a. $N = 3$, $q = 3$

$(0, 0, 3)$
$(0, 1, 2)$
$(0, 3, 0)$
$(0, 2, 1)$
$(1, 0, 2)$
$(1, 2, 0)$
$(1, 1, 1)$
$(2, 0, 1)$
$(2, 1, 0)$
$(3, 0, 0)$

b. Using the formula, we get

$$\Omega(N, q) = \binom{q+N-1}{q} = \left(\frac{(q+N-1)!}{q!(N-1)!}\right)$$

$$\rightarrow \frac{5!}{3!\,2!} = 10$$

c. A: $N_A = 2$ $\quad N_B = 2$

$q_A = 4$ $\quad q_B = 2$

(0, 4)	(0, 2)
(1, 3)	(1, 1)
(2, 2)	(2, 0)
(3, 1)	
(4, 0)	

For each of the elements in A, we need to list each element from B.

A	B	A	B	A	B
(0, 4)	(0, 2)	(0, 4)	(1, 1)	(0, 4)	(2, 0)
(1, 3)	(0, 2)	(1, 3)	(1, 1)	(1, 3)	(2, 0)
(2, 2)	(0, 2)	(2, 2)	(1, 1)	(2, 2)	(2, 0)
(3, 1)	(0, 2)	(3, 1)	(1, 1)	(3, 1)	(2, 0)
(4, 0)	(0, 2)	(4, 0)	(1, 1)	(4, 0)	(2, 0)

d. With total energy $q = 6$, knowing $q_A = 6 - b_b$, q_A can have values from 0, 6. In other words, 7 values.

$q_A = \{0, 1, 2, 3, 4, 5, 6\}$

e. See "problem_e" listed below.

f. Number of total microstates are found by multiplying $\Omega_A \cdot \Omega_B$.
For given values of q_A and q_B, we multiply the multiplicities togetter.
After, we need to iterate through q_A.
This will give alot more states.

g. See "problem_g" listed below.

h. $\ln \Omega(N, q) = \ln\left(\frac{(q+N-1)!}{q!(N-1)!}\right) \simeq \ln\left(\frac{(q+N)!}{q!\,N!}\right)$

$= \ln(q+N)! - \ln q! - \ln N!$

Apply Stirling's approximation on each term:

$\simeq (q+N)\ln(q+N) - (q+N) - [q\ln q - q] - [N\ln N - N]$

$= (q+N)\ln(q+N) \cancel{-q} \cancel{-N} - q\ln q \cancel{+q} - N\ln N \cancel{+N}$

$= (q+N)\underbrace{\ln(q+N)} - q\ln q - N\ln N$

$$\ln\left(q\left(1+\frac{N}{q}\right)\right) = \ln q + \ln\left(\frac{N}{q}+1\right)$$

$$(\text{Taylor}) = \ln q + \frac{N}{q}$$

$\to (q+N)\left[\ln q + \frac{N}{q}\right] - q\ln q - N\ln N$

$= \cancel{q\ln q} + N + N\ln q + \frac{N^2}{q} \cancel{- q\ln q} - N\ln N$

$= N\left(1 + \cancel{\frac{N}{q}} + \ln\frac{q}{N}\right) \quad \frac{N}{q} \ll 1$

$= N\left(\ln\frac{q}{N} + 1\right)$

i. $S = k \ln \Omega(N, q)$

$$\ln \Omega(N, q) = N\left(\ln \frac{q}{N} + 1\right)$$

$$= N \ln \frac{q}{N} + N$$

$$\rightarrow \Omega(N, q) = e^{N \ln \frac{q}{N} + N}$$

$$= e^{\ln\left(\frac{q}{N}\right)^N} e^N$$

$$= \left(\frac{q}{N}\right)^N e^N$$

$$= \left(\frac{e q}{N}\right)^N$$

Thus we get

$$S = Nk \ln\left(\frac{e q}{N}\right)$$

j. $T = \left(\frac{\partial S}{\partial U}\right)^{-1}$

$$S = Nk\left[\ln \frac{q}{N} + 1\right] = Nk \ln q - Nk \ln N + Nk$$

$$q = \frac{U}{\epsilon} \rightarrow S = Nk \ln \frac{U}{\epsilon} - Nk \ln N + Nk$$

$$\frac{\partial S}{\partial U} = Nk \frac{1}{U/\epsilon} \cdot \frac{1}{\epsilon} = Nk \frac{\cancel{\epsilon}}{U} \frac{1}{\cancel{\epsilon}} = \frac{Nk}{U} = \frac{1}{T}$$

$$\rightarrow T = \frac{U}{Nk} \rightarrow U = NkT$$

k. A spin has two "sides" and is independent. We can therefore write 2^N.

l. I have that $E = -S\mu B = -(S_+ - S_-)\mu B = -2S\mu B$

m. See "problem_m" listed below.

n. If we just insert N, and S_+ into the formula:

$$\Omega(N, S_+) = \frac{N!}{S_+!\underbrace{(N-S_+)}_{N_-}!} = \frac{N!}{S_+!\,S_-!}$$

If S_+ counts number of spins and N is the number of total spins, $N - S_+$ is the remaining spins, S_-.

0. We have that $2s = S_+ - S_-$ and that $N = S_+ + S_-$.

If we combine the two, we find an expression for S_+ and S_- we can use in the formula we found in the last excersice.

$S_- = N - S_+ \rightarrow 2s = S_+ - (N - S_+)$

$= S_+ - N + S_+$

$2s = 2S_+ - N$

$s = S_+ - N/2$

$\rightarrow S_+ = s + N/2$

$S_+ = N - S_- \rightarrow 2s = (N - S_-) - S_-$

$= N - S_- - S_-$

$2s = N - 2S_-$

$s = N/2 - S_-$

$\rightarrow S_- = N/2 - s$

$$\rightarrow \Omega(N, s) = \frac{N!}{(\frac{N}{2} + s)!(\frac{N}{2} - s)!}$$

p. Formula 4.90 in the compendium says that

$$P(N,n) = \frac{N!}{(N/2-n)!\,(N/2+n)!} 2^{-N}$$

which we know can be written as

$$\Omega(N,s) = C(N) e^{-\frac{s^2}{N/2}}$$

$$= C(N) e^{-\frac{2s^2}{N}}$$

where $C(N)$ is a constant.

q. We can see that the analytic curve fits well. However, I enqountered some dificulities getting the hist()-function work as I wanted. (Probably because I use Octave, and not MatLab)

r. We know from previous excercises that

$S = k \ln \Omega$ and that $\Omega(N, S_+) = \frac{N!}{S_+!\,(N-S_+)!}$

If we write

$$S = k \ln\left(\frac{N!}{S_+!\,(N-S_+)!}\right)$$

$$= k[\ln N! - (\ln S_+! + \ln(N-S_+)!]$$

$$= k[\ln N! - \ln S_+! - \ln(N-S_+)!]$$

And apply Stirling's approximation.

$$\simeq k[N \ln N - N - (S_+ \ln S_+ - S_+) - ((N-S_+)\ln(N-S_+) - (N-S_+))$$

$$= k[N \ln N \cancel{-N} - S_+ \ln S_+ \cancel{+S_+} - (N-S_+)\ln(N-S_+) \cancel{+N} \cancel{-S_+}$$

$$= k[N \ln N - S_+ \ln S_+ - (N-S_+)\ln(N-S_+)$$

5. $\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial S}{\partial U}\frac{\partial S_+}{\partial S_+} = \frac{\partial S}{\partial S_+}\frac{\partial S_+}{\partial U} = \frac{\partial S}{\partial S_+}\left(\frac{\partial U}{\partial S_+}\right)^{-1}$

From excercise 1., we have that total energy is:

$U = -2s\mu B \rightarrow -(2S_+ - N)\mu B$

which gives

$$\frac{\partial U}{\partial S_+} = \frac{\partial}{\partial S_+}\left(-2S_+\mu B + N\mu B\right) = -2\mu B$$

$$\frac{\partial S}{\partial S_+} = \frac{\partial}{\partial S_+}\left(k[N \ln N - S_+ \ln S_+ - N\ln(N-S_+) + S_+\ln(N-S_+)]\right)$$

$$= k[-\ln S_+ - S_+\frac{1}{S_+} - N\frac{1}{N-S_+}(-1) + \ln(N-S_+) + S_+\frac{1}{N-S_+}(-1)]$$

$$= k[-\ln S_+ - 1 + \frac{N}{N-S_+} + \ln(N-S_+) - \frac{S_+}{N-S_+}]$$

$$= k[\ln\left(\frac{N-S_+}{S_+}\right)$$

$$\frac{1}{T} = \frac{\partial S}{\partial S_+}\left(\frac{\partial U}{\partial S_+}\right)^{-1} = k \ln\left(\frac{N - S_+}{S_+}\right)\frac{1}{-2\mu B}$$

$$\rightarrow T = -\frac{2\mu B}{k}\frac{1}{\ln\left(\frac{N-S_+}{S_+}\right)}$$