FYS3140 Mathematical Methods, Home exam 2018

Due Monday April 16th at 10:00h (strict deadline!)

6. april 2018

Important – please read:

- **Mark your paper with your** *candidate number*, **not your name!**
- **Hand in at the physics front office – do not use devilry or email.**
- **Please sign your name on the list at the counter when handing in!**
- **Keep a copy of your paper!**
- **Clear argumentation and tidy presentation will be taken into account**

Good luck! :)

Problem 1: Differential equation

Choosing an appropriate (analytical) method, find the general solution of the differential equation

$$y''(x) + \frac{3}{x}\, y'(x) - \frac{24}{x^2}\, y(x) = 56x^6. \tag{1}$$

Problem 2: Complex analysis

This problem consists of two independent parts, so it is not necessary to solve Part A first.

Part A:

a) Construct a function that has a pole of order 3 at $z = 3 + i$, a zero of order 4 at $z = 2i$, and is otherwise analytic in the complex plane.

b) Identify and classify the isolated singularities of the function

$$f(z) = \frac{z^3 + 8}{(z-5)^3(z+2)}. \tag{2}$$

Part B:
In this part we will discover yet another neat use of complex contour integrals: The possibility to derive a highly non-trivial summation formula that allows us to evaluate a variety of infinite sums. There are no heavy calculations or new concepts in this problem. But you have to read the text carefully, apply some of the concepts and techniques you have learned, and connect the pieces.

First some definitions. Assume $f(z) = P(z)/Q(z)$ is a rational function which has *no* singularities at $z = 0, \pm 1, \pm 2, ...$. Moreover, deg($Q(z)$) $\geq$ deg($P(z)$) + 2. In addition we define the function

$$g(z) = f(z) \cdot \pi \cot(\pi z). \tag{3}$$

a) Show that the residues of $g(z)$ at $z = n$ $(n = 0, \pm 1, \pm 2, ...)$ are given by

$$Res(g; n) = f(n). \tag{4}$$

For the rest of the problem we will be working with the square contour Γ_N connecting the points $K(1+i)$, $K(-1+i)$, $K(-1-i)$ and $K(1-i)$ where $K = N + 1/2$, and N is a positive integer. In the following you will also need the fact that there exists a constant M, independent of N, such that $|\pi \cot(\pi z)| \leq M$ for all z on the contour. (You can try to prove this if you like, but that is not required here.)

b) Sketch the contour (for some convenient value of N) and prove that

$$\lim_{N\to\infty} \int_{\Gamma_N} g(z)dz = 0 \tag{5}$$

c) Using your previous results and the residue theorem, prove that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\sum(\text{Residues of } g(z) \text{ at the poles of } f(z)). \tag{6}$$

This is a rather powerful summation formula, as we will exemplify now:

d) Use the summation formula from **c)** to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} = \pi \coth \pi. \tag{7}$$

Problem 3: Dirac delta function

This problem evolves around the operator identity

$$\delta\left(f(t)\right) = \sum_i \frac{1}{|f'(t_i)|}\, \delta(t - t_i) \tag{8}$$

where t_i are the zeros of the function $f(t)$, and the derivatives $f'(t_i) \neq 0$.

a) Derive the identity (8).
Hints:
(i) The validity of (8) should be shown within an integral with an arbitrary test function $g(t)$, $\int_{-\infty}^{\infty} \delta\left(f(t)\right) g(t) dt$.
(ii) Split the integral into a sum of integrals, each including only one t_i.
(iii) Taylor expand $f(t)$ about each t_i, keeping terms up to first order.

b) Use (8) to write expressions for (i) $f(t) = t^2 - a^2$ and (ii) $f(t) = \sin t$.

c) Let us get a visual intuition for (8) by studying it graphically as follows: Use your favourite electronic plotting tool, e.g. Python. Choose a convenient delta sequence $\phi_n(t)$. For the two functions in **b)**, plot $\phi_n(f(t))$ and see that this gives the sequences of delta peaks expected from (8). Play with the value of n such that you get reasonably nice and sharp peaks (but not just lines).

d) Evaluate the integral

$$\int_{-\pi/2}^{\pi/2} \cos t\, \delta(\sin t) dt. \tag{9}$$