Mathematical Foundations of the Buera & Shin (2013) Spectral Time Iteration Implementation

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1 The Economic Environment

The economy is populated by a continuum of agents who choose to be either workers or entrepreneurs. Agents are heterogeneous in their asset holdings a and entrepreneurial ability z.

1.1 Preferences and Shocks

Agents maximize expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{1}$$

Entrepreneurial ability z follows a reset process: with probability ψ, $z_{t+1} = z_t$; with probability $1-\psi$, a new ability is drawn $z' \sim \pi(z)$ from a Pareto distribution with tail parameter η. In the pre-reform state, agents also face idiosyncratic output wedges $\tau \in \{\tau^+, \tau^-\}$, where $\mathbb{P}(\tau = \tau^+|z) = 1 - e^{-qz}$.

2 Production and Optimal Firm Decisions

An entrepreneur with ability z and assets a operates the technology:

$$y = z(k^\alpha l^{1-\alpha})^{1-\nu} \tag{2}$$

where ν represents the span-of-control parameter (entrepreneur's share). The entrepreneur faces a collateral constraint $k \leq \lambda a$.

2.1 Static Profit Maximization

The entrepreneur's problem is solved in two stages. First, given capital k, the optimal labor $l^*(k,z)$ is found by solving:

$$\max_l \{z(k^\alpha l^{1-\alpha})^{1-\nu} - wl - (r+\delta)k\} \tag{3}$$

The first-order condition with respect to l is:

$$(1-\nu)(1-\alpha)zk^{\alpha(1-\nu)}l^{(1-\alpha)(1-\nu)-1} = w \tag{4}$$

Solving for l yields the conditional labor demand:

$$l^*(k,z) = \left[\frac{(1-\nu)(1-\alpha)z}{w}\right]^{\frac{1}{1-(1-\alpha)(1-\nu)}} k^{\frac{\alpha(1-\nu)}{1-(1-\alpha)(1-\nu)}} \tag{5}$$

Substituting $l^*(k,z)$ back into the profit function gives a profit function $\pi(k)$ that is concave in k. The unconstrained capital demand k^{unc} satisfies $\pi'(k^{unc}) = r + \delta$. If $k^{unc} > \lambda a$, the entrepreneur is borrowing-constrained and sets $k^* = \lambda a$.

3 The Recursive Household Problem

The agent's state is (a, z). The total income is $y(a, z) = \max\{\pi(a, z, k^*), w\} + (1 + r)a$. The Bellman equation is:

$$V(a, z) = \max_{a'} \left\{ \frac{(y(a, z) - a')^{1-\sigma} - 1}{1 - \sigma} + \beta \text{Cont}(a', z) \right\} \tag{6}$$

where the continuation value is:

$$\text{Cont}(a', z) = \psi V(a', z) + (1 - \psi) \int V(a', z')\pi(z')dz' \tag{7}$$

3.1 Derivation of the Euler Equation and KKT Conditions

To solve the household's problem under the borrowing constraint $a_{t+1} \geq a_{min}$, we form the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma} - 1}{1 - \sigma} + \mu_t(a_{t+1} - a_{min}) \right] \tag{8}$$

Substituting $c_t = y(a_t, z_t) - a_{t+1}$, the first-order condition with respect to a_{t+1} is:

$$-c_t^{-\sigma} + \beta \mathbb{E}_t[(1 + r)c_{t+1}^{-\sigma}] + \mu_t = 0 \implies c_t^{-\sigma} = \beta(1 + r)\mathbb{E}_t[c_{t+1}^{-\sigma}] + \mu_t \tag{9}$$

The Lagrange multiplier μ_t must satisfy the Karush-Kuhn-Tucker (KKT) complementarity conditions:

$$\mu_t \geq 0, \quad a_{t+1} \geq a_{min}, \quad \mu_t(a_{t+1} - a_{min}) = 0 \tag{10}$$

When the agent is unconstrained ($\mu_t = 0$), the standard Euler equation holds. When the constraint binds ($a_{t+1} = a_{min}$), we have $\mu_t > 0$, implying $c_t^{-\sigma} > \beta(1 + r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$.

4 Spectral Discretization and Setup

The policy function $a'(a, z)$ is approximated by a bivariate Chebyshev polynomial $\hat{a}(a, z; \boldsymbol{\gamma})$.

4.1 Chebyshev Collocation Nodes

The state space is discretized using the zeros of the N-th degree Chebyshev polynomial $\xi \in [-1, 1]$:

$$\xi_k = \cos\left(\frac{2k - 1}{2N}\pi\right), \quad k \in \{1, \dots, N\} \tag{11}$$

The nodes are mapped to the physical state space (a, z). For assets, we use a log-linear mapping to cluster nodes near the borrowing constraint:

$$a_k = \exp\left(\frac{\ln(a_{max} + \text{shift}) + \ln(a_{min} + \text{shift})}{2} + \xi_k \frac{\ln(a_{max} + \text{shift}) - \ln(a_{min} + \text{shift})}{2}\right) - \text{shift} \tag{12}$$

while ability nodes z_j are mapped linearly.

5 Spectral Time Iteration Logic

5.1 Pointwise Euler Inversion

To solve for the optimal policy, we utilize a recursive operator that linearizes the Global Euler Equation. At each collocation node (a_i, z_j), we define the **Target Function** $F(a')$ as the difference between current marginal utility and expected future marginal utility:

$$F(a'; a_i, z_j, \boldsymbol{\gamma}^{(n)}) \equiv \underbrace{(y_i - a')^{-\sigma}}_{\text{Current MU}} - \beta(1 + r) \underbrace{\mathcal{E}(a'; \boldsymbol{\gamma}^{(n)})}_{\text{Continuation MU}} \tag{13}$$

where $\mathcal{E}$ is the expectation over future states $(\hat{a}_i, z')$ given the **known** continuation policy $\boldsymbol{\gamma}^{(n)}$:

$$\mathcal{E}(a'; \boldsymbol{\gamma}^{(n)}) = \psi \left(y(a', z_j) - \boldsymbol{\Phi}(a', z_j)^\top \boldsymbol{\gamma}^{(n)} \right)^{-\sigma} + (1-\psi) \sum_k \pi_k \left(y(a', z_k) - \boldsymbol{\Phi}(a', z_k)^\top \boldsymbol{\gamma}^{(n)} \right)^{-\sigma} \tag{14}$$

Since $\mathcal{E}$ is fixed for a given guess of a', we can identify the optimal a^*_{target} by direct inversion. The solver handles the KKT conditions robustly by applying the non-negativity constraint after the unconstrained inversion:

$$a^*_{target,i} = \max\{a_{min}, y_i - \left[\beta(1+r)\mathcal{E}(a'; \boldsymbol{\gamma}^{(n)})\right]^{-1/\sigma}\} \tag{15}$$

This "max" operator ensures that when the implied savings would fall below a_{min}, the agent is snapped to the constraint, implicitly satisfying $\mu_t > 0$. This formulation depends exclusively on the grid points (a_i, z_j) and the candidate savings a', with the future behavior already "baked" into the coefficients $\boldsymbol{\gamma}^{(n)}$.

5.2 Damped Coefficient Update

The algorithm computes the new policy nodes $\boldsymbol{A}^*_{new}$ by applying a damping parameter $\theta \in (0, 1]$ to ensure numerical stability:

$$\boldsymbol{A}^*_{new} = (1-\theta)\boldsymbol{A}^*_{old} + \theta \boldsymbol{A}^*_{target} \tag{16}$$

The updated spectral coefficients $\boldsymbol{\gamma}^{(n+1)}$ are then obtained by solving the linear basis transformation:

$$\boldsymbol{\gamma}^{(n+1)} = \boldsymbol{T}^{-1} \boldsymbol{A}^*_{new} \tag{17}$$

where $\boldsymbol{T}$ is the matrix of Chebyshev basis functions evaluated at the collocation nodes.

6 Distributional and Transition Dynamics

The distribution is tracked on a dense grid using a Markov transition matrix Q derived from the continuous policy $\hat{a}$.

$$\mu_{t+1}(a', z') = \sum_{a,z} \mathbb{P}(a'|a, z)\mathbb{P}(z'|z)\mu_t(a, z) \tag{18}$$

Aggregate labor L_t and capital K_t are then calculated by summing firm decisions over the distribution μ_t.

7 Convergence and General Equilibrium

The general equilibrium wage w and interest rate r are found by iterating on the price paths using Algorithm B.2. The spectral method's continuity ensures that the excess demand functions $ED_L(w, r)$ and $ED_K(w, r)$ are smooth, allowing for robust bisection even with a relatively small number of spectral nodes.