MAE0221

Probabilidade I

Pedro Gigeck Freire 10737136 16/06/2021

Provinha 08

Termo de Comprometimento

~~Sejam X e Y dois~~

Eu me comprometo a manter uma conduta ética e adequada durante a realização desta tarefa. Exemplos de conduta inadequada são fornecer e/ou receber auxílio de outras pessoas, consultar material não autorizado, entre outras.

Pedro Gigeck Freire

Sejam X e Y v.a. independentes tais que para

$r > 0$, $s > 0$, $\theta > 0$

$$X \sim Gama(r, \theta) \quad e \quad Y \sim (s, \theta)$$

Considere $T = X + Y$ e $U = X/(X+Y)$

a) Encontre a função densidade conjunta de T e U

Temos que o suporte de X é $x \in [0, \infty)$ e de Y é $y \in [0, \infty)$

Portanto o suporte de T é $[0, \infty)$ e de U é $[0, \infty)$.

Vamos determinar as transformações "inversas" de T e U

$$U = x/_{x+y} = \frac{x}{t} \Rightarrow \boxed{ut = x}$$

$$t = x + y = ut + y \Rightarrow \boxed{t(1-u) = y}$$

Tomamos

$h_1(t, u) = ut$ e $h_2(t, u) = t(1-u)$

Agora, calculamos o Jacobiano

$$J_h = \det \begin{pmatrix} \frac{\partial h_1}{\partial \tau} & \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial \tau} & \frac{\partial h_2}{\partial u} \end{pmatrix} = \det \begin{pmatrix} u & \tau \\ 1-u & -\tau \end{pmatrix} = u(-\tau) - \tau(1-u)$$

$$= -\tau u - \tau + \tau u = \boxed{-\tau}$$

Com o valor do Jacobiano, podemos aplicar o resultado visto em aula:

$$f_{T,U}(\tau,u) = f_{X,Y}(h_1(\tau,u),\ h_2(\tau,u))\,|J_h(\tau,u)|$$

$$= f_X(h_1(\tau,u)) \cdot f_Y(h_2(\tau,u))\,|J_h(\tau,u)|$$ pois X e Y são independentes

$$= f_X(u\tau)\, f_Y(\tau(1-u))\,|-\tau|$$

(pois $\tau \geq 0$)

$$= \frac{\theta^r}{\Gamma(r)} e^{-\theta(u\tau)} (u\tau)^{r-1} \mathbb{1}_{(0,\infty)}(u\tau) \cdot \frac{\theta^s}{\Gamma(s)} e^{-\theta(\tau(1-u))} (\tau(1-u))^{s-1} \mathbb{1}_{[0,\infty)}(\tau(1-u))\ \tau$$

$$= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta(u\tau+\tau(1-u))} (u\tau)^{r-1} \tau^{s-1} (-u\tau)^{s-1} \mathbb{1}_{[0,\infty)}(u\tau)\, \mathbb{1}_{[0,\infty)}(\tau(1-u))\ \tau$$

$$= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta\tau} \tau^{s-1} (u\tau)^{r-1} (-1)^{s-1} (u\tau)^{s-1} \underbrace{\mathbb{1}_{[0,\infty)}(u\tau)}_{u\geq 0,\ \tau\geq 0}\ \underbrace{\mathbb{1}_{[0,\infty)}(\tau(1-u))}_{\tau\geq 0}\ \tau$$

$$= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta\tau} \tau^{s} (u\tau)^{r+s-2} (-1)^{s-1} \mathbb{1}_{[0,\infty)}(u)\, \mathbb{1}_{[0,\infty)}(\tau)\, \mathbb{1}_{[0,\infty)}(1-u)$$

$$\boxed{= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta\tau} \tau^{s} (u\tau)^{r+s-2} (-1)^{s-1} \mathbb{1}_{[0,\infty)}(\tau)\, \mathbb{1}_{[0,1]}(u)}$$

b) Encontre e identifique as marginais:

Marginal de T

$$f_T(t) = \int_{u \in \mathbb{R}} f_{T,U}(t,u)\,du = \int_{u \in \mathbb{R}} \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta t} t^{s} (ut)^{r+s-2} (-1)^{s-1} \mathbb{1}_{[0,\infty)}(t)\,\mathbb{1}_{[0,1]}(u)\,du$$

$$= 0 + \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta t}\, t^{s}\, t^{r+s-2} (-1)^{s-1} \int_{u=0}^{1} u^{r+s-2}\,du$$

$$= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta t}\, t^{s} t^{r+s-2} (-1)^{s-1} \left(\frac{u^{r+s-1}}{r+s-1}\right)\Bigg|_{u=0}^{1}$$

$$= \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta t}\, t^{r+2s-2} (-1)^{s-1} \frac{1}{\cancel{r}+s-1} \qquad \text{para} \quad t \geq 0$$

Então sabemos que $T \sim Gama(r+s, \theta)$

Marginal de U

$$f_U(u) = \int_{t \in \mathbb{R}} f_{T,U}(t,u)\,dt = \int_{t \in \mathbb{R}} \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\theta t} t^{r+2s-2} u^{r+s-2} (-1)^{s-1} \mathbb{1}_{[0,\infty)}(t)\,\mathbb{1}_{[0,1]}(u)\,dt$$

$$= 0 + \frac{\theta^{r+s}}{\Gamma(r)\Gamma(s)} u^{r+s-2} (-1)^{s-1} \int_{t=0}^{\infty} e^{-\theta t} t^{r+2s-2}\,dt \quad \mathbb{1}_{[0,1]}(u)$$

c) T e U não são independentes, já que

$f_{T,U}(t,u) \neq f_T(t) \cdot f_U(u)$.