

Data Generation

Backward Recursion

Let's start from the last period, we assume that there is no scrap value in the dynamic problem, so $V_{T+1}(s_{t+1}, a_{t+1}) = 0$ for all $s \in S$ and $a \in \{0, 1\}$. We have:

$$\begin{aligned}\pi_T(s_T, a_T) &= a_T [-R - \nu(1)] + (1 - a_T) [-\mu \cdot s - \nu(0)] \\ V_T(s_T, a_T) &= \max_{a_T} [-R - \nu(1)] + (1 - a_T) [\mu \cdot s - \nu(0)] + \beta \mathbb{E} \left(\underbrace{V_{T+1}(s_{T+1}, a_{T+1})}_{=0} \right) \\ V_T(s_T, a_T) &= \max_{a_T} \pi_T(s_T, a_T)\end{aligned}$$

Clearly given the discrete nature of the problem, the optimal problem has the following form:

$$\begin{aligned}\sigma_T(s_T, \nu) = 1 &\iff -R - \nu_T(1) \geq -\mu \cdot s_T - \nu_T(0) \\ &\iff \mu \cdot s_T - R \geq \nu_T(1) - \nu_T(0)\end{aligned}$$

and given that $\nu(a) \sim N(0, 1)$ for $a \in \{0, 1\}$ then $\nu_T(1) - \nu_T(0) \sim N(\mu \cdot s - R, 2) \Rightarrow \Pr(a = 1|s) = 1 \Rightarrow \Phi\left(\frac{\mu \cdot s_T - R}{\sqrt{2}}\right)$.

At time $T - 1$ the problem starts being dynamic, given that the action today determines the state tomorrow:

$$V_{T-1}(s_{T-1}, a_{T-1}) = \max_{a_{T-1}} a_{T-1} [-R - \nu(1)_{T-1}] + (1 - a_{T-1}) [\mu \cdot s_{T-1} - \nu_{T-1}(0)] + \beta \mathbb{E} [V_T(s_T, a_T)]$$

and the transition is given by $s_T = 1$ if $a_{T-1} = 1$ and $s_T = \min\{s_{T-1} + 1, S^{max}\}$. Define $\tilde{s}_t \equiv \min\{s_{t-1} + 1, S^{max}\}$

$$\begin{aligned}v_{T-1}(s_{T-1}, 1) &= -R + \beta \mathbb{E} (V_T(1, a_T)) \\ &= -R + \beta \left(\Phi\left(\frac{\mu - R}{\sqrt{2}}\right) \cdot (-R) + \left(1 - \Phi\left(\frac{\mu - R}{\sqrt{2}}\right)\right) \cdot (-\mu) \right) \\ &= -R + \beta \left[\Phi\left(\frac{\mu - R}{\sqrt{2}}\right) [\mu - R] \right] \\ v_{T-1}(s_{T-1}, 0) &= -\mu \cdot s_{T-1} + \beta \mathbb{E} (V_T(\tilde{s}_{T-1}, a_T)) \\ &= -\mu \cdot s_{T-1} + \beta \left(\Phi\left(\frac{\mu \cdot \tilde{s}_T - R}{\sqrt{2}}\right) \cdot (-R) + \left(1 - \Phi\left(\frac{\mu \cdot \tilde{s}_T - R}{\sqrt{2}}\right)\right) \cdot (-\mu \cdot \tilde{s}_T) \right)\end{aligned}$$

Again given the unobserved shocks, the optimal decision rule at time $T - 1$ can be written as:

$$V_{T-1}(s_{T-1}, a_{T-1}) = \max_{a_{T-1}} (v_{T-1}(s_{T-1}, 1) + \nu(1), v_{T-1}(s_{T-1}, 0) + \nu(0))$$

which gives the decision rule:

$$\begin{aligned}\sigma_{T-1}(s_{T-1}, \nu_{T-1}) = 1 &\iff v_{T-1}(s_{T-1}, 1) + \nu(1) \geq v_{T-1}(s_{T-1}, 0) + \nu(0) \\ &\iff v_{T-1}(s_{T-1}, 1) - v_{T-1}(s_{T-1}, 0) \geq \nu(0) - \nu(1)\end{aligned}$$

so that

$$\Pr(\sigma_{T-1}(s_{T-1}, \nu) = 1) = \Phi\left(\frac{v_{T-1}(s_{T-1}, 1) - v_{T-1}(s_{T-1}, 0)}{\sqrt{2}}\right)$$

To ease the notation we omit the state variables when referring to the optimal policy. Notice that $\Pr(\sigma_{T-1} = 1)$ is of dimension $|S| \times 1$. Hence, recursively, we can define a matrix containing optimal cutoff rules. Assume we want to generate the data for T periods¹, then we define Σ as the matrix of dimension $T \times S$ containing the cutoff rules for every period:

$$\Sigma = \begin{bmatrix} \Pr(\sigma_T = 1)' \\ \Pr(\sigma_{T-1} = 1)' \\ \vdots \\ \Pr(\sigma_0 = 1)' \end{bmatrix} = \begin{bmatrix} \Phi\left(\frac{v_{T-1}(s_1, 1) - v_{T-1}(s_1, 0)}{\sqrt{2}}\right) & \dots & \Phi\left(\frac{v_{T-1}(s_5, 1) - v_{T-1}(s_5, 0)}{\sqrt{2}}\right) \\ \vdots & \dots & \vdots \\ \Phi\left(\frac{v_0(s_1, 1) - v_0(s_1, 0)}{\sqrt{2}}\right) & \dots & \Phi\left(\frac{v_0(s_5, 1) - v_0(s_5, 0)}{\sqrt{2}}\right) \end{bmatrix}$$

Forward Data Simulation

Finally, we can simulate forward N observations so that the final artificial dataset will be of dimension $N \times T$. The algorithm is the following, $\forall i \in N$:

1. Draw an initial $s_{i,0} \in S$ using a random number generator.
2. Draw a random number between 0 and 1: $p_{i,0} \sim \mathcal{U}([0, 1])$.
3. If $p_{i,0} \leq \Pr(\sigma_0 = 1, s_{i,0})$ then $a_{i,0} = 1$, otherwise $a_{i,0} = 0$.
4. If $a_{i,0} = 1 \Rightarrow s_{i,1} = 1$, otherwise $s_{i,1} = \min\{s_{i,0} + 1, S^{max}\}$.
5. Repeat steps 2-4 for T periods.

In this way, we can generate a dataset with N buses observed for T periods.

¹Assume also that $S = \{1, 2, 3, 4, 5\}$ so that $S^{max} = 5$.