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Subtractive logic

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Abstract

This paper is the first part of a work whose purpose is to investigate duality in some related frameworks (cartesian closed categories, lambda-calculi, intuitionistic and classical logics) from syntactic, semantical and computational viewpoints. We start with category theory and we show that any bicartesian closed category with coexponents is degenerated (i.e. there is at most one arrow between two objects). The remainder of the paper is devoted to logical issues. We examine the propositional calculus underlying the type system of bicartesian closed categories with coexponents and we show that this calculus corresponds to subtractive logic: a conservative extension of intuitionistic logic with a new connector (subtraction) dual to implication. Eventually, we consider first-order subtractive logic and we present an embedding of classical logic into subtractive logic. © 2001 Published by Elsevier Science B.V.

0. Introduction

This paper is the first part of a work whose purpose is to investigate duality in some related frameworks (cartesian closed categories, lambda-calculi, intuitionistic and classical logics) from syntactic, semantical and computational viewpoints.

It is rather natural to begin with category theory where duality is a built-in concept. Indeed, to any categorical notion corresponds immediately a dual notion. In particular, we give the definition of a coexponent, dual notion of the exponent of cartesian closed categories (CCC). This leads then to the definition of a bi-[CCC] (i.e. a CCC whose dual is also a CCC). This structure seems to be studied for the first time by Filinski [8] within the framework of functional languages semantics.

From a logical standpoint, if we consider the *symmetrical categorical propositional calculus* underlying the axiomatics of bi-[CCC], we easily show that this calculus

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corresponds to the logic studied by Rauszer [19, 20] (let us call it subtractive logic). This logic is a conservative extension of propositional intuitionistic logic with a new connective (subtraction) dual to implication. It is interesting to notice that subtraction allows to define a "weak negation" for which the excluded-middle law holds (but not the contradiction law). We will prove that subtraction is not definable from weak negation.

Moreover, we will see that topological and Kripke semantics of subtractive propositional logic are the same as in intuitionistic logic and simpler in the first-order framework (however, in this framework subtractive logic is no longer conservative over intuitionistic logic since the axiom scheme called DIS is provable). Duality also allows to define a very simple embedding of classical logic into subtractive logic.

This paper is organized as follows: the first section deals with category theory whereas the remainder of the paper is devoted to logical issues (Sections 2–4 are independent of Section 1). We show in the first section that any bi-[CCC] is degenerated (there is at most one arrow between two objects). As a corollary we obtain that, in the category of sets and total functions, the coexponent of two sets is in general not defined.

In Section 2, we define a symmetrical categorical propositional calculus and we show that this calculus corresponds to C. Rauszer's subtractive propositional logic [19, 20]. We give a direct proof of conservativity over intuitionistic logic using Kripke semantics. Then we prove that the deduction theorem does not hold in the symmetrical categorical propositional calculus (and consequently, there is no functionnal completeness in bi-[CCC]). To round off this section, we extend the work of Rauszer with some new properties of subtractive logic (mainly non-definability results).

In Section 3, we examine why first-order subtractive logic is no longer conservative over intuitionistic logic although it is conservative over DIS-logic (which is also called Constant Domain Logic in the literature).

In Section 4, we extend Gentzen's sequent calculus LK with subtraction (we thus obtain SLK) and then we restrict this (classical) calculus to subtractive logic. Eventually, we define a simple embedding of classical logic into subtractive logic and then we show how to translate cut-free proofs of LK.

1. Bi-CCC with coexponents

This structure seems to be studied for the first time by Filinski [8] within the framework of functional languages semantics. Filinski shows that duality in bi-[CCC] may be interpreted in an elegant way as a duality between values and continuations. He builds for that purpose a symmetrical λ-calculus in which continuations can be explicitly handled (as well as values). Then, he extends Lambek and Scott's well-known theorem which expresses the equivalence between CCC and simply typed λ-calculus [14]. However, to obtain functional completeness in bi-[CCC] (i.e. the property which states the ability to simulate λ-abstraction), Filinski is led to extend the axiomatics of bi-[CCC] by adding a new morphism (and its dual).

Surprisingly, from a logical standpoint, the type of this morphism is a generalisation of the excluded-middle axiom. We will prove in Section 2 that this extension to classical logic is already justified by logical arguments: the deduction theorem does not hold in the symmetrical categorical propositional calculus, while adding it as a rule yields classical logic.

We will here confine ourselves to the theory of bi-[CCC] (without additional morphism). We first recall the definition of a category and how duality is expressed in category theory. Then we show that any bi-[CCC] is degenerated (there is at most one arrow between two objects), although this snag was hitherto closely related to classical logic. As a corollary we obtain that, in the category of sets and total functions, the coexponent of two sets is in general not defined (more specifically, we will show that in fact it is defined if and only if either one set or the other is empty).

1.1. Categories

A *directed graph* is a structure G consisting of a collection of *objects* and a collection of *arrows* together with two applications, called *source* and *target*, which both assign an object to any arrow. A *category* may be defined as a directed graph with some extra structure: a unary application $Id : objects(\mathscr{C}) \to arrows(\mathscr{C})$ such that: $source(Id_A) = target(Id_A) = A$ for any A of $objects(\mathscr{C})$ and a partial binary function $\circ : arrows(\mathscr{C}) \times arrows(\mathscr{C}) \to arrows(\mathscr{C})$, which is defined on (g, f) if and only if $target(f) = source(g)$, and such that $target(g \circ f) = target(g)$ and $source(g \circ f) = source(f)$. Moreover, these applications must satisfy:

- $f \circ Id_A = Id_B \circ f = f$, for any $f : A \to B$.
- $(h \circ g) \circ f = h \circ (g \circ f)$, for any $f : A \to B$, $g : B \to C$ and $h : C \to D$

Notation. For any category $\mathscr{C}$, we will denote by $\mathscr{C}[A, B]$ the set of arrows (or *morphisms*) whose source is A and whose target is B.

Definition 1.1. Let $\mathscr{C}$ be a category, an arrow $i : A \to B$ is an *isomorphism* if there is an arrow $j : B \to A$ such that $j \circ i = Id_A$ and $i \circ j = Id_B$. The objects A and B are then called *isomorphic*.

1.1.1. Dual category

Given a category $\mathscr{C}$, one can define its dual category, denoted $\mathscr{C}^\perp$, in the following way: the objects of $\mathscr{C}^\perp$ are the objects of $\mathscr{C}$, the arrows of $\mathscr{C}^\perp$ are the arrows of $\mathscr{C}$ and the applications *source* and *target* of $\mathscr{C}^\perp$ are, respectively, the application *target* and *source* of $\mathscr{C}$. In other words, the arrows of $\mathscr{C}^\perp$ are obtained by *inverting* the arrows of $\mathscr{C}$, that can also be stated as $\mathscr{C}^\perp[A, B] \equiv \mathscr{C}[B, A]$. We will denote by $f^\perp$ the arrow of $\mathscr{C}^\perp$ obtained by inverting some arrow f of $\mathscr{C}$. The identity in $\mathscr{C}^\perp$ is the same as in $\mathscr{C}$. The composition of two arrows $f^\perp : B \to A$ and $g^\perp : C \to B$ is defined by $f^\perp \circ g^\perp = (g \circ f)^\perp : C \to A$. (Notice that this duality is involutive, i.e. for any category $\mathscr{C}$, $(\mathscr{C}^\perp)^\perp = \mathscr{C}$.)

1.2. Applying duality

Let us first recall familiar constructions of a bicartesian category: final and initial objects, product and coproduct (see [1] for instance). They correspond, respectively, to the set-theoretical notions of singleton and empty set, cartesian product and disjoint union. Then we consider the construction dual to the exponent.

1.2.1. Final and initial objects

Definition 1.2. In a category $\mathscr{C}$, an object $\top$ is *final* if for any object A of $\mathscr{C}$, there is a unique arrow in $\mathscr{C}[A,\top]$ (we denote by $\diamondsuit_A : A \to \top$ this unique arrow).

Definition 1.3. In a category $\mathscr{C}$, an object $\perp$ is *initial* if for any object A of $\mathscr{C}$, there is a unique arrow in $\mathscr{C}[\perp,A]$ (we denote by $\square_A : \perp \to A$ this unique arrow).

1.2.2. Product and coproduct

Definition 1.4. In any category $\mathscr{C}$, given two objects A and B an object, we call *product* of A and B an object, denoted by $A \times B$, together with two arrows $\pi_{A,B} : A \times B \to A$ and $\pi'_{A,B} : A \times B \to B$ which satisfy this property: for any object C, any arrows $f : C \to A$ and $g : C \to B$, there is a unique arrow $h : C \to A \times B$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_{A,B}"] \arrow[r, "\pi'_{A,B}"'] & B \end{tikzcd}$$

We denote by $\langle f, g \rangle$ this unique h.

Definition 1.5. A category in which any pair of objects admits a product is called a *cartesian category*.

Definition 1.6. In any category $\mathscr{C}$, we call *coproduct* of two objects A and B an object, denoted by $A \oplus B$, together with two arrows $\iota_{A,B} : A \to A \oplus B$ and $\iota'_{A,B} : B \to A \oplus B$ which satisfy this property: for any object C, any arrows $f : A \to C$ and $g : B \to C$, there is a unique arrow $h : A \oplus B \to C$ such that the following diagram commutes:

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, "\iota_{A,B}"'] & A \oplus B \arrow[u, "h"] & B \arrow[ul, "g"'] \arrow[l, "\iota'_{A,B}"] \end{tikzcd}$$

We denote by $[f, g]$ this unique h.

Notation. Given two arrows $f:A\to C$ and $g:B\to D$ of $\mathscr{C}$, as usual, we denote by $f\times g$ the arrow (unique by definition of $C\times D$) $\langle f\circ\pi_{A,B}, g\circ\pi'_{A,B}\rangle : A\times B\to C\times D$ and $f\oplus g$ the arrow (unique by the definition of $A\oplus B$) $[\iota_{C,D}\circ f, \iota'_{C,D}\circ g]: A\oplus B\to C\oplus D$.

Definition 1.7. A cartesian category in which any pair of objects admits a coproduct is called a *bicartesian category*.

1.2.3. Exponents

Given two sets A and B, let us denote by B^A the set of applications from A to B. For any set C, any application from $C\times A$ to B corresponds in a bijective way (by curryfication) to some application from C to B^A. The following definition generalizes this concept with a generic cartesian category.

Definition 1.8. In any cartesian category $\mathscr{C}$, we call *exponent* of two objects A and B an object, denoted by B^A, together with an arrow $\epsilon_{A,B} : B^A\times A\to B$ which satisfy the following property: for any object C, any arrow $f : C\times A\to B$, there is a unique arrow $h : C\to B^A$ such that the next diagram commutes:

$$\begin{tikzcd} C \arrow[d, "h"'] & C\times A \arrow[r, "f"] \arrow[d, "h\times Id_A"'] & B \\ B^A & B^A\times A \arrow[ur, "\epsilon_{A,B}"'] & \end{tikzcd}$$

We denote by $f^\star$ this unique h.

Remark. Intuitively, the object B^A represents the set of morphisms of $\mathscr{C}[A,B]$, ϵ corresponds to evaluation and * to curryfication. Just as in the category of sets, we can show that the operation $f\mapsto f^\star$ is a bijection between the set of arrows from $C\times A$ to B and the set of arrows from C to B^A. In other words (denoting the set-theoretic bijection by $\approx$)

$$\mathscr{C}[C\times A,B]\approx\mathscr{C}[C,B^A].$$

Definition 1.9. A cartesian (resp. bicartesian) category in which any pair of objects admits an exponent is called a *cartesian* (*resp. bicartesian*) *closed category*.

1.2.4. Coexponents

Applying duality again, it is possible to give a definition of coexponents. Namely, the coexponent of two objects A and B of a bicartesian category $\mathscr{C}$ can be obtained by taking the exponent of A and B in $\mathscr{C}^\perp$ (if it exists) and then inverting $\epsilon_{A,B}$ (which is an arrow of $\mathscr{C}^\perp$). We do not give any intuition of it because we will see that this definition does not correspond to any set-theoretic notion (see Corollary 1.15).

Definition 1.10. In any bicartesian category $\mathscr{C}$, we call *coexponent* of two objects A and B an object, denoted by B_A, together with an arrow $\ni_{A,B}: B\to B_A\oplus A$ which satisfy this property: for any object C, any arrow $f : B \to C \oplus A$, there is a unique arrow $h : B_A \to C$ such that the following diagram commutes:

$$\begin{tikzcd} C & C \oplus A & B \arrow[l, "f"'] \arrow[dl, "\ni_{A,B}"] \\ B_A \arrow[u, "h"] & B_A \oplus A \arrow[u, "h \oplus Id_A"] & \end{tikzcd}$$

We denote by $f^{\star}$ this unique h.

Definition 1.11. A bicartesian closed category in which any pair of objects admits a coexponent is called a *bi*-[*cartesian closed*] category.

Notation. To avoid any confusion, we will shorten bi-[cartesian closed] category in bi-[CCC] since bi-CCC is the abbreviation generally used for *bicartesian closed category* (i.e. with coproducts but without coexponents).

A bi-[CCC] is a CCC whose dual is also a CCC (and thus a bi-[CCC]). A natural question is then: *Does there exist non-trivial such structures*?. The answer is *no* since any bi-[CCC] is degenerated (i.e. there is at most one arrow between two objects).

1.3. Any bi-[CCC] is degenerated

It is known that any CCC provided with an involution is degenerated (see [11, Appendix B]). In a more general way, any attempt to extend the theory of CCC to the interpretation of classical logic results in the same pitfall [14, 22]. We show here that this phenomenon appears already within the intuitionistic framework (see Theorem 2.13). For that we will need Joyal's theorem (see [14, p. 67]).

Lemma 1.12. *In any CCC, if $\perp$ is an initial object then $(\perp \times A)$ too.*

Proof. Indeed, because $C[(\perp \times A), B] \approx C[\perp, B^A]$ (see the remark following Definition 1.8). $\square$

Theorem 1.13 (Joyal). *In any CCC, if $\perp$ is initial and if $C[A, \perp]$ is non-empty then A is initial.*

Proof. Let us show that A is isomorphic to $\perp \times A$. If f is an arrow of $C[A, \perp]$, then the inverse morphisms between f and $\perp \times A$ are $\langle f, Id_A \rangle : A \to \perp \times A$ and $\pi'_{\perp,A} : \perp \times A \to A$. Indeed, $\pi'_{\perp,A} \circ \langle f, Id_A \rangle = Id_A$ (by definition of the product) and $\langle f, Id_A \rangle \circ \pi'_{\perp,A} = Id_{\perp \times A}$ since $\perp \times A$ is initial (by Lemma 1.12).

Theorem 1.14. *Any bi-[CCC] is degenerated: there is at most one arrow between two objects.*

Proof. In all CCC, since $\top \times B$ is isomorphic to B,

$$C[B,A] \approx C[(\top \times B),A] \approx C[\top,A^B]$$

and thus by duality:

$$C[A,B] \approx C[A,(\perp \oplus B)] \approx C[A_B,\perp]$$

then by Joyal's theorem (since $\perp$ is initial), $C[A_B,\perp]$ contains at most one arrow. □

As a direct corollary, we know that in the category of sets, the coexponent of two sets is not always defined. This corollary is obvious since the category of sets is clearly not degenerated. It is, nevertheless, possible to be more specific. Let us recall first that in any cartesian category, if $\top$ is final, the exponents $B^\top$ and $\top^A$ are defined for any objects A,B (take $B^\top = B$ with $\epsilon_{\top,B} = \pi_{B,\top}$ and $\top^A = \top$ with $\epsilon_{A,\top} = \Diamond_{\top\times A}$). By duality, in any bicartesian category, if $\perp$ are initial, the coexponents $B_\perp$ and $\perp_A$ are always defined for any objects A,B (take $B_\perp = B$ with $\ni_{\perp,B} = \imath_{B,\perp}$ and $\perp_A = \perp$ with $\ni_{A,\perp} = \Box_{\perp\oplus A}$). We will show that, in the category of sets, the coexponent is defined only in these two cases:

Proposition 1.15. *In the category of sets, the coexponent B_A of two sets A and B is defined if and only if $A = \emptyset$ or $B = \emptyset$.*

Proof. We know that, since $\emptyset$ is initial in the bicartesian closed category of sets, $B_\emptyset$ and $\emptyset_A$ are defined for any A,B. Let us assume now that A and B are not empty but that B_A is defined nonetheless. That means, in particular, that $\ni_{A,B}$ is interpreted by a total function from B to $B_A \oplus A$. This function chooses, for each element b of B, a *side* of $B_A \oplus A$. But this function must satisfy the following property: for any set C and any application f from B to $C \oplus A$, $(f_\star \oplus Id_A) \circ \ni_{A,B} = f$. But since $f_\star \oplus Id_A$ leaves the side unchanged, it is enough to take some non-empty set C and some function f which chooses in b a side different from $\ni_{A,B}$ and the property is not satisfied any longer, hence the contradiction. □

Remark. In this proof, we do not use the uniqueness of $f_\star$. Consequently in the category of sets, the *weak* coexponent (i.e. without this property of uniqueness) of two sets A and B is defined, for the same reason as above, if and only if $A = \emptyset$ or $B = \emptyset$.

1.4. No functional completeness

Functional completeness is the main result that makes possible to prove the equivalence between CCC and simply typed lambda-calculus (see [14, p. 61]): this expresses the definability of λ-abstraction from categorical combinators. Informally, it can be stated as follows: if given a hypothetical arrow $x : \top \to A$ it is possible to build an arrow $\phi(x) : \top \to B$, then there is an arrow $f : A \to B$ such that $f \circ x = \phi(x)$.

A formal statement requires to define the *polynomial category* to which belongs the term $\phi(x)$. This formalism will not be necessary to show that this property does not hold in bi-[CCC]. Indeed, we will show that there does not exist any arrow $f : A \to B$ such that $f \circ x = \phi(x)$, by showing that sometimes, there exists no arrow of type $A \to B$ at all.

More formally, the result of functional completeness admits a corollary in the logic defined by the typing rules of CCC. It is this corollary, called *deduction theorem* in [14] (p. 51), expressed in the logic corresponding to bi-[CCC], that we will disprove in Section 2.5.

On the other hand, by adding two new morphisms typed, respectively, by $(B \oplus C)^A \to B^A \oplus C$ and its dual $B \times C_A \to (B \times C)_A$ and some equations which define them, it is possible of recover functional completeness [8]. It is surprising to notice that the logical interpretation of the type of the first morphism, $A \Rightarrow (B \vee C) \vdash (A \Rightarrow B) \vee C$, is a generalization of the excluded-middle axiom. We will see in Section 2.5 that this extension to classical logic had to happen, since it arises from the deduction theorem.

2. Subtractive logic

Any degenerated category corresponds exactly to a preorder. Namely, the existence of an arrow between two objects indicates if they are comparable. Reflexivity is given by the identity and transitivity by the composition. Conversely, any preorder can be seen as a degenerated category.

Since bi-[CCC] are degenerated, they are special preorders: the constructions of a cartesian closed category define a Heyting pre-algebra (see [23, p. 259]), their dual constructions define a Brouwer pre-algebra (see [4, p. 162]), and finally the whole constructions define a Heyting–Brouwer pre-algebra (see [19, p. 220]).

On the other hand, it is possible (see [14, p. 47]) to present the theory of CCC (resp. with initial object, co-product) as a type system. If we remove all that deal with arrows in this type system, we obtain a deduction system for the minimal propositional calculus (resp. intuitionistic, with disjunction). This point of view extends to the calculus with co-exponents, that we will denote from now on "$-$" and call *subtraction* (terminology which seems to be due to Skolem, see [4, p. 144]).

2.1. A symmetrical categorical propositional calculus

The formulas are built from the usual connectives and subtraction.

Identity axioms

$$A \vdash A$$

Cut rule

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}$$

Axioms for $\bot$ and $\top$

$$\bot \vdash A \qquad\qquad A \vdash \top$$

Left intro. axiom and right intro. rule of $\wedge$

$$A \wedge B \vdash A \qquad A \wedge B \vdash B \qquad\qquad \frac{C \vdash A \qquad C \vdash B}{C \vdash A \wedge B}$$

Left intro. rule and right intro. axiom for $\vee$

$$\frac{A \vdash C \qquad B \vdash C}{A \vee B \vdash C} \qquad\qquad A \vdash A \vee B \qquad B \vdash A \vee B$$

Left intro. axiom and right intro. rule of $\Rightarrow$

$$(B \Rightarrow A) \wedge B \vdash A \qquad \frac{C \wedge B \vdash A}{C \vdash B \Rightarrow A}$$

Left intro. rule and right intro. axiom for $-$

$$\frac{A \vdash B \vee C}{A - C \vdash B} \qquad A \vdash (A - B) \vee B$$

Remark. In this system, one can derive the sequent $A \wedge \neg B \vdash A - B$ (where $\neg B$ is the usual intuitionistic negation $B \Rightarrow \bot$):

$$\dfrac{\dfrac{\dfrac{A \vdash (A-B) \vee B \qquad \neg B \vdash \neg B}{A \wedge \neg B \vdash ((A-B) \vee B) \wedge \neg B}\times}{A \wedge \neg B \vdash (A-B) \vee (B \wedge \neg B)}\alpha \qquad \dfrac{A - B \vdash A - B \qquad \dfrac{B \wedge \neg B \vdash \bot \quad \bot \vdash A - B}{B \wedge \neg B \vdash A - B}}{(A-B) \vee (B \wedge \neg B) \vdash A - B}}{A \wedge \neg B \vdash A - B}$$

where α is the distributivity law (the proof of which is given in the example below) and $\times$ is the following derived rule:

$$\dfrac{\dfrac{A \wedge B \vdash A \quad A \vdash C}{A \wedge B \vdash C} \qquad \dfrac{A \wedge B \vdash B \quad B \vdash D}{A \wedge B \vdash D}}{A \wedge B \vdash C \wedge D}$$

The converse sequent $A - B \vdash A \wedge \neg B$ is not derivable (see Section 2.9) but it becomes derivable in the presence of the excluded-middle axiom, expressed here as the sequent $\top \vdash \neg B \vee B$:

$$\dfrac{\dfrac{A \vdash A \vee B}{A - B \vdash A} \qquad \dfrac{\dfrac{A \vdash \top \quad \top \vdash \neg B \vee B}{A \vdash \neg B \vee B}}{A - B \vdash \neg B}}{A - B \vdash A \wedge \neg B}$$

The equivalence, in classical logic, between $A - B$ and $A \wedge \neg B$ thus justifies the use of subtraction symbol ($A - B$ means *A and not B* and may be read *A without B* or *A minus B*). Moreover, in the usual (classical) interpretation of propositional connectives in the Boolean algebra made up of subsets of a set, the subtraction is interpreted as expected by the set-theoretic difference.

Definition 2.1. We call *symmetrical categorical propositional calculus* the previous set of deduction rules (the calculus without the rules for subtraction is called *categorical propositional calculus*).

Remark. It is known (see [14, p. 47]) that the categorical propositional calculus corresponds to intuitionistic logic. It is clear by construction that the symmetrical calculus corresponds to Heyting–Brouwer algebras. We will call this logic *propositional subtractive logic* (we will show in Theorem 2.13 that this logic is conservative over propositional intuitionistic logic).

In this deduction system, each connective admits a dual connective ($\top, \wedge, \Rightarrow$ are respectively dual of $\bot$, $\vee, -$). We can then map any propositional formula A to a formula $\bar{A}$ obtained by replacing each connective of A by its dual (and by switching the arguments if the connective is $-$ or $\Rightarrow$, since from a syntactic viewpoint, the connective dual to $-$ is actually $\Leftarrow$). The duality thus defined on formulas does not match with duality in classical Boolean logic (represented by negation). Indeed, atoms are unchanged by this translation: we will thus call it "pseudo-duality" (it is, nevertheless, possible *to supplement* this pseudo-duality by giving a duality on atoms [2]). It is easily checked that this pseudo-duality is involutive, i.e. for any formula, $\overline{(\bar{A})} = A$ and we can show the following property:

Proposition 2.2. *If a sequent $A \vdash B$ is derivable, then the sequent $\overline{B} \vdash \overline{A}$ is also derivable.*

Proof. It is clear that for each connective, the left (resp. right) introduction rule is dual to the right (resp. left) introduction of the its dual connective. This makes it possible to map any instance of a rule to its pseudo-dual instance and the pseudo-dual derivation of some derivation of $A \vdash B$ is of course a proof of $\overline{B} \vdash \overline{A}$. $\square$

Example. We give now the proofs of some distributivity laws. These laws are interesting since they are not satisfied in a generic lattice, so we need implication or subtraction to prove them (see Section 4.2.1). Implication permits to directly derive this first law as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \wedge C \vdash B \wedge C}{B \wedge C \vdash (B \wedge C) \vee (B \wedge A)}}{C \vdash B \Rightarrow ((B \wedge C) \vee (B \wedge A))} \quad \dfrac{\dfrac{B \wedge A \vdash B \wedge A}{B \wedge A \vdash (B \wedge C) \vee (B \wedge A)}}{A \vdash B \Rightarrow ((B \wedge C) \vee (B \wedge A))}}{C \vee A \vdash B \Rightarrow ((B \wedge C) \vee (B \wedge A))}}{B \wedge (C \vee A) \vdash B \wedge (B \Rightarrow ((B \wedge C) \vee (B \wedge A)))}}{B \wedge (C \vee A) \vdash (B \wedge C) \vee (B \wedge A)}$$

(the last two rules that occur above do not belong to the categorical calculus, but are easily derived). Subtraction makes it possible to prove directly this second distributivity law (which is however an intuitionistic theorem and thus provable without subtraction):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vee C \vdash B \vee C}{(B \vee C) \wedge (B \vee A) \vdash B \vee C}}{((B \vee C) \wedge (B \vee A)) - B \vdash C} \quad \dfrac{\dfrac{B \vee A \vdash B \vee A}{(B \vee C) \wedge (B \vee A) \vdash B \vee A}}{((B \vee C) \wedge (B \vee A)) - B \vdash A}}{((B \vee C) \wedge (B \vee A)) - B \vdash C \wedge A}}{(((B \vee C) \wedge (B \vee A)) - B) \vee B \vdash B \vee (C \wedge A)}}{(B \vee C) \wedge (B \vee A) \vdash B \vee (C \wedge A)}$$

2.2. *Bi-topological semantics*

A well-known semantics for intuitionistic propositional calculus consists in topological models. It is, in fact, a subclass of Heyting algebras which is sufficient to obtain a completeness theorem. We first recall topological semantics, then we present an extension of this semantics allowing to interpret subtraction (following Rauszer [19]).

2.2.1. *Topological models*

We begin with the definition of a topological space, as well as the usual concept of *interior*.

Definition 2.3. A *topological space* is given by a set X and a collection $\mathcal{O}$ of subsets of X containing $\emptyset$ and X, and closed under finite intersection and unspecified union. Any element S of $\mathcal{O}$ is called an *open set* and its complement, denoted by S^c, is a *closed set* of the topological space.

Definition 2.4. Given any topological space $\mathcal{O}$ on X, and any subset S of X, the *interior* of S, denoted by $int(S)$, is the union of all open sets included in S (hence the largest open subset of S).

We can now recall the topological semantics of a propositional formula. Each formula is interpreted by an open set.

Definition 2.5. Given a topological space $\mathcal{O}$ on X, an assignment $\mathcal{V}$ which maps each propositional atom to an open set, we define the interpretation $[\![A]\!]$ of a formula A by induction:
- $[\![A]\!] \equiv \mathcal{V}(A)$ if A is a propositional atom,
- $[\![\top]\!] \equiv X$, $[\![\bot]\!] \equiv \emptyset$,
- $[\![A \wedge B]\!] \equiv [\![A]\!] \cap [\![B]\!]$,
- $[\![A \vee B]\!] \equiv [\![A]\!] \cup [\![B]\!]$,
- $[\![A \Rightarrow B]\!] \equiv int([\![A]\!]^c \cup [\![B]\!])$.

A sequent $A \vdash B$ is *valid* in $\mathcal{O}$ iff $[\![A]\!] \subset [\![B]\!]$.

Notation. To avoid any ambiguity, we will denote by $[\![A]\!]^{\mathcal{V}}_{\mathcal{O}}$ (i.e. the topological space is denoted as a subscript and the assignment as a superscript).

Theorem 2.6 (Completeness and soundness). *A propositional formula is provable in intuitionistic logic if and only if it is valid in any topological model* (*resp. finite model*).

Proof. See [23, p. 246], for instance. □

2.2.2. Bi-topological models

To interpret implication in topological spaces, we used the concept of interior, which exists because the collection of open sets is closed under unspecified union. We would like subtraction, which is the connective dual to implication, to have a dual semantics. Unfortunately, the dual of a topological space (defined as the collection of its closed sets) is not a topological space since it is not closed under unspecified union but under unspecified intersection (it is in fact an instance of Brouwer algebras). A simple way to overcome this problem is to use bi-topological spaces, i.e. closed under unspecified union *and* unspecified intersection.

Definition 2.7. A *bi-topological space* is given by a set X and a collection $\mathcal{O}$ of subsets of X containing $\emptyset$ and X, and closed under unspecified intersection and unspecified union.

Definition 2.8. Given a bi-topological space $\mathcal{O}$ on X, and a subset S of X, the *exterior* of S, denoted by $ext(S)$, is the intersection of all the open sets containing X (hence the smallest open set containing S).

Remark. For any set X, complementation defines a well-known duality on $\mathscr{P}(X)$ considered as a Boolean algebra. This duality extends to bi-topological spaces, the dual of $(X,\mathcal{O})$ being $(X,\mathcal{O}^{\perp})$ where $\mathcal{O}^{\perp}$ is defined by:

$$\mathcal{O}^{\perp}=\{S^{\mathrm{c}}\colon\ S\in\mathcal{O}\}.$$

The concept of exterior is then dual to the concept of interior since for any set S of $\mathscr{P}(X)$:

$$(int_{\mathcal{O}^{\perp}}(S^{\mathrm{c}}))^{\mathrm{c}}=ext_{\mathcal{O}}(S),$$

since $(int_{\mathcal{O}^{\perp}}(S^{\mathrm{c}}))^{\mathrm{c}}$ is the intersection of closed sets of $\mathcal{O}^{\perp}$ (i.e. open sets of $\mathcal{O}$) which contain S, in other words $ext_{\mathcal{O}}(S)$.

2.2.3. Semantics of subtraction

We are now able to extend the topological interpretation to formulas involving subtraction. First of all let us notice that the semantics of connectives is defined independently of the assignment (as a function from $\mathcal{O}\times\mathcal{O}$ to $\mathcal{O}$ for the connectives $\wedge,\vee,\Rightarrow$ and as elements of $\mathcal{O}$ for $\top,\perp$). Indeed, for any open S,T of $\mathcal{O}$:

- $[\![\top]\!]_{\mathcal{O}}=X$, $[\![\perp]\!]_{\mathcal{O}}=\emptyset$,
- $[\![\wedge]\!]_{\mathcal{O}}(T,S)=T\cap S$,

- $[\![\vee]\!]_{\mathcal{O}}(T,S) = T \cup S$,
- $[\![\Rightarrow]\!]_{\mathcal{O}}(T,S) = int(T^{c} \cup S)$.

The (0-ary) connectives $\top$ and $\bot$ are dual (i.e. $[\![\bot]\!]_{\mathcal{O}}^{\perp} = [\![\top]\!]_{\mathcal{O}}$) since the following equation holds:

$$([\![\top]\!]_{\mathcal{O}})^{c} = [\![\top]\!]_{\mathcal{O}^{\perp}}.$$

In the same way, the (binary) connectives $\wedge$ and $\vee$ are dual (i.e. $[\![\wedge]\!]_{\mathcal{O}}^{\perp} = [\![\vee]\!]_{\mathcal{O}}$) since the following equation holds:

$$([\![\wedge]\!]_{\mathcal{O}}(S,T))^{c} = [\![\vee]\!]_{\mathcal{O}^{\perp}}(T^{c},S^{c}).$$

We naturally expect subtraction and implication semantics also to be dual, which is stated by the equation $[\![\Leftarrow]\!]_{\mathcal{O}}^{\perp} = [\![-]\!]_{\mathcal{O}}$. Consequently, for any open S,T of $\mathcal{O}$,

$$([\![-]\!]_{\mathcal{O}}(S,T))^{c} = [\![\Rightarrow]\!]_{\mathcal{O}^{\perp}}(T^{c},S^{c}).$$

Namely,

$$([\![-]\!]_{\mathcal{O}}(S,T))^{c} = int_{\mathcal{O}^{\perp}}((T^{c})^{c} \cup S^{c}) = (int_{\mathcal{O}^{\perp}}(T^{c} \cap S)^{c}) = (ext_{\mathcal{O}}(T^{c} \cap S))^{c},$$

we thus obtain the following bi-topological semantics for subtraction:

$$[\![\mathscr{A} - \mathscr{B}]\!] \equiv ext([\![\mathscr{A}]\!] \cap [\![\mathscr{B}]\!]^{c}).$$

Before studying in detail properties of bi-topological semantics, it is natural to raise the following question: *Does there exist non-trivial instance of bi-topological spaces*?. The answer is *yes* since there are bi-topological spaces which are not Boolean algebras (in a Boolean algebra every open set is also a closed set). A bi-topological space is, however, degenerated since it always consists of the final sections of some preorder. This theorem is a well-known result in topology (see [3, p. 48]):

Theorem 2.9. *A topological space* $(X,\mathcal{O})$ *is bi-topological if and only if* $\mathcal{O}$ *is the set of all final sections* (*or initial sections*) *of some preorder on* X.

Proof. It is easy to see that the set of all final sections of a preorder on X is a bi-topological space on X. Conversely, we define the following relation on X:

$$x \leqslant y \equiv \forall S \in \mathcal{O}(x \in S \Rightarrow y \in S).$$

This relation is clearly reflexive and transitive, it is thus a preorder. Let us show that the final sections of this preorder are exactly the open sets of $\mathcal{O}$. By definition of the preorder, any open set is a final section. Let us consider now a final section S of the preorder and show that S is an open set. We denote by $V(x)$ the intersection of all the open sets which contains x. It is clear that if $y \in V(x)$ then $y \geqslant x$ because any open set containing x contains $V(x)$ and thus y. Consequently, for any $x \in S$, $V(x) \subset S$. We conclude that $S = \bigcup_{x \in S} V(x)$ is open. $\square$

Remark. In the bi-topological space defined by a preorder $(X, \leqslant)$, the interior of a subset E of X, which is the largest final section included in E, can be defined by

$$x \in int(E) \quad \text{iff} \ \forall y \geqslant x (y \in E).$$

and by duality, the exterior of a set, which is the smallest final section containing E, can be defined by

$$x \in ext(E) \quad \text{iff} \ \exists y \leqslant x (y \in E).$$

2.3. Kripke's forcing

We consider now the extension of Kripke semantics to subtractive logic (following Rauszer [19]). We have shown that any bi-topological space is composed of the final sections of a preorder. However, there is an interpretation of intuitionistic logic whose models (due to Kripke) are exactly preorders together with a *forcing relation* (denoted $\Vdash$) between the nodes of the preorder and the formulas. This relation must satisfy the following property: when a node "forces" a formula, then any greater node also forces the formula. This monotony property just says that formulas are interpreted by final sections of the preorder. We will see that Kripke's forcing corresponds exactly to the bi-topological interpretation of the previous section. Let us give first the formal definition of a Kripke model (notice that we do not confine ourselves to trees, see the remark following Corollary 2.19).

Definition 2.10. A *Kripke model* is given by a preorder $(E, \leqslant)$ and an assignment $\mathscr{V}$ which maps any propositional atom to a final section of the preorder. The forcing relation is defined for any node α by induction on formulas:
- $\alpha \Vdash A \equiv \alpha \in \mathscr{V}(A)$, *if* A is a propositional atom
- $\alpha \Vdash \top$ and $\alpha \nVdash \bot$
- $\alpha \Vdash (A \vee B) \equiv \alpha \Vdash A$ or $\alpha \Vdash B$
- $\alpha \Vdash (A \wedge B) \equiv \alpha \Vdash A$ and $\alpha \Vdash B$
- $\alpha \Vdash (A \Rightarrow B) \equiv$ for any $\beta \geqslant \alpha$ ($\beta \Vdash A$ implies $\beta \Vdash B$)

Proposition 2.11. *Given a preorder* $(E, \leqslant)$ *and an assignment* $\mathscr{V}$ *which maps any propositional atom to a final section of the preorder, then for any node* α *and any formula* A:

$$\alpha \Vdash A \quad \textit{iff} \ \ \alpha \in [\![A]\!],$$

where $[\![A]\!]$ *is the interpretation of* A *in the bi-topological space defined by* $(E, \leqslant)$.

Proof. By induction, the only non-trivial case being the case of implication. If we consider that the connectives are classically interpreted in the meta-language then using the induction hypothesis we get

$$(\beta \Vdash A \ \textit{implies} \ \beta \Vdash B) \ \text{iff} \ (\beta \nVdash A \ \textit{or} \ \beta \Vdash B) \ \text{iff} \ \beta \in [\![A]\!]^{c} \cup [\![B]\!].$$

By definition, $\alpha \Vdash (A \Rightarrow B) \equiv$ *for any* $\beta \geqslant \alpha$ ($\beta \Vdash A$ *implies* $\beta \Vdash B$), and consequently:

$$\alpha \Vdash (A \Rightarrow B) \quad \text{iff} \ \ \alpha \in int([\![A]\!]^{c} \cup [\![B]\!])$$

by the remark following Theorem 2.9. □

2.3.1. Semantics of subtraction

Since a Kripke model is exactly a bi-topological model, the semantics of subtraction is then defined by

$$\alpha \Vdash (A - B) \quad \text{iff} \ \ \alpha \in ext([\![A]\!] \cap [\![B]\!]^{c}).$$

We thus obtain the following Kripke semantics of subtraction:

$$\alpha \Vdash (A - B) \equiv \text{exists } \beta \leqslant \alpha \ (\beta \Vdash A \text{ and } \beta \nVdash B).$$

Proposition 2.11 extends then by duality to subtraction. In other words, Kripke models always allow to interpret subtraction: conservativity over intuitionistic propositional logic arises from this property.

2.4. Conservativity over intuitionistic logic

We are now able to show the conservativity of propositional subtractive logic over propositional intuitionistic calculus.

Lemma 2.12 (Soundness). *Any formula provable in the symmetrical categorical propositional calculus is valid in any Kripke model.*

Proof. Let us consider a formula provable in the symmetrical categorical propositional calculus. This formula is valid in any Heyting–Brouwer algebra, and in particular in any bi-topological model (i.e. in any Kripke model). □

Theorem 2.13 (Conservativity). *Any formula containing no subtraction provable in the symmetrical categorical propositional calculus is provable in intuitionistic logic.*

Proof. Let us consider a formula provable in the symmetrical categorical propositional calculus. By Lemma 2.12, this formula is valid in any Kripke model. The completeness theorem for Kripke models (see [23, p. 254]) allows us to conclude that this sequent is provable in intuitionistic logic. □

The work of Rauszer [19] enables us to close this section with the completeness theorem for (finite) Kripke models.

Theorem 2.14 (Completeness). *Any formula valid in any Kripke model* (*resp. finite model*) *is provable in the symmetrical categorical propositional calculus.*

Proof. By the completeness theorem of Rauszer (see [19, p. 244]), any valid formula valid in all finite Kripke models is valid in all Heyting–Brouwer algebras. The result is then obtained by applying the completeness theorem for these algebras. □

2.5. No deduction theorem

We show here that the deduction theorem, which holds in the categorical propositional calculus, does not hold any longer in the symmetrical categorical propositional calculus. First of all, let us recall the statement of this theorem:

Theorem 2.15. *In the categorical propositional calculus, for any propositional formulas A and B, if there is a derivation of $\top \vdash B$ in which any leaf is either an axiom, or a sequent $\top \vdash A$, then there exists a derivation in this calculus of $A \vdash B$ (in which all leaves are axioms).*

Proof. See [14, p. 51]. □

To show that this theorem does not hold any more in the symmetrical categorical propositional calculus, we show that in the presence of subtraction, this theorem allows to derive the excluded-middle axiom. For that, we will need the following lemma (where $\neg A$ stands for $A \Rightarrow \perp$):

Lemma 2.16. *For any propositional formula A, the sequent $\neg(A \vee \neg A) \vdash \perp$ is derivable in intuitionistic logic.*

Proof. Here is a proof in the categorical propositional calculus:

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg(A \vee \neg A) \wedge A \vdash \neg(A \vee \neg A) \qquad \dfrac{A \vdash A \vee \neg A}{\neg(A \vee \neg A) \wedge A \vdash A \vee \neg A}}{\neg(A \vee \neg A) \wedge A \vdash \perp}}{\neg(A \vee \neg A) \vdash \neg A}}{\neg(A \vee \neg A) \vdash A \vee \neg A} \qquad \neg(A \vee \neg A) \vdash \neg(A \vee \neg A)}{\neg(A \vee \neg A) \vdash \perp}$$

□

Corollary 2.17. *In the symmetrical categorical propositional calculus, the deduction theorem does not hold.*

Remark. In the symmetrical categorical propositional calculus, if the deduction theorem were satisfied, so would be its dual: given a derivation of $B \vdash \perp$ where any leaf is either an axiom or a sequent $A \vdash \perp$ there should be a derivation of $B \vdash A$. Indeed, by duality there is a proof of $\top \vdash \overline{B}$ where any leaf is either an axiom, or a sequent $\top \vdash \overline{A}$. By the deduction theorem, there is a derivation of $\overline{A} \vdash \overline{B}$, and thus, again by duality, a proof of $B \vdash A$.

Proof. Let us consider the following derivation, where the proof of $\neg(A \vee \neg A) \vdash \bot$ is the one given in the proof of the previous lemma:

$$\dfrac{\dfrac{A \vee \neg A \vdash \bot}{\top \vdash \neg(A \vee \neg A)} \qquad \begin{array}{c}\vdots\\ \neg(A \vee \neg A) \vdash \bot\end{array}}{\top \vdash \bot}$$

By the remark above, if the deduction theorem were satisfied in the symmetrical categorical propositional calculus, so would be its dual. Thus let us apply this last property to the derivation of $\top \vdash \bot$ from $A \vee \neg A \vdash \bot$: it should then exist a derivation of $\top \vdash A \vee \neg A$, which contradicts the conservativity of the symmetrical categorical propositional calculus over intuitionistic logic. $\square$

Remark. The deduction theorem and its dual can be formulated in the form of deduction rules that discharge some "hypothetic sequent":

$$\dfrac{\begin{array}{c}[\top \vdash A]\\ \vdots\\ \top \vdash B\end{array}}{A \vdash B} \qquad\qquad \dfrac{\begin{array}{c}[A \vdash \bot]\\ \vdots\\ B \vdash \bot\end{array}}{B \vdash A}$$

We saw in the previous proof that adding these rules to the symmetrical categorical propositional calculus leads to classical logic. For instance, we give here a direct proof of $\neg\neg A \vdash A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A \vdash \bot]^1}{\top \vdash \neg A} \qquad [\top \vdash \neg\neg A]^2}{\top \vdash \bot}}{\top \vdash A}\,(1)}{\neg\neg A \vdash A}\,(2)$$

This calculus extends in fact the Curry–Howard isomorphism to Filinski's symmetrical lambda-calculus. Indeed, it is easy to bridge the gap between these deduction rules and the type system described in [8].

2.6. *Weak negation*

In intuitionistic logic, it is usual to define the negation $\neg A$ (we will call it *intuitionistic negation*) as $A \Rightarrow \bot$. We then obtain the following rules derived from the categorical propositional calculus (the axiom is also called the *contradiction law*):

$$\dfrac{B \wedge A \vdash \bot}{B \vdash \neg A} \qquad A \wedge \neg A \vdash \bot$$

The topological and Kripke semantics of the negation are also derived from implication and absurdity semantics:

$$[\![\neg A]\!] \equiv int([\![A]\!]^{c}) \quad \text{and} \quad \alpha \Vdash \neg A \equiv \forall \beta \geqslant \alpha (\beta \nVdash A).$$

By duality, in subtractive logic, it is possible to define a new negation, denoted $\sim$, that we will call *weak negation*, by $\sim A \equiv \top - A$ (notice that this negation is native in Rauszer's work [19]). The bi-topological semantics (i.e. Kripke semantics) of the weak negation is given by

$$[\![\sim A]\!] \equiv ext([\![A]\!]^{c}) \quad \text{and} \quad \alpha \Vdash \sim A \equiv \exists \beta \leqslant \alpha (\beta \nVdash A).$$

The weak negation satisfies the following derived rules (the axiom is also called *excluded-middle law for weak negation*) dual to those of intuitionistic negation:

$$\frac{\top \vdash A \vee B}{\sim A \vdash B} \qquad \top \vdash \sim A \vee A$$

Remark. Since the intuitionistic negation satisfies the contradiction law and the weak negation satisfies the excluded-middle law, it is enough to identify these two negations to obtain classical logic. From the semantics point of view, identifying the interior of $[\![A]\!]^{c}$ and the exterior of $[\![A]\!]^{c}$ amounts to saying that $[\![A]\!]$ is a clopen set (a set that is both a closed set and an open set). The set of clopen sets of a bi-topological space is by definition closed under complementation: it thus forms a Boolean algebra.

Before detailing some properties of weak negation, we show that this new connective is not definable from the other intuitionistic connectives. This result, which arises directly from Corollary 2.19, has as a straightforward consequence the non-definability of subtraction (which can also be obtained as a corollary of Rauszer's non-conservativity result in the first order framework, see Section 3).

Lemma 2.17. *The sequent* $\top \vdash \neg\neg A \vee \neg A$ *is not derivable in intuitionistic logic, but it is valid in any Kripke models with a greatest element.*

Proof. Let us consider a Kripke model with a greatest element σ. In this model, either $\sigma \Vdash A$ or $\sigma \nVdash A$. In the first case, no α can force $\neg A$ (since $\alpha \leqslant \sigma$), therefore any α forces $\neg\neg A$. In the second case, no α can force A (since $\alpha \leqslant \sigma$) and thus any α forces $\neg A$. In both cases, any α forces $\neg\neg A \vee \neg A$.

However, the formula $\neg\neg A \vee \neg A$ is not an intuitionistic theorem since it is not valid in the following model:

$$\begin{tikzcd} A \quad \beta \arrow[dr, dash] & & \gamma \arrow[dl, dash] \\ & \alpha & \end{tikzcd}$$

Indeed, α does not force $\neg A$ since β force A, and α does not force either $\neg\neg A$ since γ force $\neg A$. $\square$

Corollary 2.18 (Dual of Lemma 2.17). *The sequent* $\sim\sim A \wedge \sim A \vdash \bot$ *is not a theorem of intuitionistic subtractive logic, but it is valid in any Kripke model with a least element.*

Proof. Indeed, a Kripke model has a least element if and only if its dual has a greatest element. □

Corollary 2.19. *Weak negation* (*and thus subtraction*) *is not definable from the other connectives in intuitionistic propositional logic.*

Remark. Trees are no longer enough to obtain completeness in the presence of weak negation (and a fortiori in the presence of subtraction).

2.7. Properties of weak negation

We focus here on some properties of weak negation. We justify first of all the terminology of *weak negation* by showing that it is indeed weaker than the intuitionistic negation (i.e. $\vdash \neg A \Rightarrow {\sim}A$). The semantical justification is obvious, since for any set E, $int(E) \subset ext(E)$. Here is a syntactic proof:

$$\dfrac{\top \vdash A \vee {\sim}A \qquad \dfrac{\dfrac{\dfrac{A \wedge \neg A \vdash \bot \qquad \bot \vdash {\sim}A}{A \wedge \neg A \vdash {\sim}A}}{A \vdash \neg A \Rightarrow {\sim}A} \qquad \dfrac{{\sim}A \wedge \neg A \vdash {\sim}A}{{\sim}A \vdash \neg A \Rightarrow {\sim}A}}{A \vee {\sim}A \vdash \neg A \Rightarrow {\sim}A}}{\top \vdash \neg A \Rightarrow {\sim}A}$$

As a consequence (and by applying duality), one obtains the following implications:

$$\neg{\sim}A \vdash {\sim}{\sim}A \vdash A \vdash \neg\neg A \vdash {\sim}\neg A$$

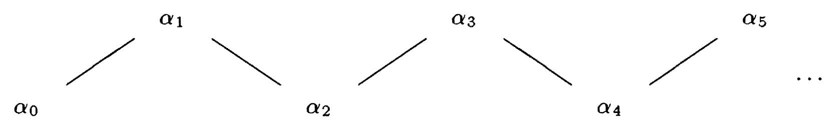
These implications are strict. Indeed, it is known that $A \vdash \neg\neg A$ is a strict implication (its converse implication gives classical logic), its dual ${\sim}{\sim}A \vdash A$ is thus also strict. In addition, if the equivalence $\neg\neg A \Leftrightarrow {\sim}\neg A$ were satisfed, since $\neg\neg\neg A \Leftrightarrow \neg A$, all formulas of the following (infinite) list of implications would be equivalent:

$$\ldots \vdash \neg{\sim}\neg{\sim}\neg{\sim}A \vdash \neg{\sim}\neg{\sim}A \vdash \neg{\sim}A$$

Yet *these implications are all strict*. Indeed, let us take $A_1 = A$ and for $i \geqslant 1$:

$$A_{2i} = {\sim}A_{2i-1},$$
$$A_{2i+1} = \neg A_{2i}.$$

Now, let us consider the (right) infinite Kripke model below:



In this model, let us assign to A the set of all nodes except α_0 (which is indeed a final section). It is easy to see that the semantics of A_{2i} is $\{\alpha_0, \ldots, \alpha_{2i-1}\}$ and the semantics of A_{2i+1} is $\{\alpha_{2i+1}, \alpha_{2i+2}, \ldots\}$. The former implications are thus strict.

2.7.1. Weak negation and intuitionistic implication

We will now use weak negation to approximate intuitionistic implication. The consequence $(\neg A \vee B) \vdash (A \Rightarrow B)$ is an usual intuitionistic result, here is some proof of it:

$$\dfrac{\dfrac{\dfrac{\neg A \wedge A \vdash \bot \quad \bot \vdash B}{\neg A \wedge A \vdash B}}{\neg A \vdash A \Rightarrow B} \quad \dfrac{B \wedge A \vdash B}{B \vdash A \Rightarrow B}}{\neg A \vee B \vdash A \Rightarrow B}$$

Weak negation makes it possible to prove $(A \Rightarrow B) \vdash (\sim A \vee B)$. We have already proved the dual sequent $(B \wedge \neg A) \vdash (B - A)$ in Section 2.1. Consequently,

$$(\neg A \vee B) \vdash (A \Rightarrow B) \vdash (\sim A \vee B)$$

And dually, we obtain the following sequents:

$$(B \wedge \neg A) \vdash (B - A) \vdash (B \wedge \sim A)$$

Again, these implications are strict since on one hand it is clear (take $A = B$) that $(A \Rightarrow B) \nvdash (\neg A \vee B)$ and on the other hand subtraction is not definable from the other connectives (see Corollary 2.19), therefore $(B - A) \nvdash (B \wedge \neg A)$. The other consequences are strict by duality.

2.8. Equiprovability versus equivalence

It is known that in propositional logic, if A is a classical theorem then $\neg\neg A$ is an intuitionistic theorem. This property does not hold any longer in subtractive logic. Indeed, we know that $\sim A \Rightarrow \neg A$ is a classical theorem but not a subtractive theorem. Moreover, since $\phi \Rightarrow \neg\neg\phi$ and $\neg\neg\neg\phi \Rightarrow \neg\phi$ are intuitionistic theorems and $\neg\neg(\phi \Rightarrow \psi)$ is equivalent to $\neg\neg\phi \Rightarrow \neg\neg\psi$ in intuitionistic logic, $\neg\neg(\sim A \Rightarrow \neg A)$ is thus equivalent to $\neg\neg\sim A \Rightarrow \neg\neg\neg A$ which is not a subtractive theorem (since $\sim A \Rightarrow \neg\neg\sim A$ and $\neg\neg\neg A \Rightarrow \neg A$ are subtractive theorems and $\sim A \Rightarrow \neg A$ is not).

We also know that the axiom $A \vdash \neg\sim A$ leads to classical logic since $\neg\sim A \vdash \sim\sim A$ is derivable (and $A \vdash \sim\sim A$ is dual to $\neg\neg A \vdash A$). However, the following striking property is satisfied in the symmetrical categorical propositional calculus:

Proposition 2.20. *In the symmetrical categorical propositional calculus, if $A \vdash B$ is a theorem then $\neg(A - B)$ is also a theorem. As a special case, if B is a theorem, $\neg\sim B$ is also a theorem.*

Proof.

$$\dfrac{\dfrac{\top \wedge (A - B) \vdash (A - B) \quad \dfrac{\dfrac{A \vdash B \quad B \vdash \bot \vee B}{A \vdash \bot \vee B}}{(A - B) \vdash \bot}}{\top \wedge (A - B) \vdash \bot}}{\top \vdash \neg(A - B)}$$

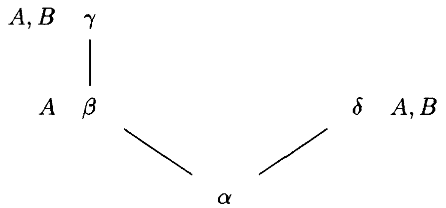
Eventually, by taking $A = \top$ in this proof, we obtain that if B is a theorem then $\neg{\sim}B$ is also a theorem. $\square$

2.9. Undefinability of subtraction from weak negation

We have already shown that subtraction is not definable from the usual intuitionistic connectives (since weak negation is not, see Corollary 2.19). We prove here that subtraction is not definable either in intuitionistic logic from the usual connectives *and weak negation*. The proof of this result which is based on the validity lemma for Kripke models is a straightforward corollary of the following proposition:

Proposition 2.21. *There is a Kripke model interpreting two propositional atoms A and B in which any formula built from $\top, \bot, A, B$ with the connectives $\vee, \wedge, \Rightarrow, \neg, \sim$ has a semantics different from the semantics of $A - B$.*

Proof. Let us consider the following Kripke model (the nodes are labelled by the forced atoms):



The semantics of $A - B$ in this model is

$$[\![A - B]\!] = ext(\{\beta,\gamma,\delta\} \cap \{\gamma,\delta\}^{c}) = ext(\{\beta,\gamma,\delta\} \cap \{\alpha,\beta\}) = ext(\{\beta\}) = \{\beta,\gamma\}$$

Let us denote by N the set of all nodes. We will show that no other formula built from $\top$, $\bot$, A, B with the connectives $\vee$, $\wedge$, $\Rightarrow$, $\neg$, $\sim$ is interpreted by $\{\beta,\gamma\}$. For that, it is enough to show that the set of final sections:

$$\mathscr{E} = \{\emptyset, \{\gamma,\delta\}, \{\beta,\gamma,\delta\}, N\}$$

(which are the respective interpretations of $\bot$, B, A and $\top$) is closed under the semantics of the connectives $\vee$, $\wedge$, $\Rightarrow$, $\neg$, $\sim$. Intuitionistic negation is dealt with as a particular case of implication. Moreover, the set $\mathscr{E}$ is clearly closed under union and intersection. It thus remains to be checked that for any elements U, V, W of $\mathscr{E}$, $int(U^{c} \cup V)$ and $ext(W^{c})$ are also elements of $\mathscr{E}$. It is clear in the cases $U = V$, $U = \emptyset$, $U = N$, $V = N$, $W = \emptyset$, $W = N$. The remaining cases are:

- $int(\{\gamma,\delta\}^{c} \cup \emptyset) = int(\{\gamma,\delta\}^{c}) = int(\{\alpha,\beta\}) = \emptyset \in \mathscr{E}$,
- $int(\{\beta,\gamma,\delta\}^{c} \cup \emptyset) = int(\{\beta,\gamma,\delta\}^{c}) = int(\{\alpha\}) = \emptyset \in \mathscr{E}$,
- $int(\{\gamma,\delta\}^{c} \cup \{\beta,\gamma,\delta\}) = int(\{\alpha,\beta\} \cup \{\beta,\gamma,\delta\}) = int(N) = N \in \mathscr{E}$,
- $int(\{\beta,\gamma,\delta\}^{c} \cup \{\gamma,\delta\}) = int(\{\alpha\} \cup \{\gamma,\delta\}) = int(\{\alpha,\gamma,\delta\}) = \{\gamma,\delta\} \in \mathscr{E}$,

- $ext(\{\gamma,\delta\}^c) = ext(\{\alpha,\beta\}) = N \in \mathscr{E}$,
- $ext(\{\beta,\gamma,\delta\}^c) = ext(\{\alpha\}) = N \in \mathscr{E}$. $\square$

3. First-order subtractive logic

Extending the symmetrical categorical propositional calculus to a first-order calculus does not raise any difficulty, we just take the usual right and left introduction rules of quantifiers, restricted to a single formula on both sides of every sequent (terms are defined as usual upon a signature Σ).

Intro. rules for $\exists$ (where x does not occur free in C)

$$\frac{B \vdash C}{\exists x B \vdash C} \qquad \frac{A \vdash B[t/x]}{A \vdash \exists x B}$$

Intro. rules for $\forall$ (where x does not occur free in C)

$$\frac{B[t/x] \vdash C}{\forall x B \vdash C} \qquad \frac{A \vdash B}{A \vdash \forall x B}$$

Notice that the rules for $\exists$ are exactly dual to those for $\forall$. Consequently, duality extends to first-order subtractive logic by inverting the quantifiers $\exists$ and $\forall$ (as we will see it is not the case in usual intuitionistic logic). That means in particular that we preserve the following property in the first-order framework: if $A \vdash B$ is derivable then $\overline{B} \vdash \overline{A}$ is also derivable. Consequently, since the sequent $\exists x A(x) \wedge B \vdash \exists x(A(x) \wedge B)$ is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{A(x) \wedge B \vdash A(x) \wedge B}{A(x) \wedge B \vdash \exists x(A(x) \wedge B)}}{A(x) \vdash B \Rightarrow \exists x(A(x) \wedge B)}}{\exists x A(x) \vdash B \Rightarrow \exists x(A(x) \wedge B)}}{\exists x A(x) \wedge B \vdash \exists x(A(x) \wedge B)}$$

its dual (usually called DIS) $\forall x(A(x) \vee B) \vdash \forall x A(x) \vee B$ is also derivable (just take the dual proof):

$$\dfrac{\dfrac{\dfrac{\dfrac{A(x) \vee B \vdash A(x) \vee B}{\forall x(A(x) \vee B) \vdash A(x) \vee B}}{\forall x(A(x) \vee B) - B \vdash A(x)}}{\forall x(A(x) \vee B) - B \vdash \forall x A(x)}}{\forall x(A(x) \vee B) \vdash \forall x A(x) \vee B}$$

However, it is known that this axiom is not intuitionistic. In the first-order framework, subtractive logic is thus not conservative over intuitionistic logic. Let us now consider this issue from a semantical standpoint.

As a consequence of non-conservativity over intuitionistic logic, we know that the usual Kripke semantics for first-order logic does not extend to subtraction. Naturally, we would like to use the interpretation given within the propositional framework. But, because of the possible growth of the domain, this interpretation is no longer necessarily well defined. A solution to this problem which moreover allows to restore a semantical duality consists in requiring the domain to be the same in all worlds.

3.1. Constant domain logic (CDL)

Let Σ be a countable first-order language. A *constant domain Kripke model* is given by a triple: a preorder $(X, \leqslant)$, a Σ-algebra $\mathscr{A}$ and an interpretation $\mathscr{I}$ which maps any n-ary predicate symbol to a function from $dom(\mathscr{A})^n$ to the final sections of $(X, \leqslant)$.

Notation. Given a Σ-algebra $\mathscr{A}$ and an assignment $\mathscr{V}$ which maps variables to elements of $dom(\mathscr{A})$, the notation $\mathscr{V}_{\mathscr{A}}$ stands for the usual canonical extension of $\mathscr{V}$ to terms.

Definition 3.1 (*Kripke's forcing*). Given a preorder $(X, \leqslant)$, a Σ-algebra $\mathscr{A}$ and an interpretation $\mathscr{I}$ and an assignment $\mathscr{V}$ which maps any free variable of A to some element of $dom(\mathscr{A})$ we define the forcing relation $\alpha \Vdash_{\mathscr{V}} A$ (where $\mathscr{V}$ is an assignment) by induction on A:

- $\alpha \Vdash_{\mathscr{V}} A(t_1, \ldots, t_n) \equiv \alpha \in \mathscr{I}(A)(\mathscr{V}_{\mathscr{A}}(t_1), \ldots, \mathscr{V}_{\mathscr{A}}(t_n))$ if A is a n-ary predicate symbol,
- $\alpha \Vdash_{\mathscr{V}} \top$ and $\alpha \nVdash_{\mathscr{V}} \bot$,
- $\alpha \Vdash_{\mathscr{V}} (A \vee B) \equiv \alpha \Vdash_{\mathscr{V}} A$ or $\alpha \Vdash_{\mathscr{V}} B$,
- $\alpha \Vdash_{\mathscr{V}} (A \wedge B) \equiv \alpha \Vdash_{\mathscr{V}} A$ and $\alpha \Vdash_{\mathscr{V}} B$,
- $\alpha \Vdash_{\mathscr{V}} (A \Rightarrow B) \equiv$ for all $\beta \geqslant \alpha$ ($\beta \Vdash_{\mathscr{V}} A$ implies $\beta \Vdash_{\mathscr{V}} B$),
- $\alpha \Vdash_{\mathscr{V}} \exists x A \equiv$ exists $a \in dom(\mathscr{A})$ ($\alpha \Vdash_{\mathscr{V} \cup \{(x,a)\}} A$),
- $\alpha \Vdash_{\mathscr{V}} \forall x A \equiv$ for all $a \in dom(\mathscr{A})$ ($\alpha \Vdash_{\mathscr{V} \cup \{(x,a)\}} A$).

Remark. The only difference between the definition above and the usual Kripke semantics for intuitionistic logic (where the domain of $\mathscr{A}$ may grow) lies in the case of $\forall$:

$$\alpha \Vdash_{\mathscr{V}} \forall x A \equiv \text{ for all } \beta \geqslant \alpha, \text{ for all } a \in dom_\beta(\mathscr{A}) \; (\beta \Vdash_{\mathscr{V}; x:=a} A).$$

Since the domain is constant, if we consider the sets of final sections of the preorder $(X, \leqslant)$ as a bitopological space $(X, \mathcal{O})$, we obtain the following presentation of constant domain Kripke's semantics:

Definition 3.2 (*Bitopological semantics*). Given a bitopological space $(X, \mathcal{O})$, a Σ-algebra $\mathscr{A}$ and an interpretation $\mathscr{I}$, we define the interpretation $[\![A]\!]_{\mathscr{V}}$ (where $\mathscr{V}$ is an assignment) of a formula A by induction:

- $[\![A(t_1, \ldots, t_n)]\!]_{\mathscr{V}} \equiv \mathscr{I}(A)(\mathscr{V}_{\mathscr{A}}(t_1), \ldots, \mathscr{V}_{\mathscr{A}}(t_n))$ if A is a n-ary predicate symbol,
- $[\![\top]\!]_{\mathscr{V}} \equiv X$, $[\![\bot]\!] \equiv \emptyset$,
- $[\![A \wedge B]\!]_{\mathscr{V}} \equiv [\![A]\!]_{\mathscr{V}} \cap [\![B]\!]_{\mathscr{V}}$,
- $[\![A \vee B]\!]_{\mathscr{V}} \equiv [\![A]\!]_{\mathscr{V}} \cup [\![B]\!]_{\mathscr{V}}$,
- $[\![A \Rightarrow B]\!]_{\mathscr{V}} \equiv int([\![A]\!]^c_{\mathscr{V}} \cup [\![B]\!]_{\mathscr{V}})$,

- $[\![\exists x A]\!]_{\mathscr{V}} \equiv \bigcup_{a \in dom(\mathscr{A})} [\![A]\!]_{\mathscr{V} \cup \{(x,a)\}}$,
- $[\![\forall x A]\!]_{\mathscr{V}} \equiv \bigcap_{a \in dom(\mathscr{A})} [\![A]\!]_{\mathscr{V} \cup \{(x,a)\}}$.

This semantics defines the so-called constant domain logic (CDL). One of the well-known results in model theory of intuitionistic logic states that the theory containing the axiom scheme DIS is complete for CDL [9, 16].

Theorem 3.3 (Soundness and completeness). *A formula is intuitionistically provable in the first order theory DIS iff it is valid in all constant domain Kripke models.*

3.2. Kripke semantics for first-order subtractive logic

We now consider subtractive logic as an extension of CDL where the semantics of subtraction is exactly the one given in the propositional framework:

$$\alpha \Vdash_{\mathscr{V}} (A - B) \equiv \text{ exists } \beta \leqslant \alpha (\beta \Vdash_{\mathscr{V}} A \text{ and } \beta \nVdash_{\mathscr{V}} B).$$

For this semantics, Rauszer proved the soundness and completeness theorems [20].

Theorem 3.4 (Soundness and completeness). *A formula is provable in first-order subtractive logic iff it is valid in all constant domain Kripke models.*

As in the propositional framework, since the subtraction is interpreted directly in CDL's models, we obtain the conservativity as a corollary.

Theorem 3.5 (Conservativity). *Subtractive logic is conservative over* CDL.

4. Embedding classical logic into subtractive logic

In the classical sequent calculus LK of Gentzen [21], it is possible to derive the following proof of the excluded-middle law:

$$\cfrac{\cfrac{A \vdash A}{A \vdash \bot, A}}{\vdash \neg A, A}$$

This proof is *corrupted* (from an intuitionistic point of view) for the following reason: we discharge an assumption (A) on some conclusion ($\bot$) whereas another conclusion (A) depends clearly on the assumption that is being discharged.

The rule in question is the right introduction of implication (or the intuitionistic negation), in the presence of multiple conclusions. In Section 4.5, we show that weak negation (studied in the previous chapter) enables to define a *weak implication* for which precisely the right introduction rule in the presence of multiple conclusions is *derivable in intuitionistic subtractive logic*. In addition, we define (in Section 4.5) a translation, using bi-topological semantics, from classical propositional logic into propositional intuitionistic subtractive logic. We also show that this translation extends directly to cut-free proofs of the sequent calculus LK. We notice then that any "classical" occurrence of the right introduction rule of implication (resp. negation) is translated into some occurrence of the right introduction rule of weak implication (resp. negation). When the whole proof has been translated, no occurrence of the right introduction rule of implication remains: the proof obtained is then clearly subtractive.

4.1. The classical calculus LK

We first recall the rules of Gentzen sequent calculus LK (see [12] for example). Again rules for subtraction are obtained by duality (the calculus obtained is called SLK). However, we can show here that there is a unique (classical) negation (i.e. intuitionistic negation and weak negation are provably equivalent) and that subtraction is definable from the classical negation, denoted by $A^{\perp}$, by $A - B \equiv A \wedge B^{\perp}$ (just as implication is definable by $A \Rightarrow B \equiv A^{\perp} \vee B$). Notice that if we take now $\neg A \equiv A \Rightarrow \perp$ and $\sim A \equiv \top - A$ we obtain derived rules, which are exactly the same as those defining classical negation.

Axiom

$$A \vdash A$$

Cut rule

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

Contraction rules

$$\frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Weakening rules

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Exchange rules

$$\frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'} \qquad \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}$$

Rules for $\perp$ and $\top$

$$\frac{\Gamma \vdash \Delta, \perp}{\Gamma \vdash \Delta, A} \qquad \frac{\Gamma, \top \vdash \Delta}{\Gamma, A \vdash \Delta}$$

Rules for classical negation

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, A^{\perp} \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}$$

Intro. rules for $\wedge$

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \wedge B}$$

Intro. rules for $\vee$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B}$$

Right intro. rules for $\Rightarrow$

$$\frac{\Gamma, B \vdash \Delta \qquad \Gamma' \vdash \Delta', A}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B}$$

Intro. rules for $\exists$ (if x does not occur free in Γ, Δ)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \qquad \frac{\Gamma' \vdash \Delta', B[t/x]}{\Gamma' \vdash \Delta', \exists x B}$$

Intro. rules for $\forall$ (if x does not occur free in Γ', Δ')

$$\frac{\Gamma, A[t/x] \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \qquad \frac{\Gamma' \vdash \Delta', B}{\Gamma' \vdash \Delta', \forall x B}$$

Remark. It is possible to give another formulation of the rules which require that upper sequents have the same contexts (additive rules). It is well known that this formulation is equivalent to the one presented above, it is enough to apply a few times the weakening rules and the contraction rules to show the equivalence. We will use equally both formulations.

4.2. The classical calculus SLK

The rules for subtraction are obtained by duality from the rules for implication. As expected, we thus obtain rules for a *sequent calculus*, since they are left and right introduction rules.

Right intro. rule of $-$

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', A - B}$$

Left intro. rule of $-$

$$\frac{\Gamma, A \vdash \Delta, B}{\Gamma, A - B \vdash \Delta}$$

Remark. Just like implication *internalizes* the deduction symbol $\vdash$ on the right (as a conclusion), subtraction *internalizes* the deduction symbol $\vdash$ on the left (as an hypothesis). More specifically, the following rules are derivable:

$$\frac{\Gamma \vdash A \Rightarrow B, \Delta}{\Gamma, A \vdash B, \Delta} \qquad\qquad \frac{\Gamma, A - B \vdash \Delta}{\Gamma, A \vdash B, \Delta}$$

Indeed,

$$\frac{\Gamma \vdash A \Rightarrow B, \Delta \qquad \dfrac{B \vdash B \qquad A \vdash A}{A, A \Rightarrow B \vdash B}}{\Gamma, A \vdash B, \Delta} \qquad \frac{\dfrac{A \vdash A \qquad B \vdash B}{A \vdash B, A - B} \qquad \Gamma, A - B \vdash \Delta}{\Gamma, A \vdash B, \Delta}$$

4.2.1. About sequents with multiple conclusions

The sequent calculus was introduced by G. Gentzen for a very specific purpose: to prove the cut elimination theorem. This system had much success thereafter since it allows a precise analysis of computational contents, in particular in classical logic (like the system LC [10] for instance). It is also often claimed that this presentation is somehow minimal: each connective is defined by its introduction rules, independently of the other connectives. This is not the case in the categorical calculus since the definition of implication is based on the conjunction (and the definition of subtraction is based on disjunction). However, *the minimalist character of this presentation is misleading*. Indeed, if we consider for example the fragment of the calculus containing only the conjunction and disjunction connectives, it is possible to derive the following proof:

$$\frac{\dfrac{\dfrac{\dfrac{B \vdash B \qquad C \vdash C}{B, C \vdash B \wedge C}}{B, C \vdash (B \wedge C) \vee (B \wedge A)} \qquad \dfrac{\dfrac{B \vdash B \qquad A \vdash A}{B, A \vdash B \wedge A}}{B, A \vdash (B \wedge C) \vee (B \wedge A)}}{B, C \vee A \vdash (B \wedge C) \vee (B \wedge A)}}{B \wedge (C \vee A) \vdash (B \wedge C) \vee (B \wedge A)}$$

whereas it is known that any lattice is not necessarily distributive. The rules for conjunction and disjunction in LK are thus more powerful than those of the categorical calculus (again, only if we confine ourselves to the fragments of these calculi containing only the conjunction and the disjunction). Actually, we used an implicit distributivity law contained in the structural rules which axiomatize the *left comma* (which represents also a conjunction). Notice that this proof is intuitionistic (the sequents have only one conclusion) and may be translated in natural deduction.

Another example, DIS, that we can prove thanks to subtraction but which is not an intuitionistic theorem (see Section 3) is now provable without subtraction.

Remark. We will use in the proof of DIS these derived rules for $\vee$:

$$\frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B} \qquad \frac{\Gamma \vdash \Delta, A \vee B}{\Gamma \vdash \Delta, A, B}$$

Proposition 4.1. *The axioms scheme DIS is provable in LK using only the structural rules and the rules for* $\vee$ *and* $\forall$.

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{A(x)\vee B \vdash A(x)\vee B}{\forall x(A(x)\vee B)\vdash A(x)\vee B}}{\forall x(A(x)\vee B)\vdash A(x),B}}{\forall x(A(x)\vee B)\vdash \forall xA(x),B}}{\forall x(A(x)\vee B)\vdash \forall xA(x)\vee B} \qquad \square$$

4.3. Intuitionistic restrictions of LK

In propositional logic, LK can be restricted to intuitionistic calculi in several ways.

- The most famous restriction is obtained by limiting the sequents to at most one conclusion: this is LJ (see [21], for example). However the rules of subtraction restricted in this way:

  $$\dfrac{\Gamma\vdash A \qquad \Gamma',B\vdash}{\Gamma,\Gamma'\vdash A-B} \qquad \dfrac{\Gamma,A\vdash B}{\Gamma,A-B\vdash}$$

  do not define any more the subtraction previously studied. Indeed, we can then derive:

  $$\dfrac{\dfrac{\dfrac{B\vdash B}{B,A\vdash B}}{B,A-B\vdash}}{A-B\vdash\neg B}$$

  and in particular $\sim B \vdash \neg B$ which is not valid in intuitionistic subtractive logic. Duality suggests the constraint to use if we want to stay in the intuitionistic framework: it is necessary *to restrict the sequents to at most one hypothesis*. But this restriction gives us a degenerated calculus or brings us back to the symmetrical categorical calculus.
- A weaker restriction of LK to an intuitionistic calculus consists in restricting only the right introduction rule of implication to a unique conclusion. It is known [6, 7] that this propositional calculus is *conservative over intuitionistic propositional logic*, we will call it LK^1. On the other hand, in the first-order framework LK^1 is conservative over DIS-logic (we already gave the proof of DIS, see Proposition 4.1).

  By duality, this restriction extends of course to the left introduction rule of subtraction: the upper sequent of the rule must have only one hypothesis. The calculus thus obtained is *conservative over subtractive logic* (see Proposition 4.3), it will be called SLK^1.

4.4. The subtractive calculus SLK^1

The rules of this calculus are those of SLK where the rules for implication and subtraction are replaced by the following ones (and the classical negation is removed):

Intro. rules for $\Rightarrow$

$$\frac{\Gamma, B \vdash \Delta \qquad \Gamma' \vdash \Delta', A}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \qquad \frac{\Gamma, B \vdash A}{\Gamma \vdash B \Rightarrow A}$$

Intro. rules for $-$

$$\frac{A \vdash \Delta, B}{A - B \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta', A - B}$$

Again, we obtain the derived rules (that are different this time) for intuitionistic negation $\neg A \equiv A \Rightarrow \bot$ and weak negation $\sim A \equiv \top - A$:

Derived intro. rule of $\neg$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta} \qquad \frac{\Gamma, A \vdash}{\Gamma \vdash \neg A}$$

Derived intro. rule of $\sim$

$$\frac{\vdash A, \Delta}{\sim A \vdash \Delta} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \sim A, \Delta}$$

4.4.1. Embedding SLK^1 into the categorical calculus

We show in this section that the (resp. subtractive) propositional calculus LK^1 is conservative over the (resp. symmetrical) categorical propositional calculus, and that the first order calculus SLK^1 is conservative over the first-order symmetrical categorical calculus.

Notation. If $\Gamma \vdash \Delta$ is the sequent $\Gamma_1, \Gamma_2, \ldots_n \vdash \Delta_1, \Delta_2, \ldots, \Delta_m$, the notation $\Gamma^\wedge \vdash \Delta^\vee$ is an abbreviation for $(\ldots((\Gamma_1 \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash (\ldots((\Delta_1 \vee \Delta_2) \vee \cdots \vee \Delta_m)$

Proposition 4.2. *A propositional sequent $\Gamma \vdash \Delta$ is derivable in* SLK^1 *iff $\Gamma^\wedge \vdash \Delta^\vee$ is derivable in the symmetrical categorical propositional calculus.*

Proof. By induction on the derivation, directly for the introduction rules of $\top$, $\wedge$, and $\Rightarrow$ and by duality for $\bot$, $\vee$, and $-$. In fact, the constraints on SLK^1 are needed only to derive the translation of the right introduction rule of implication and the left introduction rule of subtraction, and precisely theses rules are translated into themselves (see [2] for further details). $\square$

Proposition 4.3. *A first-order sequent $\Gamma \vdash \Delta$ is derivable in* SLK^1 *then $\Gamma^\wedge \vdash \Delta^\vee$ iff it is derivable in the* (*first order*) *symmetrical categorical propositional calculus.*

Proof. The only tricky case is the translation of the rules for $\forall$ and $\exists$ from SLK^1 into the first-order symmetrical categorical propositional calculus. Let us treat the case of

$\forall$ (where x occurs neither in Γ nor in Δ in the right introduction rule):

$$\frac{\Gamma^\wedge \vdash A \vee \Delta^\vee}{\Gamma^\wedge \vdash \Delta^\vee \vee \forall x A} \qquad \frac{\Gamma^\wedge \wedge A[t/x] \vdash \Delta^\vee}{\Gamma^\wedge \wedge \forall x A \vdash \Delta^\vee}$$

can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma^\wedge \vdash A \vee \Delta^\vee}{\Gamma^\wedge - \Delta^\vee \vdash A}}{\Gamma^\wedge - \Delta^\vee \vdash \forall x A}}{\Gamma^\wedge \vdash \Delta^\vee \vee \forall x A} \qquad \dfrac{\dfrac{\dfrac{\Gamma^\wedge \wedge A[t/x] \vdash \Delta^\vee}{A[t/x] \vdash \Gamma^\wedge \Rightarrow \Delta^\vee}}{\forall x A \vdash \Gamma^\wedge \Rightarrow \Delta^\vee}}{\Gamma^\wedge \wedge \forall x A \vdash \Delta^\vee}$$

The rules for $\exists$ are obtained as usual by duality. $\square$

Corollary 4.4. *In the first-order framework, the calculus* SLK^1 *is conservative over the calculus* LK^1.

Proof. Indeed, SLK^1 and LK^1 both allow to prove DIS (see Proposition 4.1) but are also both conservative over the first-order symmetrical categorical calculus. $\square$

4.5. Embedding LK into SLK^1

In any bi-topological space $\mathcal{O}$ on X, given a valuation $\mathcal{V}$, it is possible to give (independently of its bi-topological interpretation) the classical interpretation of a formula A on $\mathcal{P}(X)$ seen as a boolean algebra. In addition, we saw that the interpretation of the negation led to two bi-topological semantics (the interior or the exterior of the complement) which approximate upwards and downwards the interpretation of classical negation (i.e. the complement). This property extends to the other connectives: the interior and the exterior of the classical semantics of a connective can always be rephrased as the intuitionistic semantics of the same connective (or some derived connectives). For instance, the interior of the classical semantics of the implication is exactly the intuitionistic semantics of implication, whereas the exterior of the semantics of implication is exactly the intuitionistic semantics of *weak implication* defined by $A \rightsquigarrow B \equiv {\sim}A \vee B$ (since $ext(A^c \cup B) = ext(A^c) \cup B$).

This observation leads us to the following question: could we inductively map to any formula A, two new formulas A^- and A^+ whose topological semantics surround the *most accurately* the classical interpretation of A? We here answer this question by defining a translation of (subtractive) classical propositional logic into propositional subtractive logic. Then we show that this translation extends directly to cut-free proofs of the sequent calculus LK.

4.6. Translation of formulas

We define the following translation of the classical propositional formulas into subtractive propositional formulas. We denote by $^-$ the translation that decreases the bi-topological interpretation of the formula whereas we denote by $^+$ the translation that increases it. The classical negation of A is again denoted by $A^{\perp}$.

$$\begin{array}{ll}
A^- \equiv A, \text{ if } A \text{ is an atom}, & A^+ \equiv A, \text{ if } A \text{ is an atom}, \\
(A \wedge B)^- \equiv A^- \wedge B^-, & (A \wedge B)^+ \equiv A^+ \wedge B^+, \\
(A \vee B)^- \equiv A^- \vee B^-, & (A \vee B)^+ \equiv A^+ \vee B^+, \\
(A \Rightarrow B)^- \equiv A^+ \Rightarrow B^-, & (A \Rightarrow B)^+ \equiv {\sim}A^- \vee B^+, \\
(A - B)^- \equiv A^- \wedge \neg B^+, & (A - B)^+ \equiv A^+ - B^-, \\
(A^{\perp})^- \equiv \neg A^+, & (A^{\perp})^+ \equiv {\sim}A^-.
\end{array}$$

Remark. The translation of subtraction is given here to reveal the symmetry between the two translations. It is known that in classical logic, any propositional formula is equivalent, for example, with a formula containing only conjunctions and negations. However such formulas are provable in classical logic if and only if they are provable in intuitionistic logic. Thus a translation is interesting only when it is *faithful*. According to this criterion, if we consider a formula not containing any subtraction, this translation just substitutes any classical negation (resp. implication) by a weak or intuitionistic negation (resp. implication) according to its occurrence (either positive or negative) in the original formula.

Example. We recall that weak implication is defined by $A \rightsquigarrow B = {\sim} A \vee B$.
- the translation of the axiom $A^{\perp\perp} \Rightarrow A$ is $(A^{\perp\perp})^- \rightsquigarrow A^+ \equiv \neg {\sim} A \rightsquigarrow A$,
- the translation of Peirce axiom $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$ is: $((A \Rightarrow B) \Rightarrow A)^- \rightsquigarrow A^+ \equiv ((A \Rightarrow B)^+ \Rightarrow A) \rightsquigarrow A \equiv ((A \rightsquigarrow B) \Rightarrow A) \rightsquigarrow A$,
- the translation of $A^{\perp\perp} \vdash A$ is $\neg {\sim} A \vdash A$,
- the translation of $((A \Rightarrow B) \Rightarrow A) \vdash A$ is $((A \rightsquigarrow B) \Rightarrow A) \vdash A$.

Definition 4.5. Given a set X and an assignment $\mathscr{V}$ on $\mathscr{P}(X)$, we denote by $[\![A]\!]^K$ the interpretation of A in the boolean algebra of subsets of X. More specifically, for any propositional formula A (possibly containing subtractions):
- $[\![A]\!]^K = \mathscr{V}(A)$ if A is an atom,
- $[\![A \wedge B]\!]^K = [\![A]\!]^K \cap [\![B]\!]^K$,
- $[\![A \vee B]\!]^K = [\![A]\!]^K \cup [\![B]\!]^K$,
- $[\![A \Rightarrow B]\!]^K = ([\![A]\!]^K)^c \cup [\![B]\!]^K$,
- $[\![A - B]\!]^K = [\![A]\!]^K \cap ([\![B]\!]^K)^c$,
- $[\![A^{\perp}]\!]^K = ([\![A]\!]^K)^c$.

Of course, we have the following lemma:

Lemma 4.6. *For any propositional formula A, we have $A^+ \Leftrightarrow A \Leftrightarrow A^-$ in classical logic (where $A - B \equiv A \wedge B^{\perp}$ and ${\sim} A \equiv A^{\perp}$).*

Proof. It is easy to check that $[\![A^-]\!]^K = [\![A]\!]^K = [\![A^+]\!]^K$. $\square$

Theorem 4.7. *If a sequent $A \vdash B$ is valid in classical logic, then $A^- \vdash B^+$ is valid in intuitionistic subtractive logic.*

Proof. By induction on the formula, we show that for any bi-topological space $\mathcal{O}$ on a set X, any (intuitionistic) assignment $\mathcal{V}$ on $\mathcal{O}$:

$$[\![A^-]\!]_{\mathcal{O}} \subset [\![A]\!]^K \subset [\![A^+]\!]_{\mathcal{O}}.$$

Indeed:

- If A is an atom then $A^- = A = A^+$ and $[\![A]\!]^K = [\![A]\!]_{\mathcal{O}} = \mathcal{V}(A)$.
- $[\![(A \wedge B)^-]\!]_{\mathcal{O}} = [\![A^- \wedge B^-]\!]_{\mathcal{O}} = [\![A^-]\!]_{\mathcal{O}} \cap [\![B^-]\!]_{\mathcal{O}} \subset [\![A]\!]^K \cap [\![B]\!]^K$
  $= [\![A \wedge B]\!]^k \subset [\![A^+]\!]_{\mathcal{O}} \cap [\![B^+]\!]_{\mathcal{O}} = [\![A^+ \wedge B^+]\!]_{\mathcal{O}} = [\![(A \wedge B)^+]\!]_{\mathcal{O}}$.
- $[\![(A \Rightarrow B)^-]\!]_{\mathcal{O}} = [\![A^+ \Rightarrow B^-]\!]_{\mathcal{O}} = int([\![A^+]\!]^c \cup [\![B^-]\!]) \subset [\![A^+]\!]^c \cup [\![B^-]\!]$
  $\subset ([\![A]\!]^K)^c \cup [\![B]\!]^K = [\![A \Rightarrow B]\!]^K \subset ([\![A^-]\!]^c_{\mathcal{O}} \cup [\![B^+]\!]_{\mathcal{O}}) \subset ext([\![A^-]\!]^c_{\mathcal{O}} \cup [\![B^+]\!]_{\mathcal{O}})$
  $= ext([\![A^-]\!]^c_{\mathcal{O}}) \cup [\![B^+]\!]_{\mathcal{O}} = [\![\sim A^- \vee B^+]\!]_{\mathcal{O}} = [\![(A \Rightarrow B)^+]\!]_{\mathcal{O}}$.
- $[\![(A^\perp)^-]\!]_{\mathcal{O}} = [\![\neg A^+]\!]_{\mathcal{O}} = int([\![A^+]\!]^c) \subset ([\![A]\!]^K)^c = [\![A^\perp]\!]^K \subset [\![A^-]\!]^c_{\mathcal{O}}$
  $\subset ext([\![A^-]\!]^c_{\mathcal{O}}) = [\![\sim A^-]\!]_{\mathcal{O}}$.

The other cases are obtained by duality. We deduce then that if for any classical assignment $[\![A]\!]^K \subset [\![B]\!]^K$ then for any intuitionistic assignment $[\![A^-]\!]_{\mathcal{O}} \subset [\![A]\!]^K \subset [\![B]\!]^K \subset [\![B^+]\!]_{\mathcal{O}}$ and thus $A^- \vdash B^+$ is valid in intuitionistic logic. $\square$

Corollary 4.8. *If a formula A is a classical tautology, then A^+ is an intuitionistic tautology.*

4.7. Translation of proofs

We show in this section that any cut-free proof $\Gamma \vdash \Delta$ of LK can be turned into a (cut-free) proof of $\Gamma^+ \vdash \Delta^-$ in SLK^1.

Lemma 4.7. *In* SLK^1, *it is possible to derive the following rule*:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightsquigarrow B, \Delta}$$

Proof.

$$\cfrac{\cfrac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \sim A, B, \Delta}}{\Gamma \vdash \sim A \vee B, \Delta} \qquad \square$$

Lemma 4.8. *For any formula A, the sequent $A^- \vdash A^+$ is derivable in* SLK^1.

Proof. By induction on the formula A. The only non-trivial case is the case of implication. Let us assume that $A^- \vdash_{\mathrm{SLK}^1} A^+$ and $B^- \vdash_{\mathrm{SLK}^1} B^+$ and let us show that

$(A \Rightarrow B)^- \vdash_{\mathrm{SLK}^1} (A \Rightarrow B)^+$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots \qquad \vdots}{A^- \vdash A^+ \quad B^- \vdash B^+}}{A^-, A^+ \Rightarrow B^- \vdash B^+}}{A^+ \Rightarrow B^- \vdash \sim A^-, B^+}}{A^+ \Rightarrow B^- \vdash \sim A^- \vee B^+} \qquad \square$$

Theorem 4.9. *It is possible to translate any cut-free proof in LK of a sequent $\Gamma \vdash \Delta$, into a (cut-free) proof in* SLK[1] *of the sequent $\Gamma^- \vdash \Delta^+$ in the following way: translate each occurence of an axiom $A \vdash A$, by a proof in* SLK[1] *of $A^- \vdash A^+$, each occurrence of the left or right introduction rule of the classical negation respectively by the derived rules:*

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \sim A, \Delta} \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}$$

and eventually each occurence of the right introduction rule of implication by the derived rule:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightsquigarrow B, \Delta}$$

Proof. The proof is done by induction on the derivation of $\Gamma \vdash_{\mathrm{LK}} \Delta$. An axiom $A \vdash A$, by construction, is replaced by a proof of $A^- \vdash_{\mathrm{SLK}^1} A^+$ (see Lemma 4.8). The translation of an instance of a rule of LK is either already an instance of a rule of SLK[1] or an instance of the derived rule given by the translation. For example, the translation of an instance of the right introduction rule of implication gives:

$$\frac{\Gamma^-, A^- \vdash B^+, \Delta^+}{\Gamma^- \vdash A^- \rightsquigarrow B^+, \Delta^+}$$

and precisely $A^- \rightsquigarrow B^+ = \sim A^- \vee B^+ = (A \Rightarrow B)^+$. $\square$

Remark. The restriction to cut-free proofs is justified by the fact that an instance of the cut-rule:

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma', A \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

is translated into the following instance, which is no longer an instance of the cut rule:

$$\frac{\Gamma^- \vdash A^+, \Delta^+ \qquad \Gamma'^-, A^- \vdash \Delta'^+}{\Gamma^-, \Gamma'^- \vdash \Delta^+, \Delta'^+}$$

Of course, due to cut-elimination in LK, this translation is still valid.

4.8. First-order logic

The translation extends directly to first-order logic. However, we point out that in first-order logic, DIS is provable in the calculus SLK[1] (see Proposition 4.1).

Consequently, the translation presented here is from first-order classical logic into first-order subtractive logic.

$$(\forall xA)^- \equiv \forall xA^-, \qquad (\forall xA)^+ \equiv \forall xA^+,$$
$$(\exists xA)^- \equiv \exists xA^-, \qquad (\exists xA)^+ \equiv \exists xA^+.$$

Theorem 4.10. *Given a cut-free proof of $\Gamma \vdash \Delta$, in first-order* LK, *the proof obtained by applying the translation described in the statement of Theorem* 4.5 *is a* (*cut-free*) *proof in* SLK^1 *of the first-order sequent* $\Gamma^- \vdash \Delta^+$.

Remark. This translation of cut-free proofs of LK extends by duality to cut-free proofs of SLK. However, although we conjecture cut-elimination in SLK, we did not consider this issue in this paper.

4.9. Future work

In this paper, we have not considered computational issues. Although bi-[CCC] are degenerated, it is possible to weaken the equational theory (see [8] for an interpretation of weak initial and final objects in terms of lazy and eager evaluation) and to study corresponding rewriting systems.

Various confluent rewriting systems, based on categorical combinators and in which λ-calculus can be simulated, are known (see [13, Section 4.4]). However, extending this result to bi-[CCC] combinators, involve first to solve this problem in bi-cartesian closed categories (i.e. with co-product but without co-exponent) which already seems to raise difficulties [5]. Moreover, Mellies [15] proved that some systems of categorical combinators, with extensionality rules, even in a typed framework, does not satisfy the normalization property. The proof of this result is obtained by defining a fixed point using categorical combinators, exactly in the same way as in some λ-calculus with explicit substitutions.

Besides, we have shown that there is no functionnal completeness in bi-[CCC]. Consequently, if we want to define some λ-calculus whose type system corresponds to subtractive logic, we need to restrict the λ-abstraction (i.e. the right introduction rule of implication) of some classical λ-calculus. A forthcoming paper will be devoted to this subject (where we will use Parigot's $\lambda\mu$-calculus [17, 18] whose computational properties are better known than those of Filinski's symmetrical λ-calculus).

References

[1] A. Asperti, G. Longo, Categories, Types and Structures, MIT Press, Cambridge, MA, 1991.
[2] T. Crolard, Extension de l'isomorphisme de Curry-Howard au traitement des exceptions (application d'une étude de la dualité en logique intuitionniste). Thése de Doctorat. Université Paris 7, 1996.
[3] Á. Császár, Fondement de la topologie générale, Gauthier-Villars, Paris, 1960.
[4] H.B. Curry, Foundations of Mathematical Logic, McGraw-Hill, New York, 1963.
[5] D.J. Dougherty, R. Subrahmanyam, Equality between functionals in the presence of coproducts, in Proc. 10th Annual IEEE Symp. on Logic in Computer Science, 1995, pp. 282–291.

[6] A.G. Dragalin, Mathematical intuitionism — introduction to proof theory, in: Translations of Mathematical Monographs, Vol. 67, American Matematical Society, Providence, RI, 1988.
[7] R. Dyckhoff, Contraction-free sequent calculi for intuitionistic logic, J. Symbolic Logic 57(3) (1992) 795–807.
[8] A. Filinski, Declarative continuations: an investigation of duality in programming language semantics, in Category Theory and Comput. Sci., Lecture Notes in Computer Science, Vol. 389, Springer, Berlin, 1989, pp. 224–249.
[9] D. Gabbay, Semantical Investigations in Heyting Intuitionistic Logic, Reidel, Dordrecht, 1981.
[10] J.-Y. Girard, A new constructive logic: classical logic, Math. Struct. Comput. Sci. 1 (1991) 255–296.
[11] J.-Y. Girard, On the unity of logic, Ann. Pure Appl. Logic 59 (1993) 201–217.
[12] J.-Y. Girard, Y. Lafont, P. Taylor, Proofs and Types, Cambridge Tracts in Theoretical Computer Science, Vol. 7, Cambridge University Press, Cambridge, 1988.
[13] G. Huet, Cartesian closed categories and lambda-calculus, in: P.L. Curien, G. Cousineau, B. Robinet, (Eds.), Combinators and Functionnal Programming Languages, Vol. 242, Springer, Berlin, 1986, pp. 123–135.
[14] J. Lambek, P.J. Scott, An Introduction to Higher Order Categorical Logic, Studies in Advanced Mathematics, Vol. 7, Cambridge University Press, Amsterdam, 1986.
[15] P.A. Mellies, Typed λ-calculi with explicit substitutions may not terminate, in 2nd Internat. Conf. on Typed Lambda Calculi and Applications, LNCS, Edinburgh, UK, 1995, pp. 328–334.
[16] H. Ono, Model extension theorem and Graigs interpolation theorem for intermediate predicate logics, Rep. Math. Logic 15 (1983) 41–57.
[17] M. Parigot, $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction, in Proc. Logic Prog. and Autom. Reasoning, Lecture Notes in Computer Science, Vol. 624, Springer, Berlin, 1992, pp. 190–201.
[18] M. Parigot, Classical proofs as programs, in Computationnal Logic and Theory, Lecutre Notes in Computer Science, Vol. 713, Springer, Berlin, 1993, pp. 263–276.
[19] C. Rauszer, Semi-boolean algebras and their applications to intuitionistic logic with dual operations, in Fund. Math. 83 (1974) 219–249.
[20] C. Rauszer, An algebraic and Kripke-style approach to a certain extension of intuitionistic logic, in Dissertationes Mathematicae, Vol. 167, Institut Mathématique de l'Académie Polonaise des Sciences, 1980.
[21] M.E. Szabo, Gentzen Collected Work, North-Holland, Amsterdam, 1969.
[22] M.E. Szabo, A categorical characterization of boolean algebras, Algebra Univ. 4 (1974) 192–194.
[23] D. van Dalen, Intuitionistic logic, in Handbook of Philosophical Logic, Vol. 3, D. Reidel Publishing Company, Dordrecht, 1986, pp. 225–339.