

# FYS3140 - Home exam 2018

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## Problem 1: Differential equation

We are asked to find the general solution of the differential equation

$$y''(x) + \frac{3}{x}y'(x) - \frac{24}{x^2}y(x) = 56x^6. \quad (1)$$

Solving this equation involves two major steps; 1) find the complementary function  $f_c$ , and 2) find the particular solution. For the complementary function, we start by multiplying through by  $x^2$  to get

$$x^2y''(x) + 3xy'(x) - 24y(x) = 56x^8, \quad (2)$$

which has the form  $ax^2y'' + bxy' + cy = g(x)$  and thus is a second order non-homogeneous Cauchy-Euler differential equation. Recognizing  $a = 1, b = 3$  and  $c = -24$ , we can write

$$am(m-1) + bm + c = 0 \rightarrow m(m-1) + 3m - 24 = m^2 + 2m - 24 = 0, \quad (3)$$

which yields  $m_1 = 4$  and  $m_2 = -6$ . Since  $m_1$  and  $m_2$  are two distinct real roots the complementary function is a function on the form

$$y_c = c_1x^{m_1} + c_2x^{m_2} \rightarrow y_c = c_1x^4 + c_2x^{-6}. \quad (4)$$

For the particular solution we will use *variation of parameters*. As we can see, [1](#) is on the form  $y'' + p(x)y' + q(x)y = g(x)$ . Since  $p(x) = 3/x, q(x) = -24/x^2$  and  $g(x) = 56x^2$  are all continuous on an open interval, the particular solution can be found by

$$Y_p = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} dx, \quad (5)$$

where  $W(y_1, y_2)$  is the Wronskian of  $y_1$  and  $y_2$ .  $y_1$  and  $y_2$  is from the complementary function. Starting by finding the Wronskian of  $y_1$  and  $y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \rightarrow W = \begin{vmatrix} x^4 & x^{-6} \\ 4x^3 & -6x^{-7} \end{vmatrix} = -6x^{-7}x^4 - 4x^3x^{-6} = -10x^{-3}, \quad (6)$$

we can write

$$Y_p = -x^4 \int \frac{x^{-6} 56x^6}{-10x^{-3}} dx + x^{-6} \int \frac{x^4 56x^6}{-10x^{-3}} dx \quad (7)$$

$$= \frac{56}{10} \left( x^4 \int x^3 dx - x^{-6} \int x^{13} dx \right) \quad (8)$$

$$= \frac{56}{10} \left( \frac{x^8}{4} - \frac{x^8}{14} \right) \quad (9)$$

$$= \frac{56}{10} \left( \frac{10x^8}{56} \right) \quad (10)$$

$$= x^8 \quad (11)$$

Finally, we find our general solution by adding the complementary function and the particular solution together

$$y(x) = y_c + Y_p \rightarrow y(x) = c_1 x^4 + c_2 x^{-6} + x^8, \quad (12)$$

which also can be written as

$$y(x) = \frac{c_2}{x^6} + c_1 x^4 + x^8, \quad (13)$$

and that's my final answer.

## Problem 2: Complex analysis

### Part A:

a)

For a function that has a *pole of order 3* at  $z = 3 + i$ , a *zero of order 4* at  $z = 2i$ , we have the following function

$$f(z) = \frac{(z - 2i)^4}{(z - [3 + i])^3} \quad (14)$$

b)

We are asked to classify the isolated singularity of the function

$$f(x) = \frac{z^3 + 8}{(z - 5)^3(z + 2)}. \quad (15)$$

If we write

$$f(x) = \frac{1}{(z - 5)^3} \frac{z^3 + 8}{(z + 2)}. \quad (16)$$

Polynomial division,  $(z^3 + 8) : (z + 2)$ , yields

$$f(x) = \frac{z^2 - 2z + 4}{(z - 5)^3}, \quad (17)$$

which shows  $z = -2$  is a *removable singularity*. Now, if we write

$$\frac{1}{(z-5)^3} = \left( \frac{1}{z-5} \right)^3 = \left( -\frac{\frac{1}{5}}{1-\frac{z}{5}} \right)^3 \quad (18)$$

$$= \left( -\frac{1}{5} \frac{1}{1-\frac{z}{5}} \right)^3 = \left( -\frac{1}{5} \sum_{n=0}^{\infty} \left( \frac{z}{5} \right)^n \right)^3 \quad (19)$$

$$= \left( -\frac{1}{5} \left[ 1 + \frac{z}{5} + \frac{z^2}{25} + \frac{z^3}{25} + \dots \right] \right)^3 \quad (20)$$

$$= \left( -\frac{1}{5} - \frac{z}{25} - \frac{z^2}{125} - \frac{z^3}{675} + \dots \right)^3 \quad (21)$$

## Part B:

a)

We start writing  $\cot(z)$  as  $\cos(z)/\sin(z)$  to get

$$g(z) = f(z)\pi \cot(\pi z) = \frac{f(z)\pi \cos(\pi z)}{\sin(\pi z)}. \quad (22)$$

If we now set  $a(z) \equiv f(z)\pi \cos \pi z$  and  $b(z) \equiv \sin \pi z$  we have that  $g(z) = a(z)/b(z)$ ,  $a(n) = \text{finite constant} \neq 0$ , and  $b(n) = 0$ ,  $b'(n) \neq 0$  and thus the residue can be found by equation 6.2 in Boas

$$\text{Res}(n) = \frac{a(n)}{b'(n)} \rightarrow \text{Res}(n) = \frac{f(n)\pi \cos(\pi n)}{\pi \cos(\pi n)} = f(n), \quad (23)$$

which is what we were suppose to show.

b)

c)

d)

## Problem 3: The Dirac delta function

a)

We start by using the hint and introduce an arbitrary test function  $g(x)$ . If we write

$$\int_{-\infty}^{+\infty} \delta[f(x)]g(x)dx \quad (24)$$

Then, for every  $t_i$ , split the integral into parts, namely

$$\int_{-\infty}^{+x_0-\epsilon} \delta[f(x)]g(x)dx + \int_{-x_0-\epsilon}^{+x_0+\epsilon} \delta[f(x)]g(x)dx + \int_{x_0+\epsilon}^{+\infty} \delta[f(x)]g(x)dx \quad (25)$$

**b)**

**(i)**

For  $f(t) = t^2 - a^2$  we have  $f'(t) = 2t \rightarrow f'(1) = 2, f'(2) = 4$  yielding

$$\delta(t^2 - a^2) = \delta((t+a)(t-a)) = \sum_i \frac{\delta(t-t_i)}{|2t|} = \frac{1}{2}\delta(t-1) + \frac{1}{4}\delta(t-2) \quad (26)$$

**(i)**

For  $f(t) = \sin(t)$  we have  $f'(t) = \cos(t) \rightarrow f'(0) = 1, f'(\pi) = -1$

$$\delta(\sin(t)) = \sum_i \frac{\delta(t-t_i)}{|\cos(t)|} = \delta(t-0) + \delta(t+\pi) \quad (27)$$

**c)**

**d)**