Replication of Buera & Shin (2013) Implementation with Howard Policy Improvement and Histograms

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1 The Economic Environment

The economy consists of a continuum of infinitely lived agents of measure one. Agents are heterogeneous in their asset holdings a and entrepreneurial ability z.

1.1 Preferences and Shocks

Agents maximize expected lifetime utility:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{1}$$

Entrepreneurial ability z follows a reset process. In each period, with probability ψ, the ability remains constant: $z_{t+1} = z_t$. With probability $1-\psi$, a new ability is drawn $z' \sim \pi(z)$ from a Pareto distribution:

$$P(Z \leq z) = 1 - z^{-\eta}, \quad z \in [1, \infty) \tag{2}$$

To simulate distortions in the pre-reform state, agents are subject to idiosyncratic output wedges $\tau \in \{\tau^+, \tau^-\}$. The probability of being in the high-tax state τ^+ is conditional on z: $P(\tau = \tau^+|z) = 1 - e^{-qz}$.

2 Firm Decisions and Optimal Production

Entrepreneurs operate a technology $y = z(k^\alpha l^{1-\alpha})^{1-\nu}$. They face a collateral constraint $k \leq \lambda a$.

2.1 Analytical Solutions for Firm Decisions

Given (w, r, a, z), the entrepreneur's problem is solved in two stages. 1. **Conditional Labor Demand**: For any k, the optimal labor $l^*(k, z)$ solves the first-order condition $\partial\pi/\partial l = 0$:

$$l^*(k, z) = \left[\frac{(1-\nu)(1-\alpha)z}{w}\right]^{\frac{1}{1-(1-\alpha)(1-\nu)}} k^{\frac{\alpha(1-\nu)}{1-(1-\alpha)(1-\nu)}} \tag{3}$$

2. **Optimal Capital**: Substituting l^* into the profit function yields a concave function $\pi(k)$. The unconstrained optimal capital k^{unc} is found where $\partial\pi/\partial k = 0$. The actual capital choice is capped by the leverage limit:

$$k^*(a, z) = \min\{k^{unc}(z, w, r), \lambda a\} \tag{4}$$

The total income of an agent is $i(a, z; w, r) = \max\{\pi(a, z, k^*, l^*), w\} + (1 + r)a$.

3 State Space Discretization

The state space $\mathcal{A} \times \mathcal{Z}$ is discretized for numerical implementation.

3.1 Asset Grid

To capture the high curvature of the value function near the borrowing constraint, a power-spaced grid for $a \in [a_{min}, a_{max}]$ is used:

$$a_i = a_{min} + (a_{max} - a_{min}) \left(\frac{i}{N_a - 1} \right)^p, \quad i \in \{0, \dots, N_a - 1\} \tag{5}$$

where $p = 2$ provides a dense cluster of points at low asset levels.

3.2 Ability Grid

The Pareto distribution $\pi(z)$ is sampled into N_z bins of equal probability. Let u be a uniform random variable. The grid points are midpoints of probability intervals $[F(z_j), F(z_{j+1})]$ where F is the Pareto CDF:

$$z_j = (1 - \bar{u}_j)^{-1/\eta}, \quad \bar{u}_j = \frac{j + 0.5}{N_z} \cdot u_{max} \tag{6}$$

4 Numerical Value Function Iteration

The value function $V(a, z)$ is solved on the discretized grid using an optimized iterative process.

4.1 Initialization

The initial guess $V^{(0)}$ is set to the utility of consuming the current income forever at the lowest grid point, providing a stable starting point for the iteration:

$$V^{(0)}(a_i, z_j) = \frac{u(i(a_i, z_j) - a_0)}{1 - \beta} \tag{7}$$

4.2 Expectation Splitting

To compute the continuation value $\mathbb{E}[V(a', z')|z]$, the algorithm splits the expectation to exploit the discrete nature of the reset process. Before making decisions, the "reset" expectation $\bar{V}(a')$ is pre-calculated:

$$\bar{V}(a') = \sum_j \pi_j V(a', z_j) \tag{8}$$

The specific continuation value for state (a, z) becomes a weighted average of the current ability's value and the reset value:

$$\text{Cont}(a', z) = \psi V(a', z) + (1 - \psi)\bar{V}(a') \tag{9}$$

4.3 Howard Policy Improvement

The Bellman operator $T(V)$ is computationally expensive due to the maximization step. Howard's improvement reduces the number of maximizations by iterating on a fixed policy $\sigma(a, z)$.

1. **Policy Step**: Solve for $\sigma(a, z) = \arg\max_{a'} \{u(c) + \beta \text{Cont}(a', z)\}$.
2. **Value Step**: For a fixed σ, iterate N_{how} times:

$$V^{(m+1)}(a, z) = u(i(a, z) - \sigma(a, z)) + \beta[\psi V^{(m)}(\sigma(a, z), z) + (1 - \psi)\bar{V}^{(m)}(\sigma(a, z))] \tag{10}$$

This significantly speeds up convergence as the Value Step is a linear operation[1].

[1] In principle, the value of a fixed policy σ can be solved in closed form as $V_\sigma = (I - \beta P_\sigma)^{-1} U_\sigma$, where P_σ is the transition matrix of the state space. However, for a grid size of $N_a N_z \approx 36,000$, the memory and computational overhead of inverting a P_σ matrix with 1.3×10^9 elements makes the iterative successive approximation approach more efficient in terms of CPU cache and memory.

4.4 Possible Improvements: Continuous Choice via Interpolation

A significant enhancement to the discrete grid search is to allow for a continuous asset choice $a' \in [a_{min}, i]$. This is implemented by replacing the discrete maximization with a continuous optimizer and interpolating the continuation value:

$$a^*(a,z) = \arg \max_{a' \in [a_{min}, i]} \{u(i - a') + \beta \mathcal{I}(\text{Cont}, a', z)\} \tag{11}$$

where $\mathcal{I}$ is an interpolating function (e.g., linear or cubic spline). Mathematically, this smooths the aggregate supply and demand functions $K_d(r)$ and $L_d(w)$, eliminating the "step" artifacts inherent in discrete grids and improving the convergence of the bisection algorithms in the general equilibrium solver.

4.5 Coupled VFI for Distortions

In the pre-reform state, the economy features idiosyncratic distortions $\tau \in \{\tau^+, \tau^-\}$. The solver maintains two coupled value functions $V^+(a,z)$ and $V^-(a,z)$. The expectation $\bar{V}(a')$ now accounts for the probability $p(z) = P(\tau = \tau^+ | z)$:

$$\bar{V}(a') = \sum_j \pi_j \left[p(z_j) V^+(a', z_j) + (1 - p(z_j)) V^-(a', z_j) \right] \tag{12}$$

The decision for an agent in state (a, z, τ) depends on the specific income $i(a, z, \tau)$, but the continuation value uses the shared $\bar{V}(a')$ since τ is redrawn whenever z resets.

5 Distributional and Transition Dynamics

5.1 Monte Carlo Binned Distribution

The distribution $\mu_t(a, z)$ is tracked by simulating N_{sim} agents. At any time t, agent j has assets a_j. The mass is mapped to the asset grid via linear interpolation weights:

$$\omega_{k+1} = \frac{a_j - a_k}{a_{k+1} - a_k}, \quad \omega_k = 1 - \omega_{k+1} \tag{13}$$

This allows for calculating aggregate labor demand $L_{d,t}$ and capital demand $K_{d,t}$ as smooth functions of prices:

$$K_{d,t}(w, r) = \sum_{i,j} \mu_t(a_i, z_j) k^*(a_i, z_j, w, r) \mathbf{1}_{\{\pi > w\}} \tag{14}$$

6 Algorithm B.2: Nested Price Path Iteration

The general equilibrium transition path is computed via Algorithm B.2 (Appendix B.2), which decomposes the market clearing of capital and labor into nested price-path sequences.

6.1 Price Path Sequences

Let $\boldsymbol{w} = \{w_t\}_{t=0}^T$ and $\boldsymbol{r} = \{r_t\}_{t=0}^T$ be the sequences of wages and interest rates. The algorithm solves for fixed points of these sequences such that markets clear at every t.

6.2 Nested Iteration Structure

The implementation uses a nested loop structure:

1. **Outer Loop (Interest Rates)**: Updates the guess for the interest rate path $\boldsymbol{r}^{(n)}$.
    (a) **Inner Loop (Wages)**: Given $\boldsymbol{r}^{(n)}$, iteratively updates the wage path $\boldsymbol{w}^{(k)}$.
        i. **Backward Induction**: Solve for policies $a'_t(a, z)$ given $(w_t^{(k)}, r_t^{(n)})$ for $t = T-1, \ldots, 0$.
        ii. **Forward Simulation**: Simulate N_{sim} agents forward from μ_0 using a'_t to obtain the sequence of binned distributions $\{\mu_t\}_{t=0}^T$.

iii. **Labor Clearing**: For each t, find the wage w_t^* that clears the labor market *given the fixed distribution* μ_t:

$$L_{excess}(w_t^*|\mu_t, r_t^{(n)}) = 0 \tag{15}$$

iv. **Wage Relaxation**: Update the wage path using damping η_w:

$$w_t^{(k+1)} = \eta_w w_t^* + (1 - \eta_w) w_t^{(k)} \tag{16}$$

(b) **Capital Clearing**: For each t, find the interest rate r_t^* that clears the capital market given the converged wage and distribution:

$$K_{excess}(r_t^*|\mu_t, w_t^{(converged)}) = 0 \tag{17}$$

(c) **Interest Rate Relaxation**: Update the path using damping η_r:

$$r_t^{(n+1)} = \eta_r r_t^* + (1 - \eta_r) r_t^{(n)} \tag{18}$$

6.3 Numerical Market Clearing (Bisections)

The clearing prices w_t^* and r_t^* are found using a robust root-finding method:

- **Bracketing Scan**: Before bisecting, the price space $[P_{min}, P_{max}]$ is scanned at N_{scan} points to find a bracket $[a, b]$ where the excess demand function changes sign.
- **Bisection**: Once a bracket is found, a bisection method converges to the specific clearing price with a tolerance of 10^{-10}.

6.4 Convergence Criteria

The algorithm requires two conditions to stop:

1. **Sequence Convergence**: The maximum change in price paths between iterations must be small: $\max_t |r_t^{(n+1)} - r_t^{(n)}| < \epsilon_{seq}$.
2. **Market Clearing Gaps**: The actual excess demand residuals (calculated by re-evaluating the economy with the final prices) must satisfy $|ED_L| < \epsilon_{ED}$ and $|ED_K| < \epsilon_{ED}$.