Replication of Buera & Shin (2013)
Implementation with Spectral Time Iteration and Chebyshev Collocation

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1 Introduction

This document details the numerical implementation of the Buera-Shin (2013) algorithm using **Spectral Time Iteration**. Unlike the discrete Value Function Iteration (VFI) approach, this method approximates the policy function continuously using Chebyshev polynomials and solves the model by satisfying the Euler equation at a set of optimal collocation points.

2 The Economic Environment

The environment consists of heterogeneous agents with assets a and ability z.

2.1 Preferences and Budget Constraint

Agents solve:

$$V(a, z) = \max_{c,a'} \left\{ \frac{c^{1-\sigma} - 1}{1 - \sigma} + \beta \mathbb{E}[V(a', z')|z] \right\} \tag{1}$$

subject to:

$$c + a' = i(a, z; w, r), \quad a' \geq a_{min} \tag{2}$$

where $i(a, z; w, r) = \max\{\pi(a, z), w\} + (1 + r)a$ is the total income.

3 Spectral Approximation and Collocation

The policy function $a'(a, z)$ is approximated using a bivariate Chebyshev polynomial basis.

3.1 State Space Transformation

To utilize Chebyshev polynomials defined on $[-1, 1]$, the state space is mapped using a log-linear transformation for assets:

$$x_a = 2\frac{\log(a + \bar{a}) - \log(a_{min} + \bar{a})}{\log(a_{max} + \bar{a}) - \log(a_{min} + \bar{a})} - 1 \tag{3}$$

where $\bar{a}$ is a shift parameter to handle points near zero. Ability z is mapped linearly:

$$x_z = 2\frac{z - z_{min}}{z_{max} - z_{min}} - 1 \tag{4}$$

3.2 Approximation Basis

The savings policy is represented as:

$$\hat{a}(a, z; \boldsymbol{C}) = \sum_{i=0}^{N_a-1} \sum_{j=0}^{N_z-1} c_{ij} T_i(x_a) T_j(x_z) \tag{5}$$

where $T_k(\cdot)$ is the k-th Chebyshev polynomial. The coefficients $\boldsymbol{C}$ are determined by requiring the savings choice to satisfy the model's optimality conditions at the $N_a \times N_z$ Chebyshev nodes.

4 Time Iteration and Euler Equation Residuals

The solver identifies the optimal coefficients $\boldsymbol{C}$ by iterating on the Euler equation.

4.1 The Euler Equation

Optimality requires:

$$c_t^{-\sigma} = \beta(1+r)\mathbb{E}[c_{t+1}^{-\sigma}] \tag{6}$$

Substituting the budget constraint and the spectral approximation:

$$[i(a,z) - \hat{a}(a,z)]^{-\sigma} = \beta(1+r)\mathbb{E}\left[(i(\hat{a}, z') - \hat{a}(\hat{a}, z'))^{-\sigma}\right] \tag{7}$$

4.2 Iterative Update Logic

Given current coefficients $\boldsymbol{C}^{(n)}$:

1. At each Chebyshev node (a_i, z_j), compute the expected marginal utility:

$$\mathbb{E}[MU] = \psi MU(a', z_j, \boldsymbol{C}^{(n)}) + (1-\psi)\sum_k \pi_k MU(a', z_k, \boldsymbol{C}^{(n)}) \tag{8}$$

   where $a' = \hat{a}(a_i, z_j; \boldsymbol{C}^{(n)})$.

2. Back out the implied optimal target consumption: $c^* = [\beta(1+r)\mathbb{E}[MU]]^{-1/\sigma}$.
3. Compute the implied next-period assets: $a^*_{target} = i(a_i, z_j) - c^*$.
4. Apply the borrowing constraint: $a^*_{final} = \max\{a_{min}, a^*_{target}\}$.
5. Solve the linear system for new coefficients $\boldsymbol{C}^{(n+1)}$ such that $\hat{a}(a_i, z_j; \boldsymbol{C}^{(n+1)}) = a^*_{final}$.

5 Coupled Spectral Solver for Distorted Economies

In the pre-reform state, agents face idiosyncratic distortions $\tau \in \{\tau^+, \tau^-\}$. The solver handles this by maintaining two sets of Chebyshev coefficients, $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$, representing the policy functions for each distortion state.

5.1 Redraw Probability and Expectation

The redraw probability $p(z) = P(\tau = \tau^+|z)$ is integrated into the expected marginal utility calculation. When an agent resets their ability z (with probability $1-\psi$), both (z, τ) are redrawn:

$$\mathbb{E}_z[MU] = \sum_k \pi_k \left[p(z_k) MU(a', z_k, \boldsymbol{C}^+) + (1 - p(z_k)) MU(a', z_k, \boldsymbol{C}^-)\right] \tag{9}$$

The policy $\hat{a}(a, z, \tau)$ then depends on the current tax state, but both policies are coupled through the reset expectation.

6 Distributional Dynamics on a Dense Grid

While the policy function is continuous, the distribution $\mu(a, z)$ is represented discretely on a dense grid of $N_a^{dense} \times N_z^{dense}$ nodes to ensure accurate integration of aggregates.

6.1 Histogram Update

The distribution is updated using a transition matrix Q derived from the continuous policy $\hat{a}$ and linear interpolation:

$$\mu_{t+1} = Q(\boldsymbol{C})\mu_t \tag{10}$$

Aggregates are calculated as:

$$K_d = \sum_{a,z} \mu(a, z) k^*(a, z) \cdot \mathbb{I}(\text{entrepreneur}) \tag{11}$$

7 Algorithm B.2: Nested Price Path Iteration

The transition path is solved using the nested algorithm from Buera and Shin (2013).

7.1 Stability via Nesting

- **Inner Loop (Wage Path)**: For a fixed interest rate path $\{r_t\}$, the algorithm finds the $\{w_t\}$ path that clears the labor market.
- **Outer Loop (Interest Rate Path)**: Updates the $\{r_t\}$ path until the capital market clears.

Spectral methods provide exceptionally smooth demand functions, which significantly improves the convergence speed of the bisection and relaxation algorithms used in the nested loops compared to discrete VFI implementations.