Single-agent dynamic optimization models

In these lecture notes we consider specification and estimation of dynamic optimization models. Focus on single-agent models.

1 Rust (1987)

Rust (1987) is one of the first papers in this literature. Model is quite simple, but empirical framework introduced in this paper for dynamic discrete-choice (DDC) models is still widely applied.

Agent is Harold Zurcher, manager of bus depot in Madison, Wisconsin. Each week, HZ must decide whether to replace the bus engine, or keep it running for another week. This engine replacement problem is an example of an *optimal stopping* problem, which features the usual tradeoff: (i) there are large fixed costs associated with "stopping" (replacing the engine), but new engine has lower associated future maintenance costs; (ii) by not replacing the engine, you avoid the fixed replacement costs, but suffer higher future maintenance costs. Optimal solution is characterized by a threshold-type of rule: there is a "critical" cutoff mileage level x^* below which no replacement takes place, but above which replacement will take place.

Remark: Another well-known example of optimal stopping problem in economics is job search model: each period, unemployed worker decides whether to accept a job offer, or continue searching. Optimal policy is characterized by "reservation wage": accept all job offers with wage above a certain threshold.

1.1 Behavioral Model

At the end of each week t, HZ decides whether or not to replace engine. *Control* variable defined as:

$$i_t = \begin{cases} 1 & \text{if HZ replaces} \\ 0 & \text{otherwise.} \end{cases}$$

For simplicity. we describe the case where there is only one bus (in the paper, buses are treated as independent entities).

HZ chooses the (infinite) sequence $\{i_1, i_2, i_3, \dots, i_t, i_{t+1}, \dots\}$ to maximize discounted expected utility stream:

$$\max_{\{i_1, i_2, i_3, \dots, i_t, i_{t+1}, \dots\}} E \sum_{t=1}^{\infty} \beta^{t-1} u\left(x_t, \epsilon_t, i_t; \theta\right) \tag{1}$$

where

- x_t is the mileage of the bus at the end of week t. Assume that evolution of mileage is stochastic (from HZ's point of view) and follows

$$x_{t+1} \begin{cases} \sim G(x'|x_t) & \text{if } i_t = 0 \text{ (don't replace engine in period } t) \\ = 0 & \text{if } i_t = 1\text{: once replaced, bus is good as new} \end{cases} \tag{2}$$

  and $G(x'|x)$ is the conditional probability distribution of next period's mileage x' given that current mileage is x. HZ knows G; econometrician knows the form of G, up to a vector of parameters which are estimated.

- ϵ_t denotes shocks in period t, which affect HZ's choice of whether to replace the engine. These are the "structural errors" of the model (they are observed by HZ, but not by us), and we will discuss them in more detail below.

- Since mileage evolves randomly, this implies that even given a sequence of replacement choices $\{i_1, i_2, i_3, \dots, i_t, i_{t+1}, \dots\}$, the corresponding sequence of mileages $\{x_1, x_2, x_3, \dots, x_t, x_{t+1}, \dots\}$ is still random. The expectation in Eq. (1) is over this stochastic sequence of mileages and over the shocks $\{\epsilon_1, \epsilon_2, \dots\}$.

- The *state* variables of this problem are:

  1. x_t: the mileage. Both HZ and the econometrician observe this, so we call this the "observed state variable"
  2. ϵ_t: the utility shocks. Econometrician does not observe this, so we call it the "unobserved state variable"

Define value function:

$$V(x_t, \epsilon_t) = \max_{i_\tau,\ \tau=t+1, t+2, \dots} E_t \left[\sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u\left(x_t, \epsilon_t, i_t; \theta\right) | x_t \right]$$

where maximum is over all possible sequences of $\{i_{t+1}, i_{t+2}, \dots\}$. Note that we have imposed stationarity, so that the value function $V(\cdot)$ is a function of t only indirectly, through the value that the state variable x takes during period t.

Remark: An important distinction between empirical papers with dynamic optimization models is whether agents have infinite-horizon, or finite-horizon. Stationarity (or time homogeneity) is assumed for infinite-horizon problems, and they are solved using value function iteration. Finite-horizon problems are non-stationary, and solved by backward induction starting from the final period. Most structural dynamic models used in labor economics and macro are finite-horizon.

Using the Bellman equation, we can break down the DO problem into an (infinite) sequence of single-period decisions:

$$i_t = i^*(x_t, \epsilon_t; \theta) = \text{argmax}_i \left\{ u(x_t, \epsilon_t, i; \theta) + \beta E_{x', \epsilon' | x_t, \epsilon_t, i_t} V(x', \epsilon') \right\}$$

where the value function is

$$\begin{aligned} V(x, \epsilon) &= \max_{i=1,0} \left\{ u(x, \epsilon, i; \theta) + \beta E_{x', \epsilon' | x_t, \epsilon_t, i_t} V(x', \epsilon') \right\} \\ &= \max \left\{ u(x, \epsilon, 0; \theta) + \beta E_{x', \epsilon' | x_t, \epsilon_t, i_t} V(x', \epsilon'), u(x, \epsilon, 1; \theta) + \beta E V(0, \epsilon'). \right\} \quad (3) \\ &= \max \left\{ \tilde{V}(x_t, \epsilon_t, 1), \tilde{V}(x_t, \epsilon_t, 0) \right\}. \end{aligned}$$

In the above, we define the **choice-specific value function**

$$\tilde{V}(x_t, \epsilon_t, i_t) = \begin{cases} u(x_t, \epsilon_t, 1; \theta) + \beta E V(0, \epsilon') & \text{if } i_t = 1 \\ u(x_t, \epsilon_t, 0; \theta) + \beta E_{x', \epsilon' | x_t, \epsilon_t, i_t} V(x', \epsilon') & \text{if } i_t = 0. \end{cases}$$

We make the following parametric assumptions on utility flow:

$$u(x_t, \epsilon_t, i; \theta) = -c\left((1 - i_t) * x_t; \theta\right) - i * RC + \epsilon_{it}$$

where

- $c(\cdots)$ is the maintenance cost function, which is presumably increasing in x (higher x means higher costs)
- RC denotes the "lumpy" fixed costs of adjustment. The presence of these costs implies that HZ won't want to replace the engine every period.

- $\epsilon_{it},\ i = 0, 1$ are structural errors, which represents factors which affect HZ's replacement choice i_t in period t, but are unobserved by the econometrician. Define $\epsilon_t \equiv (\epsilon_{0t}, \epsilon_{1t})$.

  As Rust remarks (bottom, pg. 1008), you need this in order to generate a positive likelihood for your observed data. Without these ϵ's, we observe as much as HZ does, and $i_t = i^*(x_t; \theta)$, so that replacement decision should be perfectly explained by mileage. Hence, model will not be able to explain situations where there are two periods with identical mileage, but in one period HZ replaced, and in the other HZ doesn't replace.

Given additivity, we also define the *choice-specific utility flow*

$$u(x, i; \theta) \equiv u(x, \epsilon, i; \theta) - \epsilon_i$$

equal to the utility flow minus the ϵ_i.

As remarked earlier, these assumption imply a very simple type of optimal decision rule $i^*(x, \epsilon; \theta)$: in any period t, you replace when $x_t \geq x^*(\epsilon_t)$, where $x^*(\epsilon_t)$ is some optimal cutoff mileage level, which depends on the value of the shocks ϵ_t.

Parameters to be estimated are:

1. parameters of maintenance cost function $c(\cdots)$;
2. replacement cost RC;
3. parameters of mileage transition function $G(x'|x)$.

Remark: In these models, the discount factor β is typically not estimated. Essentially, the time series data on $\{i_t, x_t\}$ could be equally well explained by a myopic model, which posits that

$$i_t = \text{argmax}_{i \in \{0,1\}} \left\{ u(x_t, \epsilon_t, 0), u(x_t, \epsilon_t, 1) \right\},$$

or a forward-looking model, which posits that

$$i_t = \text{argmax}_{i \in \{0,1\}} \left\{ \tilde{V}(x_t, \epsilon_t, 0), \tilde{V}(x_t, \epsilon_t, 1) \right\}.$$

In both models, the choice i_t depends just on the current state variables x_t, ϵ_t. Indeed, Magnac and Thesmar (2002) shows that in general, DDC models are nonparametrically underidentified, without knowledge of β and $F(\epsilon)$, the distribution of the ϵ shocks. (Below, we show how knowledge of β and F, along with an additional normalization, permits nonparametric identification of the utility functions in this model.)

Intuitively, in this model, it is difficult to identify β apart from fixed costs. In this model, if HZ were myopic (ie. β close to zero) and replacement costs RC were low, his decisions may look similar as when he were forward-looking (ie. β close to 1) and RC were large. Reduced-form tests for forward-looking behavior exploit scenarios in which some variables which affect future utility are known in period t: consumers are deemed forward-looking if their period t decisions depends on these variables. (Example: Chevalier and Goolsbee (2005) examine whether students' choices of purchasing a textbook now depend on the possibility that a new edition will be released soon.)

1.2 Econometric Model

Data: observe $\{i_t, x_t\}$, $t = 1, \ldots, T$ for 62 buses. Treat buses as homogeneous and independent (ie. replacement decision on bus i is not affected by replacement decision on bus j).

Rust makes the following conditional independence assumption, on the Markovian transition probabilities in the Bellman equation above:

$$\begin{aligned} p(x', \epsilon'|x, \epsilon, i) &= p(\epsilon'|x', x, \epsilon, i) \cdot p(x'|x, e, i) \\ &= p(\epsilon'|x') \cdot p(x'|x, i). \end{aligned} \tag{4}$$

The first line is just factoring the joint density into a conditional times a marginal. The second line shows the simplifications from Rust's assumptions. Namely, two types of conditional independence: (i) given x, ϵ's are independent over time; and (ii) conditional on x and i, x' is independent of ϵ.

Likelihood function for a single bus:

$$\begin{aligned} & l\left(x_1, \ldots, x_T, i_t, \ldots, i_T | x_0, i_0; \theta\right) \\ = & \prod_{t=1}^{T} Prob\left(i_t, x_t | x_0, i_0, \ldots, x_{t-1}, i_{t-1}; \theta\right) \\ = & \prod_{t=1}^{T} Prob\left(i_t, x_t | x_{t-1}, i_{t-1}; \theta\right) \\ = & \prod_{t=1}^{T} Prob\left(i_t | x_t; \theta\right) \times Prob\left(x_t | x_{t-1}, i_{t-1}; \theta_3\right). \end{aligned} \tag{5}$$

The third line arises from the Markovian feature of the problem, and the last equality arises due to the conditional independence assumption.

Hence, the log likelihood is additively separable in the two components:

$$\log l = \sum_{t=1}^{T} \log Prob\left(i_t | x_t; \theta\right) + \sum_{t=1}^{T} \log Prob\left(x_t | x_{t-1}, i_{t-1}; \theta_3\right).$$

Given the factorization of the likelihood function above, we can estimate in two steps:

1. Estimate θ_3, the parameters of the Markov transition probabilities for mileage, conditional on non-replacement of engine (i.e., $i_t = 0$). (Recall that $x_{t+1} = 0$ wp1 if $i_t = 1$.)

We assume a discrete distribution for $\Delta x_t \equiv x_{t+1} - x_t$, the incremental mileage between any two periods:

$$\Delta x_t = \begin{cases} [0, 5000) & \text{w/prob } p \\ [5000, 10000) & \text{w/prob } q \\ [10000, \infty) & \text{w/prob } 1 - p - q \end{cases}$$

so that $\theta_3 \equiv \{p, q\}$, with $0 < p, q < 1$ and $p + q < 1$.

This first step can be executed separately from the more substantial second step.

2. Estimate θ, parameters of maintenance cost function $c(\cdots)$ and engine replacement costs.

Here, we make a further assumption that the ϵ's are distributed i.i.d. (across choices and periods), according to the Type I extreme value distribution. So this implies that in Eq. (4) above, $p(\epsilon'|x') = p(\epsilon')$, for all x'.

Expand the expression for $Prob(i_t = 1|x_t; \theta)$ equals

$$\begin{aligned} &Prob\left\{-c(0;\theta) - RC + \epsilon_{1t} + \beta EV(0,\epsilon') > -c(x_t;\theta) + \epsilon_{0t} + \beta E_{x',\epsilon'|x_t,\epsilon_t,i_t=0} V(x',\epsilon')\right\} \\ =&Prob\left\{\epsilon_{1t} - \epsilon_{0t} > c(0;\theta) - c(x_t;\theta) + \beta\left[E_{x',\epsilon'|x_t,\epsilon_t,i_t=0} V(x,\epsilon) - EV(0,\epsilon')\right] + RC\right\} \end{aligned}$$

Because of the logit assumptions on ϵ_t, the replacement probability simplifies to a multinomial logit-like expression:

$$= \frac{\exp\left(-c(0;\theta) - RC + \beta EV(0,\epsilon')\right)}{\exp\left(-c(0;\theta) - RC + \beta EV(0,\epsilon')\right) + \exp\left(-c(x_t;\theta) + \beta E_{x',\epsilon'|x_t,\epsilon_t,i_t=0} V(x',\epsilon')\right)}.$$

This is called a "dynamic logit" model, in the literature.

Using the choice-specific utility flow notation, the choice probability takes the form

$$Prob\left(i_t|x_t;\theta\right) = \frac{\exp\left(u(x_t,i_t,\theta) + \beta E_{x',\epsilon'|x_t,\epsilon_t,i_t} V(x',\epsilon')\right)}{\sum_{i=0,1}\exp\left(u(x_t,i,\theta) + \beta E_{x',\epsilon'|x_t,\epsilon_t,i} V(x',\epsilon')\right)}. \tag{6}$$

1.2.1 Estimation method for second step: Nested fixed-point algorithm

The second-step of the estimation procedures is via a "nested fixed point algorithm".

Outer loop: search over different parameter values $\hat{\theta}$.

Inner loop: For $\hat{\theta}$, we need to compute the value function $V(x;\hat{\theta})$. After $V(x,\epsilon;\hat{\theta})$ is obtained, we can compute the LL fxn in Eq. (6).

1.2.2 Computational details for inner loop

Compute value function $V(x;\hat{\theta})$ by iterating over Bellman's equation (3).

A clever and computationally convenient feature in Rust's paper is that he iterates over the *expected* value function $EV(x,i) \equiv E_{x',\epsilon'|x,i} V(x',\epsilon';\theta)$. The reason for this is that you avoid having to calculate the value function at values of ϵ_0 and ϵ_1, which are additional state variables. He iterates over the following equation (which is Eq. 4.14

in his paper):

$$EV(x,i) = \int_y \log \left\{ \sum_{j \in C(y)} \exp\left[u(y,j;\theta) + \beta EV(y,j)\right] \right\} p(dy|x,i) \tag{7}$$

Somewhat awkward notation: here "EV" denotes a function. Here x, i denotes the *previous* period's mileage and replacement choice, and y, j denote the *current* period's mileage and choice (as will be clear below).

This equation can be derived from Bellman's equation (3):

$$\begin{aligned}
V(y,\epsilon;\theta) =& \max_{j \in 0,1} \left[u(y,j;\theta) + \epsilon + \beta EV(y,j)\right] \\
\Rightarrow E_{y,\epsilon}\left[V(y,\epsilon;\theta) \mid x,i\right] \equiv EV(x,i;\theta) =& E_{y,\epsilon|x,i} \left\{ \max_{j \in 0,1} \left[u(y,j;\theta) + \epsilon + \beta EV(y,j)\right] \right\} \\
=& E_{y|x,i} E_{\epsilon|y,x,i} \left\{ \max_{j \in 0,1} \left[u(y,j;\theta) + \epsilon + \beta EV(y,j)\right] \right\} \\
=& E_{y|x,i} \log \left\{ \sum_{j=0,1} \left[u(y,j;\theta) + \beta EV(y,j)\right] \right\} \\
=& \int_y \log \left\{ \sum_{j=0,1} \left[u(y,j;\theta) + \beta EV(y,j)\right] \right\} p(dy|x,i).
\end{aligned}$$

The next-to-last equality uses the closed-form expression for the expectation of the maximum, for extreme-value variates.

Once the $EV(x,i;\theta)$ function is computed for θ, the choice probabilities $p(i_t|x_t)$ can be constructed as

$$\frac{\exp\left(u(x_t,i_t;\theta) + \beta EV(x_t,i_t;\theta)\right)}{\sum_{i=0,1} \exp\left(u(x_t,i;\theta) + \beta EV(x_t,i;\theta)\right)}.$$

The value iteration procedure: The expected value function $EV(\cdots;\theta)$ will be computed for each value of the parameters θ. The computational procedure is iterative.

Let τ index the iterations. Let $EV^\tau(x,i)$ denote the expected value function during the τ-th iteration. (We suppress the functional dependence of EV on θ for convenience.) Let the values of the state variable x be discretized into a grid of points, which we denote $\vec{r}$.

- $\tau = 0$: Start from an initial guess of the expected value function $EV(x, i)$. Common way is to start with $EV(x, i) = 0$, for all $x \in \vec{r}$, and $i = 0, 1$.
- $\tau = 1$: Use Eq. (7) and $EV^0(x; \theta)$ to calculate, at each $x \in \vec{r}$, and $i \in \{0, 1\}$.

$$\begin{aligned} EV^1(x,i) = &\int_y \log\left\{\sum_{j\in C(y)} \exp\left[u(y,j;\theta) + \beta EV^0(y,j)\right]\right\} p(dy|x,i) \\ =&p \cdot \int_x^{x+5000} \log\left\{\sum_{j\in C(y)} \exp\left[u(y,j;\theta) + \beta EV^0(y,j)\right]\right\} dy + \\ &q \cdot \int_{x+5000}^{x+10000} \log\{\cdots\}\, dy + (1-p-q) \cdot \int_{x+10000}^{\infty} \log\{\cdots\}\, dy. \end{aligned}$$

Now check: is $EV^1(x, i)$ close to $EV^0(x, i)$? One way is to check whether

$$sup_{x,i}|EV^1(x,i) - EV^0(x,i)| < \eta$$

where η is some very small number (eg. 0.0001). If so, then you are done. If not, then

- Interpolate to get $EV^1(\cdot, i)$ at all points $x \notin \vec{r}$.
- Go to next iteration $\tau = 2$.

2 Hotz-Miller approach: avoid numeric dynamic programming

- One problem with Rust approach to estimating dynamic discrete-choice model very computer intensive. Requires using numeric dynamic programming to compute the value function(s) for every parameter vector θ.
- Alternative method of estimation, which avoids explicit DP. Present main ideas and motivation using a simplified version of Hotz and Miller (1993), Hotz, Miller, Sanders, and Smith (1994).
- For simplicity, think about Harold Zurcher model.

- What do we observe in data from DDC framework? For bus i, time t, observe:
  - $\{\tilde{x}_{it}, d_{it}\}$: observed state variables $\tilde{x}_{it}$ and discrete decision (control) variable d_{it}. For simplicity, assume d_{it} is binary, $\in \{0, 1\}$

    Let $i = 1, \ldots, N$ index the buses, $t = 1, \ldots, T$ index the time periods.
  - For Harold Zurcher model: $\tilde{x}_{it}$ is mileage on bus i in period t, and d_{it} is whether or not engine of bus i was replaced in period t.
  - Given renewal assumptions (that engine, once repaired, is good as new), define transformed state variable x_{it}: mileage since last engine change.
  - Unobserved state variables: ϵ_{it}, *i.i.d.* over i and t. Assume that distribution is known (Type 1 Extreme Value in Rust model)

- In the following, let quantities with hats ˆs denote objects obtained just from data.

  Objects with tildes ˜s denote "predicted" quantities, obtained from both data and calculated from model given parameter values θ.

- From this data alone, we can estimate (or "identify"):
  - Transition probabilities of observed state and control variables: $G(x'|x, d)$[1], estimated by conditional empirical distribution

$$\hat{G}(x'|x,d) \equiv \begin{cases} \sum_{i=1}^{N} \sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}(x_{it}=x, d_{it}=0)} \cdot \mathbf{1}\left(x_{i,t+1} \leq x', x_{it} = x, d_{it} = 0\right), & \text{if } d = 0 \\ \sum_{i=1}^{N} \sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}(d_{it}=1)} \cdot \mathbf{1}\left(x_{i,t+1} \leq x', d_{it} = 1\right), & \text{if } d = 1. \end{cases}$$

  - Choice probabilities, conditional on state variable: $\text{Prob}\,(d = 1|x)$[2], estimated by

$$\hat{P}(d = 1|x) \equiv \sum_{i=1}^{N} \sum_{t=1}^{T-1} \frac{1}{\sum_i \sum_t \mathbf{1}\left(x_{it} = x\right)} \cdot \mathbf{1}\left(d_{it} = 1, x_{it} = x\right).$$

    Since $\text{Prob}\,(d = 0|x) = 1 - \text{Prob}\,(d = 1|x)$, we have $\hat{P}(d = 0|x) = 1 - \hat{P}(d = 1|x)$.

[1] By stationarity, note we do not index the G function explicitly with time t.

[2] By stationarity, note we do not index this probability explicitly with time t.

- Let $\tilde{V}(x, d; \theta)$ denote the choice-specific value function, minus the error term ϵ_d. That is,

$$\tilde{V}(x, d; \theta) \equiv \tilde{V}(x, \epsilon, d; \theta) - \epsilon_d.$$

- With estimates of $\hat{G}(\cdot|\cdot)$ and $\hat{p}(\cdot|\cdot)$, as well as a parameter vector θ, you can "estimate" the choice-specific value functions by constructing the sum

$$\begin{aligned}\tilde{V}(x, d=1; \theta) =& u(x, d=1; \theta) + \beta E_{x'|x,d=1} E_{d'|x'} E_{\epsilon'|d',x'} \left[u(x', d'; \theta) + \epsilon' \right. \\ & \left. + \beta E_{x''|x',d'} E_{d''|x''} E_{\epsilon'|d'',x''} \left[u(x'', d''; \theta) + \epsilon'' + \beta \cdots\right]\right]\end{aligned}$$

$$\begin{aligned}\tilde{V}(x, d=0; \theta) =& u(x, d=0; \theta) + \beta E_{x'|x,d=0} E_{d'|x'} E_{\epsilon'|d',x'} \left[u(x', d'; \theta) + \epsilon' \right. \\ & \left. + \beta E_{x''|x',d'} E_{d''|x''} E_{\epsilon'|d'',x''} \left[u(x'', d''; \theta) + \epsilon'' + \beta \cdots\right]\right].\end{aligned}$$

  Here $u(x, d; \theta)$ denotes the per-period utility of taking choice d at state x, *without* the additive logit error. Note that the observation of $d'|x'$ is crucial to being able to forward-simulate the choice-specific value functions. Otherwise, $d'|x'$ is multinomial with probabilities given by Eq. (8) below, and is impossible to calculate without knowledge of the choice-specific value functions.

- In practice, "truncate" the infinite sum at some period T:

$$\begin{aligned}\tilde{V}(x, d=1; \theta) =& u(x, d=1; \theta) + \beta E_{x'|x,d=1} E_{d'|x'} E_{\epsilon''|d',x'} \left[u(x', d'; \theta) + \epsilon' \right. \\ & + \beta E_{x''|x',d''} E_{d''|x''} E_{\epsilon'|d'',x''} \left[u(x'', d''; \theta) + \epsilon'' + \cdots \right. \\ & \left. \left. \beta E_{x^T|x^{T-1},d^{T-1}} E_{d^T|x^T} E_{\epsilon^T|d^T,x^T} \left[u(x^T, d^T; \theta) + \epsilon^T\right]\right]\right]\end{aligned}$$

  Also, the expectation $E_{\epsilon|d,x}$ denotes the expectation of the ϵ conditional on choice d being taken, and current mileage x. For the logit case, there is a closed form:

$$E[\epsilon|d, x] = \gamma - \log(Pr(d|x))$$

  where γ is Euler's constant (0.577...) and $Pr(d|x)$ is the choice probability of action d at state x.

  Both of the other expectations in the above expressions are observed directly from the data.

- Both choice-specific value functions can be simulated by (for $d = 1, 2$):

$$\tilde{V}(x, d; \theta) \approx = \frac{1}{S} \sum_s \Big[u(x, d; \theta) + \beta \Big[u(x'^s, d'^s; \theta) + \gamma - \log(\hat{P}(d'^s | x'^s)) + \beta \Big[u(x''^s, d''^s; \theta) + \gamma - \log(\hat{P}(d''^s | x''^s)) + \beta \cdots \Big] \Big] \Big]$$

  where

  - $x'^s \sim \hat{G}(\cdot | x, d)$
  - $d'^s \sim \hat{p}(\cdot | x'^s)$, $x''^s \sim \hat{G}(\cdot | x'^s, d'^s)$
  - &etc.

  In short, you simulate $\tilde{V}(x, d; \theta)$ by drawing S "sequences" of (d_t, x_t) with a initial value of (d, x), and computing the present-discounted utility correspond to each sequence. Then the simulation estimate of $\tilde{V}(x, d; \theta)$ is obtained as the sample average.

- Given an estimate of $\tilde{V}(\cdot, d; \theta)$, you can get the *predicted choice probabilities*:

$$\tilde{p}(d = 1 | x; \theta) \equiv \frac{\exp\left(\tilde{V}(x, d = 1; \theta)\right)}{\exp\left(\tilde{V}(x, d = 1; \theta)\right) + \exp\left(\tilde{V}(x, d = 0; \theta)\right)} \tag{8}$$

  and analogously for $\tilde{p}(d = 0 | x; \theta)$. Note that the predicted choice probabilities are different from $\hat{p}(d|x)$, which are the actual choice probabilities computed from the actual data. The predicted choice probabilities depend on the parameters θ, whereas $\hat{p}(d|x)$ depend solely on the data.

- One way to estimate θ is to minimize the distance between the predicted conditional choice probabilities, and the actual conditional choice probabilities:

$$\hat{\theta} = \text{argmin}_\theta || \hat{\mathbf{p}}(d = 1 | x) - \tilde{\mathbf{p}}\,(d = 1 | x; \theta)\,||$$

  where $\mathbf{p}$ denotes a vector of probabilities, at various values of x.

- Another way to estimate θ is very similar to the Berry/BLP method. We can calculate directly from the data.

$$\hat{\delta}_x \equiv \log \hat{p}(d = 1 | x) - \log \hat{p}(d = 0 | x)$$

Given the logit assumption, from equation (8), we know that

$$\log \tilde{p}(d=1|x) - \log \tilde{p}(d=0|x) = \left[\tilde{V}(x, d=1) - \tilde{V}(x, d=0)\right].$$

Hence, by equating $\tilde{V}(x, d=1) - \tilde{V}(x, d=0)$ to $\hat{\tilde{\delta}}_x$, we obtain an alternative estimator for θ:

$$\bar{\theta} = \text{argmin}_\theta || \hat{\tilde{\delta}}_x - \left[\tilde{V}(x, d=1; \theta) - \tilde{V}(x, d=0; \theta)\right] ||.$$

2.1 A useful representation for the discrete state case

For the case when the state variables S are discrete, the value function is just a vector, and it turns out that, given knowledge of the CCP's $P(Y|S)$, solving the value function is just equivalent to solving a system of linear equations. This was pointed out in Pesendorfer and Schmidt-Dengler (2007) and Aguirregabiria and Mira (2007), and we follow the treatment in the latter paper. Specifically:

- Assume that choices Y and state variables S are all discrete. $|S|$ is cardinality of state space S.
- Per-period utilities:
$$u(Y, S, \epsilon_Y) = \bar{u}(Y, S) + \epsilon_Y$$
where ϵ_Y, for $y = 1, \ldots Y$, are i.i.d. extreme value distributed with unit variance.
- Parameters Θ. The discount rate β is treated as known and fixed.

  $\bar{V}(S; \Theta)$, which is the vector (each element denotes a different value of S) for the value function at the parameter Θ, is given by

$$\bar{V}(S; \Theta) = (I - \beta F)^{-1} \left\{ \sum_{\tilde{Y} \in (0,1)} P(\tilde{Y}) * [\bar{u}(\tilde{Y}) + \epsilon(\tilde{Y})] \right\} \tag{9}$$

  where

  – '*' denotes elementwise multiplication

– F is the $|S|$-dimensional square matrix with (i,j)-element equal to

$$Pr(S'=j|S=i) \equiv \sum_{Y=(0,1)} P(Y|S=i) \cdot Pr(S'=j|S=i,Y). \tag{10}$$

– $P(\tilde{Y})$ is the $|S|$-vector consisting of elements $Pr(\tilde{Y}|S)$.

– $\bar{u}(\tilde{Y})$ is the $|S|$-vector of per-period utilities $\bar{u}(\tilde{Y};S)$.

– $\epsilon(\tilde{Y})$ is an $|S|$-vector where each element is $E[\epsilon_{\tilde{Y}}|\tilde{Y},S]$. For the logit assumptions, the closed-form is

$$E[\epsilon_{\tilde{Y}}|\tilde{Y},S] = \text{Euler's constant} - \log(P(\tilde{Y}|S)).$$

Euler's constant is 0.57721.

2.2 Semiparametric identification of DDC Models

We can also use the Hotz-Miller estimation scheme as a basis for an argument regarding the identification of the underlying DDC model. In Markovian DDC models, without unobserved state variables, the Hotz-Miller routine exploits the fact that the Markov probabilities $x', d'|x, d$ are identiified directly from the data, which can be factorized into

$$f(x',d'|x,d) = \underbrace{f(d'|x')}_{\text{CCP}} \cdot \underbrace{f(x'|x,d)}_{\text{state law of motion}} . \tag{11}$$

In this section, we argue that once these "reduced form" components of the model are identified, the remaining parts of the models – particularly, the per-period utility functions – can be identified without any further parametric assumptions. These arguments are drawn from Magnac and Thesmar (2002) and Bajari, Chernozhukov, Hong, and Nekipelov (2007).

We make the following assumptions, which are standard in this literature:

1. Agents are optimizing in an infinite-horizon, stationary setting. Therefore, in the rest of this section, we use primes $'$'s to denote next-period values.

2. Actions D are chosen from the set $\mathcal{D} = \{0, 1, \ldots, K\}$.

3. The state variables are X.

4. The per-period utility from taking action $d \in \mathcal{D}$ in period t is:

$$u_d(X_t) + \epsilon_{d,t}, \ \forall d \in \mathcal{D}.$$

   The $\epsilon_{d,t}$'s are utility shocks which are independent of X_t, and distributed i.i.d with known distribution $F(\epsilon)$ across periods t and actions d. Let $\vec{\epsilon}_t \equiv (\epsilon_{0,1}, \epsilon_{1,t}, \ldots, \epsilon_{K,t})$.

5. From the data, the "conditional choice probabilities" CCP's

$$p_d(X) \equiv \text{Prob}(D = 1|X),$$

   and the Markov transition kernel for X, denoted $p(X'|D, X)$, are identified.

6. $u_0(S)$, the per-period utility from $D = 0$, is normalized to zero, for all X.

7. β, the discount factor, is known.[3]

Following the arguments in Magnac and Thesmar (2002) and Bajari, Chernozhukov, Hong, and Nekipelov (2007), we will show the nonparametric identification of $u_d(\cdot)$, $d = 1, \ldots, K$, the per-period utility functions for all actions except $D = 0$.

The Bellman equation for this dynamic optimization problem is

$$V(X, \vec{\epsilon}) = \max_{d \in \mathcal{D}} \left(u_d(X) + \epsilon_d + \beta E_{X', \vec{\epsilon}'|D,X} V(X', \vec{\epsilon}') \right)$$

where $V(X, \vec{\epsilon})$ denotes the value function. We define the choice-specific value function as

$$V_d(X) \equiv u_d(X) + \beta E_{X', \vec{\epsilon}'|D,X} V(X', \vec{\epsilon}').$$

Given these definitions, an agent's optimal choice when the state is X is given by

$$y^*(X) = \text{argmax}_{y \in \mathcal{D}} \left(V_d(X) + \epsilon_d \right).$$

[3]Magnac and Thesmar (2002) discuss the possibility of identifying β via exclusion restrictions, but we do not pursue that here.

Hotz and Miller (1993) and Magnac and Thesmar (2002) show that in this setting, there is a known one-to-one mapping, $q(X) : \mathbb{R}^K \to \mathbb{R}^K$, which maps the K-vector of choice probabilities $(p_1(X), \ldots, p_K(X))$ to the K-vector $(\Delta_1(X), \ldots, \Delta_K(X))$, where $\Delta_d(X)$ denotes the difference in choice-specific value functions

$$\Delta_d(X) \equiv V_d(X) - V_0(X).$$

Let the i-th element of $q(p_1(X), \ldots, p_K(X))$, denoted $q_i(X)$, be equal to $\Delta_i(X)$. The known mapping q derives just from $F(\epsilon)$, the known distribution of the utility shocks.

Hence, since the choice probabilities can be identified from the data, and the mapping q is known, the value function differences $\Delta_1(X), \ldots, \Delta_K(X)$ is also known.

Next, we note that the choice-specific value function also satisfies a Bellman-like equation:

$$\begin{aligned} V_d(X) &= u_d(X) + \beta E_{X'|X,D}\left[E_{\vec{\epsilon}'} \max_{d' \in \mathcal{D}} (V_{d'}(X') + \epsilon'_d)\right] \\ &= u_d(X) + \beta E_{X'|X,D}\left[H(\Delta_1(X'), \ldots, \Delta_K(X')) + V_0(X')\right] \end{aligned} \tag{12}$$

where $H(\cdots)$ denotes McFadden's "social surplus" function, for random utility models (cf. Rust (1994, pp. 3104ff)). Like the q mapping, H is a known function, which depends just on $F(\epsilon)$, the known distribution of the utility shocks.

Using the assumption that $u_0(X) = 0, \ \forall X$, the Bellman equation for $V_0(X)$ is

$$V_0(X) = \beta E_{X'|X,D}\left[H(\Delta_1(X'), \ldots, \Delta_K(X')) + V_0(X')\right]. \tag{13}$$

In this equation, everything is known (including, importantly, the distribution of $X'|X, D$), except the $V_0(\cdot)$ function. Hence, by iterating over Eq. (13), we can recover the $V_0(X)$ function. Once $V_0(\cdot)$ is known, the other choice-specific value functions can be recovered as

$$V_d(X) = \Delta_d(X) + V_0(X), \ \forall y \in \mathcal{D}, \ \forall X.$$

Finally, the per-period utility functions $u_d(X)$ can be recovered from the choice-specific value functions as

$$u_d(X) = V_d(X) - \beta E_{X'|X,D}\left[H(\Delta_1(X'), \ldots, \Delta_K(X')) + V_0(X')\right], \ \forall y \in \mathcal{D}, \ \forall X,$$

where everything on the right-hand side is known.

REMARK: For the case where $F(\epsilon)$ is the Type 1 Extreme Value distribution, the social surplus function is

$$H(\Delta_1(X), \ldots, \Delta_K(X)) = \log\left[1 + \sum_{d=1}^{K} \exp(\Delta_d(X))\right]$$

and the mapping q is such that

$$q_d(X) = \Delta_d(X) = \log(p_d(X)) - \log(p_0(X)), \ \forall d = 1, \ldots K,$$

where $p_0(X) \equiv 1 - \sum_{d=1}^{K} p_d(X)$. ■

3 Model with persistence in unobservables ("unobserved state variables")

Up to now, we consider models satisfying Rust's "conditional independence" assumption on the ϵ's. This rules out persistence in unobservables, which can be economically meaningful.

In this section, we consider estimation of a more general model, in which ϵ's can be serially correlated over time: now ϵ is a nontrivial unobserved state variables (it cannot just be integrated out, as in the Eq. (7) above).

For convenience: in this section we use x to denote observed state variable, ϵ to denote unobserved state variable, and i to denote the control variable.

Now we allow ϵ to evolve as a controlled first-order Markov process, evolving according to: $F(\epsilon'|\epsilon, x, d)$.

We continue to assume, as before, that the policy function is $i(x, \epsilon)$, and that the state evolution does not depend explicitly on ϵ: next period's state evolves as in $x'|x, d$.

Given these assumption, the analogue of Eq. (4) is:

$$\begin{aligned} p(x', \epsilon'|x, \epsilon, i) &= p(\epsilon'|x', x, \epsilon, i) \cdot p(x'|x, \epsilon, i) \\ &= p(\epsilon'|x', \epsilon, i) \cdot p(x'|x, i). \end{aligned} \tag{14}$$

Now the likelihood function for a single bus:

$$\begin{aligned} & l\left(x_1, \ldots, x_T, i_t, \ldots, i_T | x_0, i_0; \theta\right) \\ = & \prod_{t=1}^{T} Prob\left(i_t, x_t | x_0, i_0, \ldots, x_{t-1}, i_{t-1}; \theta\right) \\ = & \prod_{t=1}^{T} Prob\left(i_t | x_t, x_0, i_0, \ldots, x_{t-1}, i_{t-1}; \theta\right) \times Prob\left(x_t | x_{t-1}, i_{t-1}; \theta_3\right). \end{aligned} \tag{15}$$

Note that because of the serially correlated ϵ's, there is still dependence between (say) i_t and i_{t-2}, even after conditioning on (x_{t-1}, i_{t-1}): compare the last lines of Eq. (5) and Eq. (15).

Now, because of serially correlation in the ϵ's, the $Prob\left(i_t | x_t, x_0, i_0, \ldots, x_{t-1}, i_{t-1}; \theta\right)$ no longer has a closed form. Thus, we consider simulating the likelihood function. Note that simulation is part of the "outer loop" of nested fixed point estimation routine. So at the point when we simulate, we already know the policy function $i(x, \epsilon; \theta)$ and choice-specific value functions $\tilde{V}(x, \epsilon, i; \theta)$ for each possible choice i.

3.1 "Crude" frequency simulator

Since (x, i, ϵ) all evolve together, we cannot just draw the ϵ process separately from (i, x). Thus for simulation purposes, it is most convenient to go back to the full likelihood function (the first line of Eq. (15):

$$l(x_1, \ldots, x_T, i_1, \ldots, i_T | x_0, i_0, \epsilon_0; \theta).$$

Note that because the ϵ's are serially correlated, we also need to condition on an initial value ϵ_0 (which, for simplicity, we assume $\epsilon_0 = 0$, or that it is known).

For the case where (i, x) are both discrete (as in Rust's paper), the likelihood is the joint probability

$$Pr(\epsilon_1, \ldots, \epsilon_T, X_1, \ldots, X_T : \ X_t = x_t, i(X_t, \epsilon_t; \theta) = i_t, \ \forall t = 1, \ldots, T).$$

In the above, $X_1, \ldots, X_T$ (capital letters) denotes the mileage random variable, and $x_1, \ldots, x_T$ (lower case) denotes the observed mileage in the dataset. Let $F(\epsilon' | x, i, \epsilon; \gamma)$

denote the transition density for the ϵ's, and $G(X'|X, i; \theta_3)$ the transition probability for the X's. Then the above probability can be expressed as the integral:

$$\int \cdots \int \prod_t \mathbf{1}(X_t = x_t)\mathbf{1}(i(X_t, \epsilon_t; \theta) = i_t) \prod_t dF(\epsilon_t|\epsilon_{t-1}, X_{t-1}, i(X_{t-1}, \epsilon_{t-1}; \gamma))dG(X_t|X_{t-1}, i(X_{t-1}, \epsilon_t); \theta_3).$$

We can simulate this integral by drawing sequences of (X_t, ϵ_t). For each simulation draw $s = 1, \ldots, S$, we take as initial values x_0, i_0, ϵ_0. Then:

- Generate $(\epsilon_1^s, x_1^s, i_1^s)$:
  1. Generate $\epsilon_1^s \sim F(\epsilon|x_0, i_0, \epsilon_0)$
  2. Generate $x_1^s \sim G(x|x_0, i_0)$
  3. Subsequently compute $i_1^s = i(x_1^s, \epsilon_1^s; \theta)$
- Generate $(\epsilon_2^s, x_2^s, i_2^s)$:
  1. Generate $\epsilon_2^s \sim F(\epsilon|x_1^s, i_1^s, \epsilon_1^s)$
  2. Generate $x_2^s \sim G(x|x_1^s, i_1^s)$
  3. Subsequently compute $i_2^s = i(x_2^s, \epsilon_2^s; \theta)$

... and so on, up to $(\epsilon_T^s, x_T^s, i_T^s)$

Then, for the case where (i, x) are both discrete (which is the case in Rust's paper), we can approximate

$$l\left(x_1, \ldots, x_T, i_t, \ldots, i_T|x_0, i_0; \theta\right) \approx \frac{1}{S} \sum_s \prod_{t=1}^{T} \mathbf{1}(x_t^s = x_t) \cdot \mathbf{1}(i_t^s = i_t).$$

That is, the simulated likelihood is the frequency of the simulated sequences which match the observed sequence.

Gory details This is a "crude" frequency simulator. Clearly, if T is long, or S is modest, the simulated likelihood is likely to be zero. What is commonly done in practice is to smooth the indicator functions in this simulated likelihood.

The two indicators can be smoothed in different ways. If x takes lots of different values, then $\mathbf{1}(x_t^s = x_t)$ can be smoothed as $\Lambda((x_t - x_t^s)^2/h)$, where h is a small number, and $\Lambda(s)$ denotes a decreasing function which is equal to 1 when $s = 0$, and equal to zero as $s \to +\infty$. (One example could be $2 - 2\Phi(s)$, where $\Phi(s)$ denotes the standard normal CDF.)

The indicator for $\mathbf{1}(i_t^s = i_t)$ is more complicated. In dynamic programming problems, the event of i_t implies that the choice specific value function $\tilde{V}(x_t, \epsilon_t, i_t)$ is larger than all other choices $\tilde{V}(x_t, \epsilon_t, \hat{i})$, for $\hat{i} \neq i_t$. (Recall that simulation is done in the outer loop of the estimation procedure, so that the choice-specific value functions are known when we are doing simulation.) When we simulate (i, x, ϵ), the simulated choice i_t^s denotes the choice i with the maximum simulated value function:

$$\tilde{V}(x_t^s, \epsilon_t^s, i_t^s) > \tilde{V}(x_t^s, \epsilon_t^s, \hat{i}), \text{ for } \hat{i} \neq i_t^s.$$

Hence, the event $\mathbf{1}(i_t^s = i_t)$ is equivalent to

$$\mathbf{1}(\tilde{V}(x_t^s, \epsilon_t^s, i_t) > \tilde{V}(x_t^s, \epsilon_t^s, \hat{i}), \text{ for } \hat{i} \neq i_t).$$

Define

$$\Delta_t^s \equiv V(x_t^s, \epsilon_t^s, i_t) - \max_{\hat{i} \neq i_t} V(x_t^s, \epsilon_t^s, \hat{i}).$$

Then we could smooth $\mathbf{1}(i_t = i_t^s)$ by $\kappa(\Delta_t^s)$, where $\kappa(s)$ is an increasing function which tends to zero as $s \to -\infty$, and tends to 1 as $s \to +\infty$. A good example of this is $\kappa(s) = \Phi(s)$, the standard normal CDF.

3.2 Importance sampling approach: Particle filtering

Another approach is to employ importance sampling in simulating the likelihood function. This is not straightforward, given the across-time dependence between (i_t, x_t, ϵ_t). Here, we consider a new simulation approach, called *particle filtering*. It is a recursive approach to simulate dependent sequences of random variables. The presentation here draws from Fernandez-Villaverde and Rubio-Ramirez (2007).

We need to introduce some notation, and be more specific about features of the model. Let:

- $y_t \equiv \{i_t, x_t\}$. $y^t \equiv \{y_t, \ldots, y_t\}$. These are the observed sequences in the data.

- Evolution of utility shocks: $\epsilon_t|\epsilon_{t-1} \sim f(\epsilon'|\epsilon)$. Eliminate dependence of ϵ_t on x_t, i_t.

- Evolution of mileage: $x_t = h(x_{t-1}, i_{t-1}, w_t)$. Mileage x_t evolves deterministically, according to the h function. w_t is a new "mileage shock", which is assumed to be distributed iid over time, according to density $g(w)$. An example of $h(\cdots)$ is

$$x_t = \begin{cases} x_{t-1} + w_t & \text{if } i_{t-1} = 0 \\ w_t & \text{otherwise.} \end{cases}$$

  Assume $h(\cdots)$ is invertible in third argument, and define $\phi(x_{t-1}, i_{t-1}, x_t)$ as this inverse function, satisfying

$$x_t = h(x_{t-1}, i_{t-1}, \phi(x_{t-1}, i_{t-1}, x_t)).$$

- As before, the policy function is $i_t = i(x_t, \epsilon_t)$.

- Let $\epsilon^t \equiv \{\epsilon_1, \ldots, \epsilon_t\}$.

- The initial values y_0 and ϵ_0 are known.

Go back to the factorized likelihood:

$$\begin{aligned} l(y^T|y_0, \epsilon) &= \prod_{t=1}^{T} l(y_t|y^{t-1}, y_0, \epsilon_0) \\ &= \prod_t \int l(y_t|\epsilon^t, y^{t-1}) p(\epsilon^t|y^{t-1}) d\epsilon^t \\ &\approx \prod_t \frac{1}{S} \sum_s l(y_t|\epsilon^{t|t-1,s}, y^{t-1}) \end{aligned} \tag{16}$$

where in the second to last line, we omit conditioning on (y_0, ϵ_0) for convenience. In the last line, $\epsilon^{t|t-1,s}$ denotes simulated draws of ϵ^t from $p(\epsilon^t|y^{t-1})$.

Consider the two terms in the last line:

- The first term:

$$\begin{aligned} l(y_t|\epsilon^t, y^{t-1}) &= p(i_t, x_t|\epsilon^t, y^{t-1}) \\ &= p(i_t|x_t, \epsilon^t, y^{t-1}) \cdot p(x_t|\epsilon^t, y^{t-1}) \\ &= p(i_t|x_t, \epsilon_t) \cdot p(x_t|i_{t-1}, x_{t-1}) \\ &= \mathbf{1}(i(x_t, \epsilon_t) = i_t) \cdot g(\phi(x_{t-1}, i_{t-1}, x_t)) \cdot \frac{\partial h}{\partial w}\Big|_{w=\phi(x_{t-1}, i_{t-1}, x_t)} \end{aligned} \tag{17}$$

  where the last two terms in the last line is the Jacobian of the transformation from w to x. This term can be explicitly calculated, for a given value of ϵ_t.

- The second term is, generally, not obtainable in closed form. So numerical integration not feasible. The particle filtering algorithm permits us to draw sequences of ϵ^t from $p(\epsilon^t|y^{t-1})$, for every period t. Hence, the last line of (16) can be approximated by simulation.

Particle filtering proposes a recursive approach to draw sequences from $p(\epsilon^t|y^{t-1})$, for every t. Easiest way to proceed is just to describe the algorithm.

First period, $t = 1$: In order to simulate the integral corresponding to the first period, we need to draw from $p(\epsilon^1|y^0, \epsilon_0)$. This is easy. We draw $\left\{\epsilon^{1|0,s}\right\}_{s=1}^{S}$, according to $f(\epsilon'|\epsilon_0)$. The notation $\epsilon^{1|0,s}$ makes explicitly that the ϵ is a draw from $p(\epsilon^1|y^0, \epsilon_0)$. Using these S draws, we can evaluate the simulated likelihood for period 1, in Eq. (16). We are done for period $t = 1$.

Second period, $t = 2$: We need to draw from $p(\epsilon^2|y^1)$. Factorize this as:

$$p(\epsilon^2|y^1) = p(\epsilon^1|y^1) \cdot p(\epsilon_2|\epsilon^1). \tag{18}$$

Recall our notation that $\epsilon^2 \equiv \{\epsilon_1, \epsilon_2\}$. Consider simulating from each term separately:

- Getting a draw from $p(\epsilon^1|y^1)$, given that we already have draws $\left\{\epsilon^{1|0,s}\right\}$ from $p(\epsilon^1|y_0)$, from the previous period $t = 1$, is the heart of particle filtering.

  We use the principle of importance sampling: by Bayes' Rule,

$$p(\epsilon^1|y^1) \propto p(y_1|\epsilon^1, y^0) \cdot p(\epsilon^1|y^0)$$

Hence, if our desired sampling density is $p(\epsilon^1|y^1)$, but we actually have draws $\left\{\epsilon^{1|0,s}\right\}$ from $p(\epsilon^1|y^0)$, then the importance sampling weight for the draw $\epsilon^{1|0,s}$ is proportional to $\tau_1^s \equiv p(y_1|\epsilon^{1|0,s}, y^0)$.

The SIR (Sampling/Importance Resampling) algorithm in Rubin (1988) proposes that, making S draws with replacement from the samples $\left\{\epsilon^{1|0,s}\right\}_{s=1}^S$, using weights proportional to τ_1^s, yields draws from the desired density $p(\epsilon^1|y^1)$, which we denote $\{\epsilon^{1,s}\}_{s=1}^S$. This is the **filtering** step.

- For the second term in Eq. (18): we simply draw one ϵ_2^s from $f(\epsilon'|\epsilon^{1,s})$, for each draw $\epsilon^{1,s}$ from the filtering step. This is the **prediction** step.

By combining the draws from these two terms, we have $\left\{\epsilon^{2|1,s}\right\}_{s=1}^S$, which is S drawn sequences from $p(\epsilon^2|y^1)$. Using these S draws, we can evaluate the simulated likelihood for period 2, in Eq. (16).

Third period, $t = 3$: start again by factoring

$$p(\epsilon^3|y^2) = p(\epsilon^2|y^2) \cdot p(\epsilon_3|\epsilon^2). \tag{19}$$

As above, drawing from the first term requires filtering the draws $\left\{\epsilon^{2|1,s}\right\}_{s=1}^S$, from the previous period $t = 2$, to obtain draws $\{\epsilon^{2,s}\}_{s=1}^S$. Given these draws, draw $\epsilon_3^s \sim f(\epsilon'|\epsilon^{2,s})$ for each s.

And so on. By the last period $t = T$, you have

$$\left\{\left\{\epsilon^{t|t-1,s}\right\}_{s=1}^S\right\}_{t=1}^T.$$

Hence, the factorized likelihood in Eq. (16) can be approximated by simulation as

$$\prod_t \frac{1}{S}\sum_s l(y_t|\epsilon^{t|t-1,s}, y^{t-1}).$$

Note the difference between this recursive simulator, and the crude simulator described previously. The crude simulator draws S sequences, and essentially assigns zero weight to all sequences which do not match the observed sequence in the data. In contrast, in particle filtering, in each period t, we just keep sequences where predicted

choices match observed choice *that period*. This will lead to more accurate evaluation of the likelihood. Note that S should be large enough (relative to the sequence length T) so that the filtering step does not end up assigning almost all weight to one particular sequence $\epsilon^{t|t-1,s}$ in any period t.

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