Enumerating The Rationals

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- Naive approach
- Reduced rationals
- Stern-Brocot Tree
- Calkin-Wilf Tree
- Deforested Calkin-Wilf

CoLists and CoTrees

- CoData
  - CoLists

    ```
    Cons : (Q, CoList) -> CoList Q
    ```

  - CoTrees

    ```
    Node : (CoTree Q, Q, CoTree Q) -> CoTree Q

    unfold f s =
      let (x, v, y) = f s
      in Node (unfold f x, v, unfold f y)
    ```

Naive Approach

- step : Q -> (Q, Q, Q)
  step $\frac{n}{d} = (\frac{n+1}{d}, \frac{n}{d}, \frac{n}{d+1})$
- tree : CoTree Q
  tree $=$ unfold step $\frac{1}{1}$

Reduced rationals

```
gcd : Q -> N
gcd p/q =
  if m < n then gcd (m, m - n) else
  if m > n them gcd (m - n, n) else m
```

- `[Bool]`: Finite bit strings, gcd execution traces

```
igcd : Q -> (N, [Bool])
igcd p/q =
  if m < n then step True  $ igcd (m, m - n) else
  if m > n them step False $ igcd (m - n, n) else m
     where step b (d, bs) = (d, b : bs)
```

- pgcd : $Q \to$ [Bool]
- ungcd : (N, [Bool]) $\to Q$

Reduced rationals (Cont'd)

- Enumerate all rationals:
  1. Enumerate all bit finite bit strings (without dupes)
  2. Map λ p . ungcd (1, p)
  3. Done

Stern-Brocot Tree

- efficient *trie* representation of finite bit strings
- search tree property for every finite pruning

Stern-Brocot Tree (Cont'd)

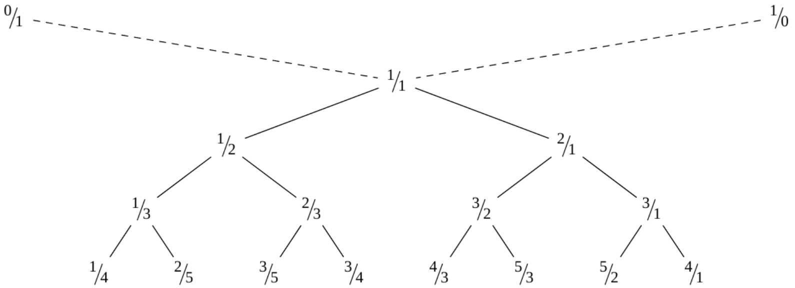

Fig. 1. The first few levels of the Stern-Brocot tree.

- step : (Q, Q) $\to$ ((Q, Q), Q, (Q, Q))
  step $(\frac{a}{b}, \frac{c}{d}) = \text{let } m = \frac{a+c}{b+d} \text{ in } ((\frac{a}{b}, m), m, (m, \frac{c}{d}))$
- tree : CoTree Q
  tree = unfold step $(\frac{0}{1}, \frac{1}{0})$

Things to prove

- $\forall q \in Q$: ungcd (igcd q) $=$ q
  - the process really is reversible
- $\forall p, q \in Q$: (pgcd p $=$ pgcd q) $\Rightarrow (p \sim q)$
  - bit string assignment is injective when identifying equivalent fractions
- $\forall p \in [Bool]$: `pgcd ( ungcd (1, p) ) = p`
  - every bit string corresponds to at least one fraction
- $\forall p \in [Bool]$: `pgcd ( lookup p tree ) = p`
  - every fraction in the tree is at the correct node

Problems with Stern-Brocot Tree

- Unfolding requires the "pseudo-rationals" $\frac{0}{1}$ and $\frac{1}{0}$ as input;

Problems with Stern-Brocot Tree

- Unfolding requires the “pseudo-rationals” $\frac{0}{1}$ and $\frac{1}{0}$ as input;
- Relation between Q-reduction and path in the Stern-Brocot tree requires both to use equivalent implementations of `gcd`

Conclusions

Calkin-Wilf Tree

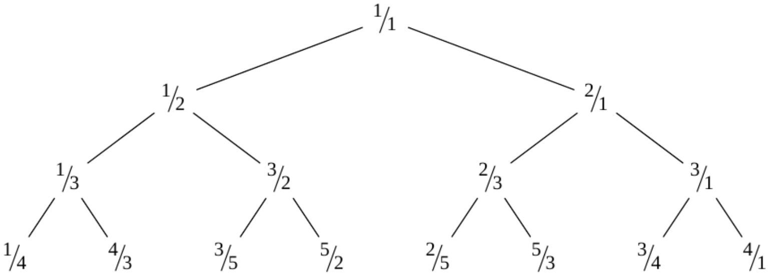

Fig. 2. The first few levels of the Calkin-Wilf tree.

- step : Q $\rightarrow$ (Q, Q, Q)
  step $\frac{m}{n} = (\frac{m}{m+n}, \frac{m}{n}, \frac{m+n}{n})$
- tree : Cotree Q
  tree = unfold step $\frac{1}{1}$