Eigenvalues of tridiagonal and symmetric matrices

Mallo Giuseppe, Nespolo Elena

Abstract

The goal of this paper is to determine the eigenvalues of the matrix using the Gershgorin's theorems. Indeed when considering a matrix with particular proprieties, such as tridiagonality and simmetry, it is possible to extract valuable insights about the matrix, such as the range within which the eigenvalues lie. These insights then lead us to accurately identifying the eigenvalues.

1 Problem overview

The considered matrix $\mathbb{R}^{n\times n}$ is a symmetric and tridiagonal matrix, hence the superdiagonal and the subdiagonal share the same values and the form of the matrix will be:

$$A = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \dots & & \\ \beta_2 & \alpha_2 & \beta_2 & 0 & & \\ 0 & \beta_3 & \alpha_3 & \beta_3 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & 0 & \beta_{n-2} & \alpha_{n-1} & \beta_{n-2} \\ & & & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}$$

In particular, we will use the matrix in $\mathbb{R}^{n\times n}$:

$$A = \begin{bmatrix} 2 & 1 & 0 & \dots & & \\ 1 & 2 & 1 & 0 & & \\ 0 & 1 & 2 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 2 & 1 \\ & & & 0 & 1 & 2 \end{bmatrix}$$

We can immediately see that this matrix is irreducible because all the elements of the subdiagonal and of the superdiagonal are ones, therefore nonzero.

It is also diagonally dominant by rows (and by columns) because for each row the sum of the absolute values of the elements that don't belong to the main diagonal are always less than the absolute value of the element of the diagonal.

When A possesses these two characteristics, it is guaranteed to be nonsingular, as established by the following theorem:

Theorem 1 *Let $A \in \mathbb{R}^{n\times n}$ is diagonally dominant and irreducible, then A is nonsingular.*

Moreover, if the diagonal elements of A are all positive, then the real parts of the eigenvalues are positive.

Finally, if A is symmetric, then the eigenvalues are positive and therefore definite positive.

This can also be demonstrated using Gershgorin's theorem, which not only confirms the positivity of the eigenvalues but also provides a bounded range for their values.

For any matrix, we can define the Gershgorin's circle as:

$$K_i = \left\{ z \in \mathbb{C} : |z - a_{i,i}| \leq \sum_{j=1, j\neq i}^{n} |a_{i,j}| \right\} \quad i = 1 \dots n$$

In our case, we get only two circles:

$$K_i = \{z \in \mathbb{C} : |z - 2| \leq 2\} \quad i = 2 \dots n-1$$
$$K_i = \{z \in \mathbb{C} : |z - 2| \leq 1\} \quad i = 1, n$$

This claims that the minimum value the real part of an eigenvalue of A can reach is 0, while the maximum is 4, as show in the Figure 1.

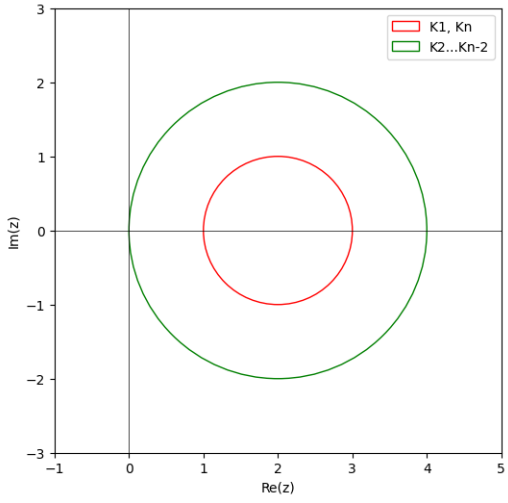

Figure 1: Gershgorin's circles for A

Theorem 2 *Let A be an irreducible matrix, then any eigenvalue of A lying on the boundary of a Gershgorin's circle must lie on the boundary of all the Gershgorin's circles.*

Furthermore, after claiming the previously stated theorem, we conclude that the eigenvalues of A don't belong to the boundaries of any of the two calculated circles. In particular, it is included the value of 0 and it confirms that A is non singular.

This result, combined with the fact that A is positive definite, leads to the conclusion that the eigenvalues lie within $(0,1) \cup (1,3) \cup (3,4)$.

2 Computing eigenvalues with the Newton Method

The Newton method is a numerical method used to solve optimization problems, as well as to solve non-linear equations or systems of equations. However, the Newton method is very sensitive to the starting point, which is why we will use the interval provided by the Gershgorin circles to find a subinterval from which to select a starting point for the Newton method that is close enough to the solution.

The question is, how do we obtain the subinterval from which to choose the initial point?

We will make use of some quantities that are useful for this purpose.

First, we use polynomials of increasing degree (from 0 to n, where n is the number of eigenvalues), each equal to the truncation of the characteristic polynomial at a given degree. Below, we list them:

- $p_0(\lambda) = 1$
- $p_1(\lambda) = \alpha_1 - \lambda$
- ...
- $p_i(\lambda) = (\lambda_i - \alpha_i)p_{i-1}(\lambda) - \beta_i^2 p_{i-2}(\lambda)$
- ...

As i=$2 \dots n$

The first order derivatives of those polynomials are:

- $p_0'(\lambda) = 0$
- $p_1'(\lambda) = -1$
- ...
- $p_i'(\lambda) = (\lambda_i - \alpha_i)p_{i-1}'(\lambda) - \beta_i^2 p_{i-2}'(\lambda) - p_{i-1}(\lambda)$
- ...

Then, we choose a common point λ_* for all the polynomials on which to evaluate them. Each polynomial will return a scalar value with a specific sign. If we order the values returned by the degree of the polynomial in increasing order, we can count how many times a scalar changes sign compared to the first non-zero value preceding it. This count will be equal to a fundamental quantity, which is W.

Now, thanks to a theorem, we can state that:

Theorem 3 *Let* $[a, b] \in \mathbb{R}$. *Then, the number of eigenvalues in* $[a, b)$ *is* $W(b) - W(a)$.

This theorem gives us the possibility to identify a subinterval where a single eigenvalue will be contained. Under the assumption that the roots of the characteristic polynomial (and therefore, the eigenvalues) of matrix A are simple, distinct, and ordered in this way:

$$\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_i > \ldots > \lambda_n$$

Supposing we want to find the k-th largest eigenvalue, we can apply the following algorithm.

Given k, a_0 and b_0

$a \leftarrow a_0$

$b \leftarrow b_0$

while $W(b) - W(a) > 1$

$m \leftarrow 1/2$

if $W(m) < n - k + 1$

$a \leftarrow m$

else

$b \leftarrow m$

With this pseudocode, we note that as long as $W(b) - W(a)$ is strictly greater than 1, it means that we can improve and therefore narrow the interval of interest even further. The way to narrow it down is by varying one of the two endpoints at each iteration. To do this, we calculate the midpoint m of the current interval $[a, b]$.

If $W(m) < n-k+1$, it means that the k-th largest eigenvalue we are looking for will not be found before m. In that case, m will definitely become the lower bound of the new subinterval that we will use for the next iteration. Otherwise, it will be the upper bound.

After finding a sufficiently narrow interval where the k-th largest eigenvalue lies, we can choose its midpoint as the starting point for the Newton's method.

The functioning of Newton's method can be explained with this pseudocode:

Given x_0, a tolerance tol, a maximum number of iterations $maxIter$ and the characteristic polynomial of A that is $p_n(\lambda)$

$x \leftarrow x_0$

$x_{new} \leftarrow 0$

$i \leftarrow 1$

while $(i < maxIter$ and $|p_n(x)| > tol)$

$x_{new} \leftarrow x - \frac{p_n(x)}{p'_n(x)}$

$x_{new} \leftarrow x$

$i \leftarrow i + 1$

return x

This method will converge to the k-th largest eigenvalue, as the interval we use is narrow enough and contains only it.

3 Conclusion

We apply this method to the matrix A of dimension 10.

With the first step that relates to the finding of an subinterval containing only 1 eigenvalues, the calculations provide us with the following split of the interval $(0, 4)$:

$$0 < \lambda_{10} \leq 0.25 \leq \lambda_9 \leq 0.5 \leq \lambda_8 < 1 < \lambda_7 \leq 1.5 \leq \lambda_6 \leq 2 \leq \lambda_5 \leq 2.5 \leq \lambda_4 < 3 < \lambda_3 \leq 3.5 \leq \lambda_2 \leq 3.75 \leq \lambda_1 < 4$$

Then the obtained eigenvalues are:

$$0.081, 0.317, 0.69, 1.169, 1.715, 2.285, 2.831, 3.31, 3.683, 3.919$$

The eigenvalues are obtained with at most 4 iterations because Newton method has a quadratic convergence if the initial guess is obtained near the solution. Hence this approach is really fast and competitive but the strict hypotheses about the form of the matrix are a significative disadvantage for this method since it cannot be applied very often.