

FYS3140 - Home exam 2018

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Problem 1: Differential equation

We are asked to find the general solution of the differential equation

$$y''(x) + \frac{3}{x}y'(x) - \frac{24}{x^2}y(x) = 56x^6. \quad (1)$$

Solving this equation involves two major steps; 1) find the complementary function f_c , and 2) find the particular solution. For the complementary function, we start by multiplying through by x^2 to get

$$x^2y''(x) + 3xy'(x) - 24y(x) = 56x^8, \quad (2)$$

which has the form $ax^2y'' + bxy' + cy = g(x)$ and thus is a second order non-homogeneous Cauchy-Euler differential equation. Recognizing $a = 1, b = 3$ and $c = -24$, we can write

$$am(m-1) + bm + c = 0 \rightarrow m(m-1) + 3m - 24 = m^2 + 2m - 24 = 0, \quad (3)$$

which yields $m_1 = 4$ and $m_2 = -6$. Since m_1 and m_2 are two distinct real roots the complementary function is a function on the form

$$y_c = c_1x^{m_1} + c_2x^{m_2} \rightarrow y_c = c_1x^4 + c_2x^{-6}. \quad (4)$$

For the particular solution we will use *variation of parameters*. As we can see, [1](#) is on the form $y'' + p(x)y' + q(x)y = g(x)$. Since $p(x) = 3/x, q(x) = -24/x^2$ and $g(x) = 56x^2$ are all continuous on an open interval, the particular solution can be found by

$$Y_p = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx, \quad (5)$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2 . y_1 and y_2 is from the complementary function. Starting by finding the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow W = \begin{vmatrix} x^4 & x^{-6} \\ 4x^3 & -6x^{-7} \end{vmatrix} = -6x^{-7}x^4 - 4x^3x^{-6} = -10x^{-3}, \quad (6)$$

we can write

$$Y_p = -x^4 \int \frac{x^{-6} 56x^6}{-10x^{-3}} dx + x^{-6} \int \frac{x^4 56x^6}{-10x^{-3}} dx \quad (7)$$

$$= \frac{56}{10} \left(x^4 \int x^3 dx - x^{-6} \int x^{13} dx \right) \quad (8)$$

$$= \frac{56}{10} \left(\frac{x^8}{4} - \frac{x^8}{14} \right) \quad (9)$$

$$= \frac{56}{10} \left(\frac{10x^8}{56} \right) \quad (10)$$

$$= x^8 \quad (11)$$

Finally, we find our general solution by adding the complementary function and the particular solution together

$$y(x) = y_c + Y_p \rightarrow y(x) = c_1 x^4 + c_2 x^{-6} + x^8, \quad (12)$$

which also can be written as

$$y(x) = \frac{c_2}{x^6} + c_1 x^4 + x^8, \quad (13)$$

and that's my final answer.

Problem 2: Complex analysis

Part A:

a)

For a function that has a *pole of order 3* at $z = 3 + i$, a *zero of order 4* at $z = 2i$, we have the following function

$$f(z) = \frac{(z - 2i)^4}{(z - [3 + i])^3} \quad (14)$$

b)

We are asked to classify the isolated singularity of the function

$$f(x) = \frac{z^3 + 8}{(z - 5)^3(z + 2)}. \quad (15)$$

If we write

$$f(x) = \frac{1}{(z - 5)^3} \frac{z^3 + 8}{(z + 2)}. \quad (16)$$

Polynomial division, $(z^3 + 8) : (z + 2)$, yields

$$f(x) = \frac{z^2 - 2z + 4}{(z - 5)^3}, \quad (17)$$

which shows $z = -2$ is a *removable singularity*. Now, if we write

$$\frac{1}{(z-5)^3} = \left(\frac{1}{z-5} \right)^3 = \left(-\frac{\frac{1}{5}}{1-\frac{z}{5}} \right)^3 \quad (18)$$

$$= \left(-\frac{1}{5} \frac{1}{1-\frac{z}{5}} \right)^3 = \left(-\frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{z}{5} \right)^n \right)^3 \quad (19)$$

$$= \left(-\frac{1}{5} \left[1 + \frac{z}{5} + \frac{z^2}{25} + \frac{z^3}{25} + \dots \right] \right)^3 \quad (20)$$

$$= \left(-\frac{1}{5} - \frac{z}{25} - \frac{z^2}{125} - \frac{z^3}{675} + \dots \right)^3 \quad (21)$$

Part B:

a)

We start writing $\cot(z)$ as $\cos(z)/\sin(z)$ to get

$$g(z) = f(z)\pi \cot(\pi z) = \frac{f(z)\pi \cos(\pi z)}{\sin(\pi z)}. \quad (22)$$

If we now set $a(z) \equiv f(z)\pi \cos \pi z$ and $b(z) \equiv \sin \pi z$ we have that $g(z) = a(z)/b(z)$, $a(n) = \text{finite constant} \neq 0$, and $b(n) = 0$, $b'(n) \neq 0$ and thus the residue can be found by equation 6.2 in Boas

$$Res(n) = \frac{a(n)}{b'(n)} \rightarrow Res(n) = \frac{f(n)\pi \cos(\pi n)}{\pi \cos(\pi n)} = f(n), \quad (23)$$

which is what we were suppose to show.

b)

$N = 1/2$ yields $K = 1$, so let's go for that! With $K = 1$, we get the following contour

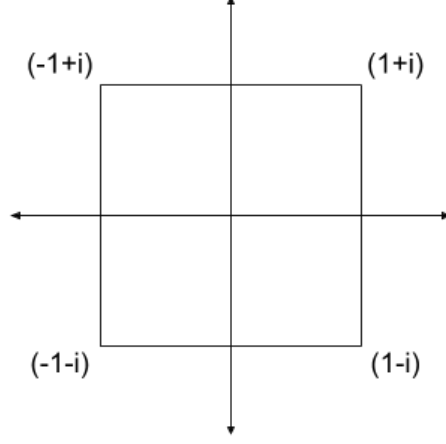


Figure 1: Contour for $N = 1/2$.

$$\lim_{N \rightarrow \infty} \int_{\Gamma_N} g(z) dz = \lim_{N \rightarrow \infty} \int_{\Gamma_N} \frac{P(z)}{Q(z)} \pi \cot \pi z dz = 0 \quad (24)$$

If we use the fact that $y(y) = P(z)/Q(z)$, where $\deg(Q(z)) \geq \deg(P(z)) + 2$. For $N \rightarrow \infty$, $K \approx N$, in which the contour length will approach infinity. An integral on this form tends to zero. we have that $\pi \cot(\pi z) \leq M$, in other words that $\pi \cot \pi z$ is less then or equal a *constant* M , we can write

$$M \lim_{N \rightarrow \infty} \int_{\Gamma_N} \frac{P(z)}{Q(z)} dz = 0. \quad (25)$$

c)

d)

Problem 3: The Dirac delta function

a)

We will use two usefull identeties in this proof;

$$\int_a^b \delta(t)g(x) = \begin{cases} 0 & : \notin (a, b) \\ g(x) & : \in (a, b) \end{cases}. \quad (26)$$

$$\int_a^b \delta(ft)g(x) = \frac{1}{|x|} \int \delta(x)g(x)dx \quad (27)$$

We start by using the hint and introduce an arbitrary test function $g(t)$. Taking the integral of this yields

$$\int_{-\infty}^{+\infty} \delta[f(x)]g(t)dt, \quad (28)$$

which can be written as a sum of three integrals

$$\int_{-\infty}^{t_0-\epsilon} \delta[f(t)]g(t)dt + \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f(t)]g(t)dt + \int_{t_0+\epsilon}^{+\infty} \delta[f(t)]g(t)dt. \quad (29)$$

By the definition of the delta function, we know that $f(t)$ only have a zero in the middle term and thus the first and last integral is zero which yields

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f(t)]g(t)dt. \quad (30)$$

Expanding $f(t)$ centered at t_0 up to the first order yields $f(t_0) + f'(t_0)(t - t_0)$ and thus, with $f(t_0) = 0$, we have

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f'(t_0)(t - t_0)]g(t)dt \quad (31)$$

By 27, we can write

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \frac{1}{|f'(t_0)|} \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t - t_0)g(t)dt \quad (32)$$

On a generalized form we can summarize over all t_i which yields

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \sum_i \frac{1}{|f'(t_i)|} \int_{t_i-\epsilon}^{t_i+\epsilon} \delta(t - t_i)g(t)dt \quad (33)$$

b)

(i)

For $f(t) = t^2 - a^2$ we have $f'(t) = 2t \rightarrow f'(a) = 2a, f'(-a) = -2a$ yielding

$$\delta(t^2 - a^2) = \delta((t + a)(t - a)) = \sum_i \frac{\delta(t - t_i)}{|2t|} = \frac{1}{2a}\delta(t - a) + \frac{1}{2a}\delta(t + a) \quad (34)$$

(i)

For $f(t) = \sin(t)$ we have $f'(t) = \cos(t) \rightarrow f'(0) = 1, f'(\pi) = -1$

$$\delta(\sin(t)) = \sum_i \frac{\delta(t - t_i)}{|\cos(t)|} = \delta(t - 0) + \delta(t + \pi) \quad (35)$$

c)

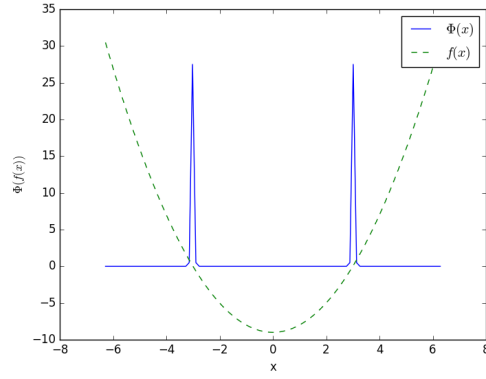


Figure 2: As expected we see two peaks at the roots $a \pm 3$.

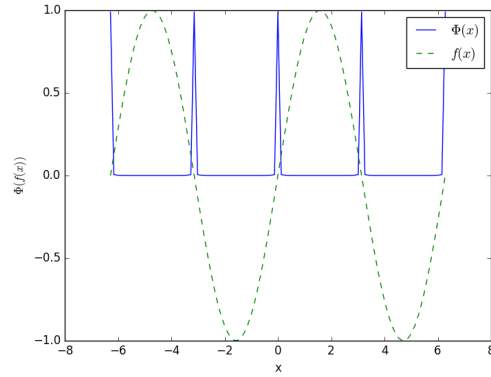


Figure 3: As expected we see periodical peaks at $n = 0, \pi, 2\pi, \dots$

d)

We start with the integral

$$\int_{-\pi/2}^{+\pi/2} \cos t \delta[\sin t] dt, \quad (36)$$

which can be written as

$$\int_{-\pi/2}^{+\pi/2} \cos t \sum_i \frac{\delta(t - t_i)}{|\cos t_i|} dt. \quad (37)$$

We are integrating from $-\pi/2$ to $\pi/2$. In this interval, $\sin t$ is only zero at $t = 0$ which yields

$$\int_{-\pi/2}^{+\pi/2} \cos t \frac{\delta(t)}{|\cos 0|} dt = \int_{-\pi/2}^{+\pi/2} \cos t \delta(t) dt = 1. \quad (38)$$