Replication of Buera & Shin (2013) Implementation with Howard Policy Improvement and Histograms

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February 4, 2026

1 The Economic Environment

The economy consists of a continuum of infinitely lived agents of measure one. Agents are heterogeneous in their asset holdings a and entrepreneurial ability z.

1.1 Preferences and Shocks

Agents maximize expected lifetime utility:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \tag{1}$$

Entrepreneurial ability z follows a reset process. In each period, with probability ψ, the ability remains constant: $z_{t+1} = z_t$. With probability $1 - \psi$, a new ability is drawn $z' \sim \pi(z)$ from a Pareto distribution:

$$P(Z \leq z) = 1 - z^{-\eta}, \quad z \in [1, \infty) \tag{2}$$

To simulate distortions in the pre-reform state, agents are subject to idiosyncratic output wedges $\tau \in \{\tau^+, \tau^-\}$. The probability of being in the high-tax state τ^+ is conditional on z: $P(\tau = \tau^+ | z) = 1 - e^{-qz}$.

2 Firm Decisions and Optimal Production

Entrepreneurs operate a technology $y = z(k^\alpha l^{1-\alpha})^{1-\nu}$. They face a collateral constraint $k \leq \lambda a$.

2.1 Analytical Solutions for Firm Decisions

Given (w, r, a, z), the entrepreneur's problem is solved in two stages. 1. **Conditional Labor Demand**: For any k, the optimal labor $l^*(k, z)$ solves the first-order condition $\partial \pi / \partial l = 0$:

$$l^*(k, z) = \left[\frac{(1-\nu)(1-\alpha)z}{w} \right]^{\frac{1}{1-(1-\alpha)(1-\nu)}} k^{\frac{\alpha(1-\nu)}{1-(1-\alpha)(1-\nu)}} \tag{3}$$

2. **Optimal Capital**: Substituting l^* into the profit function yields a concave function $\pi(k)$. The unconstrained optimal capital k^{unc} is found where $\partial \pi / \partial k = 0$. The actual capital choice is capped by the leverage limit:

$$k^*(a, z) = \min\{k^{unc}(z, w, r), \lambda a\} \tag{4}$$

The total income of an agent is $i(a, z; w, r) = \max\{\pi(a, z, k^*, l^*), w\} + (1 + r)a$.

3 State Space Discretization

The state space $\mathcal{A} \times \mathcal{Z}$ is discretized for numerical implementation.

3.1 Asset Grid

To capture the high curvature of the value function near the borrowing constraint, a power-spaced grid for $a \in [a_{min}, a_{max}]$ is used:

$$a_i = a_{min} + (a_{max} - a_{min}) \left(\frac{i}{N_a - 1} \right)^p, \quad i \in \{0, \ldots, N_a - 1\} \tag{5}$$

where $p = 2$ provides a dense cluster of points at low asset levels.

3.2 Ability Grid

The Pareto distribution $\pi(z)$ is sampled into N_z bins of equal probability. Let u be a uniform random variable. The grid points are midpoints of probability intervals $[F(z_j), F(z_{j+1})]$ where F is the Pareto CDF:

$$z_j = (1 - \bar{u}_j)^{-1/\eta}, \quad \bar{u}_j = \frac{j + 0.5}{N_z} \cdot u_{max} \tag{6}$$

4 Numerical Value Function Iteration

The value function $V(a, z)$ is solved on the discretized grid using an optimized iterative process.

4.1 Initialization

The initial guess $V^{(0)}$ is set to the utility of consuming the current income forever at the lowest grid point, providing a stable starting point for the iteration:

$$V^{(0)}(a_i, z_j) = \frac{u(i(a_i, z_j) - a_0)}{1 - \beta} \tag{7}$$

4.2 Expectation Splitting

To compute the continuation value $\mathbb{E}[V(a', z')|z]$, the algorithm splits the expectation to exploit the discrete nature of the reset process. Before making decisions, the "reset" expectation $\bar{V}(a')$ is pre-calculated:

$$\bar{V}(a') = \sum_j \pi_j V(a', z_j) \tag{8}$$

The specific continuation value for state (a, z) becomes a weighted average of the current ability's value and the reset value:

$$\mathbb{E}[V(a', z')|z] = \psi V(a', z) + (1 - \psi)\bar{V}(a') \tag{9}$$

4.3 Howard Policy Improvement

The Bellman operator $T(V)$ is computationally expensive due to the maximization step. Howard's improvement reduces the number of maximizations by iterating on a fixed policy $\sigma(a, z)$.

1. **Policy Step**: Solve for $\sigma(a, z) = \arg\max_{a'}\{u(c) + \beta\mathbb{E}[V(a', z')|z]\}$.

2. **Value Step**: For a fixed σ, iterate N_{how} times:

$$V^{(m+1)}(a, z) = u(i(a, z) - \sigma(a, z)) + \beta[\psi V^{(m)}(\sigma(a, z), z) + (1 - \psi)\bar{V}^{(m)}(\sigma(a, z))] \tag{10}$$

This significantly speeds up convergence as the Value Step is a linear operation[1].

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[1]In principle, the value of a fixed policy σ can be solved in closed form as $V_\sigma = (I - \beta P_\sigma)^{-1} U_\sigma$, where P_σ is the transition matrix of the state space. However, for a grid size of $N_a N_z \approx 36,000$, the memory and computational overhead of inverting a P_σ matrix with 1.3×10^9 elements makes the iterative successive approximation approach more efficient in terms of CPU cache and memory.

4.4 Possible Improvements: Continuous Choice via Interpolation

A significant enhancement to the discrete grid search is to allow for a continuous asset choice $a' \in [a_{min}, i]$. Restricted discrete grids $\{a_0, \ldots, a_{N_a-1}\}$ introduce "step" artifacts and high-frequency noise in aggregate excess demand functions. By employing interpolation, we solve for a^* as a continuous variable:

$$a^*(a, z) = \arg \max_{a' \in [a_{min}, i]} \{u(i - a') + \beta \mathbb{E}[V(a', z')|z]\} \tag{11}$$

There are three primary computational pathways to implementing this continuous choice:

1. Direct Root-finding (High Accuracy) We solve the First-Order Condition $(i - a^*)^{-\sigma} = \beta \frac{\partial \mathbb{E}[V(a', z')|z]}{\partial a'}$ using a 1D root-finder (e.g., Newton-Raphson or Brent's method). This involves:

- Evaluating the return function $u(c)$ exactly.
- Interpolating the continuation value $V(a')$ using advanced schemes like Akima cubic splines or Hermite polynomials to ensure smooth derivatives.

This provides the highest precision but is computationally demanding as it requires multiple function evaluations per state-space point.

2. Linear Interpolation & Algebraic Inversion (High Speed) Assuming linear interpolation of the value function between grid points a_k and a_{k+1}, the derivative $\partial V / \partial a$ becomes constant within each interval. This allows for a quasi-algebraic solution:

$$a^*_{interp} = i(a, z) - \left[\beta \frac{V_{k+1} - V_k}{a_{k+1} - a_k}\right]^{-1/\sigma} \tag{12}$$

This is significantly faster than root-finding but less accurate, as it approximates the return function's slope as piece-wise constant.

3. Fine-Grid Subdivision (GPU Friendly) We subdivide each grid interval $[a_k, a_{k+1}]$ into n sub-intervals. We then evaluate the return function and interpolated continuation value at these $n \cdot N_a$ points. While conceptually still a grid search, the fine mesh approximates continuity. This approach is highly parallelizable and "GPU friendly," as it replaces serial root-finding branching with massive vector evaluations.

Impact on General Equilibrium Regardless of the method, the primary benefit is the smoothing of aggregate capital supply $K_s(r)$ and labor demand $L_d(w)$ functions. Continuous choice eliminates the "step" oscillations in GE price loops, allowing for higher precision and faster convergence of Algorithm B.2.

4.5 Coupled VFI for Distortions

In the pre-reform state, the economy features idiosyncratic distortions $\tau \in \{\tau^+, \tau^-\}$. The solver maintains two coupled value functions $V^+(a, z)$ and $V^-(a, z)$. The expectation $\bar{V}(a')$ now accounts for the probability $p(z) = P(\tau = \tau^+|z)$:

$$\bar{V}(a') = \sum_j \pi_j \left[p(z_j)V^+(a', z_j) + (1 - p(z_j))V^-(a', z_j)\right] \tag{13}$$

The decision for an agent in state (a, z, τ) depends on the specific income $i(a, z, \tau)$, but the continuation value uses the shared $\bar{V}(a')$ since τ is redrawn whenever z resets.

5 Distributional and Transition Dynamics

5.1 Monte Carlo Binned Distribution

The distribution $\mu_t(a, z)$ is tracked by simulating N_{sim} agents. At any time t, agent j has assets a_j. The mass is mapped to the asset grid via linear interpolation weights:

$$\omega_{k+1} = \frac{a_j - a_k}{a_{k+1} - a_k}, \quad \omega_k = 1 - \omega_{k+1} \tag{14}$$

This allows for calculating aggregate labor demand $L_{d,t}$ and capital demand $K_{d,t}$ as smooth functions of prices:

$$K_{d,t}(w, r) = \sum_{i,j} \mu_t(a_i, z_j) k^*(a_i, z_j, w, r) \mathbf{1}_{\{\pi > w\}} \tag{15}$$

6 Algorithm B.2: Nested Price Path Iteration

The general equilibrium transition path is computed via Algorithm B.2 (Appendix B.2), which decomposes the market clearing of capital and labor into nested price-path sequences.

6.1 Price Path Sequences

Let $\boldsymbol{w} = \{w_t\}_{t=0}^T$ and $\boldsymbol{r} = \{r_t\}_{t=0}^T$ be the sequences of wages and interest rates. The algorithm solves for fixed points of these sequences such that markets clear at every t.

6.2 Nested Iteration Structure

The implementation uses a nested loop structure:

1. **Outer Loop (Interest Rates)**: Updates the guess for the interest rate path $\boldsymbol{r}^{(n)}$.

    (a) **Inner Loop (Wages)**: Given $\boldsymbol{r}^{(n)}$, iteratively updates the wage path $\boldsymbol{w}^{(k)}$.

       i. **Backward Induction**: Solve for policies $a'_t(a, z)$ given $(w_t^{(k)}, r_t^{(n)})$ for $t = T - 1, \ldots, 0$.
       ii. **Forward Simulation**: Simulate N_{sim} agents forward from μ_0 using a'_t to obtain the sequence of binned distributions $\{\mu_t\}_{t=0}^T$.
       iii. **Labor Clearing**: For each t, find the wage w_t^* that clears the labor market *given the fixed distribution μ_t*:

$$L_{excess}(w_t^* | \mu_t, r_t^{(n)}) = 0 \tag{16}$$

       iv. **Wage Relaxation**: Update the wage path using damping η_w:

$$w_t^{(k+1)} = \eta_w w_t^* + (1 - \eta_w) w_t^{(k)} \tag{17}$$

    (b) **Capital Clearing**: For each t, find the interest rate r_t^* that clears the capital market given the converged wage and distribution:

$$K_{excess}(r_t^* | \mu_t, w_t^{(converged)}) = 0 \tag{18}$$

    (c) **Interest Rate Relaxation**: Update the path using damping η_r:

$$r_t^{(n+1)} = \eta_r r_t^* + (1 - \eta_r) r_t^{(n)} \tag{19}$$

6.3 Numerical Market Clearing (Bisections)

The clearing prices w_t^* and r_t^* are found using a robust root-finding method:

- **Bracketing Scan**: Before bisecting, the price space $[P_{min}, P_{max}]$ is scanned at N_{scan} points to find a bracket $[a, b]$ where the excess demand function changes sign.

- **Bisection**: Once a bracket is found, a bisection method converges to the specific clearing price with a tolerance of 10^{-10}.

6.4 Convergence Criteria

The algorithm requires two conditions to stop:

1. **Sequence Convergence**: The maximum change in price paths between iterations must be small: $\max_t |r_t^{(n+1)} - r_t^{(n)}| < \epsilon_{seq}$.

2. **Market Clearing Gaps**: The actual excess demand residuals (calculated by re-evaluating the economy with the final prices) must satisfy $|ED_L| < \epsilon_{ED}$ and $|ED_K| < \epsilon_{ED}$.

In the implementation, the default tolerances are $\epsilon_{seq} = 2 \times 10^{-4}$ for sequence convergence and $\epsilon_{ED} = 2 \times 10^{-3}$ for market clearing gaps. The relaxation parameters are $\eta_w = 0.35$ for wages and $\eta_r = 0.20$ for interest rates.

7 Aggregate Variables and TFP

Once the equilibrium prices (w^*, r^*) and stationary distribution μ^* are computed, aggregate quantities are calculated as:

$$K = \sum_{i,j} \mu(a_i, z_j) \cdot k^*(a_i, z_j) \cdot \mathbf{1}_{\{\pi > w\}} \tag{20}$$

$$L_d = \sum_{i,j} \mu(a_i, z_j) \cdot l^*(a_i, z_j) \cdot \mathbf{1}_{\{\pi > w\}} \tag{21}$$

$$Y = \sum_{i,j} \mu(a_i, z_j) \cdot y(a_i, z_j, k^*, l^*) \cdot \mathbf{1}_{\{\pi > w\}} \tag{22}$$

$$s_e = \sum_{i,j} \mu(a_i, z_j) \cdot \mathbf{1}_{\{\pi > w\}} \tag{23}$$

where s_e is the share of entrepreneurs and $L_s = 1 - s_e$ is the labor supply. Total Factor Productivity (TFP) is then computed as:

$$\text{TFP} = \frac{Y}{\left(K^\alpha L_s^{1-\alpha}\right)^{1-\nu}} \tag{24}$$

This measure captures the efficiency losses from misallocation due to financial frictions.

8 Calibration Parameters

The model is calibrated following Buera & Shin (2013), Table 1:

Parameter	Symbol	Value
Risk aversion	σ	1.5
Discount factor	β	0.904
Capital share	α	0.33
Span of control	$1 - \nu$	0.79
Depreciation	δ	0.06
Pareto tail	η	4.15
Ability persistence	ψ	0.894
Collateral constraint	λ	1.35
Pre-reform distortions:		
Tax wedge	τ^+	0.57
Subsidy wedge	τ^-	-0.15
Correlation parameter	q	1.55