MAC0325 Combinatorial Optimization
Midterm

Pedro Gigeck Freire
10737136

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Exercise 1

Let P^* be an optimal solution for (1.1), i.e., an (r, s)-path in D of minimum cost. Let $k^* \coloneqq c(P^*)$ be the optimal value of (1.1).

Define the function

$$R : [k^*] \to \mathcal{P}(V)$$

$$i \mapsto R_i \coloneqq \{v \in V : \text{ there is a } (r, v)\text{-path } P \text{ in } D \text{ such that } c(P) < i\}$$

Informally, $v \in R_i$ means that the cost between r and v is smaller then i.

Proposition 1. *Let $i \in [k^*]$ and let $v \in R_i$. Then $v \in R_j$ for any $i \leq j \leq k^*$*

Proof. Since $v \in R_i$ there is a (r, v)-path P with $c(P) < i$. But then, if $j \geq i$, $c(P) < j$, thus $v \in R_j$ □

Proposition 2. *Let $j, t \in \mathbb{N}$ with $j, t \geq 1$, let $a \in A$ such that $a \in \delta^{out}(R_t)$ and let $(u, v) \coloneqq \varphi(a)$. Then*

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{out}(R_i)}(a) = j \implies a \in \delta^{out}(R_i) \textit{ for each } i \in \{t - j + 1, ..., t\}$$

Proof. Since $a \in \mathbb{1}_{\delta^{\text{out}}(R_t)}$ we have that $u \in R_t$ and $v \notin R_t$. And $v \notin R_i$ for any $i \in [t]$, because if v was in any R_i then it would be in R_t by Proposition 1.

Take the set $S \coloneqq \{\delta^{\text{out}}(R_{t-j+2}), ..., \delta^{\text{out}}(R_t)\}$, we have that $|S| = j - 1$, so if a was only in the sets of S, the sum of incidence vectors would be at most $j - 1$.

Thus there is at least one R_k with $k \leq t - j + 1$ such that $a \in \delta^{\text{out}}(R_k)$, but then $u \in R_k$, and by Proposition 1, for every $i \geq t - j + 1 \geq k$, $u \in R_i$. But we know that $v \notin R_i$, so $a \in \delta^{\text{out}}(R_i)$. And this proves our proposition. □

Lemma 1. *k^* is a feasible solution of (1.2).*

Proof. Since all arcs have natural costs, we have that

$$k^* \in \mathbb{N} \tag{1.3}$$

Moreover, take the (r, r)-path $\langle r \rangle$ with cost $0 < i$, so that $r \in R_i$. And suppose that $s \in R_i$ than there would be a (r, s)-path P' with cost $c(P') < i \leq k^* = c(P^*)$ which contradicts the optimality of P^*, thus $s \notin R_i$. Indeed,

$$r \in R_i \text{ and } s \notin R_i \text{ for each } i \in [k^*] \tag{1.4}$$

Now, let $t \in \mathbb{N}$. We will prove by induction on t that

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

The base case t $= 0$ holds because there is no $i \in [0]$, then

$$\sum_{i \in [0]} \mathbb{1}_{\delta^{\text{out}}(R_i)} = 0 \leq c$$

(Induction Hypothesis) Suppose that

$$\sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c$$

Now, suppose, by the sake of contradiction, that there is at least one arc $a \in A$, with $(u, v) \coloneqq \varphi(a)$, such that

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) > c(a)$$

We have

$$\begin{aligned}
0 &\leq c(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \text{ (by induction hypothesys)} \\
&< \sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \\
&= \mathbb{1}_{\delta^{\text{out}}(R_t)}(a) \\
&\leq 1
\end{aligned}$$

Summing up, we got

$$0 < \mathbb{1}_{\delta^{\text{out}}(R_t)}(a) \leq 1 \implies a \in \delta^{\text{out}}(R_t)$$

And

$$0 \leq c(a) - \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) < 1 \implies$$

$$c(a) = \sum_{i \in [t-1]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \implies$$

$$c(a) + 1 = \sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a)$$

Finally, by Proposition 2, taking $j = c(a)+1$ we have that $a \in \delta^{\text{out}}(R_{t-(c(a)+1)+1})$. Which means that $u \in R_{t-c(a)}$, so there is a (r, u)-path P of cost $c(P) < t - c(a)$

So we can create the (r, v)-path $P' \coloneqq P \cdot \langle u, a, v \rangle$ of cost

$$c(P') = c(P) + c(a) < t - c(a) + c(a) = t$$

This shows that $v \in R_t$, a contradiction because $a \in \delta^{\text{out}}(R_t)$.
Hence, there can not be any arc $a \in A$ with

$$\sum_{i \in [t]} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) > c(a)$$

So,

$$\sum_{i \in [k^*]} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c \tag{1.5}$$

Hence, by (1.3), (1.4) and (1.5), k^* is a feasible solution of problem (1.2). □

Theorem 1. *k^* is the optimal value of (1.2).*

Proof. Let $k' \in \mathbb{N}$ be the optimal value of (6.2). Let $R' : [k'] \to \mathcal{P}(V)$ be a function that satisfies the constraints of (6.2). In particular

$$\sum_{i \in [k']} \mathbb{1}_{\delta^{\text{out}}(R_i)} \leq c \tag{1.6}$$

Since $r \in R'_i$ and $s \notin R'_i$ for each $i \in [k']$, then any (r, s)-path P in D has one arc traversed by R_i, i.e, there is $a \in A(P)$ such that $a \in \delta^{\text{out}}(R_i)$ (Exercise 3.1 from lectures). This implies that

$$\begin{aligned}
k^* = c(P^*) &= \sum_{a \in A(P^*)} c(a) \\
&\geq \sum_{a \in A(P^*)} \sum_{i \in [k']} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) && \text{(by (1.6))} \\
&\geq \sum_{a \in A(P^*)} \sum_{(i \in [k'],\ a \in \delta^{\text{out}}(R_i))} \mathbb{1}_{\delta^{\text{out}}(R_i)}(a) \\
&\geq k' && \text{(by (exercise 3.1))}
\end{aligned}$$

So $k' \leq k^*$ and we know by Lemma 1 that k^* is a feasible value of (6.2). Hence k^* is the optimal value of (6.2). □

Exercise 2

(i)

Proposition 3. *Let $S \subseteq V$ be a stable set. Then $V \setminus S$ is a vertex cover of G.*

Proof. Suppose, by the sake of contradiction that $V \setminus S$ is not a vertex cover, i.e., there is an edge $e \in E$ such that $e \cap (V \setminus S) = \emptyset$. Then we can use some set manipulation and get (in the right there is the manipulation used)

$$\begin{aligned}
e \cap (V \setminus S) = \emptyset &\implies \\
(e \cap V) \setminus (e \cap S) = \emptyset &\implies && (A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)) \\
e \setminus (e \cap S) = \emptyset &\implies \\
(e \setminus e) \cup (e \setminus S) = \emptyset &\implies && (A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)) \\
e \setminus S = \emptyset &\implies \\
e \subseteq S &\implies \\
e \in E[S]
\end{aligned}$$

But this last statement contradicts the fact that S is stable. Thus $V \setminus S$ is a vertex cover.

□

Proposition 4. *Let $K \subseteq V$ be a vertex cover of G. Then $S := V \setminus K$ is a stable set.*

Proof. The proof is the other direction of the proof¨ we just made. Suppose, by the sake of contradiction that S is not a stable set, i.e., there is an edge $e \in E[S]$. Then

$$\begin{aligned}
e \in E[S] &\implies \\
e \subseteq S &\implies \\
e \setminus S = \emptyset &\implies \\
(e \setminus e) \cup (e \setminus S) = \emptyset &\implies \\
e \setminus (e \cap S) = \emptyset &\implies && (A \setminus B) \cup (A \setminus C) = (A \setminus (B \cap C)) \\
(e \cap V) \setminus (e \cap S) = \emptyset &\implies && (e = e \cap V) \\
e \cap (V \setminus S) = \emptyset &\implies && ((A \cap B) \setminus (A \cap C)) = A \cap (B \setminus C) \\
e \cap (V \setminus (V \setminus K)) = \emptyset &\implies \\
e \cap (V \cap K) = \emptyset &\implies && (A \setminus (A \setminus B) = A \cap B) \\
e \cap K = \emptyset
\end{aligned}$$

But this last statement contradicts the fact that K is a vertex cover. Thus S is a stable set. □

Theorem 2. $\alpha(G) + \tau(G) = |V|$

Proof. Let K^* be a vertex cover of size $\tau(G)$. Let $S^* := V \setminus K^*$. By Proposition 4, S^* is a stable set and its size is $|S^*| = |V| - \tau(G)$.

Let S' be a maximum stable set, i.e., $|S'| = \alpha(G)$. Now let $K' := V \setminus S'$. By Proposition 3, K' is a vertex cover.

Since $K*$ is minimum

$$
\begin{aligned}
|K'| \geq |K^*| &\implies \\
|K'| - |V| \geq |K^*| - |V| &\implies \\
|V| - |K'| \leq |V| - |K^*| &\implies \\
|S'| \leq |S^*| &
\end{aligned}
$$

But S' is the maximum stable set, so $|S^*| = \alpha(G)$, then $\alpha(G) = |V| - \tau(G) \implies \alpha(G) + \tau(G) = |V|$. □

(ii)

Since G is a bipartite graph, by Konig's matching theorem (8.8 from lectures) we have that $\nu(G) = \tau(G)$.

We claim that $\rho(G) = \alpha(G)$.

Let $S \subseteq V$ be a stable set of size $\alpha(G)$ and let $F \subseteq E$ be an edge cover of size $\rho(G)$ (we know it exists because there is no isolated vertex).

For each vertex $v \in S$ there is one edge $f \in F$ such that $v \in f$ but every distinct vertex require one distinct edge in the cover, because S is stable, so there is no edge that covers two vertices in S. Thus $|F| \geq |S|$.

Now suppose there are two edges $e, f \in F$ that covers the same vertex in S, say $e := \{s, k\}, f := \{s, l\}$, for $s \in S$ and $k, l \notin S$

By Lemma 1 we know that $V \setminus S$ is a minumum vertex cover, but then we could substitute $\{k, l\}$ by just $\{s\}$ and obtain a smaller vertex cover, which is a contradiction.

Thus there is one and only one edge in F for each vertex in S. And then $|F| = |S|$, i.e., $\rho(G) = \alpha(G)$ and then $\nu(G) + \rho(G) = \tau(G) + \alpha(G) = |V|$, by item (i).

(iii)

(I ran out of time)

Exercise 3

Necessity:

Let $\{M_1, M_2, ..., M_k\}$ be k disjoint perfect matchings in G. Then each vertex $v \in V$ is saturated by each one of the M_i matchings by one distinct edge. It means that there are at least k edges that saturates v, i.e.,

$$|\delta(\{v\})| \geq k \text{ for each } v \in V \tag{3.1}$$

For any $R \in V$ we have that the number of edges of $\delta(R)$ is equal to the number of edges that leaves (or we could say enter) any vertex, except for the edges that joins two vertices in S, i.e.,

$$|\delta(R)| = \sum_{v \in S} |\delta(\{v\})| - |E[S]| \text{ for each } S \subseteq V \tag{3.2}$$

From this, taking $R = V$ we have that

$$\begin{aligned}
0 = |\delta(V)| &= \sum_{v \in V} |\delta(\{v\})| - |E[V]| \\
&= \sum_{v \in V} |\delta(\{v\})| - |E| \\
&\geq \sum_{v \in V} k - |E| && \text{(by 3.1)} \\
&= |V|k - |E| \\
&\geq |U|k - |E| \\
&= tk - |E|
\end{aligned}$$

Thus

$$|E| \geq tk \tag{3.3}$$

Finally, let $P \subseteq U$ and $Q \subseteq W$, the number of edges between P and Q is $E[P \cup Q]$ because there is no edge between two vertices of P nor between two vertices of Q, so every edge in $E[P \cup Q]$ needs to be between them.

Now, summing up everything, we got

$$\begin{aligned}
E[P \cup Q] &= \sum_{v \in P \cup W} |\delta(\{v\})| - |\delta(P \cup W)| && \text{(by 3.2)} \\
&\geq \sum_{v \in P \cup W} |\delta(\{v\})| - |E| \\
&\geq \sum_{v \in P \cup W} |\delta(\{v\})| - tk && \text{(by 3.3)} \\
&= \sum_{v \in P} |\delta(\{v\})| + \sum_{v \in Q} |\delta(\{v\})| - tk && \text{(because } P \text{ and } Q \text{ are disjoint)} \\
&\geq \sum_{v \in P} k + \sum_{v \in Q} k - tk && \text{(by 3.1)} \\
&= |P|k + |Q|k - tk \\
&= k(|P| + |Q| - t)
\end{aligned}$$

This concludes the proof for necessity.

Sufficiency:

We will prove, by induction on k that if there are at least $k(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$, than G has k perfect matchings.

The base case $k = 0$ holds, because every graph has 0 perfect matchings (if it has more, than it still has (at least) 0).

Now suppose that if G is a graph in which there are at least $(k - 1)(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$, then G has $k - 1$ perfect matchings. (Induction Hypothesis)

We can add edges to G, so that there there are at least $k(|P| + |Q| - t)$ edges between any subsets $P \subseteq U$, $Q \subseteq W$.

By the induction hypothesys, we know that there are at least $k - 1$ perfect matchings, we just need to find another one. But we claim that these edges that were added saturates all vertices, and so there is a new perfect matching, disjoint of the others.

Let $u \in U$, take $P = \{u\} \subseteq U$ and $Q = W$. Then we have that the number of edges between P and W is at least $k(|P| + |Q| - t) = k(1 + t - t) = k$.

Thus for any vertex $u \in U$ there are at least k edges between u and W. By an analogous argument we can show that there are at least k edges from any vertex $w \in W$ to the set U.

So every vertex in U and W is an end point of at least one edge that is not covered by any matching, thus these edges contains a new perfect matching.

Hence G has k disjoint perfect matchings.

This ends the proof for sufficiency.