FYS3140 - Home exam 2018

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Problem 1: Differential equation

We are asked to find the general solution of the differential equation

$$y^{''}(x) + \frac{3}{x}y^{'}(x) - \frac{24}{x^2}y(x) = 56x^6. \tag{1}$$

Solving this equation involves two major steps; 1) find the complementary function f_c, and 2) find the particular solution. For the complementary function, we start by multiplying through by x^2 to get

$$x^2y^{''}(x) + 3xy^{'}(x) - 24y(x) = 56x^8, \tag{2}$$

which has the form $ax^2y^{''} + bxy^y + cy = g(x)$ and thus is a second order non-homogeneous Cauchy-Euler differential equation. Reconizing $a = 1, b = 3$ and $c = -24$, we can write

$$am(m-1) + bm + c = 0 \rightarrow m(m-1) + 3m - 24 = m^2 + 2m - 24 = 0, \tag{3}$$

which yields $m_1 = 4$ and $m_2 = -6$. Since m_1 and m_2 are two distinct real roots the complementary function is a function on the form

$$y_c = c_1x^{m_1} + c_2x^{m^2} \rightarrow y_c = c_1x^4 + c_2x^{-6}. \tag{4}$$

For the particular solution we will use *variation of parameters*. As we can see, 1 is on the form $y^{''} + p(x)y^{'} + q(x)y = g(x)$. Since $p(x) = 3/x, q(x) = -24/x^2$ and $g(x) = 56x^2$ are all continous on an open interval, the particular solution can be found by

$$Y_p = -y_1 \int \frac{y_2g(x)}{W(y_1, y_2)}dx + y_2 \int \frac{y_1g(x)}{W(y_1, y_2)}dx, \tag{5}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. y_1 and y_2 is from the complementary function. Starting by finding the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^{'} & y_2^{'} \end{vmatrix} \rightarrow W = \begin{vmatrix} x^4 & x^{-6} \\ 4x^3 & -6x^{-7} \end{vmatrix} = -6x^{-7}x^4 - 4x^3x^{-6} = -10x^{-3}, \tag{6}$$

we can write

$$
\begin{aligned}
Y_p &= -x^4 \int \frac{x^{-6} 56x^6}{-10x^{-3}} dx + x^{-6} \int \frac{x^4 56x^6}{-10x^{-3}} dx && (7) \\
&= \frac{56}{10} \left(x^4 \int x^3 \ dx - x^{-6} \int x^{13} \ dx \right) && (8) \\
&= \frac{56}{10} \left(\frac{x^8}{4} - \frac{x^8}{14} \right) && (9) \\
&= \frac{56}{10} \left(\frac{10x^8}{56} \right) && (10) \\
&= x^8 && (11)
\end{aligned}
$$

Finnaly, we find our general solution by adding the complementary function and the particular solution togetter

$$
y(x) = y_c + Y_P \rightarrow y(x) = c_1 x^4 + c_2 x^{-6} + x^8, \tag{12}
$$

which also can be written as

$$
y(x) = \frac{c_2}{x^6} + c_1 x^4 + x^8, \tag{13}
$$

and that's my final answer.

Problem 2: Complex analysis

Part A:

a)

For a function that has a *pole of order 3* at $z = 3+i$, a *zero of order 4* at $z = 2i$, we have the following function

$$
f(z) = \frac{(z - 2i)^4}{(z - [3 + i])^3} \tag{14}
$$

b)

We are asked to classify the isolated singularity of the function

$$
f(x) = \frac{z^3 + 8}{(z - 5)^3 (z + 2)}. \tag{15}
$$

If we write

$$
f(x) = \frac{1}{(z - 5)^3} \frac{z^3 + 8}{(z + 2)}. \tag{16}
$$

Polynomial division, $(z^3 + 8) : (z + 2)$, yields

$$
f(x) = \frac{z^2 - 2z + 4}{(z - 5)^3}, \tag{17}
$$

which shows $z=-2$ is a *removable singularity*. Now, if we write

$$\frac{1}{(z-5)^3} = \left(\frac{1}{z-5}\right)^3 = \left(-\frac{\frac{1}{5}}{1-\frac{z}{5}}\right)^3 \tag{18}$$

$$= \left(-\frac{1}{5}\frac{1}{1-\frac{z}{5}}\right)^3 = \left(-\frac{1}{5}\sum_{n=0}^{\infty}\big(\frac{z}{5}\big)^n\right)^3 \tag{19}$$

$$= \left(-\frac{1}{5}\left[1+\frac{z}{5}+\frac{z^2}{25}+\frac{z^3}{25}+...\right]\right)^3 \tag{20}$$

$$= \left(\left[-\frac{1}{5}-\frac{z}{25}-\frac{z^2}{125}-\frac{z^3}{675}+...\right]\right)^3 \tag{21}$$

Part B:

a)

b)

c)

d)

Problem 3: The Dirac delta function

a)

b)

c)

a)