News, Noise, and Fluctuations: An Empirical Exploration

By Blanchard, L'Huiller and Lorenzoni (AER 2013)

Replication by Piero De Dominicis

Contents

1 Introduction

In this replication, I replicate most of the figure in Section 1 and 2 of Blanchard et al. [2013]. Together with the figures, I try to give a comprehensive explanation of the methods adopted by the authors. Where possible, qualitative explanations come with the analytical expressions needed to derive the numerical results obtained in the paper. Figures in Sections 2 and 3 of this Replication Paper follows exactly the original code made available by the authors. Most of the work in Section 4, relative to the Maximum Likelihood Estimator and the Kalman Smoother, is done independently by me following some online material and the book from Kilian and Lütkepohl [2017]. All the codes have been written in Python.

2 A General Representative Agent Dynamic Linear Model with Signal Extraction

I formulate and show the analytical solution of a general linear model with signal extraction following the notation used in the appendix of the paper. Uncertainty in the model is captured by the exogenous state vector $\mathbf{x_t}$ that is modelled as follows:

$$\mathbf{x_t} = \mathbf{A}\mathbf{x_{t-1}} + \mathbf{B}\mathbf{v_t}$$

where $\mathbf{v_t} = (\epsilon_t, \eta_t, \nu_t)'$ is the vector of normally distrubuted shocks[1], $\mathbf{x_t} = (x_t, x_{t-1}, z_t)'$ is the vector of unobserved states and the matrix $\mathbf{A}$ and $\mathbf{B}$ are given by:

$$\mathbf{A} = \begin{bmatrix} 1+\rho & -\rho & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

The representative agent observes

$$\mathbf{s_t} = \mathbf{C}\mathbf{x_t} + \mathbf{D}\mathbf{v_t}$$

where $\mathbf{s_t} = (a_t, s_t)'$ and the matrices $\mathbf{C}$ and $\mathbf{D}$ are given by:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We now derive the expression for the Kalman Filter. For a generic random variable X we denote $\hat{x}_{t|t-1} \equiv \mathbb{E}\left[x_t | I_t - 1\right]$, where I_{t-1} are the information the agent has at time $t-1$, and $P_{t|t-1} = \mathbb{E}\left[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'\right]$ as the variance of the forecast error [2]. Normality of the errors give us the conditional distribution $x_{t|t-1} \sim N\left(\hat{x}_{t|t-1}, P_{t|t-1}\right)$.

[1] That is $\epsilon_t \sim N(0, \sigma^2_\epsilon)$, $\eta_t \sim N(0, \sigma^2_\eta)$ and $\nu_t \sim N(0, \sigma^2_\nu)$.
[2] Such that $P_{t-1|t-1}$ is the variance of the nowcast error.

First we derive the best forecasts for the unobserved states:

$$\begin{aligned}
\mathbf{\hat{x}_{t|t-1}} &= \mathbb{E}\left[\mathbf{x_t}|I_t-1\right] = \mathbb{E}\left[\mathbf{Ax_{t-1}}+\mathbf{Bv_t}|I_{t-1}\right] = \mathbf{A\hat{x}_{t-1|t-1}} \\
\mathbf{P_{t|t-1}} &\equiv Var\left[\mathbf{x}_t|I_{t-1}\right] = Var\left[\mathbf{Ax_{t-1}}+\mathbf{Bv_t}|I_{t-1}\right] = \mathbf{AP_{t-1|t-1}A'}+\mathbf{B\Sigma_V B'}
\end{aligned}$$

where $\mathbf{\Sigma_V} = diag(\sigma^{\mathbf{2}}_{\mathbf{v}})$ and $\sigma^{\mathbf{2}}_{\mathbf{v}} = \left(\sigma^2_\epsilon, \sigma^2_\eta, \sigma^2_\nu\right)'$. The prediction equations for the observables are:

$$\begin{aligned}
\mathbf{\hat{s}_{t|t-1}} &= \mathbb{E}\left[\mathbf{s}_t|I_{t-1}\right] = \mathbb{E}\left[\mathbf{Cx_t}+\mathbf{Dv_t}|I_{t-1}\right] = \mathbf{C\hat{x}_{t|t-1}} = \mathbf{CA\hat{x}_{t-1|t-1}} \\
Var\left[\mathbf{s}_t|I_{t-1}\right] &= Var\left[\mathbf{Cx_t}+\mathbf{Dv_t}|I_{t-1}\right] = \mathbf{CP_{t|t-1}C'}+\mathbf{D\Sigma_V D'}
\end{aligned}$$

Moreover we can write:

$$\begin{aligned}
\mathbf{x_t} &= \mathbf{\hat{x}_{t|t-1}} + (\mathbf{x_t} - \mathbf{\hat{x}_{t|t-1}}) \\
\mathbf{s_t} &= \mathbf{C}(\mathbf{\hat{x}_{t|t-1}} + (\mathbf{x_t} - \mathbf{\hat{x}_{t|t-1}})) + \mathbf{Dv_t}
\end{aligned}$$

and, due to normality of the errors, we can collect the terms to obtain:

$$\begin{bmatrix}\mathbf{x_t}\\ \mathbf{s_t}\end{bmatrix} \sim N\left(\begin{bmatrix}\mathbf{\hat{x}_{t|t-1}}\\ \mathbf{C\hat{x}_{t|t-1}}\end{bmatrix}, \begin{bmatrix}\mathbf{P_{t|t-1}} & \mathbf{P_{t|t-1}C'}\\ \mathbf{CP_{t|t-1}} & \mathbf{CP_{t|t-1}C'}+\mathbf{D\Sigma_v D'}\end{bmatrix}\right)$$

The updating equation are simply given by the conditional mean and variance for $\mathbf{x_{t|t}}$, given by:

$$\begin{aligned}
\mathbf{\hat{x}_{t|t}} &= \mathbf{\hat{x}_{t|t-1}} + \mathbf{P_{t|t-1}C'}\left[\mathbf{CP_{t|t-1}C'}+\mathbf{D\Sigma_V D'}\right]^{-\mathbf{1}}\left(\mathbf{s_t} - \mathbf{C\hat{x}_{t|t-1}}\right) \\
\mathbf{P_{t|t}} &= \mathbf{P_{t|t-1}} - \mathbf{P_{t|t-1}C'}\left[\mathbf{CP_{t|t-1}C'}+\mathbf{D\Sigma_V D'}\right]^{-\mathbf{1}}\mathbf{CP_{t|t-1}}
\end{aligned}$$

given that $\mathbf{x_{t|t}} \sim \mathbf{N}\left(\mathbf{\hat{x}_{t|t}}, \mathbf{P_{t|t}}\right)$. We can define $\mathbf{K} \equiv \mathbf{P_{t|t-1}C'}\left[\mathbf{CP_{t|t-1}C'}+\mathbf{D\Sigma_V D'}\right]^{-\mathbf{1}}$ and we get:

$$\begin{aligned}
\mathbf{\hat{x}_{t|t}} &= \mathbf{\hat{x}_{t|t-1}} + \mathbf{K}\left(\mathbf{s_t} - \mathbf{C\hat{x}_{t|t-1}}\right) \\
\mathbf{P_{t|t}} &= \mathbf{P_{t|t-1}} - \mathbf{KCP_{t|t-1}}
\end{aligned}$$

Finally, we can write the agents' expectations $\mathbf{\hat{x}_{t|t}}$ in recursive form as

$$\begin{aligned}
\mathbf{\hat{x}_{t|t}} &= \mathbf{A\hat{x}_{t-1|t-1}} + \mathbf{K}\left(\mathbf{s_t} - \mathbf{CA\hat{x}_{t-1|t-1}}\right) \\
&= (\mathbf{I} - \mathbf{KC})\mathbf{A\hat{x}_{t-1|t-1}} + \mathbf{Ks_t} \\
&= (\mathbf{I} - \mathbf{KC})\mathbf{A\hat{x}_{t-1|t-1}} + \mathbf{KCAx_{t-1}} + (\mathbf{KCB}+\mathbf{KD})\,\mathbf{v_t}
\end{aligned}$$

where in the last expression we used the defined processes for $\mathbf{s_t}$ and $\mathbf{x_t}$. To make our model as general as possible, we add an endogenous state vector $\mathbf{y_t}$. Suppose this endogenous state has the following solution:

$$\mathbf{y_t} = \mathbf{Py_{t-1}} + \mathbf{Qs_t} + \mathbf{R\hat{x}_{t|t}}$$

and by substituting expression for $\mathbf{s_t}$ and $\mathbf{\hat{x}_{t|t}}$ we get:

$$\begin{aligned}
\mathbf{y_t} &= \mathbf{P y_{t-1}} + \mathbf{Q}\left(\mathbf{C x_t} + \mathbf{D v_t}\right) + \mathbf{R}\left[(\mathbf{I}-\mathbf{KC})\mathbf{A}\mathbf{\hat{x}_{t-1|t-1}} + \mathbf{KCA x_{t-1}} + (\mathbf{KCB}+\mathbf{KD})\,\mathbf{v_t}\right] \\
&= \mathbf{P y_{t-1}} + \mathbf{Q}\left(\mathbf{C}\left[\mathbf{A x_{t-1}} + \mathbf{B v_t}\right] + \mathbf{D v_t}\right) + \mathbf{R}\left[(\mathbf{I}-\mathbf{KC})\mathbf{A}\mathbf{\hat{x}_{t-1|t-1}} + \mathbf{KCA x_{t-1}} + (\mathbf{KCB}+\mathbf{KD})\,\mathbf{v_t}\right] \\
&= \mathbf{P y_{t-1}} + (\mathbf{QCA}+\mathbf{RKCA})\,\mathbf{x_{t-1}} + \mathbf{R}(\mathbf{I}-\mathbf{KC})\mathbf{A}\mathbf{\hat{x}_{t-1|t-1}} + (\mathbf{QD}+\mathbf{R}\left[\mathbf{KCB}+\mathbf{KD}\right])\,\mathbf{v_t}
\end{aligned}$$

Then we can set up the econometrician's Kalman filter by using the dynamic system for $\left(\mathbf{x_t}, \mathbf{\hat{x}_{t|t}}, \mathbf{y_t}\right)$ and the observation equation $\mathbf{s_t^E} = \mathbf{T}\left[\mathbf{y_t} \quad \mathbf{s_t}\right]$.
Define $\mathbf{x_t^E} = \left(\mathbf{x_t}, \mathbf{\hat{x}_{t|t}}, \mathbf{y_t}\right)$, the dynamic system for $\mathbf{x_t^E}$ is given by:

$$\mathbf{x_t^E} = \begin{bmatrix} \mathbf{x_t} \\ \mathbf{\hat{x}_{t|t}} \\ \mathbf{y_t} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{KCA} & (\mathbf{I}-\mathbf{KC})\mathbf{A} & \mathbf{0} \\ \mathbf{QCA}+\mathbf{RKCA} & \mathbf{R}(\mathbf{I}-\mathbf{KC})\mathbf{A} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{x_{t-1}} \\ \mathbf{\hat{x}_{t-1|t-1}} \\ \mathbf{y_{t-1}} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{KCB}+\mathbf{KD} \\ \mathbf{QD}+\mathbf{R}\left(\mathbf{KCB}+\mathbf{KD}\right) \end{bmatrix} \mathbf{v_t}$$

2.1 A Simple Model

The model in section 1 of the paper is a specific case of the above general model. In particular, there are no endogenous states $\mathbf{y_t} = \mathbf{0}$, so that the above solution becomes:

$$\mathbf{x_t^E} = \begin{bmatrix} \mathbf{x_t} \\ \mathbf{\hat{x}_{t|t}} \\ \mathbf{y_t} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{KCA} & (\mathbf{I}-\mathbf{KC})\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{=\mathbf{H}} \begin{bmatrix} \mathbf{x_{t-1}} \\ \mathbf{\hat{x}_{t-1|t}} \\ \mathbf{y_{t-1}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B} \\ \mathbf{KCB}+\mathbf{KD} \\ \mathbf{0} \end{bmatrix}}_{=\mathbf{W}} \mathbf{v_t}$$

that is $\mathbf{x_t^E} = \mathbf{H x_{t-1}^E} + \mathbf{W v_t}$. To find the matrices $\mathbf{H}$ and $\mathbf{W}$, we need to find the matrix $\mathbf{K}$ which can be obtained through recursive iteration. In particular, starting from an initial value for $\mathbf{P}_{1|0} = \mathbf{I}_3$ we can define a recursive structure for $\mathbf{P}_{t+1|t}$ and $\mathbf{K^{(t)}}$ in the following way:

1. using $\mathbf{P_{1|0}}$, we can find $\mathbf{K^{(1)}} \equiv \mathbf{P_{t|t-1}C'}\left[\mathbf{C P_{t|t-1} C'} + \mathbf{D \Sigma_V D'}\right]^{-1}$
2. using $\mathbf{K^{(1)}}$, we can find $\mathbf{P_{1|1}} = \mathbf{P_{1|0}} - \mathbf{KCP_{1|0}}$;
3. using $\mathbf{P_{1|1}}$, we can find $\mathbf{P_{2|1}}$ using the formula for the variance of the forecast error, that is $\mathbf{P_{2|1}} = \mathbf{A P_{1|1} A'} + \mathbf{B \Sigma_V B'}$;
4. if $||\mathbf{P_{2|1}} - \mathbf{P_{1|0}}|| < \varepsilon$ for ε close to zero stop, otherwise repeat step 1 using $\mathbf{P_{2|1}}$

Once the above algorithm has converged, we can get a value for $\mathbf{K}$. Once we have $\mathbf{K}$, we can construct the matrices $\mathbf{H}$ and $\mathbf{W}$ that we can use to compute the filtered expectations. The observation equation can be represented as:

$$\mathbf{s_t^E} = \begin{bmatrix} \mathbf{a_t} \\ \mathbf{c_t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{1-\rho} & \frac{-\rho}{1-\rho} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x_t} \\ \mathbf{\hat{x}_{t|t}} \\ \mathbf{y_t} \end{bmatrix}$$

which can be written as $\mathbf{s_t^E} = \mathbf{T}\mathbf{x_t^E}$, where in this case the matrix $\mathbf{T}$ is determined through the equations $a_t = x_t + z_t$ and $c_t = \frac{1}{1-\rho}\left(\mathbb{E}_t\left[x_t\right] - \rho\mathbb{E}_t\left[x_{t-1}\right]\right)$. The expectations are those derived through the Kalman Filter above. Finally, we derive the Impulse Reponse Functions of this model and plot them in Figure 1.

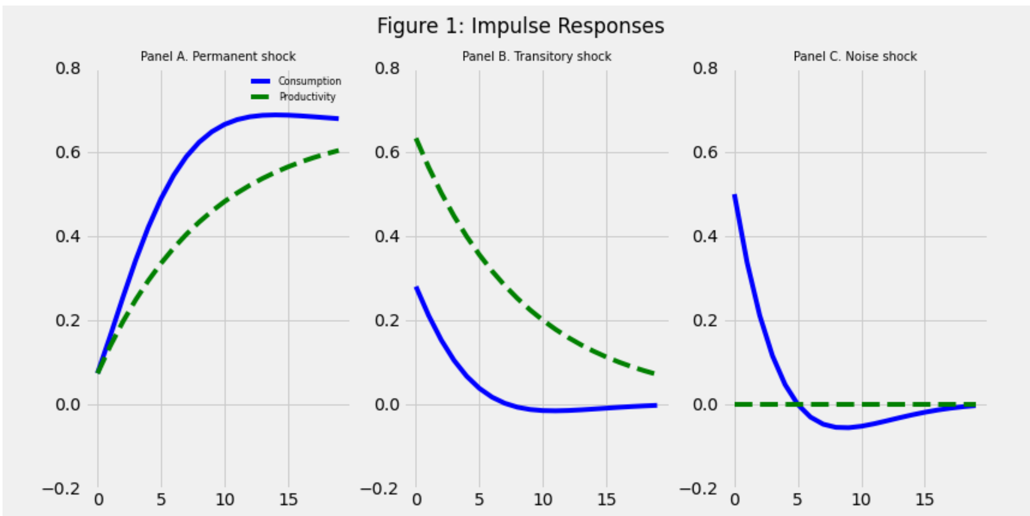

Figure 1: This figure replicates **Figure 1** in Blanchard et al. [2013]. Time unit is the quarter and the impulses are one standard deviation positive shokcs. The persistence parameter is $\rho = 0.89$, the standard deviation of the shock to the permanent technology component is $\sigma_\epsilon = 0.07$, whereas the shock to the transitory component of technology is 0.63. Finally, the standard deviation of the noise shock is $\sigma_\nu = 0.89$.

A permanent shock in ϵ_t slowly increases both productivity and consumption. Consumers take some time before realizing that the shock was indeed permanent due to the high volatility of transitory and noise shocks. In reaction to a transitory η_t shock, consumers initially adjust their consumption, but then they learn that the shock was only transitory and so gradually adjust towards zero. Similarly, a shock to the noise component ν_t, consumption initially increases and then returns normal over time, with a negative adjustment after consumers have learnt that the shock was only noise.

3 Identification and Estimation

In this section we show that it is not possible to use simple semistructural identification assumptions to estimate the economy's responses to shocks. We assume that the true data generating process is the simple toy-model used in the previous section and generate an artificial dataset. The reduced form VAR representation for c_t and a_t, in the case that the econometrician does not observe the signal, is:

$$
\begin{aligned}
c_t &= c_{t-1} + u_t^c \\
a_t &= \rho a_{t-1} + (1-\rho)c_{t-1} + u_t^a
\end{aligned}
$$

where u_t^c and u_t^a are innovations with respect to the econometrician information set. Interestingly, lagged consumption enter in the equation for productivity. The reason is that consumption embeds the additional information on x_t that the consumer obtain from observing the signal, therefore it cointains information useful to predict productivity.
Suppose we run a reduced form VAR in c_t and a_t. In general we can not obtain the three original structural shocks (ϵ, η, ν) from the two reduced form innovation (u_t^c, u_t^a). We are able to recover the structural shocks only in two particular cases:

1. First, when the signal is perfectly informative ($\sigma_\nu = 0$)) the two reduced form equation can be rewritten as:

   $$
   \begin{aligned}
   c_t &= c_{t-1} + \frac{1}{1-\rho}\epsilon_t \\
   a_t &= \rho a_{t-1} + (1-\rho)c_{t-1} + \epsilon_t + \eta_t
   \end{aligned}
   $$

   with consumption being affected only by the permanent shock, and productivity by both, it is enough to impose long-run restrictions on the variance covariance matrix to recover the structural shocks.

2. Second, when the signal is perfectly uninformative ($\sigma_\nu \to \infty$). In this case, the random walk assumption on a_t implies that:

   $$
   \begin{aligned}
   c_t &= c_{t-1} + u_t \\
   a_t &= a_{t-1} + u_t
   \end{aligned}
   $$

   In this case, the decomposition between temporary and permanent shock is irrelevant given that it is not possible to separate them with any information. In other words, the innovation on productivity can be directly interpreted as the unique permanent shock.

We now want to verify whether long-run identification restrictions can be used to separate the effect of the permanent shock ϵ_t from the $combined$ effect of the temporary shock η_t and the noise shock ν_t.
Hence we generate $N = 1000$ data points for c_t and a_t from out toy-model, plotted in Figure 2, and run a structural VAR with long-run restrictions à la Blanchard and Quah [1989] to identify a permanent and a temporary shock.

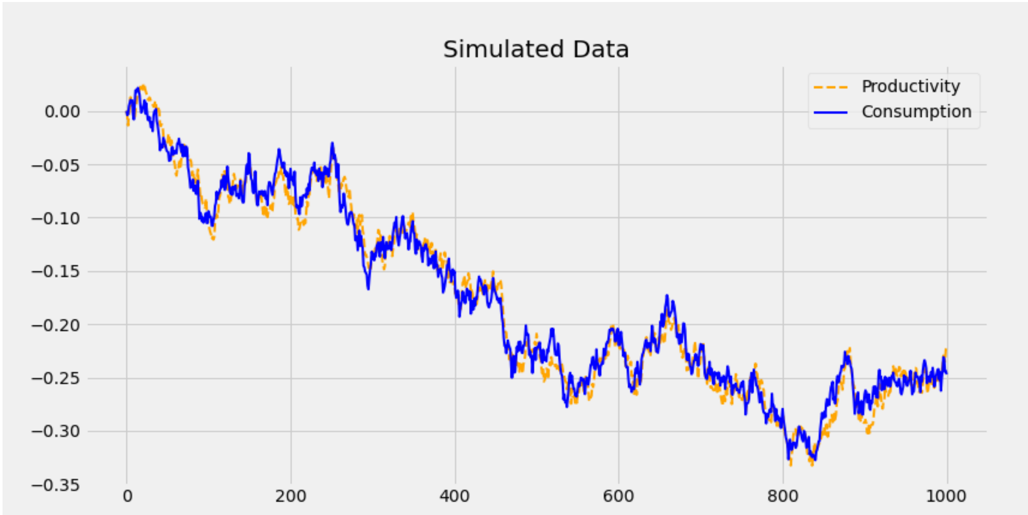

Figure 2: Clearly both series follow a common trend in the long-run. This suggests the presence of cointegration relationship.

3.1 Blanchard and Quah 1989

In Blanchard and Quah [1989], they impose a structure imposed by economic theory in order to identify the structural shocks using a reduced form VAR. Consider for example a general bivariate VMA model:

$$X_t = A(0)\varepsilon_t + A(1)\varepsilon_{t-1} + \dots$$

BQ 1989 impose a long-run restriction to identify the structural shocks, in particular they impose $\sum_{j=1}^{\infty} a_{11} = 0$, that is they shut down the long-run effect of a structural shock on a specific variable.

With stationary series, we can express X_t using the Wold-moving average representation:

$$X_t = v_t + C(1)v_{t-1} + \dots + C(k-1)v_{t-k} + \dots$$

with $Var(v) = \Omega$, where v_t are the innovations of the reduced form VAR. Thanks to the above representation, we can write $v_t = A_0\varepsilon_t$, which implies that $C(1)v_{t-1} = A(1)\varepsilon_{t-1} = C(1)A_0\varepsilon_{t-1}$. Hence in the case of a bivariate VAR the relation between reduced form errors and structural shocks is given by:

$$\begin{bmatrix} v_t^{X_1} \\ v_t^{X_2} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{bmatrix}$$

Using information from the data we can obtain identifying restriction:

$$\begin{bmatrix} v_t^{X_1} \\ v_t^{X_2} \end{bmatrix} \begin{bmatrix} v_t^{X_1} \\ v_t^{X_2} \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{bmatrix} \begin{bmatrix} \varepsilon_t^1 \\ \varepsilon_t^2 \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}'$$

In the above expression, the LHS is identified in the data, whereas the RHS is unknown. Therefore we still need an identifying assumption, that is the orthogonality condition in the variance of the structural shock, $Var(\varepsilon) = \mathbf{I}$, that implies

$$\begin{bmatrix} v_t^{X_1} \\ v_t^{X_2} \end{bmatrix} \begin{bmatrix} v_t^{X_1} \\ v_t^{X_2} \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}'$$

The above matrices give us a system of three equations, therefore we need an additional restriction to identify the matrix A_0. In levels, we have for the first row of the Wold

$$X_1 = v_{1t} + (I_1 + C_1(1))v_{1t-1} + \ldots + (I_1 + C_1(1) + \ldots + C_1(k))v_{t-k} + \ldots$$

and the Long Run effects are

$$LR = (I + C(1) + C(2) + \ldots)A_0 = (I - C)^{-1}A_0$$

such that the last restriction is $LR_{11} = 0$.

3.2 LR-SVAR Restictions with Cointegration

In this section, we follow the same notation of Kilian and Lütkepohl [2017]. Given that the two series seem to be cointegrated, we cast the model in VECM form and impose Long-Run restrictions on the VECM model remaining agnostic about the number of cointegration relationship, following the approach of the paper. Define $y_t = (a_t\ c_t)'$, then we can cast the model in VECM as

$$\Delta y_t = \alpha\beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + \ldots + \Gamma_{p-1}\Delta y_{t-p+1} + \epsilon_t$$

To understand where we are imposing identifying restrictions, it is useful to represent the above VECM in levels by using its Granger representation:

$$y_t = \Xi \sum_{i=1}^{t} \epsilon_i + \Xi^*(L)\epsilon_t + y_0^*$$

where $\Xi = \Gamma_\perp \left[\alpha'_\perp \left(I_2 - \sum_{i=1}^{p-1}\Gamma_i\right)\Gamma_\perp\right]\alpha'_\perp$, $\Xi^*(L)\epsilon_t = \sum_{j=0}^{\infty}\Xi_j^*\epsilon_{t-j}$ and $y_0^* = (a_0^*\ c_0^*)$ are the initial values for productivity and consumption. Since the structural shocks are obtained from the reduced-form errors using a linear transormation, that is $\varepsilon_t = A_0\epsilon_t$, we can replace ϵ_t by $A_0^{-1}\varepsilon_t$ in the Granger rapresentation to obtain

$$y_t = \Psi \sum_{i=1}^{t} \varepsilon_i + \Xi^*(L)A_0^{-1}\varepsilon_t + y_0^*$$

and given that $\Xi^*(L)A_0^{-1}$ coefficients approach zero as $j \to \infty$, then $\Psi = \Xi A_0^{-1}$ represents the matrix of long-run effects or permanent effects of the structural shocks on the level of the variables y_t. Restrictions on the long-run effects of the shocks can be imposed directly on Ψ. Now, in the presence of unit roots, so that when both variables are $I(1)$ and the cointegrating rank r of the system is equal to zero, then the Ψ matrix is nonsingular and we immediately specify long-run restrictions by specifying a lower-triangular long-run effects matrix. Infact, when this is the case, then the matrices Ξ and Ψ simplifies to:

$$\Xi = \left(I_2 - \sum_{i=1}^{p-1} \Gamma_i \right)^{-1}$$
$$\Psi = \left(I_2 - \sum_{i=1}^{p-1} \Gamma_i \right)^{-1} A_0^{-1}$$

Defining $\Gamma(L) = I_2 - \sum_{i=1}^{p-1} \Gamma_i L^i$ so that $\Gamma(1) = I_2 - \sum_{i=1}^{p-1} \Gamma_i$, then we have

$$\begin{aligned} \Psi\Psi' &= \Gamma(1)^{-1} A_0^{-1} A_0'^{-1} \Gamma(1)'^{-1} \\ &= \Gamma(1)^{-1} \Sigma_\epsilon \Gamma(1)'^{-1} \end{aligned}$$

which can be computed from the reduced form. Using the Cholesky decomposition of $\Psi\Psi'$ we get the desired long-run restriction, hence

$$A_0 = \left[\Gamma(1) chol(\Psi\Psi') \right]^{-1}$$

Moreover in this case, we can express Δy_t as

$$\Delta y_t = \Gamma_1 \Delta y_{t-1} + \ldots + \Gamma_{p-1} \Delta y_{t-p+1} + \epsilon_t$$

and thanks to the structural form representation $A_0 \Gamma(L) \Delta y_t = \varepsilon_t$, the structural impulse responses are obtained from the structural MA representation

$$\Delta y_t = \Gamma(L)^{-1} A_0^{-1} \varepsilon_t$$

and the long-run effects on the levels y_t are the *cumulated* $\Psi = \Gamma(1)^{-1} A_0^{-1}$.

3.3 VECM Estimation

First of all, we can run the regression $a_t = \alpha + \theta_c c_t + u_t$ and obtain the Error Correction Term $\hat{u}_t$. Starting from a VAR with two variables and $p+1$ lags, the VECM we estimate is as follows:

$$\begin{bmatrix} \Delta a_t \\ \Delta c_t \end{bmatrix} = \begin{bmatrix} \Gamma_0^a \\ \Gamma_0^c \end{bmatrix} + \begin{bmatrix} \Gamma_1^a & \Gamma_2^a \dots & \Gamma_{2p}^a \\ \Gamma_2^c & \Gamma_2^c \dots & \Gamma_{2p}^c \end{bmatrix} \begin{bmatrix} \Delta a_{t-1} \\ \vdots \\ \Delta a_{t-p} \\ \Delta c_{t-1} \\ \vdots \\ \Delta c_{t-p} \end{bmatrix} + \begin{bmatrix} \Gamma_{2p+1}^a \\ \Gamma_{2p+1}^c \end{bmatrix} \cdot \hat{u}_t + \begin{bmatrix} \epsilon_t^a \\ \epsilon_t^c \end{bmatrix}$$

and considering that we are running a model with $p = 4$ we have[3]

$$\begin{aligned} \Delta a_t = \Gamma_0^a + \Gamma_1^a \Delta a_{t-1} + \Gamma_2^a \Delta a_{t-2} + \Gamma_3^a \Delta a_{t-3} + \Gamma_4^a \Delta a_{t-4} + \dots \\ \dots + \Gamma_5^a \Delta c_{t-1} + \Gamma_5^a \Delta c_{t-1} + \Gamma_6^a \Delta c_{t-2} + \Gamma_7^a \Delta c_{t-3} + \Gamma_8^a \Delta c_{t-4} + \Gamma_9^a \hat{u}_{t-1} + \epsilon_t^a \\ \Delta c_t = \Gamma_0^c + \Gamma_1^c \Delta a_{t-1} + \Gamma_2^c \Delta a_{t-2} + \Gamma_3^c \Delta a_{t-3} + \Gamma_4^c \Delta a_{t-4} + \dots \\ \dots + \Gamma_5^c \Delta c_{t-1} + \Gamma_5^c \Delta c_{t-1} + \Gamma_6^c \Delta c_{t-2} + \Gamma_7^c \Delta c_{t-3} + \Gamma_8^c \Delta c_{t-4} + \Gamma_9^c \hat{u}_{t-1} + \epsilon_t^c \end{aligned}$$

We can use standard OLS to estimate the vectors $\hat{\mathbf{\Gamma}}^{\mathbf{a}}$ and $\hat{\mathbf{\Gamma}}^{\mathbf{c}}$ and use the estimates to recover the variance-covariance matrix

$$\hat{\mathbf{\Sigma}}_\epsilon = \begin{bmatrix} (\hat{\epsilon}_t^a)' \hat{\epsilon}_t^a & (\hat{\epsilon}_t^a)' \hat{\epsilon}_t^c \\ (\hat{\epsilon}_t^c)' \hat{\epsilon}_t^a & (\hat{\epsilon}_t^c)' \hat{\epsilon}_t^c \end{bmatrix} \cdot \frac{1}{N-p}$$

3.4 IRFs at Infinite Horizon

To compute the LR-IRF we need the steady state values for $\Delta\bar{a}$ and $\Delta\bar{c}$. Starting at arbitrary values we compute the following recursion:

- compute $\hat{u}_{t-1} = x_{t-1} - \hat{\alpha} - \hat{\theta}_c c_{t-1}$ using the estimated coefficient from the cointegration regression;
- using the updated Error Correction term, obtain a new value for Δa_t and Δc_t using the estimated VECM;

repeating the above routine for an arbitrary large T[4] we get $\Delta\bar{a}$ and $\Delta\bar{c}$.
After having estimated the steady state values, we can compute the IRF at "infinite" horizon. To compute the IRF at "infinite horizion" we start from the computed $\Delta\bar{a}$ and $\Delta\bar{c}$ and proceed with the same recursion defined above in order to generate a series of impulse response functions. In particular, we simulate a series of Error Correction Terms and compute the new values for Δa_t and Δc_t using the estimated VECM.

[3] I have generalized the code such that the number of lagged difference in the VECM can be changed, the original code does not allow for such a feature.

[4] $T = 500$ in the code.

In this way, we can get estimates of the *cumulated long-run impulse response* matrix $\hat{\Gamma}(1)$ and construct our Blanchard and Quah identification matrix

$$\hat{A}_0 = \left[\hat{\Gamma}(1)chol\left(\hat{\Gamma}(1)\hat{\Sigma}_\epsilon\hat{\Gamma}(1)'^{-1}\right)\right]^{-1}$$

Finally, the long-run effects on the levels y_t are given by the cumulating $\hat{\Gamma}(1)^{-1}\hat{A}_0^{-1}$, which are reported in Figure 3.

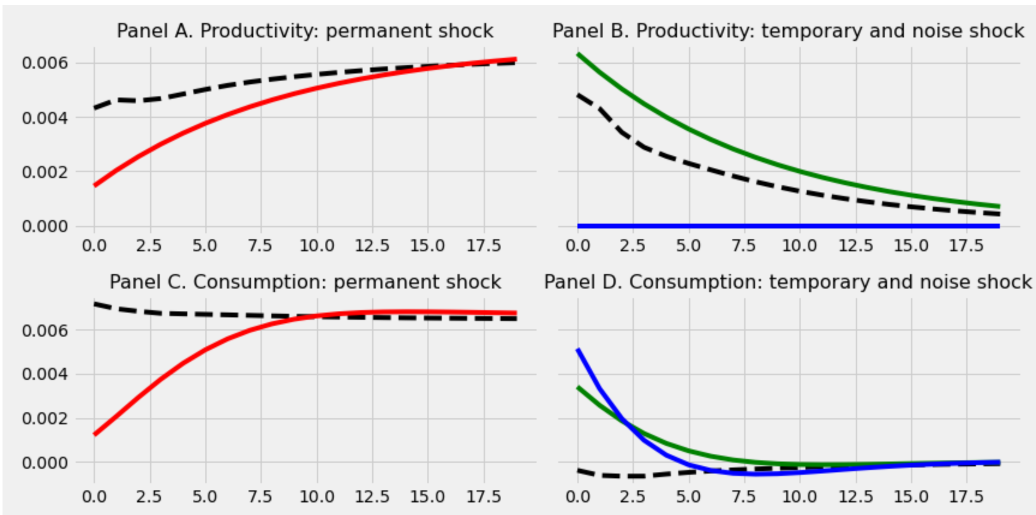

Figure 3: This figure replicates **Figure 2** in Blanchard et al. [2013].

The two BQ shocks are represented by the dashed black lines, whereas the solid lines show the three original shocks in the model. From Panel A and Panel C, we can see that the BQ shocks have relatively larger effects when we consider permanent effects. This effect is stronger for consumption than productivity. Overall, the BQ shocks fail to capture the gradual ajustment of the original ϵ_t shock (solid red color). Panel B and Panel D show instead the responses to the transitory technology η_t shock (solid green color) and the noise shock ν_t (solid blue color). The productivity response to the BQ transitory shock is fairly close to the original transitory productivity η_t shock, whereas the consumption response to the BQ transitory shock fail to replicate the true responses in the original model.

4 Maximum Likelihood

In this section, we use the same data used in the paper, that is made of quarterly observations from $1970 : 1q$ to $1980 : 1q$. Productivity a_t is measured as the logarithm of the ratio of GDP to employment and consumption c_t is the logarithm of the ratio of NIPA consumption to population. In the context of our application, we demean the data and cumulatively sum them to transform them in levels (Figure 4).

As derived in Section 2, we have that the vector of observables $\mathbf{s_t} = (a_t, c_t)'$, conditional on

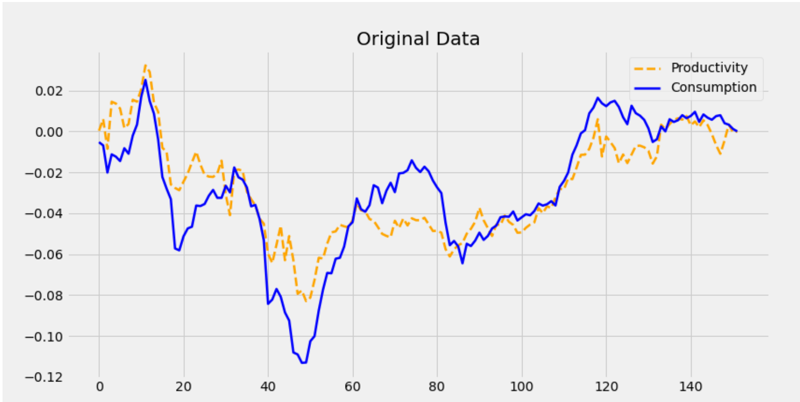

Figure 4: Original data series were in first differences. Time unit is always quarterly. Data have been first demeaned, and then cumulatively summed to obtained the orginal series in levels.

information at $t-1$, is distributed as

$$\mathbf{s_t}|\mathbf{I_{t-1}} \sim N\left(\mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}, \mathbf{CP_{t|t-1}C'} + \mathbf{D\Sigma_v D'}\right)$$

which implies that $f(\mathbf{s_t}|I_{t-1}) = N(\mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}, \mathbf{CP_{t|t-1}C'} + \mathbf{D\Sigma_v D'})$. With T observations for $\mathbf{s_t}$ we can derive the log-likelihood function

$$\begin{aligned}
L\left(\{\mathbf{s_t}\}_{\mathbf{t=1}}^{\mathbf{T}}, \theta\right) &= \prod_{t=1}^{T} L(\mathbf{s_t}|\mathbf{I_{t-1}}, \theta) \\
\log L\left(\{\mathbf{s_t}\}_{\mathbf{t=1}}^{\mathbf{T}}, \theta\right) &= \sum_{t=1}^{T} \log L(\mathbf{s_t}|\mathbf{I_{t-1}}, \theta) \\
&= \sum_{t=1}^{T} \log \left(\frac{\exp\left(\frac{1}{2}\left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)' \mathbf{Q_t} \left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)\right)}{\sqrt{(2\pi)^2} |\mathbf{Q_t}|} \right) \\
&= -T \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log |\mathbf{Q_t}| - \frac{1}{2} \sum_{t=1}^{T} \left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)' \mathbf{Q_t}^{-1} \left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)
\end{aligned}$$

where $\mathbf{Q_t} = \mathbf{CP_{t|t-1}C'} + \mathbf{D\Sigma_v D'}$ and we used the fact that $f_s\left(a_t, c_t\right) = \frac{\exp\left(\frac{1}{2}\left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)' \mathbf{Q_t}\left(\mathbf{s_t} - \mathbf{C}\hat{\mathbf{x}}_{\mathbf{t|t-1}}\right)\right)}{\sqrt{(2\pi)^2} |\mathbf{Q_t}|}$. Therefore, to compute the likelihood, we use the observation equation $\mathbf{s_t^E} = \mathbf{Tx_t^E}$, the transition $\mathbf{x_t^E} = \mathbf{Hx_{t-1}^E} + \mathbf{Wv_t}$ and the updating equations. Starting from an initial condition for $\mathbf{P_{1|0}} = \mathbf{I_9}$ and $\hat{\mathbf{x}}_{\mathbf{1|0}} = \mathbf{0}$ we compute the likelihood recursively, given a vector of paramters $\theta^{(\mathbf{k})} = \left(\rho^{(k)}, \sigma_u^{(k)}, \sigma_\nu^{(k)}\right)'$:

- using $\hat{\mathbf{x}}_{\mathbf{t|t-1}}$ compute $\hat{\mathbf{s}}_{\mathbf{t|t-1}} = \mathbf{T}\left(\theta^{(\mathbf{k})}\right) \hat{\mathbf{x}}_{\mathbf{t|t-1}}$;
- compute the forecast error using $\mathbf{u_t} = \mathbf{s_t} - \hat{\mathbf{s}}_{\mathbf{t|t-1}}$, where $\mathbf{s_t}$ are the actual observations;
- using $P_{t|t-1}$ compute the variance of the forecast error $\mathbf{Q_t} = \mathbf{T}\left(\theta^{(\mathbf{k})}\right) \mathbf{P_{t|t-1}} \mathbf{T}\left(\theta^{(\mathbf{k})}\right)'$;

- compute $(\mathbf{u_t})' \mathbf{Q_t}^{-1} (\mathbf{u_t})$ and compute the log-likelihood value for period t;
- compute $\mathbf{K_t} = \mathbf{P_{t|t-1}} \mathbf{T} \left(\theta^{(\mathbf{k})}\right)' \mathbf{Q_t^{-1}}$;
- use the updating equation to get

$$\begin{aligned}
\hat{\mathbf{x}}_{\mathbf{t|t}} &= \hat{\mathbf{x}}_{\mathbf{t|t-1}} + \mathbf{K_t u_t} \\
\mathbf{P_{t|t}} &= \mathbf{P_{t|t-1}} - \mathbf{K_t T}\left(\theta^{(\mathbf{k})}\right) \mathbf{P_{t|t-1}}
\end{aligned}$$

- finally compute the econometrician's filtered expectations as

$$\begin{aligned}
\hat{\mathbf{x}}_{\mathbf{t+1|t}} &= \mathbf{H}\left(\theta^{(\mathbf{k})}\right) \hat{\mathbf{x}}_{\mathbf{t|t}} \\
\mathbf{P_{t+1|t}} &= \mathbf{H}\left(\theta^{(\mathbf{k})}\right) \mathbf{P_{t|t}} \mathbf{H}\left(\theta^{(\mathbf{k})}\right)' + \mathbf{W}\left(\theta^{(\mathbf{k})}\right) \mathbf{\Sigma_v} \left(\theta^{(\mathbf{k})}\right) \mathbf{W}\left(\theta^{(\mathbf{k})}\right)'
\end{aligned}$$

Maximizing the likelihood we get the optimal vector of parameters θ^*.

4.1 Kalman Smoother

Once we have estimated model's parameters using structural estimation, we can recover information on the unobservable state and shocks. We do that by using the Kalman smoother, exploiting the fact that the econometrician has access to the whole sample. We draw inference on $\hat{\mathbf{x}}_{\mathbf{t|T}} = \mathbb{E}\left[\mathbf{x_t}|\mathbf{s_T}\right]$ computing again the filtered quantatities using the estimated optimal parameters θ^*. In this way we obtain estimates for $\hat{\mathbf{x}}_{\mathbf{t|t}}, \hat{\mathbf{x}}_{\mathbf{t|t-1}}, \mathbf{P_{t|t}}, \mathbf{P_{t|t-1}}$.
Using the estimates we can compute backward the smoothed estimates as follows:

$$\begin{aligned}
\hat{\mathbf{x}}_{\mathbf{T-1|T}} &= \hat{\mathbf{x}}_{\mathbf{T-1|T-1}} + \mathbf{J_{T-1}}\left(\hat{\mathbf{x}}_{\mathbf{T|T}} - \hat{\mathbf{x}}_{\mathbf{T|T-1}}\right) \\
\hat{\mathbf{x}}_{\mathbf{T-t-1|T}} &= \hat{\mathbf{x}}_{\mathbf{T-t-1|T-t-1}} + \mathbf{J_{T-t-1}}\left(\hat{\mathbf{x}}_{\mathbf{T-t|T-t}} - \hat{\mathbf{x}}_{\mathbf{T-t|T-t-1}}\right) \quad \forall t \leq T-1
\end{aligned}$$

where $\mathbf{J_{T-1}} = \mathbf{P_{T-1|T-1}} \mathbf{H}' \mathbf{P_{T|T-1}}$.
Finally, we can get the smoothed series for the consumers's real time expectations regarding long-term productivity as

$$\begin{aligned}
\lim_{j\to\infty} \mathbb{E}_t\left[x_{t+j}\right] &= \frac{1}{1-\rho}\left(\mathbb{E}_t\left[x_t\right] - \rho\mathbb{E}_t\left[x_{t-1}\right]\right) \\
x_{(t+\infty|t)|T} &= \frac{1}{1-\rho}\left(x_{(t|t)|T} - \rho x_{(t-1|t)|T}\right)
\end{aligned}$$

and compare it to the same expression computed using $x_{t|T}$ and $x_{t-1|T}$ (Figure 5). The upper subplot of figure 5, shows in solid red the econometrician's smoothed estimates of x_t, wheres in dashed blue we plotted the econometrician's smoothed estimate of the consumers' real time estimate of the same variable. In the lower subplot of figure 5, we plotted the smoothed series for the consumers' real time expectations regarding long-run productivity and we compare it using the econometrician's smoothed estimate of the consumers' real time estimate of the same variable, which uses only X_t and X_{t-1}. As we can see from the graph, when we consider expectations about long-run productivity, the model is able to generate larger short-run consumption volatility.

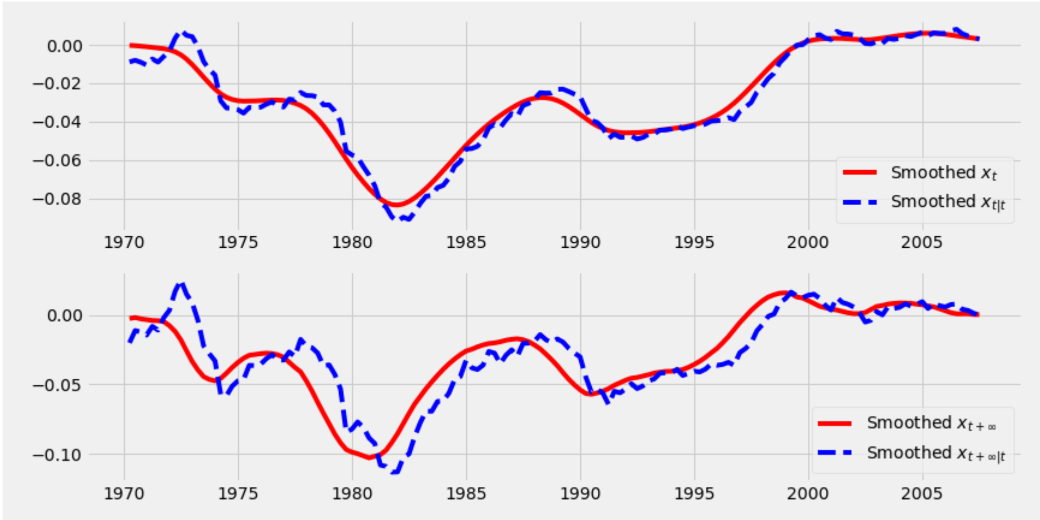

Figure 5: This figure replicates **Figure 3** in Blanchard et al. [2013]. Observations are quarterly.

In Figure 6, we report the RMSE of the smoothed estimates of x_t and z_t when data up to $t+j$ are available, for $j = 0, 1, 2\ldots$. The RMSE is calculated as the square root of $\mathbb{E}_t\left[(x_t - \mathbb{E}_{t+j}[x_t])^2\right]$. To calculate the RMSE we follow the approach of the paper, that is we compute the RMSE at the steady state of the Kalman filter.

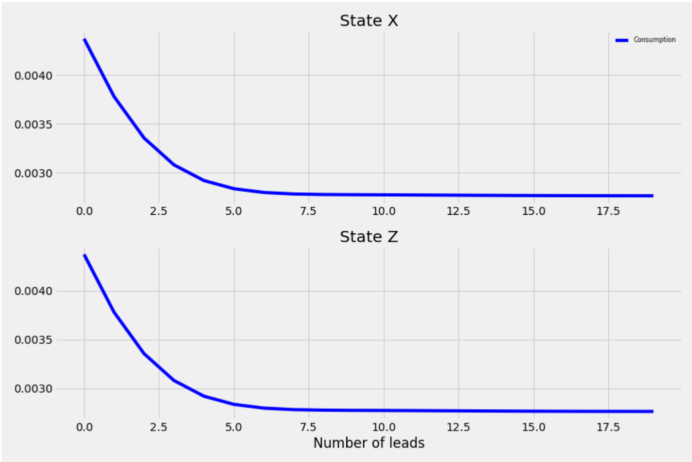

Figure 6: This figure replicates **Figure 8** in the Online Appendix of Blanchard et al. [2013]. RMSE of the estimated states at time t using data up to $t+j$

This is equivalent to assuming that the forecaster has data from $-\infty$ to $t+j$ and its information coincides with the consumers' ones. The RMSE is 0.44 for estimates using data up to data t and becomes 0.28 when we can use all possible future data. Indeed, from Figure 6, we can see that information after period $t+5$ do not decrease the RMSE further.

Using the just estimated Kalman smoother, we can derive estimates for the structural shocks ϵ_t, η_t and ν_t, which we plot in Figure 7. Notice that the smoothed estimates for the permanent shocks in consecutive quarters tend to be highly correlated, as the econometrician does not know to which quarter to attribute an observed permanent change in productivity. This apparent autocorrelation in ϵ_t is merely the result of the econometrician's information.

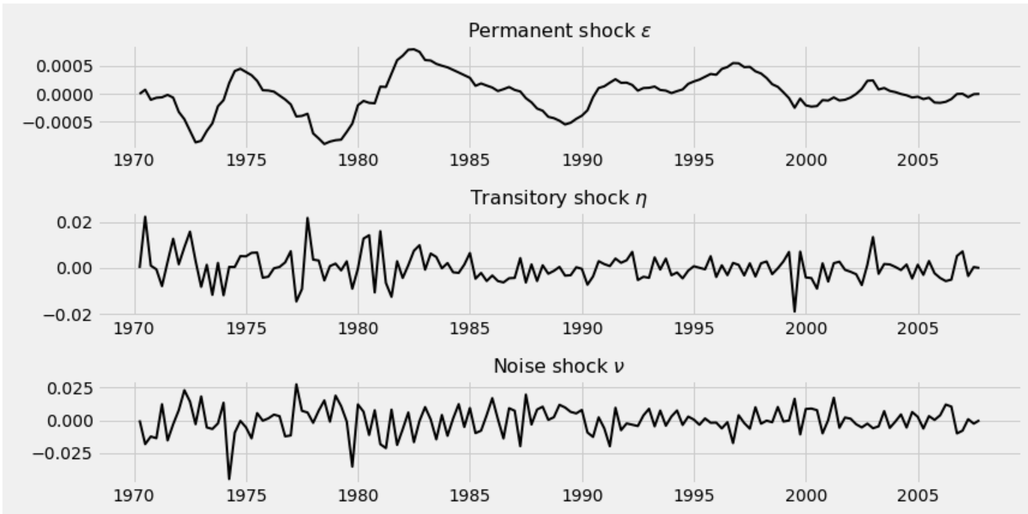

Figure 7: This figure replicates **Figure 4** in Blanchard et al. [2013]. Each plots represent the smoothed estimated of the structural shocks. Observations are quartely.

Next, we show that RMSE for the ϵ_t shock is very high. In Figure 8, we report the RMSE at the steady state for the structural shocks estimated from the Kalman Smoother. Each RMSE is normalized by dividing it by the ex ante standard deviation of the respective shock. The fact that all the shocks are different from zero at $j = 0$ implies that the model is not invertible, therefore even with an infinite data set, the econometrician would not be able to precisely recover the shocks.

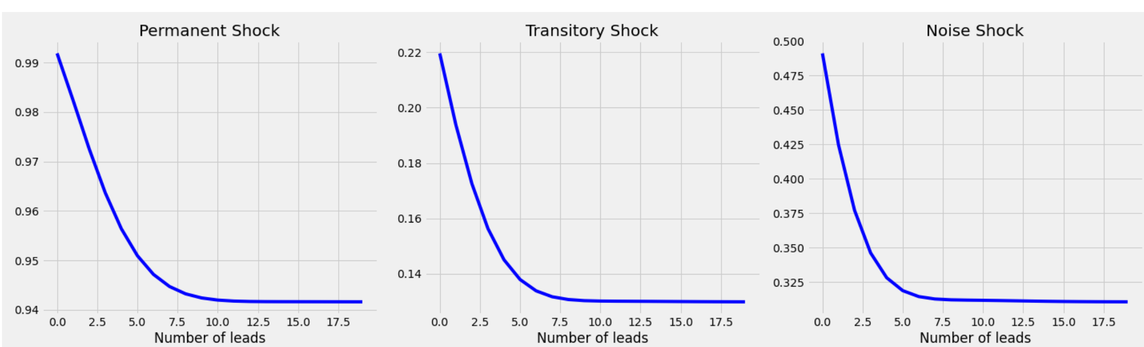

Figure 8: This figure replicates **Figure 9** in the Online Appendix of Blanchard et al. [2013]. Normalized RMSE of the estimated shocks at time t using data up to $t + j$

From Figure 8, we can observe that indeed the RMSE for the permanent shock is the highest, hence ϵ_t is the poorliest estimated structural shock. Even when using data from $-\infty$ to $t + j$, the residual variance is about 94% of the prior uncertainty on the shock. On the other hand, the transitory shock η_t is estimated with a higher precision, whereas the noise shock ν_t is estimated with a relatively higher error.

Bottom line, the econometrician is able to precisely recover the state x_t, but not the structural shocks. Why? The econometrician is not able to pin down in which quarter the change in productivity occurred, but can precisely estimate the cumulated effect of permanent productivity changes by looking at productivity growth over longer horizons.

Another evidence that we can draw from Figure 7 is that noise shocks seem to be quite important in terms of generating fluctuations in the observed data. Indeed we can explore the implications of MLE estimated paramters by performing a variance covariance decomposition, so we can check the contribution of the three shocks to explain the forecast error variance at different horizons. Table 1 confirms that *noise shocks are the major source of short-run volatility*. In particular, it account for more than 70 percent of consumption volatility at a 1-quarter horizon and more than 50 percent at one-year horizon. On the other hand, permanent technology shocks have larger effect in the long-run, explaining almost 85 percent after 3 years.

Table 1: Variance Decomposition of Consumption

Quarter	Perm. Tech.	Trans.tech.	Noise
1	0.016	0.235	0.749
4	0.269	0.198	0.533
8	0.683	0.087	0.229
12	0.832	0.046	0.122

Notes: This table replicates **Table 4** in Blanchard et al. [2013].

Similarly to Table 1, we report in Figure 9 the implied variance covariance values for both consumption and productivity. For consumption, Figure 9 confirms the numbers in Table 1. For productivity, the variance-covariance decomposition of the forecast error variance is implied by our assumptions on the structure of the model. In particular, we can see how noise shocks do not affect the forecast error variance for productivity, whereas most of the short-term variance is explained by temporary technology shock. In the long-run, after more or less 13 quarter, the forecast error variance becomes mostly explained by permanent technology shocks.

5 Conclusion

In this paper, I replicate part of the paper Blanchard et al. [2013]. With this replication, I try to clarify all the analytical expressions and computational details needed to obtain the results presented in the original paper. While most of the replication follow the original codes step-by-step, the section on maximum likelihood is coded autonomously as the authors originally solved the model using Dynare. In this replication, Section 3 of the original paper is missing. The small scale DSGE presented in Section 3 is again solved in Dynare using Bayesian methods,

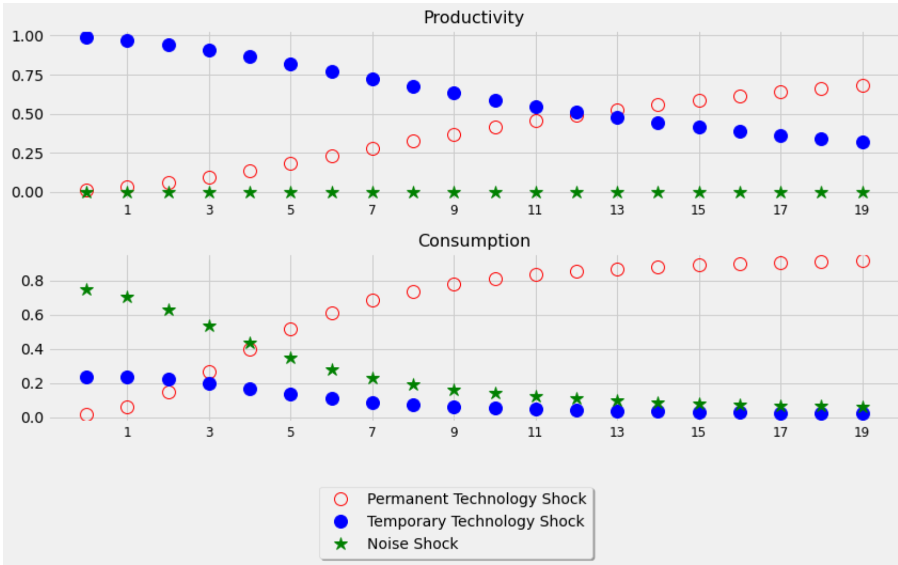

Figure 9: This figure plots the variance-covariance decomposition of consumption and productivity. The decomposition of changes in consumption is the same reported in Table 1.

therefore the replication in another programming language would be an arduous, although extremely instructive, exercise. The replication of this section is left for future work.

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