Mathematical Appendix: Spectral Policy Solver

Documentation for Buera & Shin (2010) v2 Implementation

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1 The Recursive Problem

The agent faces an infinite-horizon consumer-entrepreneur problem. The Bellman equation is given by:

$$V(a, z) = \max_{c,a'} \frac{c^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_{z'}[V(a', z')] \tag{1}$$

Subject to the budget constraint:

$$c + a' = \max\{\pi(a, z), w\} + (1 + r)a \tag{2}$$

where a is assets, z is entrepreneurial productivity, $\pi(a, z)$ is the profit from entrepreneurship, w is the equilibrium wage, and r is the interest rate.

2 The Optimality Condition

The First-Order Condition with respect to a' (the Euler Equation) is:

$$u'(c) = \beta(1 + r)\mathbb{E}_{z'}[u'(c')] \tag{3}$$

Using the CRRA utility $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, we have $u'(c) = c^{-\sigma}$. Solving for current consumption c:

$$c = \left(\beta(1 + r)\mathbb{E}_{z'}[(c')^{-\sigma}]\right)^{-1/\sigma} \tag{4}$$

3 Spectral Approximation

The policy function $a' = f(a, z)$ is approximated using a bivariate Chebyshev expansion:

$$\hat{f}(a, z; \Theta) = \sum_{i=0}^{N_a-1} \sum_{j=0}^{N_z-1} \theta_{i,j} T_i(\phi_a(a)) T_j(\phi_z(z)) \tag{5}$$

where:

- $\Theta = \{\theta_{i,j}\}$ is the matrix of `coeffs`.
- $T_i(x)$ are Chebyshev polynomials of the first kind.
- $\phi_a(a)$ and $\phi_z(z)$ are linear mappings from the physical domain to $[-1, 1]$.

4 Implementation: Fixed-Point Update

The routine `solve_policy_bivariate_update` performs the following steps:

1. **Collocation**: Defines the problem at the $N_a \times N_z$ Chebyshev nodes (a_n, z_m).

2. **Quadrature**: The expectation $\mathbb{E}_{z'}[u'(c')]$ is computed as:

$$\mathbb{E}[MU'] = \psi \cdot u'(c'(a', z_m)) + (1 - \psi) \int_{z'} u'(c'(a', z'))dG(z') \tag{6}$$

   The integral is discretized using a fine Pareto quadrature grid.

3. **Iterative Step**: Given the current coefficients $\Theta^{(k)}$, a target value f_{target} is calculated for each node.

4. **Projection**: The updated coefficients are found by projecting the targets back onto the Chebyshev basis:

$$\Theta^{(k+1)} = \mathbf{T}^{-1} \cdot \mathbf{f}_{target} \tag{7}$$