

# Mathematical Foundations of the Buera & Shin (2013) Spectral Time Iteration Implementation

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## 1 The Economic Environment

The economy is populated by a continuum of agents who choose to be either workers or entrepreneurs. Agents are heterogeneous in their asset holdings  $a$  and entrepreneurial ability  $z$ .

### 1.1 Preferences and Shocks

Agents maximize expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \quad (1)$$

Entrepreneurial ability  $z$  follows a reset process: with probability  $\psi$ ,  $z_{t+1} = z_t$ ; with probability  $1 - \psi$ , a new ability is drawn  $z' \sim \pi(z)$  from a Pareto distribution with tail parameter  $\eta$ . In the pre-reform state, agents also face idiosyncratic output wedges  $\tau \in \{\tau^+, \tau^-\}$ , where  $\mathbb{P}(\tau = \tau^+ | z) = 1 - e^{-qz}$ .

## 2 Production and Optimal Firm Decisions

An entrepreneur with ability  $z$  and assets  $a$  operates the technology:

$$y = z(k^\alpha l^{1-\alpha})^{1-\nu} \quad (2)$$

where  $\nu$  represents the span-of-control parameter (entrepreneur's share). The entrepreneur faces a collateral constraint  $k \leq \lambda a$ .

### 2.1 Static Profit Maximization

The entrepreneur's problem is solved in two stages. First, given capital  $k$ , the optimal labor  $l^*(k, z)$  is found by solving:

$$\max_l \{z(k^\alpha l^{1-\alpha})^{1-\nu} - wl - (r + \delta)k\} \quad (3)$$

The first-order condition with respect to  $l$  is:

$$(1 - \nu)(1 - \alpha)zk^{\alpha(1-\nu)}l^{(1-\alpha)(1-\nu)-1} = w \quad (4)$$

Solving for  $l$  yields the conditional labor demand:

$$l^*(k, z) = \left[ \frac{(1 - \nu)(1 - \alpha)z}{w} \right]^{\frac{1}{1 - (1 - \alpha)(1 - \nu)}} k^{\frac{\alpha(1 - \nu)}{1 - (1 - \alpha)(1 - \nu)}} \quad (5)$$

Substituting  $l^*(k, z)$  back into the profit function gives a profit function  $\pi(k)$  that is concave in  $k$ . The unconstrained capital demand  $k^{unc}$  satisfies  $\pi'(k^{unc}) = r + \delta$ . If  $k^{unc} > \lambda a$ , the entrepreneur is borrowing-constrained and sets  $k^* = \lambda a$ .

### 3 The Recursive Household Problem

The agent's state is  $(a, z)$ . The total income is  $y(a, z) = \max\{\pi(a, z, k^*), w\} + (1 + r)a$ . The Bellman equation is:

$$V(a, z) = \max_{a'} \left\{ \frac{(y(a, z) - a')^{1-\sigma} - 1}{1 - \sigma} + \beta \text{Cont}(a', z) \right\} \quad (6)$$

where the continuation value is:

$$\text{Cont}(a', z) = \psi V(a', z) + (1 - \psi) \int V(a', z') \pi(z') dz' \quad (7)$$

#### 3.1 Derivation of the Euler Equation

To solve the household's problem, we substitute  $c_t = y(a_t, z_t) - a_{t+1}$  into the value function:

$$V(a_t, z_t) = \max_{a_{t+1}} \left\{ \frac{(y(a_t, z_t) - a_{t+1})^{1-\sigma} - 1}{1 - \sigma} + \beta \mathbb{E}[V(a_{t+1}, z_{t+1}) | z_t] \right\} \quad (8)$$

The first-order condition with respect to  $a_{t+1}$  is:

$$-(y(a_t, z_t) - a_{t+1})^{-\sigma} + \beta \mathbb{E} \left[ \frac{\partial V(a_{t+1}, z_{t+1})}{\partial a_{t+1}} \right] = 0 \quad (9)$$

From the Envelope Theorem, the derivative of the value function with respect to assets is:

$$\frac{\partial V(a_t, z_t)}{\partial a_t} = (y(a_t, z_t) - a_{t+1})^{-\sigma} \cdot \frac{\partial y(a_t, z_t)}{\partial a_t} = (1 + r)c_t^{-\sigma} \quad (10)$$

Leading to the stochastic Euler equation:  $c_t^{-\sigma} = \beta(1 + r)\mathbb{E}_t[c_{t+1}^{-\sigma}]$ .

### 4 Spectral Discretization and Setup

The policy function  $a'(a, z)$  is approximated by a bivariate Chebyshev polynomial  $\hat{a}(a, z; \gamma)$ .

#### 4.1 Chebyshev Collocation Nodes

The state space is discretized using the zeros of the  $N$ -th degree Chebyshev polynomial  $\xi \in [-1, 1]$ :

$$\xi_k = \cos \left( \frac{2k - 1}{2N} \pi \right), \quad k \in \{1, \dots, N\} \quad (11)$$

The nodes are mapped to the physical state space  $(a, z)$ . For assets, we use a log-linear mapping to cluster nodes near the borrowing constraint:

$$a_k = \exp \left( \frac{\ln(a_{max} + \text{shift}) + \ln(a_{min} + \text{shift})}{2} + \xi_k \frac{\ln(a_{max} + \text{shift}) - \ln(a_{min} + \text{shift})}{2} \right) - \text{shift} \quad (12)$$

while ability nodes  $z_j$  are mapped linearly.

### 5 Spectral Time Iteration Logic

#### 5.1 Pointwise Euler Inversion

To solve for the optimal policy, we utilize a recursive operator that linearizes the Global Euler Equation. At each collocation node  $(a_i, z_j)$ , we define the **Target Function**  $F(a')$  as the difference between current marginal utility and expected future marginal utility:

$$F(a'; a_i, z_j, \gamma^{(n)}) \equiv \underbrace{(y_i - a')^{-\sigma}}_{\text{Current MU}} - \beta(1 + r) \underbrace{\mathcal{E}(a'; \gamma^{(n)})}_{\text{Continuation MU}} \quad (13)$$

where  $\mathcal{E}$  is the expectation over future states  $(\hat{a}_i, z')$  given the **known** continuation policy  $\gamma^{(n)}$ :

$$\mathcal{E}(a'; \gamma^{(n)}) = \psi \left( y(a', z_j) - \Phi(a', z_j)^\top \gamma^{(n)} \right)^{-\sigma} + (1 - \psi) \sum_k \pi_k \left( y(a', z_k) - \Phi(a', z_k)^\top \gamma^{(n)} \right)^{-\sigma} \quad (14)$$

Since  $\mathcal{E}$  is fixed for a given guess of  $a'$ , we can identify the optimal  $a_{target}^*$  by direct inversion:

$$a_{target,i}^* = \max\{a_{min}, y_i - [\beta(1+r)\mathcal{E}(a'; \gamma^{(n)})]^{-1/\sigma}\} \quad (15)$$

This formulation depends exclusively on the grid points  $(a_i, z_j)$  and the candidate savings  $a'$ , with the future behavior already "baked" into the coefficients  $\gamma^{(n)}$ .

## 5.2 Damped Coefficient Update

The algorithm computes the new policy nodes  $\mathbf{A}_{new}^*$  by applying a damping parameter  $\theta \in (0, 1]$  to ensure numerical stability:

$$\mathbf{A}_{new}^* = (1 - \theta)\mathbf{A}_{old}^* + \theta\mathbf{A}_{target}^* \quad (16)$$

The updated spectral coefficients  $\gamma^{(n+1)}$  are then obtained by solving the linear basis transformation:

$$\gamma^{(n+1)} = \mathbf{T}^{-1}\mathbf{A}_{new}^* \quad (17)$$

where  $\mathbf{T}$  is the matrix of Chebyshev basis functions evaluated at the collocation nodes.

## 6 Distributional and Transition Dynamics

The distribution is tracked on a dense grid using a Markov transition matrix  $Q$  derived from the continuous policy  $\hat{a}$ .

$$\mu_{t+1}(a', z') = \sum_{a, z} \mathbb{P}(a'|a, z)\mathbb{P}(z'|z)\mu_t(a, z) \quad (18)$$

Aggregate labor  $L_t$  and capital  $K_t$  are then calculated by summing firm decisions over the distribution  $\mu_t$ .

## 7 Convergence and General Equilibrium

The general equilibrium wage  $w$  and interest rate  $r$  are found by iterating on the price paths using Algorithm B.2. The spectral method's continuity ensures that the excess demand functions  $ED_L(w, r)$  and  $ED_K(w, r)$  are smooth, allowing for robust bisection even with a relatively small number of spectral nodes.