FYS3140 - Home exam 2018

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Problem 1: Differential equation

We are asked to find the general solution of the differential equation

$$y^{''}(x) + \frac{3}{x}y^{'}(x) - \frac{24}{x^2}y(x) = 56x^6. \tag{1}$$

Solving this equation involves two major steps; 1) find the complementary function f_c, and 2) find the particular solution. For the complementary function, we start by multiplying through by x^2 to get

$$x^2y^{''}(x) + 3xy^{'}(x) - 24y(x) = 56x^8, \tag{2}$$

which has the form $ax^2y^{''} + bxy^{y} + cy = g(x)$ and thus is a second order non-homogeneous Cauchy-Euler differential equation. Reconizing $a = 1, b = 3$ and $c = -24$, we can write

$$am(m-1) + bm + c = 0 \rightarrow m(m-1) + 3m - 24 = m^2 + 2m - 24 = 0, \tag{3}$$

which yields $m_1 = 4$ and $m_2 = -6$. Since m_1 and m_2 are two distinct real roots the complementary function is a function on the form

$$y_c = c_1x^{m_1} + c_2x^{m^2} \rightarrow y_c = c_1x^4 + c_2x^{-6}. \tag{4}$$

For the particular solution we will use *variation of parameters*. As we can see, 1 is on the form $y^{''} + p(x)y^{'} + q(x)y = g(x)$. Since $p(x) = 3/x, q(x) = -24/x^2$ and $g(x) = 56x^2$ are all continous on an open interval, the particular solution can be found by

$$Y_p = -y_1 \int \frac{y_2g(x)}{W(y_1, y_2)}dx + y_2 \int \frac{y_1g(x)}{W(y_1, y_2)}dx, \tag{5}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. y_1 and y_2 is from the complementary function. Starting by finding the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} \rightarrow W = \begin{vmatrix} x^4 & x^{-6} \\ 4x^3 & -6x^{-7} \end{vmatrix} = -6x^{-7}x^4 - 4x^3x^{-6} = -10x^{-3}, \tag{6}$$

we can write

$$
\begin{aligned}
Y_p &= -x^4 \int \frac{x^{-6} 56x^6}{-10x^{-3}} dx + x^{-6} \int \frac{x^4 56x^6}{-10x^{-3}} dx && (7)\\
&= \frac{56}{10}\left(x^4 \int x^3 \; dx - x^{-6} \int x^{13} \; dx \right) && (8)\\
&= \frac{56}{10}\left(\frac{x^8}{4} - \frac{x^8}{14} \right) && (9)\\
&= \frac{56}{10}\left(\frac{10x^8}{56} \right) && (10)\\
&= x^8 && (11)
\end{aligned}
$$

Finnaly, we find our general solution by adding the complementary function and the particular solution togetter

$$
y(x) = y_c + Y_P \rightarrow y(x) = c_1 x^4 + c_2 x^{-6} + x^8, \tag{12}
$$

which also can be written as

$$
y(x) = \frac{c_2}{x^6} + c_1 x^4 + x^8, \tag{13}
$$

and that's my final answer.

Problem 2: Complex analysis

Part A:

a)

For a function that has a *pole of order 3* at $z = 3+i$, a *zero of order 4* at $z = 2i$, we have the following function

$$
f(z) = \frac{(z-2i)^4}{(z-[3+i])^3} \tag{14}
$$

b)

We are asked to classify the isolated singularity of the function

$$
f(x) = \frac{z^3+8}{(z-5)^3(z+2)}. \tag{15}
$$

If we write

$$
f(x) = \frac{1}{(z-5)^3} \frac{z^3+8}{(z+2)}. \tag{16}
$$

Polynomial division, $(z^3+8) : (z+2)$, yields

$$
f(x) = \frac{z^2-2z+4}{(z-5)^3}, \tag{17}
$$

which shows $z = -2$ is a *removable singularity*. Now, if we write

$$\frac{1}{(z-5)^3} = \left(\frac{1}{z-5}\right)^3 = \left(-\frac{\frac{1}{5}}{1-\frac{z}{5}}\right)^3 \tag{18}$$

$$= \left(-\frac{1}{5}\frac{1}{1-\frac{z}{5}}\right)^3 = \left(-\frac{1}{5}\sum_{n=0}^{\infty}\left(\frac{z}{5}\right)^n\right)^3 \tag{19}$$

$$= \left(-\frac{1}{5}\left[1+\frac{z}{5}+\frac{z^2}{25}+\frac{z^3}{25}+...\right]\right)^3 \tag{20}$$

$$= \left(-\frac{1}{5}-\frac{z}{25}-\frac{z^2}{125}-\frac{z^3}{675}+...\right)^3 \tag{21}$$

Part B:

a)

We start writing $\cot(z)$ as $\cos(z)/\sin(z)$ to get

$$g(z) = f(z)\pi\cot(\pi z) = \frac{f(z)\pi\cos(\pi z)}{\sin(\pi z)}. \tag{22}$$

If we now set $a(z) \equiv f(z)\pi\cos\pi z$ and $b(z) \equiv \sin\pi z$ we have that $g(z) = a(z)/b(z)$, $a(n) =$ finite constant $\neq 0$, and $b(n) = 0$, $b^{'}(n) \neq 0$ and thus the residue can be found by equation 6.2 in Boas

$$Res(n) = \frac{a(n)}{b^{'}(n)} \rightarrow Res(n) = \frac{f(n)\pi\cos(\pi n)}{\pi\cos(\pi n)} = f(n), \tag{23}$$

which is what we were suppose to show.

b)

$N = 1/2$ yields $K = 1$, so let's go for that! With $K = 1$, we get the following contour

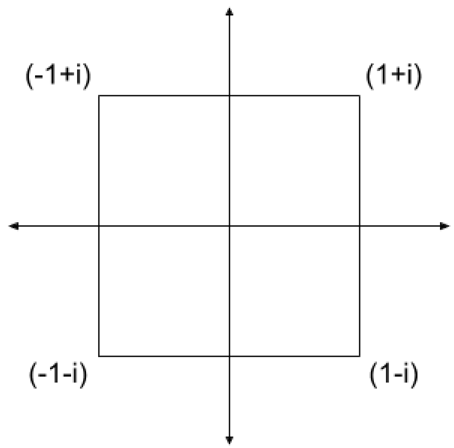

Figure 1: Contour for $N = 1/2$.

$$\lim_{N\to\infty} \int_{\Gamma_N} g(z)dz = \lim_{N\to\infty} \int_{\Gamma_N} \frac{P(z)}{Q(z)} \pi \cot \pi z dz = 0 \tag{24}$$

If we use the fact that $y(y) = P(z)/Q(z)$, where $deg(Q(z)) \geq deg(P(z)) + 2$. For $N \to \infty$, $K \approx N$, in which the contour length will approach infinity. An integral on this form tends to zero. we have that $\pi \cot(\pi z) \leq M$, in other words that $\pi \cot \pi z$ is less then or equal a *constant* M, we can write

$$M \lim_{N\to\infty} \int_{\Gamma_N} \frac{P(z}{Q(z)} dz = 0. \tag{25}$$

c)

d)

$$\pi \cot \pi = \pi \left(\frac{1}{\pi} - \frac{\pi}{3} - \frac{\pi^3}{45} - ... \right) \tag{26}$$

$$= \left(1 - \frac{\pi^2}{3} - \frac{\pi^4}{45} - ... \right) \tag{27}$$

$$\tag{28}$$

Problem 3: The Dirac delta function

a)

We will use two usefull identeties in this proof;

$$\int_a^b \delta(t)g(x) = \begin{cases} 0 & :\notin (a,b) \\ g(x) & :\in (a,b) \end{cases}. \tag{29}$$

$$\int_a^b \delta(ft)g(x) = \frac{1}{|x|}\int \delta(x)g(x)dx \tag{30}$$

We start by using the hint and introduce an arbitrary test funciton $g(t)$. Taking the intergal of this yields

$$\int_{-\infty}^{+\infty} \delta[f(x)]g(t)dt, \tag{31}$$

which can be written as a sum of three intergrals

$$\int_{-\infty}^{t_0-\epsilon} \delta[f(t)]g(t)dt + \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f(t)]g(t)dt + \int_{t_0+\epsilon}^{+\infty} \delta[f(t)]g(t)dt. \tag{32}$$

By the definition of the delta function, we know that $f(t)$ only have a zero in the middle term and thus the first and last integral is zero which yields

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f(t)]g(t)dt. \tag{33}$$

Expanding $f(t)$ centered at t_0 up to the first order yields $f(t_0) + f'(t_0)(t - t_0)$ and thus, with $f(t_0) = 0$, we have

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta[f'(t_0)(t-t_0)]g(t)dt \tag{34}$$

By 28, we can write

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \frac{1}{|f'(t_0)|}\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t-t_0)g(t)dt \tag{35}$$

On a generalized form we can summarize over all t_i which yields

$$\int_{-\infty}^{+\infty} \delta[f(t)]g(t)dt = \sum_i \frac{1}{|f'(t_i)|}\int_{t_i-\epsilon}^{t_i+\epsilon} \delta(t-t_i)g(t)dt \tag{36}$$

b)

(i)

For $f(t) = t^2 - a^2$ we have $f'(t) = 2t \to f'(a) = 2a,\ f'(-a) = -2a$ yielding

$$\delta(t^2 - a^2) = \delta((t+a)(t-a)) = \sum_i \frac{\delta(t-t_i)}{|2t|} = \frac{1}{2a}\delta(t-a) + \frac{1}{2a}\delta(t+a) \tag{37}$$

(i)

For $f(t) = \sin(t)$ we have $f'(t) = cos(t) \rightarrow f'(0) = 1$ $f'(\pi) = -1$

$$\delta(\sin(t)) = \sum_i \frac{\delta(t - t_i)}{|cos(t)|} = \delta(t - 0) + \delta(t + \pi) \tag{38}$$

c)

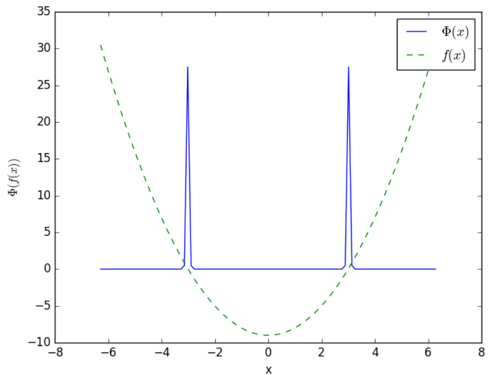

Figure 2: As expected we see two peeks at the roots $a \pm 3$.

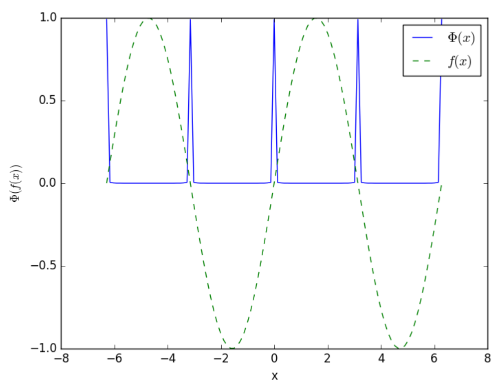

Figure 3: As expected we see periodical peeks at $n = 0, \pi, 2\pi, ...$

d)

We start with the integral

$$\int_{-\pi/2}^{+\pi/2} \cos t \delta[\sin t] dt, \tag{39}$$

which can we written as

$$\int_{-\pi/2}^{+\pi/2} \cos t \sum_i \frac{\delta(t - t_i)}{|\cos t_i|} dt. \tag{40}$$

We are integrating from $-\pi/2$ to $\pi/2$. In this interval, $\sin t$ is only zero at $t = 0$ which yields

$$\int_{-\pi/2}^{+\pi/2} \cos t \frac{\delta(t)}{|\cos 0|} dt = \int_{-\pi/2}^{+\pi/2} \cos t \delta(t) dt = 1. \tag{41}$$