FYS2140 - Home exam 2018

Canditate: 15199

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Problem 1

Particle nature of the photon

a1.

Comptons scattering forumla can be written on the form

$$\Delta\lambda = \lambda_s - \lambda_i = \lambda_C(1 - \cos\Theta), \tag{1}$$

where λ_i is the incident photon, λ_s is the scattered photon, $\cos\Theta$ is the angle between the horizontal and the scattered photon, $\lambda_C = h/(m_e c) = 2.43 \times 10^{-3} nm$ is the *Compton wavelength for the electron* where h is Plancks constant, m_c is the electron mass and c is the speed of light. Figure 1 shows a schematic of the setup.

a2.

Using Comptons formula, we can find the angle by

$$\Theta = \cos^{-1}\big(1 - \frac{\Delta\lambda}{\lambda_C}\big) = \cos^{-1}\big(1 - \frac{0.0749\ nm - 0.0709\ nm}{0.00243\ nm}\big) \approx 130^\circ, \tag{2}$$

which seems like a reasonable result concidering the graphs given in the problem.

a3.

$$\frac{\lambda_C}{\lambda_i} \tag{3}$$

Visible light yields wavelengths in the range 380 nm − 750 nm. X-rays have shorter wavelengths and thus

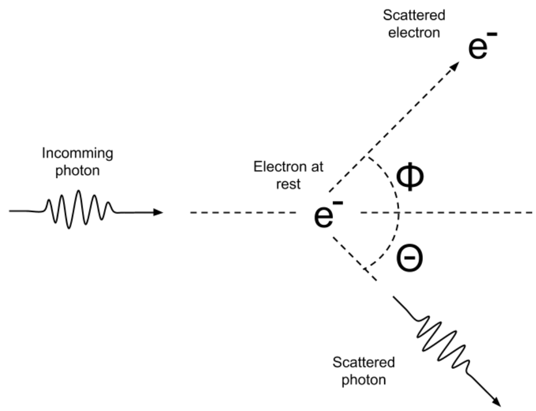

Figure 1: Compton scattering.

Wave nature of the neutron

b1.

Average energy is given by

$$\langle E \rangle = \frac{3}{2} k_B T, \tag{4}$$

where T is the temperatur in Kelvin and k_B is Boltzmanns constant. $25^\circ = 298K$ and thus

$$\rightarrow \langle E \rangle = 1.5 \times (1.38 \times 10^{-23} m^2 kg s^{-2} K^{-1})(298.15K) \approx 6.17 \times 10^{-21} J. \tag{5}$$

Momentum is found from the particles kinetic energy

$$\langle E \rangle = \frac{1}{2} m v^2 \rightarrow \langle p \rangle = \sqrt{2 \langle E \rangle m_n}, \tag{6}$$

where m_n is the neutron mass

$$\rightarrow \langle p \rangle = \sqrt{1 * 6.17 \times 10^{-21} J \times 1.67 \times 10^{-27} kg} \approx 4.54 \times 10^{-24}. \tag{7}$$

Finally we can find the wavelength by using the *deBroglie relation*

$$\lambda = \frac{h}{p} \rightarrow \lambda = \frac{6.32 \times 10^{-34} J}{4.54 \times 10^{-24}} \approx 1.30 \times 10^{-10} m \approx 0.1 nm \tag{8}$$

,

where h is Plancks constant.

b2.

Bragg diffraction is given by

$$n\lambda = 2d\sin\Theta, \tag{9}$$

where n is a positive integer and λ is the *wavelength of the incident wave*. Maximum intensity is given at $n = 1$, and thus

$$\Theta_{max} = \sin^{-1}\frac{\lambda}{2d} \tag{10}$$

We want $|\Delta\lambda|/\lambda \leq 10\% = 1/10$.
By using (1) and (9) we get write

$$\frac{|\Delta\lambda|}{\lambda} = \frac{\lambda_C(1-\cos\Theta)}{2d\sin\Theta} \leq \frac{1}{10} \tag{11}$$

$$\rightarrow \frac{h(1-\cos\Theta)}{mc2d\sin\Theta} \leq \frac{1}{10} \rightarrow \tan\frac{\Theta}{2} \leq \frac{mcd}{5h} \rightarrow \Theta = 2\tan^{-1}\frac{mcd}{5h} \tag{12}$$

Problem 2

Radioactive α-decay

a.

We find A by *normalizing the wavefunction.* $|\Psi(x,0)| = \Psi^*\Psi$, where Ψ^* is the complex conjugate and thus, the imaginary part will vannish and the integral becomes

$$|\Psi(x,0)|^2 = |A|^2\int_{-\infty}^{+\infty} e^{-(x-x_0)/2a^2} = 1. \tag{13}$$

Substituting $u = x - x_0 \rightarrow du = dx$ and $\lambda = 1/(2\pi a^2)$ yields the following standard integral that can be found in Rottmann

$$|A|^2\int_{-\infty}^{+\infty} e^{-\lambda u}du = |A|^2\sqrt{\frac{\pi}{\lambda}} = |A|^2\sqrt{2\pi a^2} = 1 \rightarrow A = \left(\frac{1}{2\pi a^2}\right)^{1/4} \tag{14}$$

which gives

$$\Psi(x,0) = \left(\frac{1}{\sqrt{2\pi a^2}}\right)^{1/4} e^{-(x-x_0)/4a^2}e^{ikx} \tag{15}$$

b.

We need to find general solutions for the three regions for which the particle can exist. As figure 2 shows, we have that $0 \leq E \leq V_0 \rightarrow E > 0$, so we're dealing with *scattering states.*

For $(0 \leq x \leq x_1),\ V(x) = 0$ and the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi, \quad k \equiv \frac{\sqrt{2mE}}{\hbar} \tag{16}$$

which general solution is

$$\psi(x) = Ae^{ikx} + Be^{-ikx}. \tag{17}$$

For $(x_1 \leq x \leq x_2),\ V(x) = V_0 \rightarrow E < V_0$, so we will get complex solutions and thus the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -q^2\psi, \quad q \equiv i\kappa,\ \kappa \equiv \frac{\sqrt{2m(V_0 - E)}}{\hbar} \tag{18}$$

which general solution is

$$\psi(x) = Ce^{iqx} + De^{-iqx} = Ce^{-\kappa x} + De^{\kappa x} \rightarrow Ce^{\kappa x} + De^{-\kappa x}. \tag{19}$$

(I switched the constants C and D to make the equation consistant with my scetch.)

For $(x \geq x_2)$, the general solution is simular to the left region, however, the latter term in the sum prepresents a wave comming in from the right and thus is omited from the equation

$$\psi(x) = Fe^{ikx}. \tag{20}$$

To summarize, the general solution of the three regions are

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & , \quad 0 \leq x \leq x_1 \\ Ce^{\kappa x} + De^{-\kappa x} & , \quad x_1 \leq x \leq x_e \\ Fe^{ikx} & , \quad x \geq x_1 \end{cases} \tag{21}$$

c.

We need to look at the bounadary conditions for $\psi(x)$ and $d\psi/dx$. The derivatives of ψ will have the following form

$$\frac{d\psi(x)}{dx} = \begin{cases} ik(Ae^{ikx} - Be^{-ikx}) & , \quad 0 \leq x \leq x_1 \\ q(Ce^{\kappa x} + De^{-\kappa x}) & , \quad x_1 \leq x \leq x_e \\ ikFe^{ikx} & , \quad x \geq x_1 \end{cases} \tag{22}$$

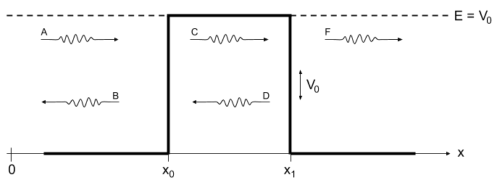

Figure 2: Potential barriere as given in the problem.

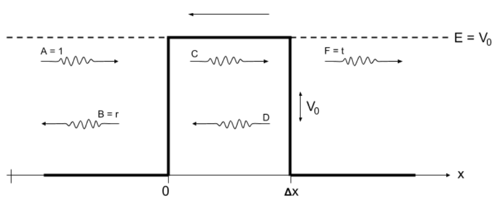

Figure 3: To find boundary conditions, we offset the potential barriere by $-x_1$. We also set $A = 1$, the *incomming wave*, $B = r$, the *reflected wave* and $F = t$, the *transmitted wave*.

Because the barriere is not located at $x = 0$, we need to offset it by $-x_1$ which will give the following boundary conditions

$$\psi_1(0) = \psi_2(0), \tag{23}$$
$$\psi_2(\Delta x) = \psi_3(\Delta x), \tag{24}$$
$$\frac{d}{dx}\psi_1(0) = \frac{d}{dx}\psi_2(0) \text{ and} \tag{25}$$
$$\frac{d}{dx}\psi_2(\Delta x) = \frac{d}{dx}\psi_3(\Delta x), \tag{26}$$
$$\tag{27}$$

which will give

$$1 + r = C + D, \tag{28}$$
$$Ce^{q\Delta x} + De^{-q\Delta x} = t, \tag{29}$$
$$ik(1 - r) = q(C - D) \text{ and} \tag{30}$$
$$q(Ce^{q\Delta x} - De^{-q\Delta x}) = ikte^{ik\Delta x}, \tag{31}$$
$$\tag{32}$$

where I have set $A = 1$, the *incomming wave*, $B = r$, the *reflected wave* and $F = t$, the *transmitted wave*.

Now we are left with four equations. We need to solve for t. This is straight forward and left as an exercice to the reader as it is trivial (I'm just kidding! Here it is!).

We start with (29), solving for C

$$C = \frac{te^{ik\Delta x} - De^{-q\Delta x}}{e^{q\Delta x}} \tag{33}$$

Inserting (33) into (31) yields

$$q(te^{ik\Delta x} - 2De^{-q\Delta x}) = ikte^{ik\Delta x}. \tag{34}$$

Solving for t yields

$$t = \frac{2qDe^{-z\Delta x}}{z^*}, \tag{35}$$

where we have introduced the variables $z \equiv q + ik$ and $q^* \equiv q - ik$.
Next, eliminate r from (28) and (30) to get

$$ik(1 - r) = q(C - D), \tag{36}$$

which after some manipulations yields

$$C = \frac{Dz^* + 2ik}{z}. \tag{37}$$

Setting (33) = (37) yields

$$te^{-z^*\Delta x} - De^{-2q\Delta x} = \frac{Dz^* + 2ik}{z}, \tag{38}$$

wich gives, by solving for D

$$D = \frac{tze^{-z^*\Delta x} - 2ik}{z^* + ze^{-2q\Delta x}}. \tag{39}$$

Put (39) into (35)

$$tz^* = \frac{tze^{-z^*\Delta x} - 2ik}{z^* + ze^{-2q\Delta x}} e^{-z\Delta x} \tag{40}$$

which, if we solve for t yields

$$t = \frac{2ike^{-z\Delta x}}{ze^{-(z^*+z)\Delta x} - \frac{z^*}{2q}\left(z^* + ze^{-2q\Delta x}\right)} \tag{41}$$

$$= \frac{1}{\frac{ze^{-z^*\Delta x}}{2ik} - \frac{z^*e^{-z\Delta x}}{4ikq}\left(z^* + ze^{-2q\Delta x}\right)} \tag{42}$$

$T = |t|^2$ and thus

$$T = \left(\frac{1}{\frac{ze^{-z^*\Delta x}}{2ik} - \frac{z^* e^{-z\Delta x}}{4ikq}(z^* + ze^{-2q\Delta x})} \right)^2 \tag{43}$$

I have skipped many steps, however, my calculations by hand is attached at the end of the document.

d.

```python

import numpy as np
import matplotlib.pyplot as plt

# Defining some constants
V_0 = 34*1e6            # eV
dx = 17*1e-15           # m
m_alpha = 3.7273*1e9    # eV*c**2
hbar = 6.5821*1e9       # eV*s

# Define a range of energies (eV)
E = np.linspace(-1e-22,1e-22, 1000)
T = 1./(1.0 + ((V_0**2)/(4*E*(V_0-E)))*np.sinh(dx*np.sqrt(2*m_alpha
    *(V_0-E))/hbar)**2)

#
plt.xlabel('E')
plt.ylabel('T(E)')
plt.plot(E, T)
#plt.legend(["ħ=1","ħ=5","ħ=7"])
plt.title("Transmision coefficient")
plt.show()
```

e.

f.

```python

from numpy import *
from matplotlib.pyplot import *


E0p = 938.27    # Rest enegy for the proton [MeV]
hbarc = 0.1973  # MeV pm
c = 3.0E2       # Speed of light [pm / as]


def Psi0(x):
    x0 = 0.100  # pm
    a = 0.005   # pm
    l = 100.0e3 # 1 / pm

    A = (1./(2*pi*a**2))**0.25
    K1 = exp(-(x-x0)**2 / (4.*a**2))
    K2 = exp(1j*l*x)

```

```
    return A * K1 * K2

def V(x):
    potential = zeros(len(x))
    potential[x>=0] = 34        # [MeV]
    return potential

nx = 801    # Number of points in x direction
np = 1e2    # Only plot every np-th calculation (for performance)
dx = 0.001  # Distance between points

a = -0.5*nx*dx
b = 0.5*nx*dx
x = linspace(a,b,nx)

dPsidx2 = zeros(nx).astype(complex64)
Psi = Psi0(x)

T = 0.012   # How long the simulation will run
dt = 1e-7   # Distance between time steps

V_x = V(x)

ion()

figure()
line, = plot(x, abs(Psi)**2)
draw()

c1 = (1j*hbarc*c) / (2.*E0p)
c2 = -(1j*c) / hbarc

t = 0
c = 1

k1 = (1j*hbarc*c) / (2.*E0p)

while t<T:
    # Calculate the derivative
    dPsidx2[1:nx-1] = (Psi[2:nx] - 2*Psi[1:nx-1] + Psi[0:nx-2]) / dx**2

    # .. new Psi
    Psi = Psi + dt * ( c1 * dPsidx2 + c2 * V_x * Psi)

    if c==np:
        line.set_ydata(abs(Psi)**2)
        draw()
        c = 0

    t += dt
    c += 1

ioff()

show()
```

g.