# FYS2140 - Home exam 2018

Canditate: 15199

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## Problem 1

### Particle nature of the photon

#### a1.

Comptons scattering formula can be written on the form

$$\Delta\lambda = \lambda_s - \lambda_i = \lambda_C(1 - \cos\Theta), \tag{1}$$

where $\lambda_i$ is the incident photon, $\lambda_s$ is the scattered photon, $\cos\Theta$ is the angle between the horizontal and the scattered photon, $\lambda_C = h/(m_e c) = 2.43 \times 10^{-3} nm$ is the *Compton wavelength for the electron* where $h$ is Planck's constant, $m_c$ is the electron mass and $c$ is the speed of light. Figure 1 shows a schematic of the setup.

#### a2.

Using Comptons formula, we can find the angle by

$$\Theta = \cos^{-1}\big(1 - \frac{\Delta\lambda}{\lambda_C}\big) = \cos^{-1}\big(1 - \frac{0.0749\ nm - 0.0709\ nm}{0.00243\ nm}\big) \approx 130^\circ, \tag{2}$$

which shows the measuremets was performed at $135^\circ$, taking uncertainties into consideration.

#### a3.

The top to the left can be explained in the following way. A change in wavelength is due to electric interactions between the photon and the weakest bounded electrons, in which we idealize the process as it was a collision between a photon and an electron. If, on the other hand the electron is heavily bounded to the atom or the incomming electromagnetic wave is not strong enough to release the electron, we can look at the process as a photon colliding with an atom, in which the atom will experience a recoil effect. If we concider the material carbon, our reaction is explained by

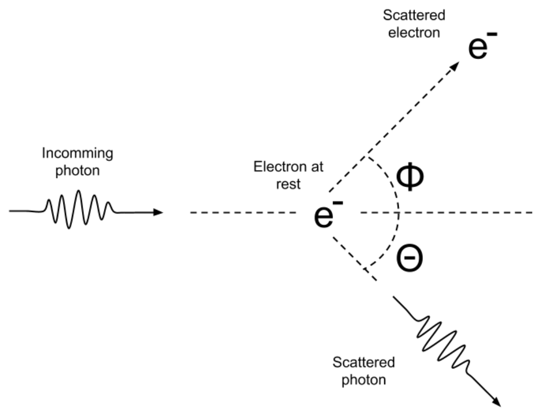


Figure 1: Compton scattering.

$$\gamma + C \rightarrow \gamma + C, \tag{3}$$

where we replace the mass of the electron by the mass of a carbaon atom, which is 22000 times heavier. This will give $\lambda_C \approx 0.1\ fm$, which within the wavelength of x-rays gives differences hardly noticable.

Visible light yields wavelengths in the range 380 $nm - 750$ $nm$. If we look at yellow light ($\approx 589$ $nm$), we have that $\lambda_C / \lambda_{yellow} \approx 10^{-6}$. Because $\lambda_C$ describe the largest possible $\Delta\lambda$, we would hardly notice any difference in wavelenght.

## Wave nature of the neutron

### b1.

Average energy is given by

$$\langle E \rangle = \frac{3}{2} k_B T, \tag{4}$$

where $T$ is the temperature in Kelvin and $k_B$ is Boltzmann's constant. $25^\circ = 298K$ and thus

$$\rightarrow \langle E \rangle = 1.5 \times (1.38 \times 10^{-23} m^2 kg s^{-2} K^{-1})(298.15K) \approx 6.17 \times 10^{-21} J. \tag{5}$$

Momentum is found from the particles kinetic energy

$$\langle E \rangle = \frac{1}{2} m v^2 \rightarrow \langle p \rangle = \sqrt{2 \langle E \rangle m_n}, \tag{6}$$

where $m_n$ is the neutron mass

$$\rightarrow \langle p \rangle = \sqrt{1 * 6.17 \times 10^{-21} J \times 1.67 \times 10^{-27} kg} \approx 4.54 \times 10^{-24}. \qquad (7)$$

Finally we can find the wavelength by using the *deBroglie relation*

$$\lambda = \frac{h}{p} \rightarrow \lambda = \frac{6.32 \times 10^{-34} J}{4.54 \times 10^{-24}} \approx 1.30 \times 10^{-10} m \approx 0.1 nm \qquad (8)$$

,

where $h$ is Planck's constant.

**b2.**

*Bragg diffraction* is given by

$$n\lambda = 2d \sin \Theta, \qquad (9)$$

where $n$ is a positive integer and $\lambda$ is the *wavelength of the incident wave*. Maximum intensity is given at $n = 1$, and thus

$$\Theta_{max} = \sin^{-1} \frac{\lambda}{2d}. \qquad (10)$$

What we want is $|\Delta\lambda|/\lambda \leq 1/10$. By using (1) and (9) we get write

$$\frac{|\Delta\lambda|}{\lambda} = \frac{\lambda_C(1 - \cos\Theta)}{2d \sin\Theta} \leq \frac{1}{10} \qquad (11)$$

$$\rightarrow \frac{h(1 - \cos\Theta)}{mc2d \sin\Theta} \leq \frac{1}{10} \rightarrow \tan\frac{\Theta}{2} \leq \frac{mcd}{5h} \rightarrow \Theta = 2\tan^{-1}\frac{mcd}{5h} \qquad (12)$$

# Problem 2

## Radioactive $\alpha$-decay

**a.**

We find $A$ by *normalizing the wavefunction.* $|\Psi(x, 0)| = \Psi^*\Psi$, where $\Psi^*$ is the complex conjugate and thus, the imaginary part will vanish and the integral becomes

$$|\Psi(x, 0)|^2 = |A|^2 \int_{-\infty}^{+\infty} e^{-(x-x_0)/2a^2} = 1. \qquad (13)$$

Substituting $u = x - x_0 \rightarrow du = dx$ and $\lambda = 1/(2\pi a^2)$ yields the following standard integral that can be found in Rottmann

$$|A|^2 \int_{-\infty}^{+\infty} e^{-\lambda u} du = |A|^2 \sqrt{\frac{\pi}{\lambda}} = |A|^2 \sqrt{2\pi a^2} = 1 \rightarrow A = \left(\frac{1}{2\pi a^2}\right)^{1/4} \qquad (14)$$

which gives

$$\Psi(x,0) = \left(\frac{1}{\sqrt{2\pi a^2}}\right)^{1/4} e^{-(x-x_0)/4a^2} e^{ikx}. \tag{15}$$

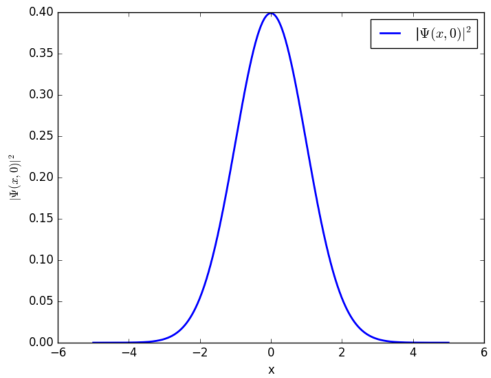


Figure 2: Plot $|\Psi(x,0)|^2$.

Energy for the stationary Schrödinger equation is defined by

$$E = \frac{\hbar^2 k^2}{2m_\alpha} \to \frac{(6.5821 \times 10^{-15} eVs \times 1.38 \times 10^{-15}\ m^{-1})}{2 \times 3.27 \times eVc^{-2}} \tag{16}$$

**b.**

We need to find general solutions for the three regions for which the particle can exist. As figure 2 shows, we have that $0 \leq E \leq V_0 \to E > 0$, so we're dealing with *scattering states*.

For $(0 \leq x \leq x_1)$, $V(x) = 0$ and the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \to \frac{d^2\psi}{dx^2} = -k^2\psi, \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \tag{17}$$

which general solution is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}. \tag{18}$$

For $(x_1 \leq x \leq x_2)$, $V(x) = V_0 \to E < V_0$, so we will get complex solutions and thus the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \to \frac{d^2\psi}{dx^2} = -q^2\psi, \quad q \equiv i\kappa, \ \kappa \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar} \tag{19}$$

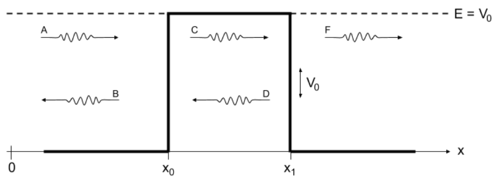


Figure 3: Potential barrier as given in the problem.

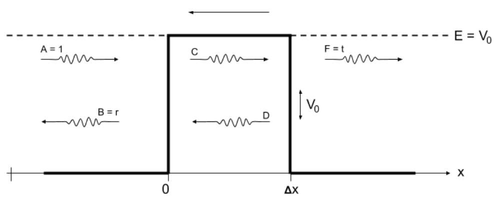


Figure 4: To find boundary conditions, we offset the potential barrier by $-x_1$. We also set $A = 1$, the *incoming wave*, $B = r$, the *reflected wave* and $F = t$, the *transmitted wave*.

which general solution is

$$\psi(x) = Ce^{iqx} + De^{-iqx} = Ce^{-\kappa x} + De^{\kappa x} \to Ce^{\kappa x} + De^{-\kappa x}. \tag{20}$$

(I switched the constants C and D to make the equation consistent with my sketch.)

For $(x \geq x_2)$, the general solution is similar to the left region, however, the latter term in the sum represent a wave coming in from the right and thus is omitted from the equation

$$\psi(x) = Fe^{ikx}. \tag{21}$$

To summarize, the general solution of the three regions are

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & , \quad 0 \leq x \leq x_1 \\ Ce^{\kappa x} + De^{-\kappa x} & , \quad x_1 \leq x \leq x_e \\ Fe^{ikx} & , \quad x \geq x_1 \end{cases} \tag{22}$$

**c.**

We need to look at the boundary conditions for $\psi(x)$ and $d\psi/dx$. The derivatives of $\psi$ will have the following form

$$
\frac{d\psi(x)}{dx} = \begin{cases} ik(Ae^{ikx} - Be^{-ikx}) & , \quad 0 \le x \le x_1 \\ q(Ce^{\kappa x} + De^{-\kappa x}) & , \quad x_1 \le x \le x_e \\ ikFe^{ikx} & , \quad x \ge x_1 \end{cases} \tag{23}
$$

Because the barrier is not located at $x = 0$, we need to offset it by $-x_1$ which will give the following boundary conditions

$$
\begin{aligned}
\psi_1(0) &= \psi_2(0), & (24)\\
\psi_2(\Delta x) &= \psi_3(\Delta x), & (25)\\
\frac{d}{dx}\psi_1(0) &= \frac{d}{dx}\psi_2(0) \text{ and} & (26)\\
\frac{d}{dx}\psi_2(\Delta x) &= \frac{d}{dx}\psi_3(\Delta x), & (27)\\
& & (28)
\end{aligned}
$$

which will give

$$
\begin{aligned}
1 + r &= C + D, & (29)\\
Ce^{q\Delta x} + De^{-q\Delta x} &= t, & (30)\\
ik(1-r) &= q(C - D) \text{ and} & (31)\\
q(Ce^{q\Delta x} - De^{-q\Delta x}) &= ikte^{ik\Delta x}, & (32)\\
& & (33)
\end{aligned}
$$

where I have set $A = 1$, the *incoming wave*, $B = r$, the *reflected wave* and $F = t$, the *transmitted wave*.

Now we are left with four equations. We need to solve for $t$. This is straight forward and left as an exercise to the reader as it is trivial (I'm just kidding! Here it is!).

We start with (30), solving for $C$

$$
C = \frac{te^{ik\Delta x} - De^{-q\Delta x}}{e^{q\Delta x}} \tag{34}
$$

Inserting (34) into (32) yields

$$
q(te^{ik\Delta x} - 2De^{-q\Delta x}) = ikte^{ik\Delta x}. \tag{35}
$$

Solving for $t$ yields

$$
t = \frac{2qDe^{-z\Delta x}}{z^*}, \tag{36}
$$

where we have introduced the variables $z \equiv q + ik$ and $q^* \equiv q - ik$.

Next, eliminate $r$ from (29) and (31) to get

$$ik(1-r) = q(C-D), \tag{37}$$

which after some manipulations yields

$$C = \frac{Dz^* + 2ik}{z}. \tag{38}$$

Setting (34) = (38) yields

$$te^{-z^*\Delta x} - De^{-2q\Delta x} = \frac{Dz^* + 2ik}{z}, \tag{39}$$

which gives, by solving for $D$

$$D = \frac{tze^{-z^*\Delta x} - 2ik}{z^* + ze^{-2q\Delta x}}. \tag{40}$$

Put (40) into (36)

$$tz^* = \frac{tze^{-z^*\Delta x} - 2ik}{z^* + ze^{-2q\Delta x}} e^{-z\Delta x} \tag{41}$$

which, if we solve for $t$ yields

$$t = \frac{2ike^{-z\Delta x}}{ze^{-(z^*+z)\Delta x} - \frac{z^*}{2q}\left(z^* + ze^{-2q\Delta x}\right)} \tag{42}$$

$$= \frac{1}{\frac{ze^{-z^*\Delta x}}{2ik} - \frac{z^* e^{-z\Delta x}}{4ikq}\left(z^* + ze^{-2q\Delta x}\right)} \tag{43}$$

$T = |t|^2$ and thus

$$T = \left(\frac{1}{\frac{ze^{-z^*\Delta x}}{2ik} - \frac{z^* e^{-z\Delta x}}{4ikq}\left(z^* + ze^{-2q\Delta x}\right)}\right)^2 \tag{44}$$

I have skipped many steps, however, my calculations by hand is attached at the end of the document.

**d.**

```

import numpy as np
import matplotlib.pyplot as plt

# Defining some constants
V_0 = 34*1e6              # eV
dx = 17*1e-15             # m
m_alpha = 3.7273*1e9      # eV*c**2
hbar = 6.5821*1e9         # eV*s

# Define a range of energies (eV)
```

```
E = np.linspace(-1e-22,1e-22, 1000)
T = 1./(1.0 + ((V_0**2)/(4*E*(V_0-E)))*np.sinh(dx*np.sqrt(2*m_alpha
    *(V_0-E))/hbar)**2)

#
plt.xlabel('E')
plt.ylabel('T(E)')
plt.plot(E, T)
#plt.legend(["ħ=1","ħ=5","ħ=7"])
plt.title("Transmision coefficient")
plt.show()
```

**e.**

**f.**

```

from numpy import *
from matplotlib.pyplot import *


E0p = 3727.3     # Rest enegy for the alpha partivle [MeV]
hbarc = 0.1973   # MeV pm
c = 3.0E2        # Speed of light [pm / as]


def Psi0(x):
    '''
    Initial state for a traveling gaussian wave packeet.
    '''
    x0 = -0.100   # pm
    a = 0.005    # pm
    l = 100.0e3 # 1 / pm

    A = (1./(2*pi*a**2))**0.25

    K1 = exp(-(x-x0)**2 / (4.*a**2))
    K2 = exp(1j*l*x)

    return A * K1 * K2

def V(x):
    potential = zeros(len(x))
    #potential[x>=0 && x<=1] = 34          # [MeV]
    for i in x:
        if i>=0 and i<=1:
            pot
    return potential

nx = 1001     # Number of points in x direction
np = 1e2     # Only plot every np-th calculation (for performance)
dx = 0.001   # Distance between points

#x1 = -0.5*nx*dx
x1 = -0.5*nx*dx
x2 = 0.5*nx*dx
x = linspace(x1,x2,nx)

```

```
dPsidx2 = zeros(nx).astype(complex64)
Psi = Psi0(x)

T = 0.012   # How long the simulation will run
dt = 1e-7   # Distance between time steps

V_x = V(x)

ion()

figure()
line, = plot(x, abs(Psi)**2)
draw()

c1 = (1j*hbarc*c) / (2.*E0p)
c2 = -(1j*c) / hbarc

t = 0
c = 1

k1 = (1j*hbarc*c) / (2.*E0p)

while t<T:
    # Calculate the derivative
    dPsidx2[1:nx-1] = (Psi[2:nx] - 2*Psi[1:nx-1] + Psi[0:nx-2]) / dx**2

    # .. new Psi
    Psi = Psi + dt * ( c1 * dPsidx2 + c2 * V_x * Psi)

    if c==np:
        line.set_ydata(abs(Psi)**2)
        draw()
        c = 0

    t += dt
    c += 1

ioff()

show()
```

**g.**