $x_1{}^*$ and $x_2{}^*$, form a *stable attractor* of period 2.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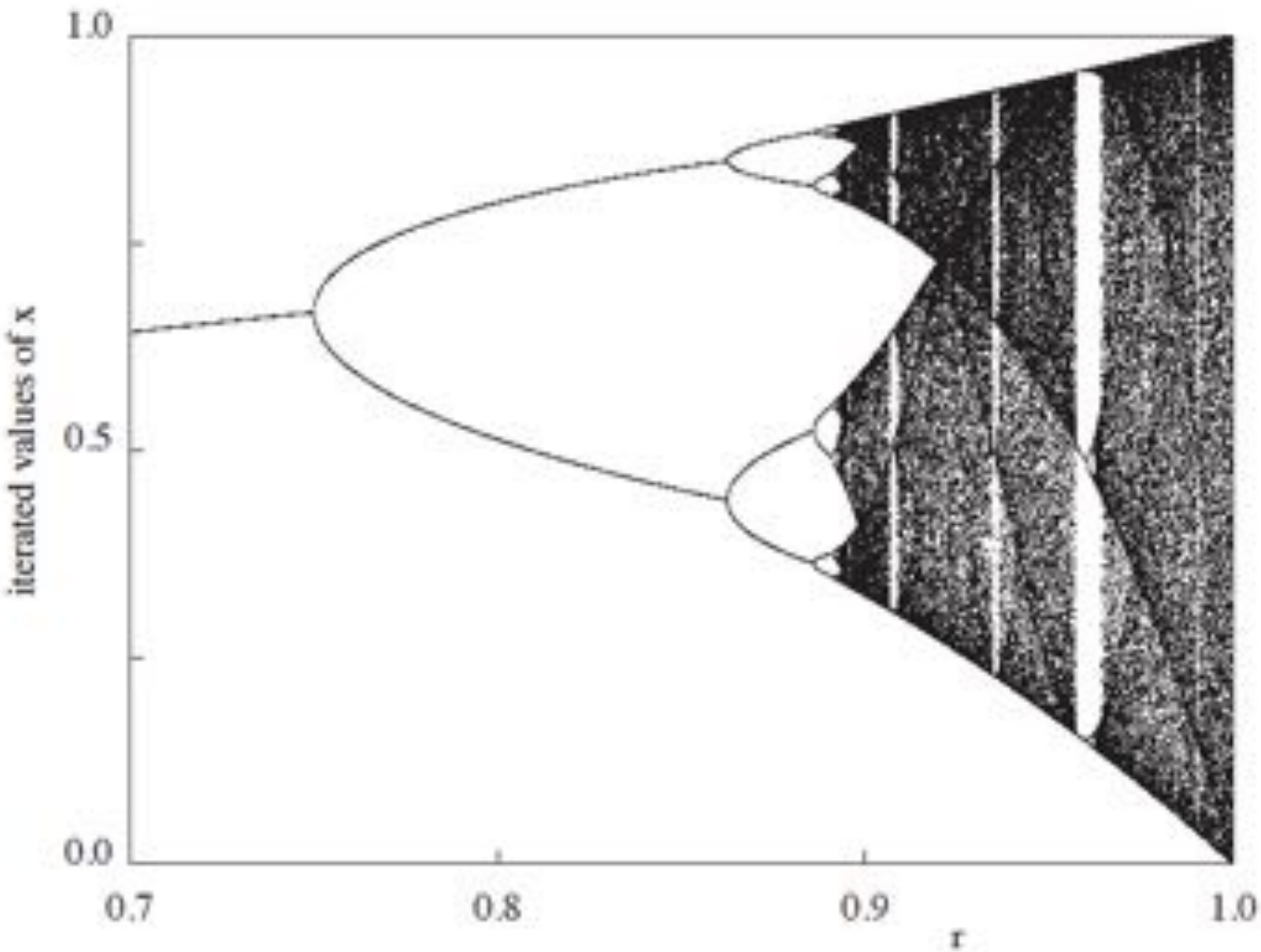


Figure 6.2: Bifurcation diagram of the logistic map. For each value of r, the iterated values of x_n are plotted after the first 1000 iterations are discarded. Note the transition from periodic to chaotic behavior and the narrow windows of periodic behavior within the region of chaos.

a. Use **BifurcateApp** to identify period 2, period 4, and period 8 behavior as can be seen in Figure 6.2. Choose **ntransient** ≥ 1000. It might be necessary to "zoom in" on a portion of the plot. How many period doublings can you find?

b. Change the scale so that you can follow the iterations of x from period 4 to period 16 behavior. How does the plot look on this scale in comparison to the original scale?

c. Describe the shape of the trajectory near the bifurcations from period 2 to period 4, period 4 to 8, etc. These bifurcations are frequently called *pitchfork bifurcations*.

The bifurcation diagram in Figure 6.2 indicates that the period doubling behavior ends at $r \approx 0.892$. This value of r is known very precisely and is given by $r = r_\infty = 0.892486417967\ldots$ At $r = r_\infty$, the sequence of period doublings accumulate to a trajectory of infinite period. In Problem 6.3 we explore the behavior of the trajectories for $r > r_\infty$.

Problem 6.3. Chaotic behavior

a. For $r > r_\infty$, two initial conditions that are very close to one another can yield very different trajectories after a few iterations. As an example, choose $r = 0.91$ and consider $x_0 = 0.5$ and 0.5001. How many iterations are necessary for the iterated values of x to differ by more than ten percent? What happens for $r = 0.88$ for the same choice of seeds?

b. The accuracy of floating point numbers retained on a digital computer is finite. To test the effect of the finite accuracy of your computer, choose $r = 0.91$ and $x_0 = 0.5$ and compute the trajectory for 200 iterations. Then modify your program so that after each iteration, the operation `x = x/10` is followed by `x = 10*x`. This combination of operations truncates the last digit that your computer retains. Compute the trajectory again and compare your results. Do you find the same discrepancy for $r < r_\infty$?

c. What are the dynamical properties for $r = 0.958$? Can you find other windows of periodic behavior in the interval $r_\infty < r < 1$?

Period Doubling

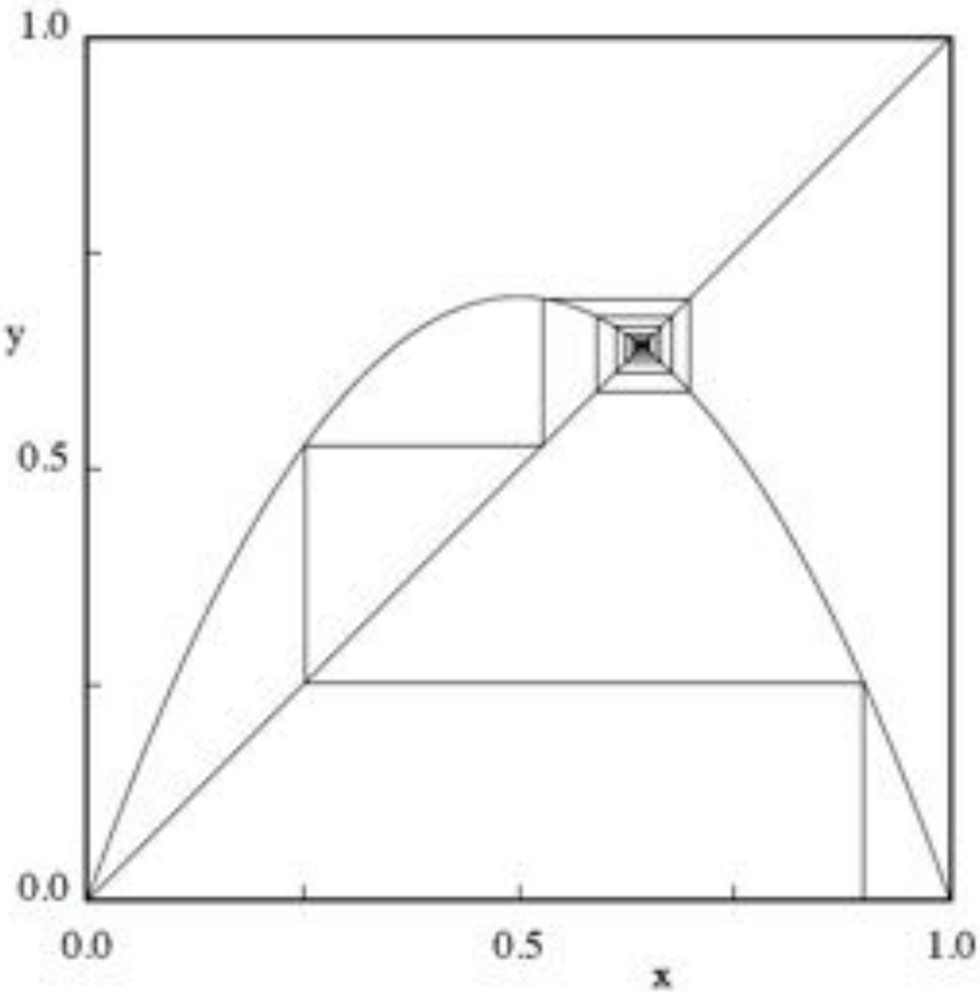
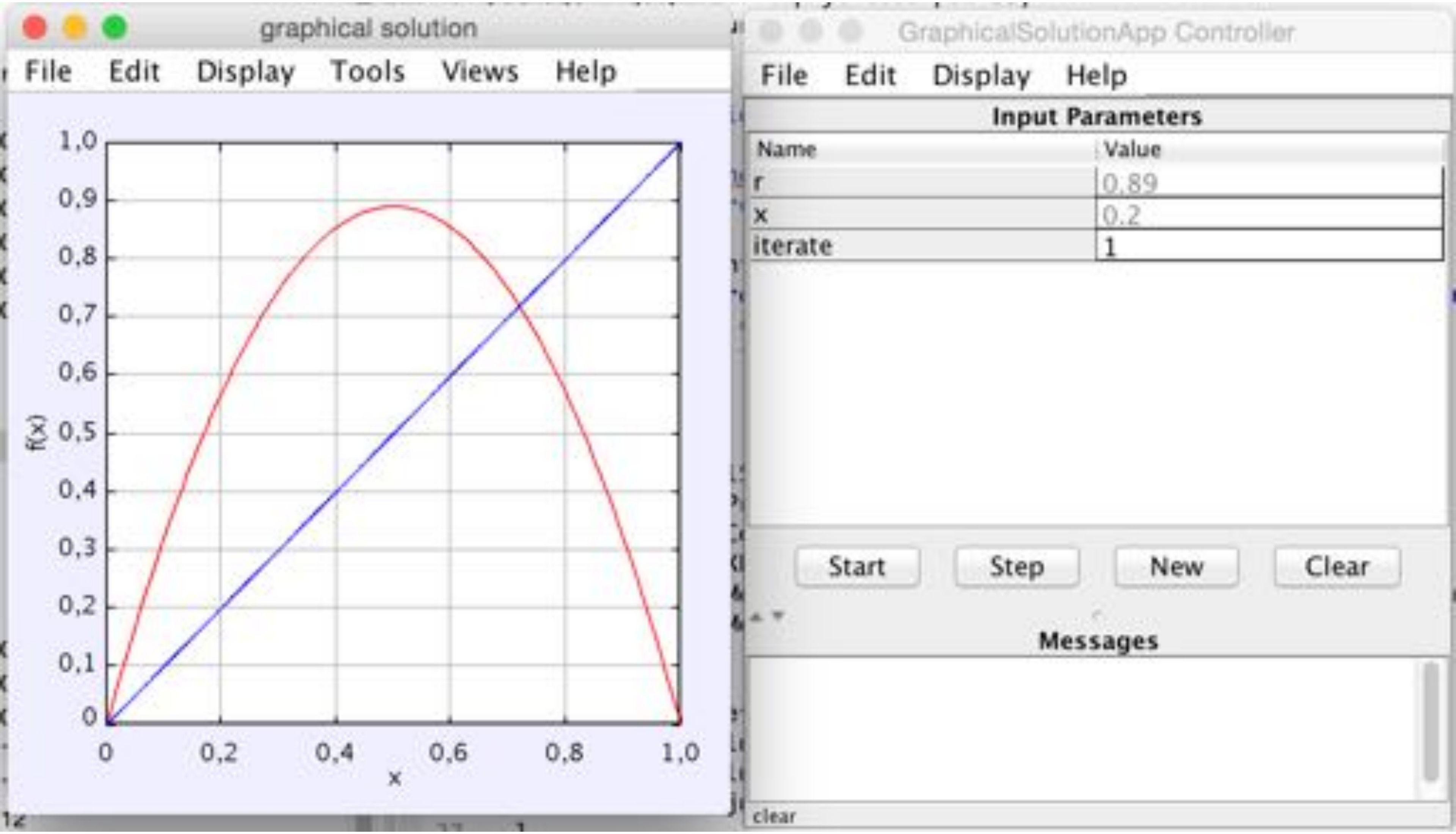


Figure 6.3: Graphical representation of the iteration of the logistic map (6.5) with $r = 0.7$ and $x_0 = 0.9$. Note that the graphical solution converges to the fixed point $x^* \approx 0.643$.

This graphical method is illustrated in Figure 6.3 for $r = 0.7$ and $x_0 = 0.9$. If we begin with any x_0 (except $x_0 = 0$ and $x_0 = 1$), the iterations will converge to the fixed point $x^* \approx 0.643$. It would be a good idea to repeat the procedure shown in Figure 6.3 by hand. For $r = 0.7$, the fixed point is stable (an attractor of period 1). In contrast, no matter how close x_0 is to the fixed point at $x = 0$, the iterates diverge away from it, and this fixed point is unstable.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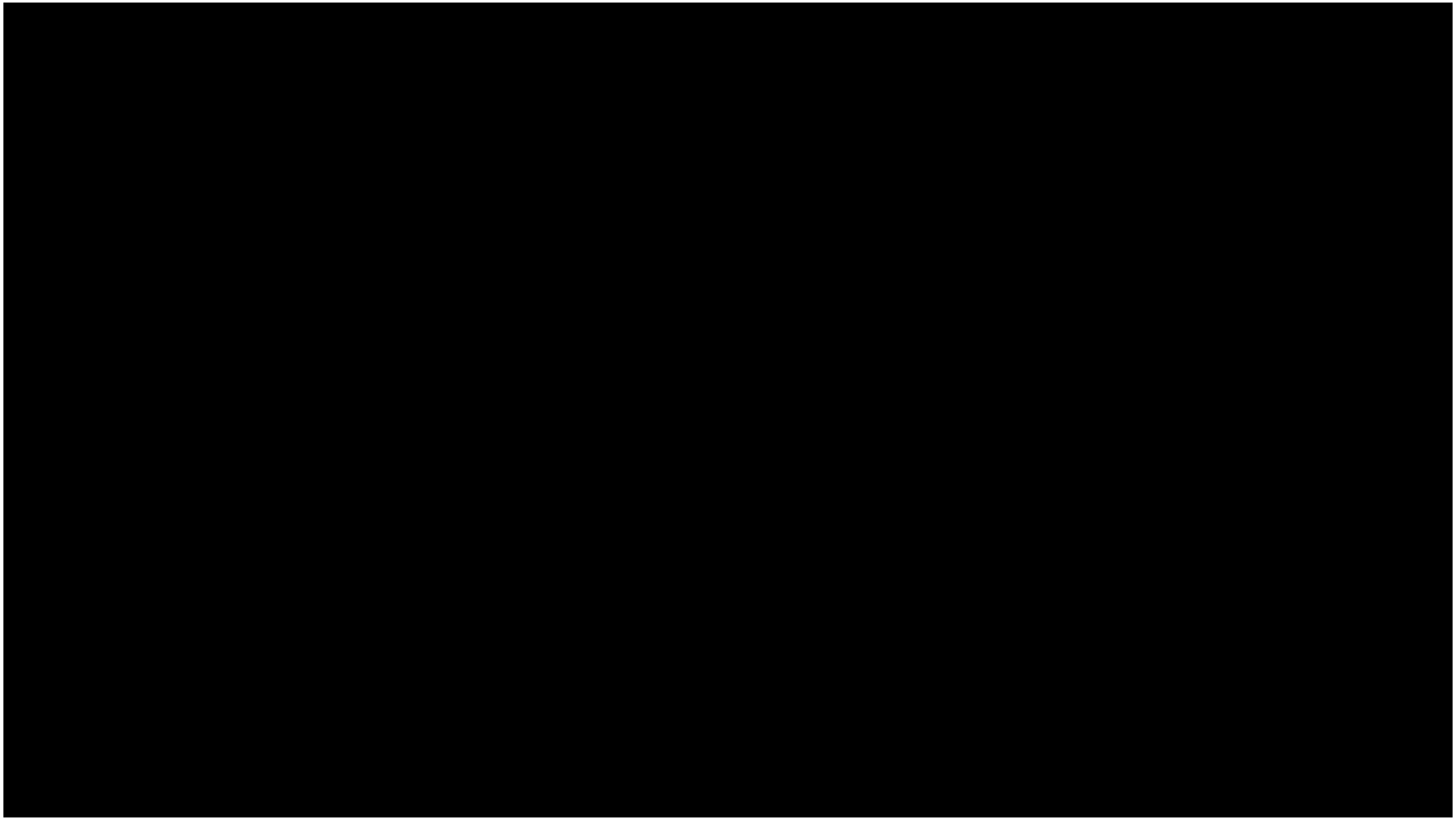
CHAOS AND PHYSICAL PHENOMENA



Higher-Dimensional Models

$$x_{n+1} = y_n + 1 - a{x_n}^2$$
$$y_{n+1} = bx_n.$$

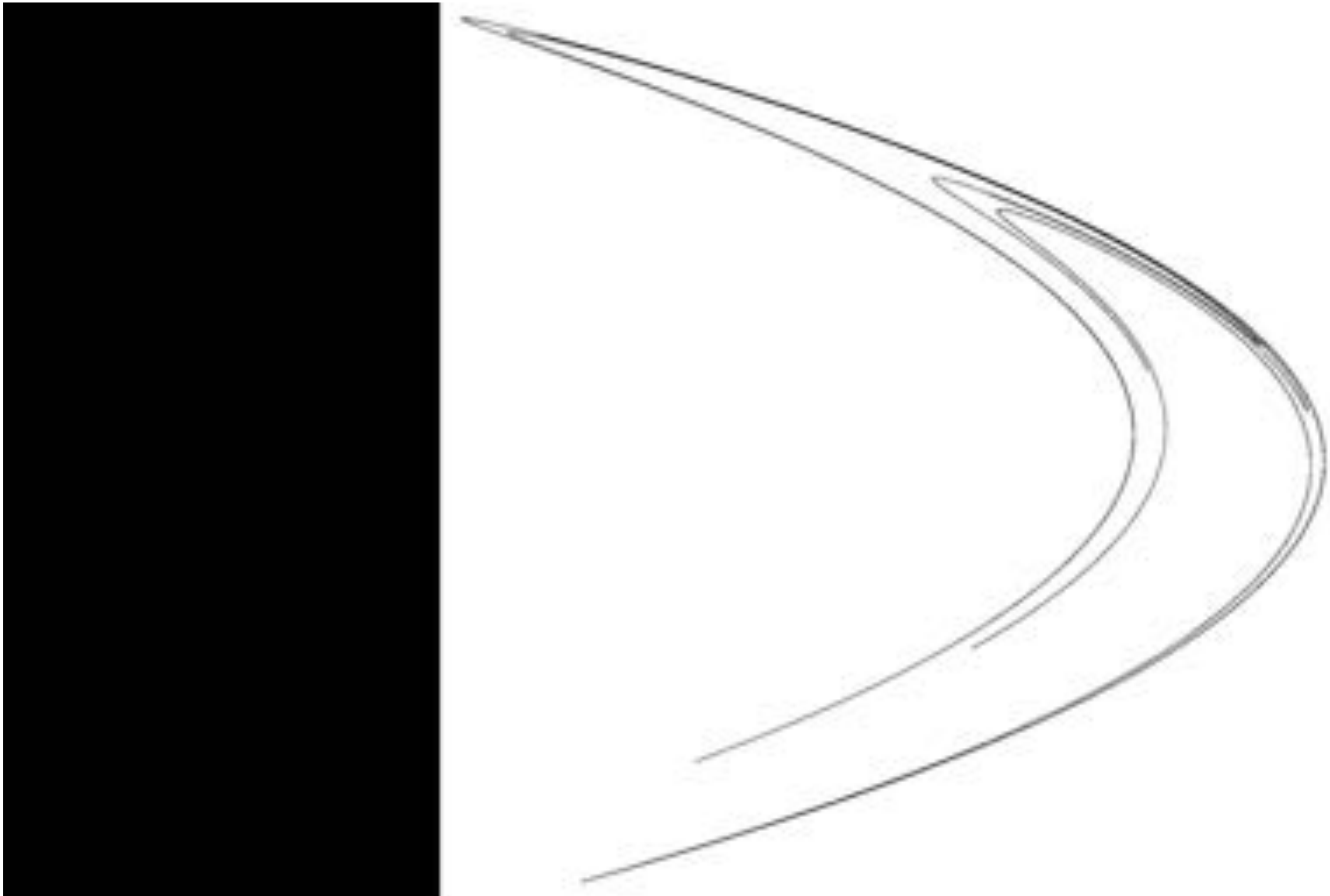
Henon maps

Problem 6.15. The Hénon map

a. Write a program to iterate (6.32) for $a = 1.4$ and $b = 0.3$ and plot 10^4 iterations starting from $x_0 = 0, y_0 = 0$. Make sure you compute the new value of y using the old value of x and not the new value of x. Do not plot the initial transient. Look at the trajectory in the region defined by $|x| \leq 1.5$ and $|y| \leq 0.45$. Make a similar plot beginning from the second initial condition, $x_0 = 0.63135448, y_0 = 0.18940634$. Compare the shape of the two plots. Is the shape of the two curves independent of the initial conditions?

b. Increase the scale of your plot so that all points in the region $0.50 \leq x \leq 0.75$ and $0.15 \leq y \leq 0.21$ are shown. Begin from the second initial condition and increase the number of computed points to 10^5. Then make another plot showing all points in the region $0.62 \leq x \leq 0.64$ and $0.185 \leq y \leq 0.191$. If time permits, make an additional enlargement and plot all points within the box defined by $0.6305 \leq x \leq 0.6325$ and $0.1889 \leq y \leq 0.1895$. You will have to increase the number of computed points to order 10^6. What is the structure of the curves within each box? Does the attractor appear to have a similar structure on smaller and smaller length scales? The region of points from which the points cannot escape is the basin of the Hénon attractor. The attractor is the set of points to which all points in the basin are attracted. That is, two trajectories that begin from different conditions will eventually lie on the attractor.

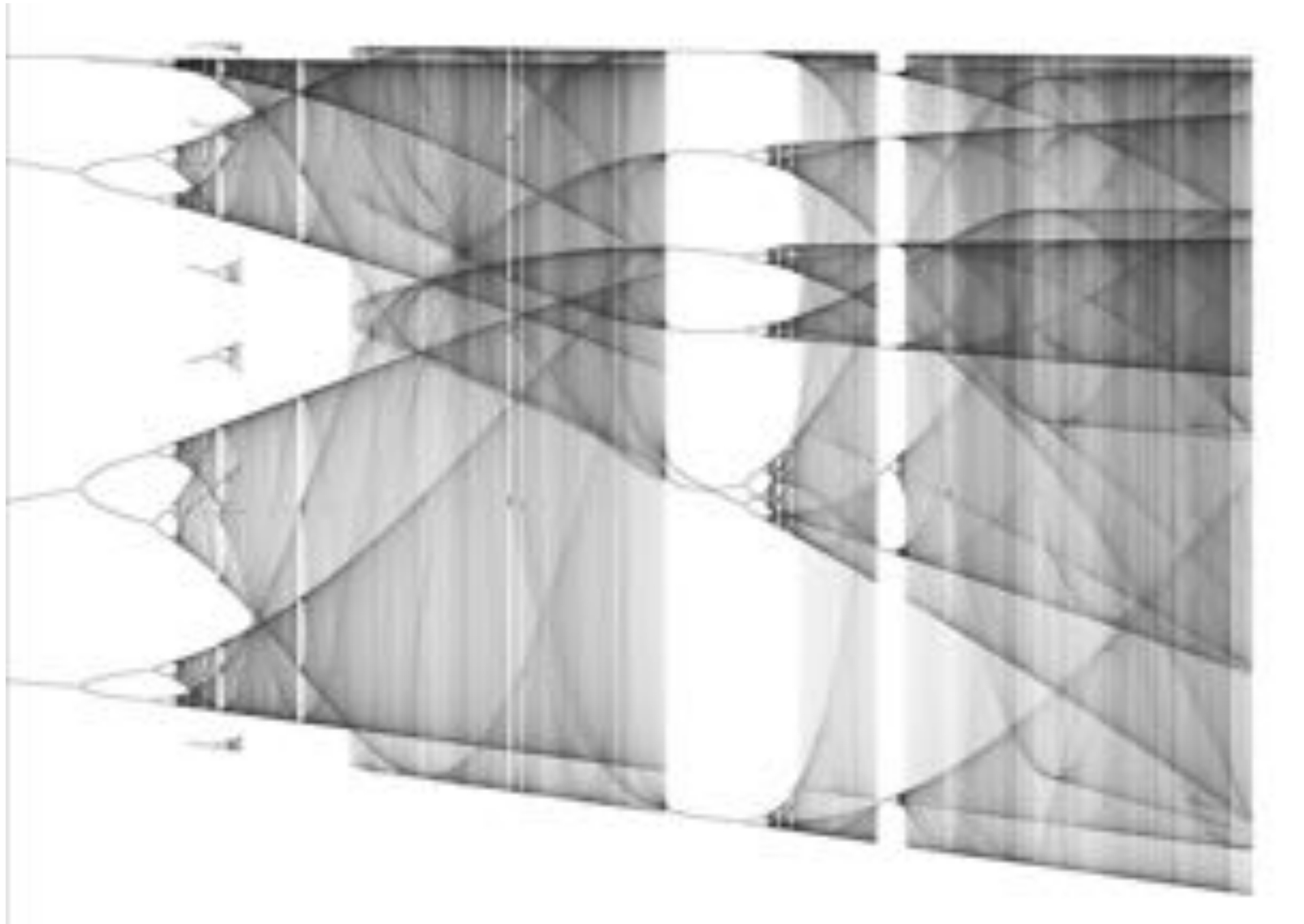
c. Determine if the system is chaotic, that is, sensitive to initial conditions. Start two points very close to each other and watch their trajectories for a fixed time. Choose different colors for the two trajectories.



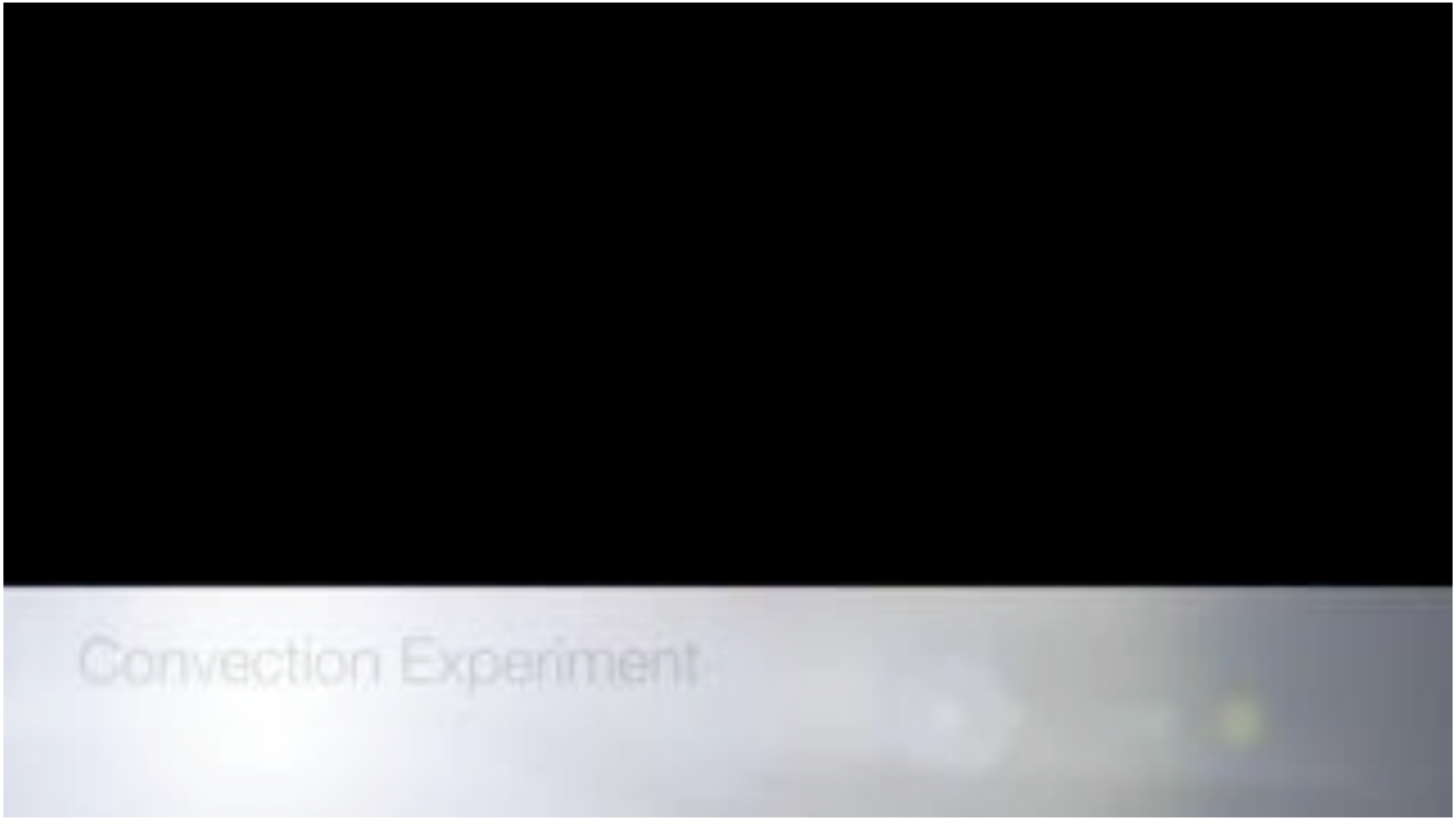
Hénon attractor for $a = 1.4$ and $b = 0.3$

https://en.wikipedia.org/wiki/H%C3%A9non_map

https://en.wikipedia.org/wiki/H%C3%A9non_map#/media/File:Henon_Multifractal_Map_movie.gif https://en.wikipedia.org/wiki/H%C3%A9non_map



FLUID DYNAMICS: CONVECTION


Convection Experiment

Higher-Dimensional Models

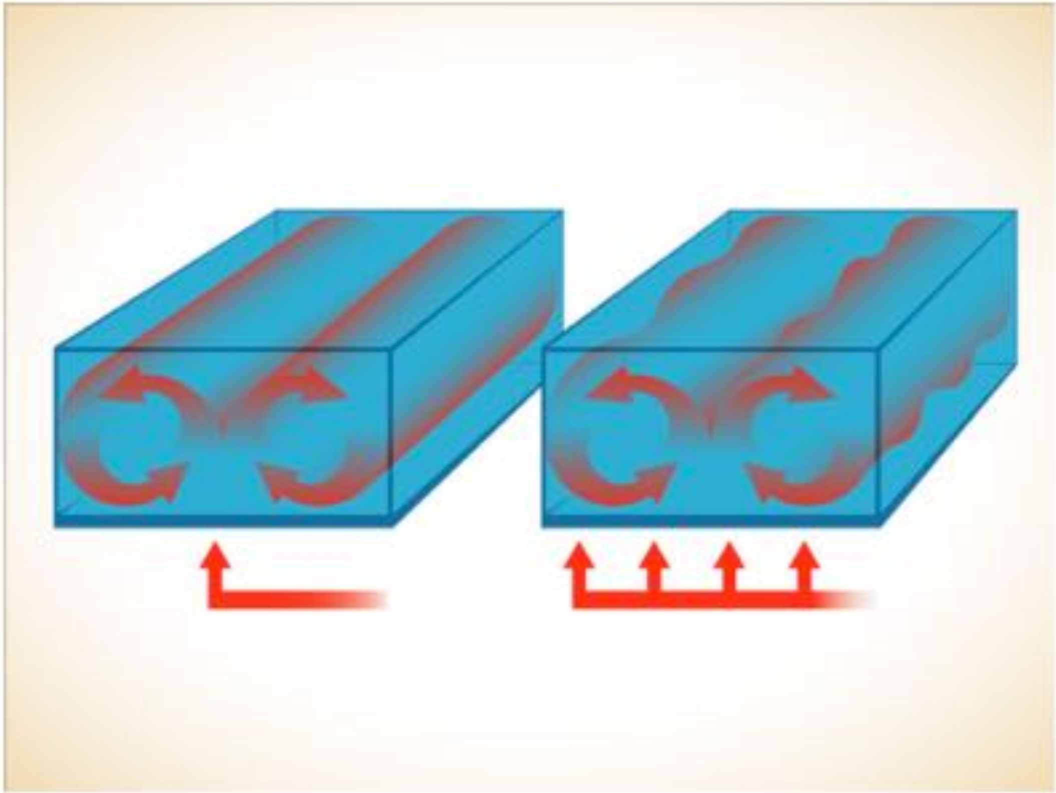
Lorenz maps

$$\frac{dx}{dt} = -\sigma x + \sigma y \tag{6.33a}$$

$$\frac{dy}{dt} = -xz + rx - y \tag{6.33b}$$

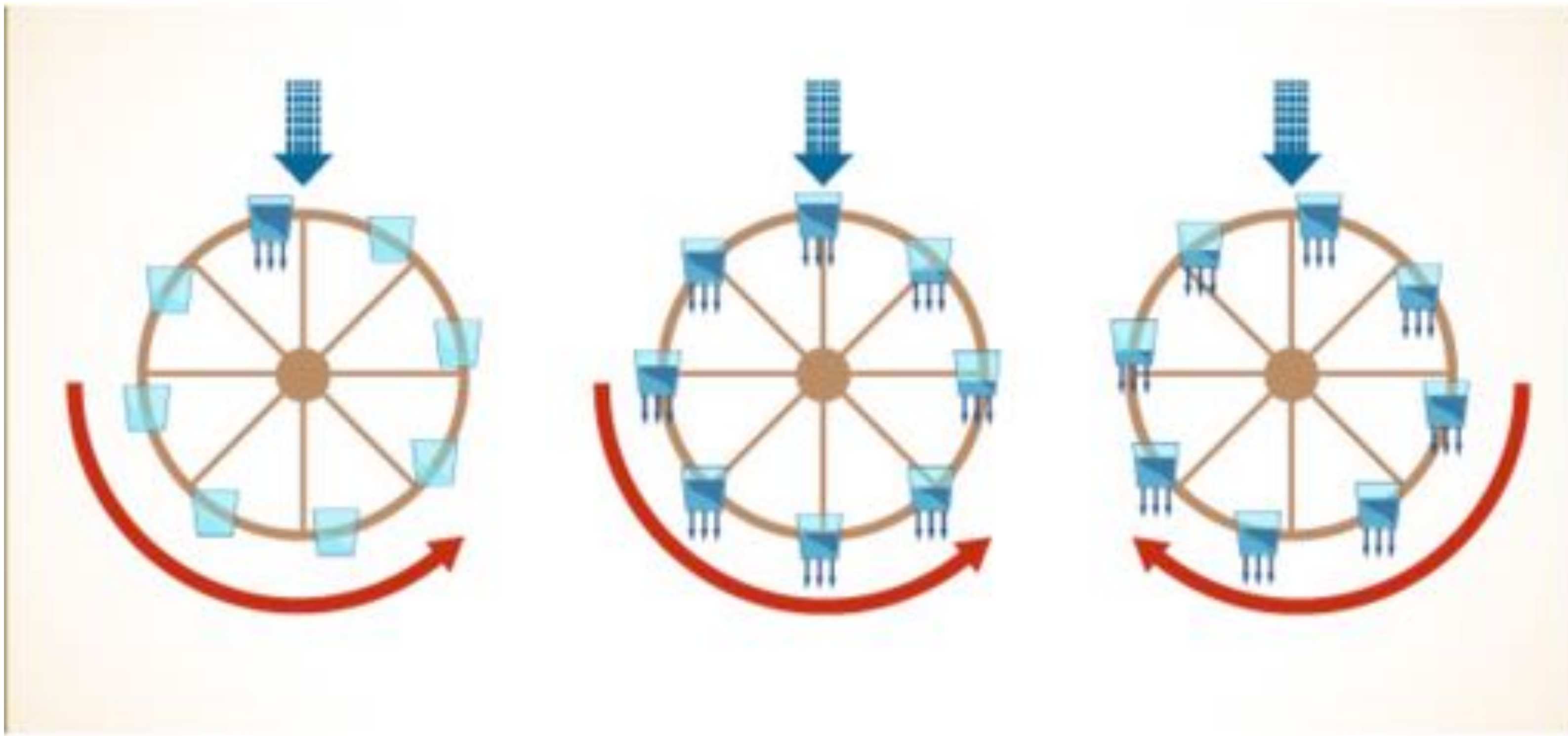
$$\frac{dz}{dt} = xy - bz, \tag{6.33c}$$

where x is a measure of the fluid flow velocity circulating around the cell, y is a measure of the temperature difference between the rising and falling fluid regions, and z is a measure of the difference in the temperature profile between the bottom and the top from the normal equilibrium temperature profile. The dimensionless parameters σ, r, and b are determined by various fluid properties, the size of the Raleigh-Benard cell, and the temperature difference in the cell. Note that the variables x, y, and z have nothing to do with the spatial coordinates, but are measures of the state of the system. Although it is not expected that you will understand the relation of the Lorenz equations to convection, we have included these equations here to reinforce the idea that simple sets of equations can exhibit chaotic behavior.



A ROLLING FLUID. When a liquid or gas is heated from below, the fluid tends to organize itself into cylindrical rolls *(left)*. Hot fluid rises on one side, loses heat, and descends on the other side—the process of convection. When the heat is turned up further (right), an instability sets in, and the rolls develop a wobble that moves back and forth along the length of the cylinders. At even higher temperatures, the flow becomes wild and turbulent.

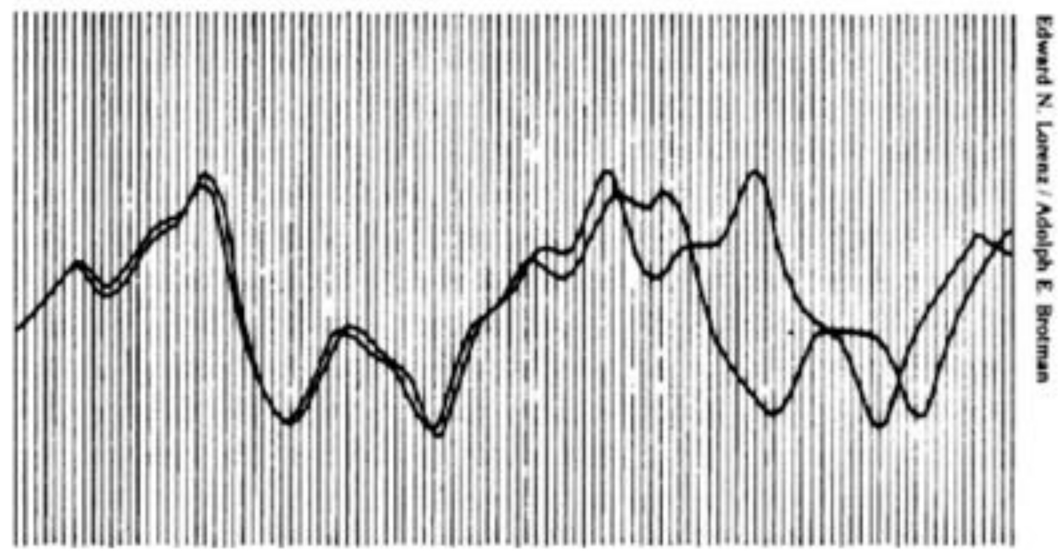
Simulation (https://www.youtube.com/watch?v=OM0l2YPVMf8)



Edward Norton Lorenz

Edward Norton Lorenz

Born	May 23, 1917 West Hartford, Connecticut, United States
Died	April 16, 2008 (aged 90) Cambridge, Massachusetts, United States
Residence	United States
Fields	Mathematics and Meteorology
Institutions	Massachusetts Institute of Technology
Alma mater	Dartmouth College (BA, 1938) Harvard University (MA, 1940) Massachusetts Institute of

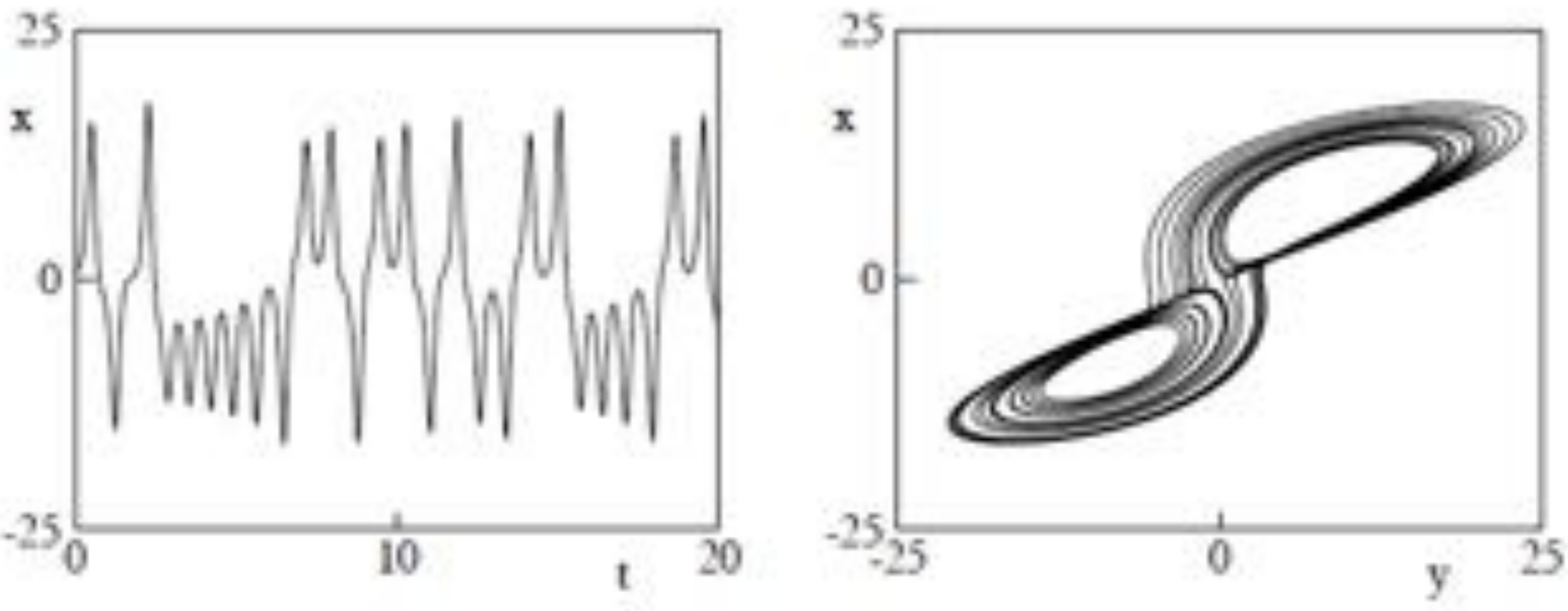


HOW TWO WEATHER PATTERNS DIVERGE. From nearly the same starting point, Edward Lorenz saw his computer weather produce patterns that grew farther and farther apart until all resemblance disappeared. (From Lorenz's 1961 printouts.)

The application of the Lorenz equations to weather prediction has led to a popular metaphor known as the *butterfly effect*. This metaphor is made even more meaningful by inspection of Figure 6.10. The "butterfly effect" often is ascribed to Lorenz (see Hilborn). In a 1963 paper he remarked that:

> "One meteorologist remarked that if the theory were correct, one flap of a seagull's wings would be enough to alter the course of the weather forever."

By 1972, the sea gull had evolved into the more poetic butterfly and the title of his talk was "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"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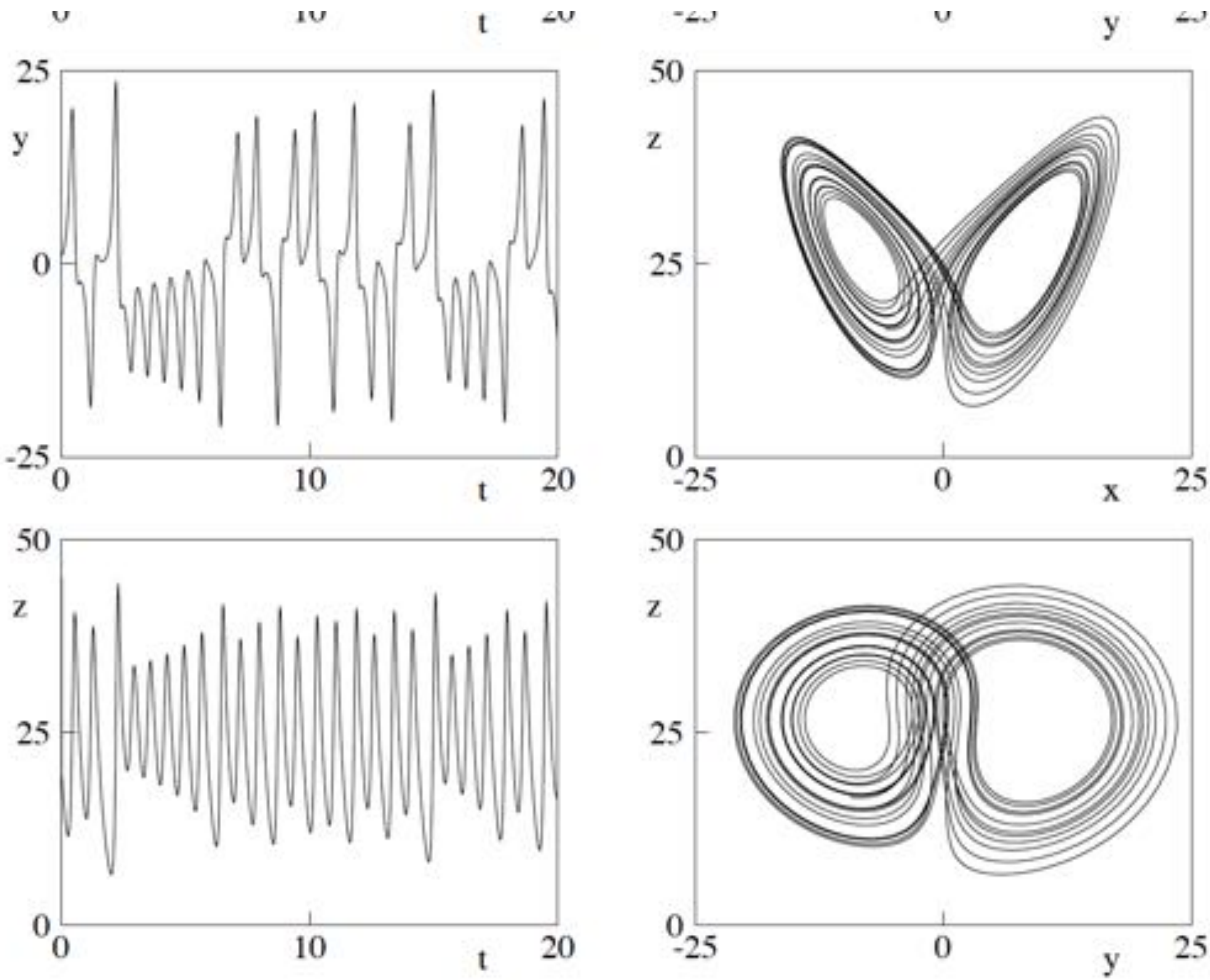

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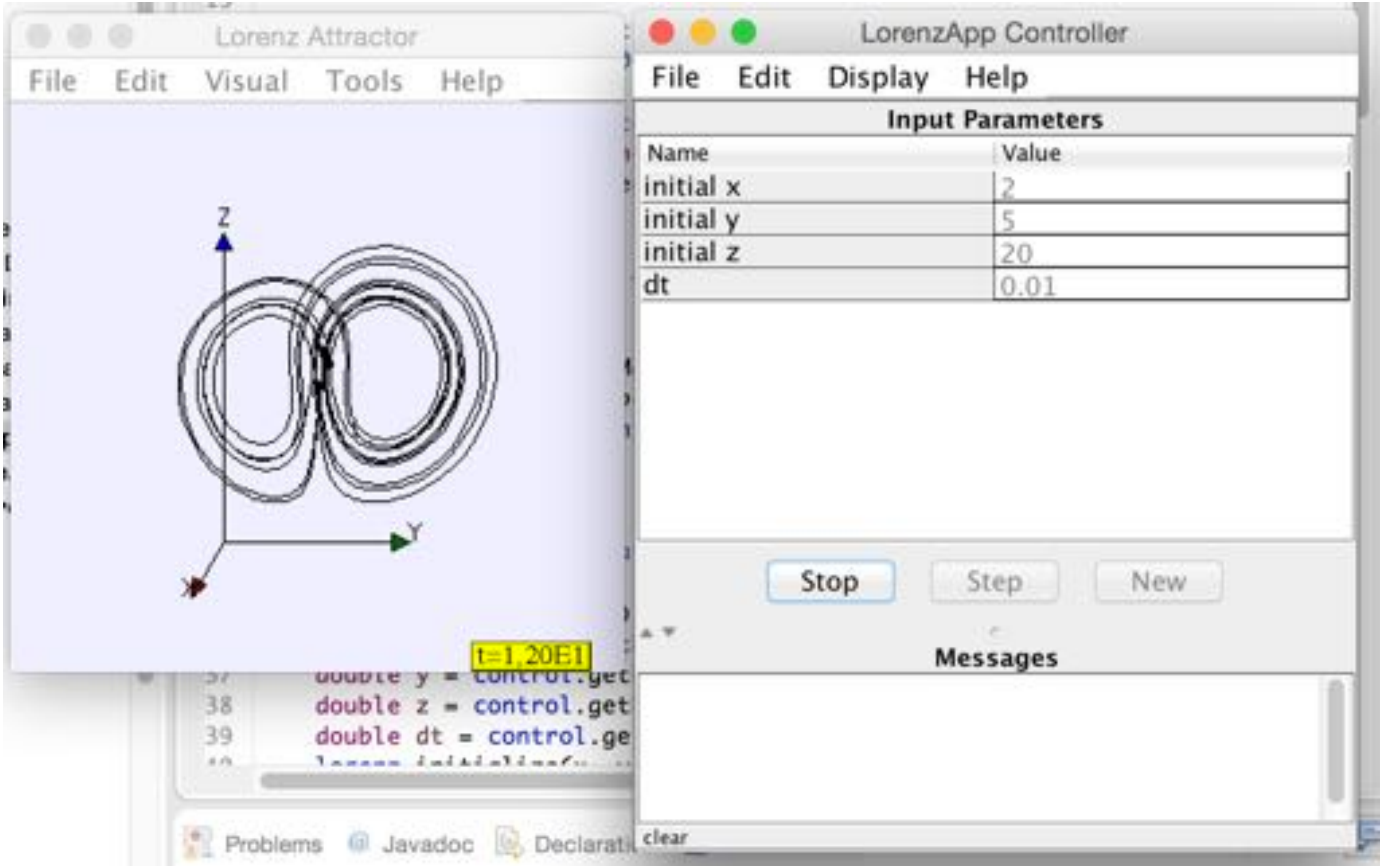

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38 double z = control.get
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Exercícios

- Implementar jupyters para exercícios 1, 2 e 3 da figura no próximo slide.


1 — $x_{n+1} = f(x_n) = 4r\, x_n(1 - x_n)$
2- Mapa de Lorenz
usando
3- Mapa
$y_{n+1} = b x_n$

Exercício 1

- Mapa logístico
- 3 tipos de gráficos, a), b), c) no quadro branco.
- Mapa de bifurcação, plotar x_min até x_max depois da convergência ao ponto fixo, eg x_9000 até x_10000

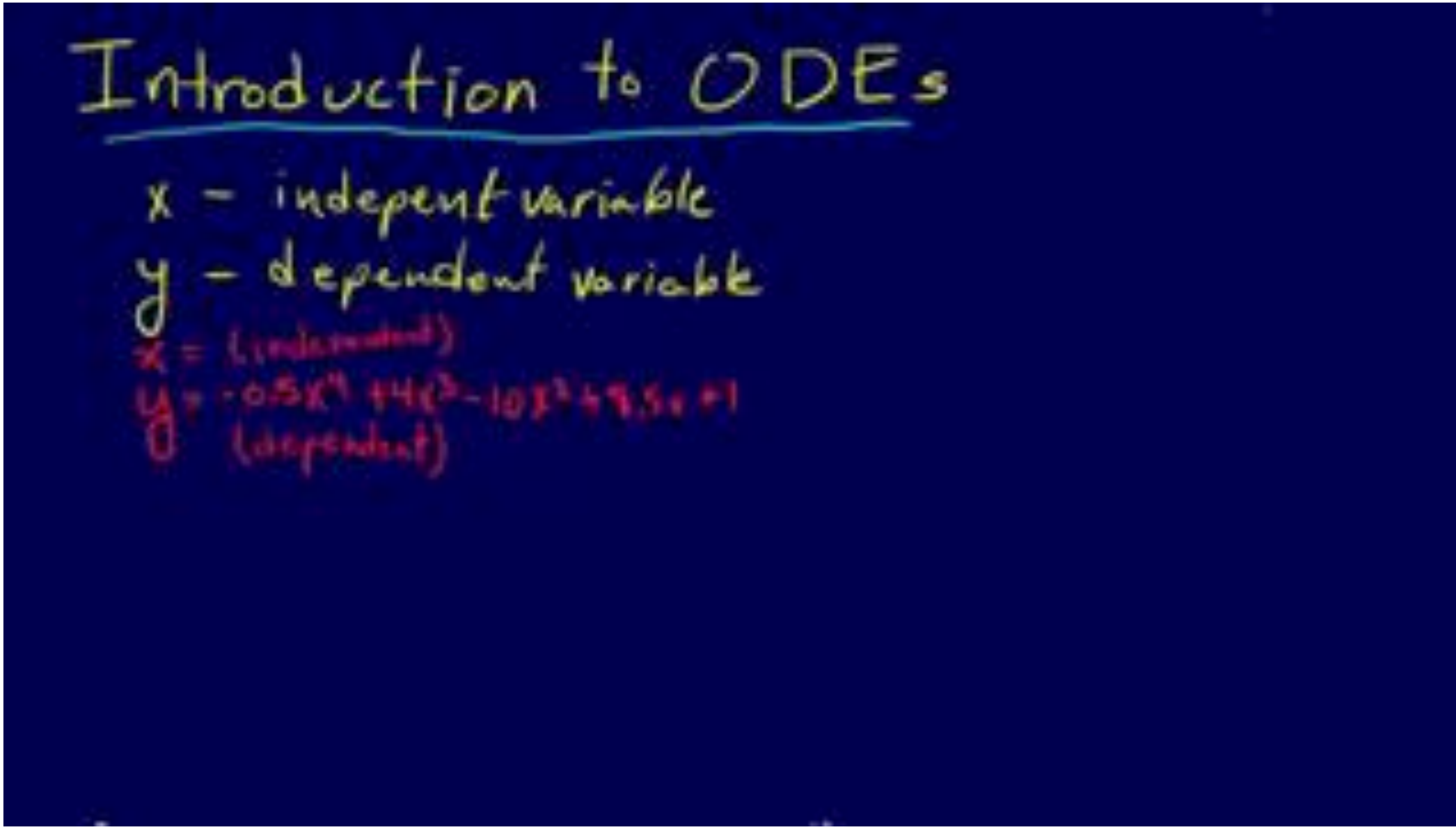
Exercício 2

- Equações de Lorenz usando Algoritmo de Euler
- 4 tipos de gráficos como no quadro do slide acima:
    a) x(t), y(t), z(t)
    b) x X y, x X z, y X z
    c) x X y X z
    d) x,y X t, x,z X t, y,z X t

Exercício 3

- Mapa de Lorenz, dois tipos de gráficos:
  a) x(n), y(n), x X y
  b) Mapa de bifurcações como o mapa logístico

RUNGE-KUTTA



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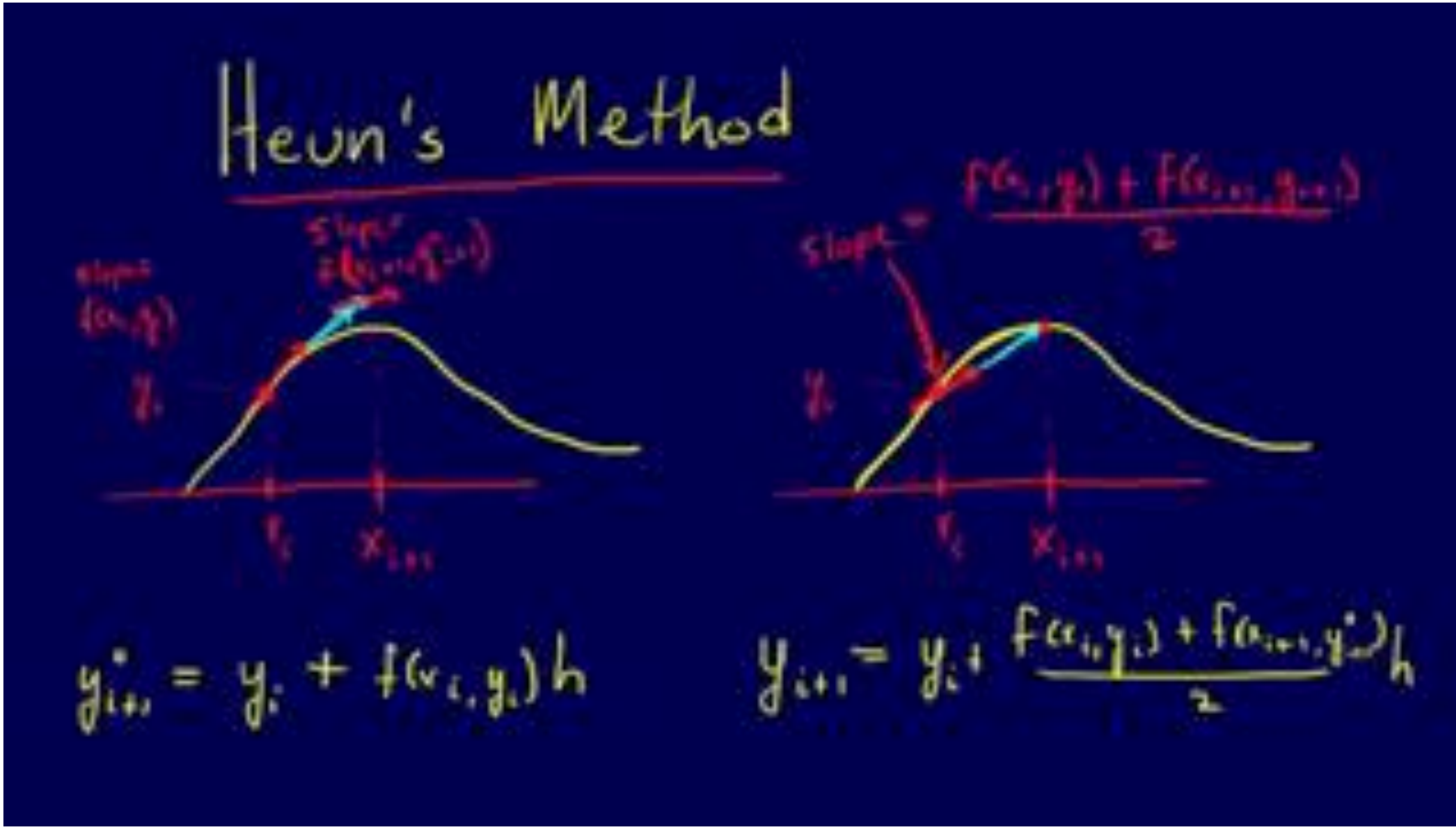

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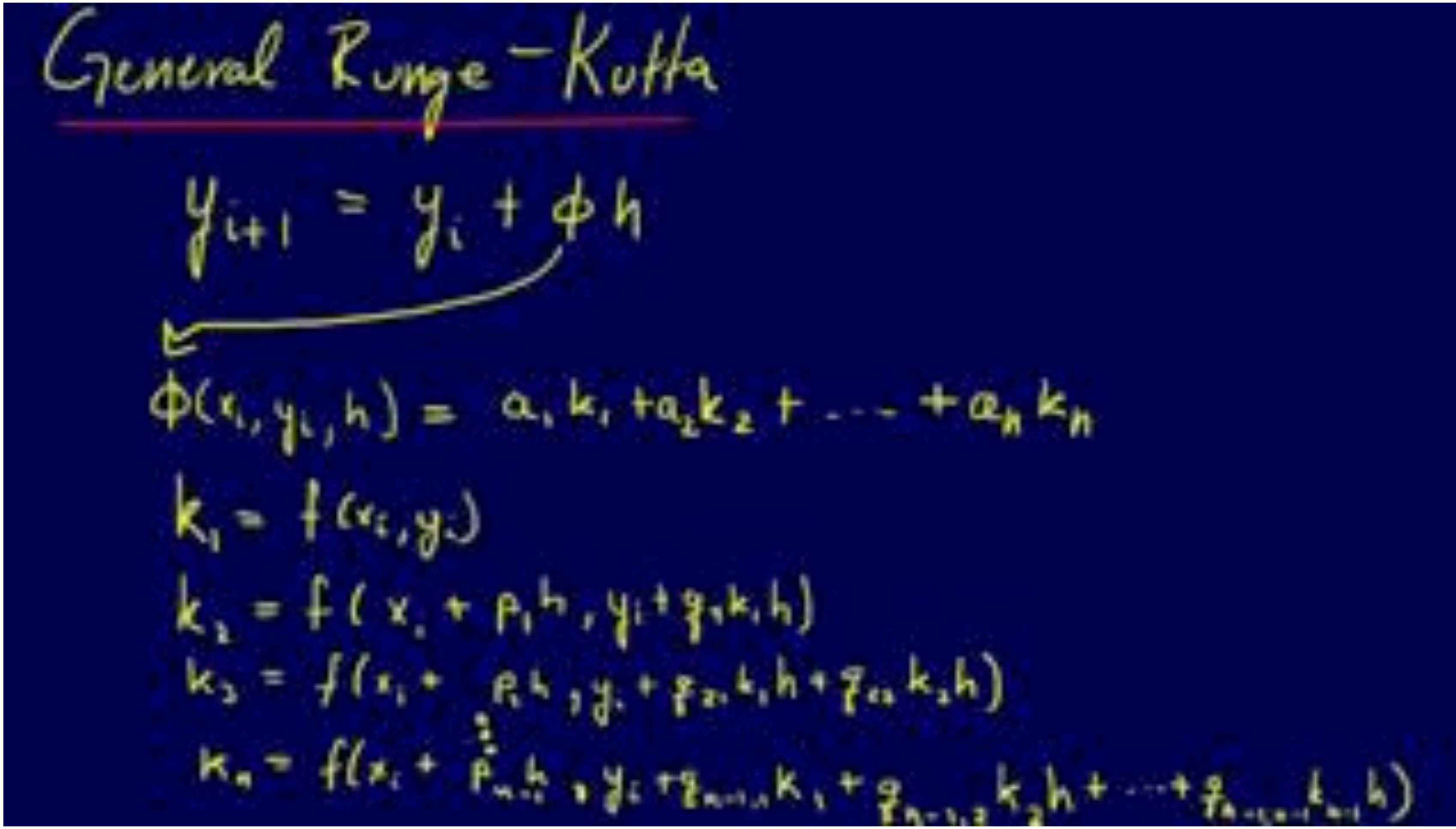

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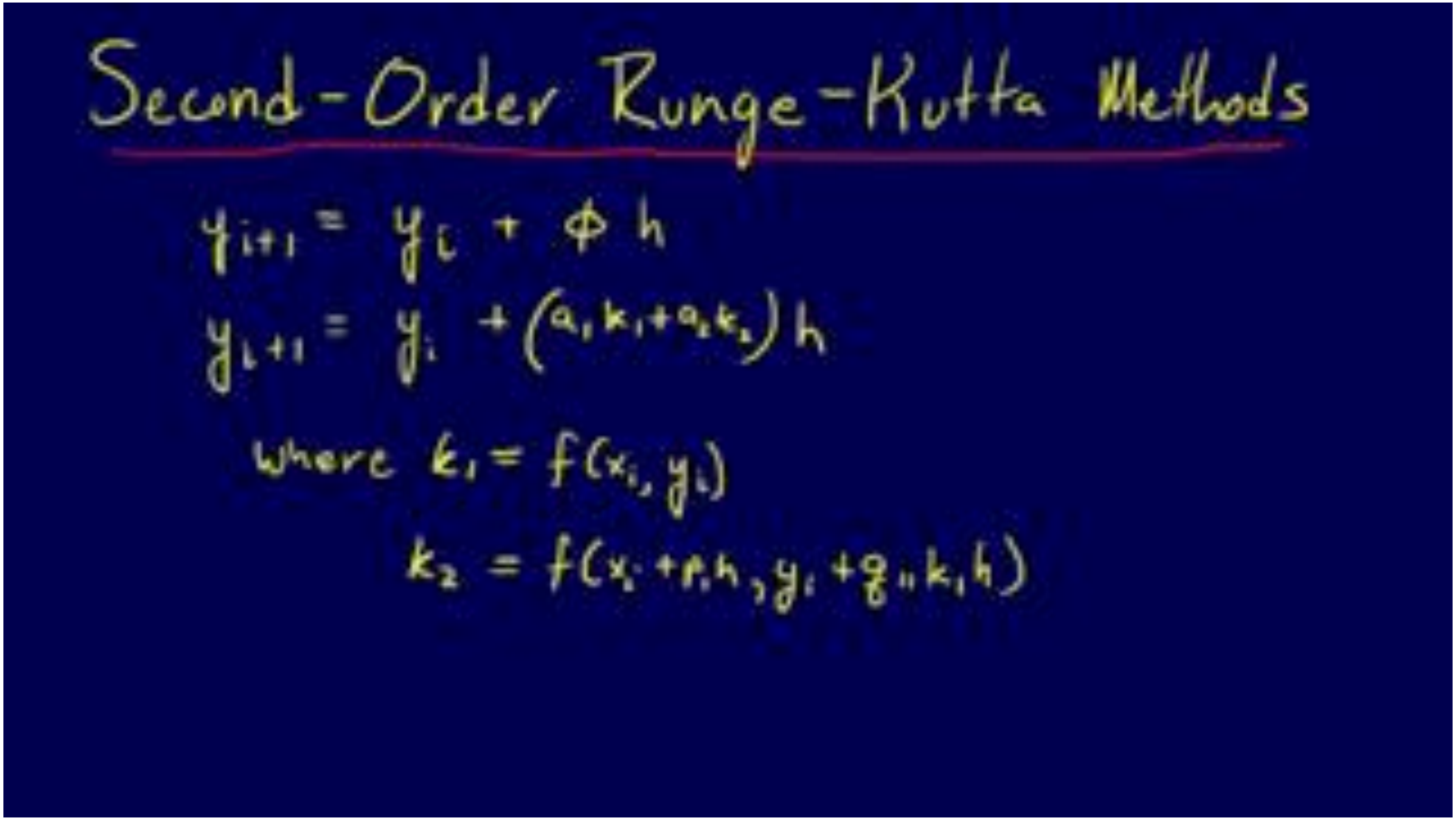

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Runge-Kutta

$$\frac{dv}{dt} = a(t), \tag{3.31a}$$

$$\frac{dx}{dt} = v(t), \tag{3.31b}$$

$$v_{n+1} = v_n + a_n \Delta t + O\big((\Delta t)^2\big), \tag{3.32a}$$

$$x_{n+1} = x_n + v_n \Delta t + \frac{1}{2} a_n (\Delta t)^2 + O\big((\Delta t)^3\big). \tag{3.32b}$$

Runge-Kutta

$$v_{n+1} = v_n + a_n \Delta t \tag{3.33a}$$

$$x_{n+1} = x_n + v_n \Delta t. \qquad \text{(Euler algorithm)} \tag{3.33b}$$

$$v_{n+1} = v_n + a_n \Delta t, \tag{3.34a}$$

$$x_{n+1} = x_n + v_{n+1} \Delta t. \qquad \text{(Euler-Cromer algorithm)} \tag{3.34b}$$

Runge-Kutta

$$v_{n+1} = v_n + a_n \Delta t, \tag{3.35a}$$

$$x_{n+1} = x_n + \frac{1}{2}(v_{n+1} + v_n)\Delta t. \qquad \text{(midpoint algorithm)} \tag{3.35b}$$

Note that if we substitute (3.35a) for v_{n+1} into (3.35b), we obtain

$$x_{n+1} = x_n + v_n \Delta t + \frac{1}{2} a_n \, \Delta t^2. \tag{3.36}$$

Runge-Kutta

The most common finite difference method for solving ordinary differential equations is the *Runge-Kutta* method. To explain the many algorithms based on this method, we consider the solution of the first-order differential equation

$$\frac{dx}{dt} = f(x, t). \tag{3.52}$$

$$y_{n+1} = y_n + (c_1 r_1 + c_2 r_2 + c_3 r_3 + c_4 r_4)\Delta t. \tag{3.53}$$