MAC0325 Combinatorial Optimization

Assignment 3

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Exercise 1

Let $u : A \to \mathbb{R} \cup \{+\infty\}$ be a capacity function given by $u(a) \coloneqq +\infty$ for each $a \in A$, and let $\beta \coloneqq 1$.

We claim that the Shortest Walk Problem (SWP) on (D, c, r, s) is homomorphically equivalent to the Min-Cost Flow Problem (MCFP) on (D, c, u, r, s, β)

First, let us set some notation.

Let $X \coloneqq \{W : W \text{ is a } rs\text{-walk in } D\}$ be the feasible set of SWP on (D, c, r, s) and let $Y \coloneqq \{f \in R_+^A : f \leq u \text{ and } B_D f = \beta(e_s - e_r)\}$ be the feasible set of MCFP on (D, c, u, r, s, β).

Proposition 1. Let $\varphi : X \to Y$ be a function given by

$$\varphi(W) \coloneqq \mathbb{1}_W, \text{ for each } W \in X. \tag{1.1}$$

Then, SWP $\xrightarrow{\varphi}$ MCFP.

Proof. Let $W \in X$. We have.

$$\begin{aligned} c^\intercal \varphi(W) &= c^\intercal \mathbb{1}_W && \text{by (1.1)} \\ &= c(W) && \text{by (11.13) from the lectures} \end{aligned}$$

Indeed, the walk W and the flow obtained $\varphi(W)$ have the same objective value. Hence φ is a homomorphism. □

Proposition 2. Let $f \in Y$ be a rs-flow in D. Then there is a walk $W \in X$ such that $c^\intercal f \geq c(W)$.

Proof. If G has any negative cycle $C \coloneqq \langle v_0, a_1, v_1, ..., a_l, v_l \rangle$, then there is a walk R from r to v_0 and a walk S from v_0 to s, because $r \rightsquigarrow v \rightsquigarrow s$ for each $v \in V$.

So one can take a rs-walk W with cost arbitrarily low, by taking

$$W \coloneqq R \cdot (\prod_{i \in [k]} C) \cdot S$$

for some $k \in \mathbb{N}$ arbitrarily high, so that $c^\intercal f \geq c(W)$.

Now suppose G has no negative cycles.

By the exercise 18.10 from the lectures (Decomposition of Flows), there are: a set $\mathcal{C}$ of cycles of G, a set $\mathcal{P}$ of rs-paths of G, a vector $y \in R_+^{\mathcal{C}}$ and a vector $x \in R_+^{\mathcal{P}}$ such that

$$f = \sum_{C \in \mathcal{C}} y_C \mathbb{1}_C + \sum_{P \in \mathcal{P}} x_P \mathbb{1}_P, \tag{1.2}$$

$$\mathbb{1}^\intercal x = value(f) = \beta = 1 \tag{1.3}$$

Now, let $W \in \mathcal{P}$ attain a path of minimun cost, i.e.,

$$c(W) = \min_{P \in \mathcal{P}} c(P) \implies$$

$$c(W) \geq c(P) \text{ for each } P \in \mathcal{P} \tag{1.4}$$

Then

$$\begin{aligned}
c^\intercal f &= c^\intercal (\sum_{C \in \mathcal{C}} y_C \mathbb{1}_C + \sum_{P \in \mathcal{P}} x_P \mathbb{1}_P) && \text{by (1.2)} \\
&= c^\intercal \sum_{C \in \mathcal{C}} y_C \mathbb{1}_C + c^\intercal \sum_{P \in \mathcal{P}} x_P \mathbb{1}_P \\
&= \sum_{C \in \mathcal{C}} y_C (c^\intercal \mathbb{1}_C) + \sum_{P \in \mathcal{P}} x_P (c^\intercal \mathbb{1}_P) \\
&\geq \sum_{C \in \mathcal{C}} y_C + \sum_{P \in \mathcal{P}} x_P (c^\intercal \mathbb{1}_P) && \text{since } G \text{ has no negative cycles} \\
&\geq \sum_{P \in \mathcal{P}} x_P (c^\intercal \mathbb{1}_P) && \text{since } y \geq 0 \\
&= \sum_{P \in \mathcal{P}} x_P \sum_{a \in A} c(a) \mathbb{1}_P(a) \\
&= \sum_{P \in \mathcal{P}} x_P \sum_{a \in A} c(a) [a \in A(P)] \\
&= \sum_{P \in \mathcal{P}} x_P c(P) \\
&\geq \sum_{P \in \mathcal{P}} x_P c(W) && \text{by (1.4)} \\
&= c(W) \sum_{P \in \mathcal{P}} x_P \\
&= c(W) \mathbb{1}^\intercal x \\
&= c(W) && \text{by (1.3)}
\end{aligned}$$

□

Note that Proposition 2 builds an implicit homomorphism from MCFP on (D, c, u, r, s, β) to SWP on (D, c, r, s).

Thus, since Proposition 1 builds a homomorphism from SWP to MCFP, these two problems are homomorphically equivalent.

Exercise 3

We will build a new graph by adding a new arc from s to r with cost 0 and lower and upper capacity equal to β.

Let a' be a new arc (not already in A), let $A' := A \cup \{a'\}$ and let $\varphi' : A' \to V \times V$ be an extension of φ by setting $\varphi'(a') := (s, r)$.

Let $D' := (V, A', \varphi')$. Let $c' \in \mathbb{R}^{A'}$ be an extension of c by setting

$$c'(a') := 0 \tag{3.1}$$

Let $u' : A' \to \mathbb{R}_+ \cup \{+\infty\}$ be an extension of u by setting

$$u'(a') := \beta \tag{3.2}$$

And let $l' \in R_+^{A'}$ be defined by

$$l'(a) := \begin{cases} 0 & \text{if } a \in A \\ \beta & \text{if } a = a' \end{cases}$$

Now, we may build our isomorphism.

Let $X := \{f \in R_+^A : f \leq u \text{ and } B_D f = \beta(e_s - e_r)\}$ be the set of feasible rs-flows in D (with respect to u).

Let $Y := \{f' \in R_+^{A'} : l' \leq f' \leq u'\}$ be the set of feasible circulations in D' (with respect to l' and u').

3.1 The Homomorphism Let $\psi : X \to Y$ be a function that maps a flow in D to its correspondent circulation in D', i.e.

$$\psi(f) := f' \tag{3.3}$$

with

$$f'(a) := \begin{cases} f(a) & \text{if } a \in A \\ \beta & \text{if } a = a' \end{cases} \quad \text{for each } a \in A' \tag{3.4}$$

Note that ψ is well defined, since f' is indeed a feasible circulation with respect to l' and u' by the definition of these vectors. For each $a \in A$, we have

$$l'(a) = 0 \leq f(a) = f'(a) = f(a) \leq u(a) = u'(a)$$

and for $a = a'$ we have

$$l'(a') = f'(a') = u'(a') = \beta.$$

Lemma 1. ψ, as defined in 3.1, is a homomorphism from the Min-Cost Flow Problem on (D, c, u, r, s, β) to the Min-Cost Circulation Problem on (D', c', l', u').

Proof.

$$\begin{aligned}
c'^{\intercal}\psi(f) &= c'^{\intercal} f' && \text{by (3.3)} \\
&= \sum_{a \in A'} c'(a) f'(a) \\
&= \sum_{a \in A} c'(a) f'(a) + c'(a') && \text{by definition of } A' \\
&= \sum_{a \in A} c'(a) f'(a) && \text{by (3.1)} \\
&= \sum_{a \in A} c(a) f(a) && \text{since } c' \text{ and } f' \text{ are extensions} \\
&= c^{\intercal} f
\end{aligned}$$

Thus, $\psi(f)$ and f have the same objective value, so ψ is a homomorphism. □

Proposition 3. ψ, as defined in 3.1, is injective.

Proof. Let $f, g \in X$ such that $\psi(f) = \psi(g)$. We have

$$\begin{aligned}
\psi(f) = \psi(g) &\implies f' = g' && \text{by (3.3)} \\
&\implies f'(a) = g'(a) \text{ for each } a \in A' \\
&\implies f'(a) = g'(a) \text{ for each } a \in A && \text{because } A \subset A' \\
&\implies f(a) = g(a) \text{ for each } a \in A && \text{by (3.4)} \\
&\implies f = g
\end{aligned}$$

Hence ψ is injective. □

Proposition 4. ψ, as defined in 3.1, is surjective.

Proof. Let $f' \in Y$ and set $f \coloneqq f'|_A$. We claim that $f \in X$.

First, we have that f is a rs-flow in D that respects

$$0 = l'(a) \leq f(a) \leq u'(a) = u(a) \text{ for each } a \in A$$

Now, note that $f'(a') = \beta$, because

$$\beta = l'(a) \leq f'(a) \leq u'(a) = \beta$$

Then, $B_D f = \beta(e_s - e_r)$, since $B_D f' = 0$, so that when we remove the arc a' the only affected vertices are s and r, so the excess of flow in these vertices is $f'(a') = \beta$ in s and $-\beta$ in r.

These facts shows that $f \in X$.

Thus, it is straighfoward from the definition of ψ and the fact that $f'(a) = \beta$ that $f' = \psi(f)$

Hence, each $f' \in Y$ is the image of some $f \in X$, so ψ is surjective. □

Lemma 2. ψ, as defined in 3.1, is a bijection.

Proof. Immeadiate from Propositions 3 and 4. □

Lemma 3. Let ψ be defined as in 3.1. Then the inverse function $\psi^{-1} : Y \to X$ is a homomorphism from the Min-Cost Circulation Problem on (D', c', l', u') to the Min-Cost Flow Problem on (D, c, u, r, s, β).

Proof. As we showed in the proof of Proposition 4, for each $f' \in Y$ we have

$$\psi^{-1}(f') = f'|_A. \tag{3.5}$$

So, for each circulation $f' \in Y$

$$\begin{aligned}
c^{\mathsf{T}}\psi^{-1}(f') &= c^{\mathsf{T}} f'|_A && \text{by (3.5)} \\
&= \sum_{a \in A} c(a) f'(a) \\
&= \sum_{a \in A} c'(a) f'(a) && \text{by definition of } c' \\
&= \sum_{a \in A} c'(a) f'(a) + c'(a') f'(a) && \text{by (3.1)} \\
&= \sum_{a \in A'} c'(a) f'(a) && \text{by definition of } A' \\
&= c'^{\mathsf{T}} f'
\end{aligned}$$

Hence f' and $\psi^{-1}(f')$ is a homomorphism. □

Theorem 1. ψ, as defined in 3.1, is a isomorphism.

Proof. Immediate from Lemmas 1, 2 and 3. □

Furthermore, note that if u and β are integral, then u' and l' are both integral, and if the flow $f \in X$ is integral, than the circulation $\psi(f)$ is also integral by the definition of ψ.

Exercise 14

We start with $M_0 = \emptyset$ and $y_0 = 0$

$\mathbf{t = 0}$ (Matching update)
$P_0 = \langle 3, 9 \rangle$
$M_1 = \{\{3, 9\}\}$

$\mathbf{t = 1}$ (Dual update)
$K_1 = \{3\}$
$d_1 = (1, 1, 0, 1, 1, 0, 0, 0, 0, 0)$
$\lambda_1 = 1$
$y_2 = (1, 1, 0, 1, 1, 0, 0, 0, 0, 0)$

$\mathbf{t = 2}$ (Dual update)
$K_2 = \{9\}$
$d_2 = (1, 1, 1, 1, 1, 0, 0, 0, -1, 0)$
$\lambda_2 = 1$
$y_3 = (2, 2, 1, 2, 2, 0, 0, 0, -1, 0)$

$\mathbf{t = 3}$ (Matching update)
$P_3 = \langle 1, 9, 3, 10 \rangle$
$M_4 = \{\{1, 9\}, \{3, 10\}\}$

$\mathbf{t = 4}$ (Dual update)
$K_4 = \{1, 3\}$
$d_4 = (0, 1, 0, 1, 1, 0, 0, 0, 0, 0)$
$\lambda_4 = 2$
$y_5 = (2, 4, 1, 4, 4, 0, 0, 0, -1, 0)$

$\mathbf{t = 5}$ (Dual update)
$K_5 = \{3, 9\}$
$d_5 = (1, 1, 0, 1, 1, 0, 0, 0, -1, 0)$
$\lambda_5 = 1$
$y_6 = (3, 5, 1, 5, 5, 0, 0, 0, -2, 0)$

$\mathbf{t = 6}$ (Dual update)
$K_6 = \{9, 10\}$
$d_6 = (1, 1, 1, 1, 1, 0, 0, 0, -1, -1)$
$\lambda_6 = 1$
$y_7 = (4, 6, 2, 6, 6, 0, 0, 0, -3, -1)$

$\mathbf{t = 7}$ (Matching update)
$P_7 = \langle 2, 10, 3, 6 \rangle$
$M_8 = \{\{2, 10\}, \{3, 6\}, \{1, 9\}\}$

t = 8 (Dual update)
$K_8 = \{9, 10, 3\}$
$d_8 = (1, 1, 0, 1, 1, 0, 0, 0, -1, -1)$
$\lambda_8 = 2$
$y_9 = (6, 8, 2, 8, 8, 0, 0, 0, -5, -3)$

t = 9 (Dual update)
$K_9 = \{6, 9, 10\}$
$d_9 = (1, 1, 1, 1, 1, -1, 0, 0, -1, -1)$
$\lambda_9 = 3$
$y_{10} = (9, 11, 5, 11, 11, -3, 0, 0, -8, -6)$

t = 10 (Matching update)
$P_{10} = \langle 5, 9, 1, 6, 3, 8 \rangle$
$M_{11} = \{\{2, 10\}, \{5, 9\}, \{1, 6\}, \{3, 8\}\}$

t = 11 (Dual update)
$K_{11} = \{1, 2, 3, 5\}$
$d_{11} = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$
$\lambda_{11} = 4$
$y_{12} = (9, 11, 5, 15, 11, -3, 0, 0, -8, -6)$

t = 12 (Dual update)
$K_{12} = \{1, 3, 5, 10\}$
$d_{12} = (0, 1, 0, 1, 0, 0, 0, 0, 0, -1)$
$\lambda_{12} = 1$
$y_{13} = (9, 12, 5, 16, 11, -3, 0, 0, -8, -7)$

t = 13 (Dual update)
$K_{13} = \{3, 6, 9, 10\}$
$d_{13} = (1, 1, 0, 1, 1, -1, 0, 0, -1, -1)$
$\lambda_{13} = 1$
$y_{14} = (10, 13, 5, 17, 12, -4, 0, 0, -9, -8)$

t = 14 (Dual update)
$K_{14} = \{6, 8, 9, 10\}$
$d_{14} = (1, 1, 1, 1, 1, -1, 0, -1, -1, -1)$
$\lambda_{14} = 3$
$y_{15} = (13, 16, 8, 20, 15, -7, 0, -3, -12, -11)$

t = 15 (Matching update)
$P_{15} = \langle 4, 10, 2, 6, 1, 8, 3, 7 \rangle$
$M_{16} = \{\{4, 10\}, \{2, 6\}, \{1, 8\}, \{3, 7\}, \{5, 9\}\}$

Thus, the optimal value found was 39 and the solutions found were

$$M = \{\{4, 10\}, \{2, 6\}, \{1, 8\}, \{3, 7\}, \{5, 9\}\}$$
$$y = (13, 16, 8, 20, 15, -7, 0, -3, -12, -11)$$