FYS3140 - Home exam 2018

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Problem 1: Differential equation

We are asked to solve the differential equation

$$y^{''}(x) + \frac{3}{x}y^{'}(x) - \frac{24}{x^2}y(x) = 56x^6. \tag{1}$$

Multiplying through by x^2 yields

$$x^2y^{''}(x) + 3xy^{'}(x) - 24y(x) = 56x^8, \tag{2}$$

which has the form $ax^2y^{''} + bxy^{y} + cy = g(x)$ and thus is a second order non-homogeneous Cauchy-Euler differential equation.

Solving this equation involves two major steps; 1) find the complementary function f_c, and 2) find the particular solution.

For the complementary function, we reconize that $a = 1, b = 3$ and $c = -24$, and write

$$am(m-1) + bm + c = 0 \rightarrow m(m-1) + 3m - 24 = m^2 + 2m - 24 = 0, \tag{3}$$

which yields $m_1 = 4$ and $m_2 = -6$. Since m_1 and m_2 are two distinct real roots the complementary function is a function on the form

$$y_c = c_1x^{m_1} + c_2x^{m^2} \rightarrow y_c = c_1x^4 + c_2x^{-6}. \tag{4}$$

For the particular solution we will use the variation of parameters. Given $y^{''} + p(x)y^{'} + q(x)y = g(x)$. Since $p(x) = 3/x, q(x) = -24/x^2$ and $g(x) = 56x^2$ are all continous on an open interval, the particular solution can be found by

$$Y_p = -y_1\int\frac{y_2g(x)}{W(y_1,y_2)}dx + y_2\int\frac{y_1g(x)}{W(y_1,y_2)}dx, \tag{5}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. y_1 and y_2 is from the complementary function. Starting by finding the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} \rightarrow W = \begin{vmatrix} x^4 & x^{-6} \\ 4x^3 & -6x^{-7} \end{vmatrix} = -6x^{-7}x^4 - 4x^3x^{-6} = -10x^{-3}, \tag{6}$$

we can write

$$Y_p = -x^4 \int \frac{x^{-6} 56x^6}{-10x^{-3}} dx + x^{-6} \int \frac{x^4 56x^6}{-10x^{-3}} dx \tag{7}$$

$$= \frac{56}{10} \left(x^4 \int x^3 \ dx - x^{-6} \int x^{13} \ dx \right) \tag{8}$$

$$= \frac{56}{10} \left(\frac{x^8}{4} - \frac{x^8}{14} \right) \tag{9}$$

$$= \frac{56}{10} \left(\frac{10x^8}{56} \right) \tag{10}$$

$$= x^8 \tag{11}$$

Finnaly, we find our general solution by adding the complementary function and the particular solution togetter

$$y(x) = y_c + Y_P \rightarrow y(x) = c_1 x^4 + c_2 x^{-6} + x^8, \tag{12}$$

which also can be written as

$$y(x) = \frac{c_2}{x^6} + c_1 x^4 + x^8, \tag{13}$$

and that's my final answer.

Problem 2: Complex analysis

Part A:

a)

For a function that has a *pole of order 3* at $z = 3+i$, a *zero of order 4* at $z = 2i$, we have the following function

$$f(z) = \frac{(z - 2i)^4}{(z - [3 + i])^3} \tag{14}$$

b)

Part B:

a)

b)

c)

d)

Problem 3: The Dirac delta function

a)

b)

c)

a)