Mathematical Foundations of the Buera & Shin (2013) Spectral Time Iteration Implementation

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1 The Economic Environment

The economy is populated by a continuum of agents who choose to be either workers or entrepreneurs. Agents are heterogeneous in their asset holdings a and entrepreneurial ability z.

1.1 Preferences and Shocks

Agents maximize expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma} \tag{1}$$

Entrepreneurial ability z follows a reset process: with probability ψ, $z_{t+1} = z_t$; with probability $1 - \psi$, a new ability is drawn $z' \sim \pi(z)$ from a Pareto distribution with tail parameter η. In the pre-reform state, agents also face idiosyncratic output wedges $\tau \in \{\tau^+, \tau^-\}$, where $\mathbb{P}(\tau = \tau^+|z) = 1 - e^{-qz}$.

2 Production and Optimal Firm Decisions

An entrepreneur with ability z and assets a operates the technology:

$$y = z(k^\alpha l^{1-\alpha})^{1-\nu} \tag{2}$$

where ν represents the span-of-control parameter (entrepreneur's share). The entrepreneur faces a collateral constraint $k \leq \lambda a$.

2.1 Static Profit Maximization

The entrepreneur's problem is solved in two stages. First, given capital k, the optimal labor $l^*(k, z)$ is found by solving:

$$\max_l \{z(k^\alpha l^{1-\alpha})^{1-\nu} - wl - (r + \delta)k\} \tag{3}$$

The first-order condition with respect to l is:

$$(1 - \nu)(1 - \alpha)zk^{\alpha(1-\nu)} l^{(1-\alpha)(1-\nu)-1} = w \tag{4}$$

Solving for l yields the conditional labor demand:

$$l^*(k, z) = \left[\frac{(1 - \nu)(1 - \alpha)z}{w}\right]^{\frac{1}{1-(1-\alpha)(1-\nu)}} k^{\frac{\alpha(1-\nu)}{1-(1-\alpha)(1-\nu)}} \tag{5}$$

Substituting $l^*(k, z)$ back into the profit function gives a profit function $\pi(k)$ that is concave in k. The unconstrained capital demand k^{unc} satisfies $\pi'(k^{unc}) = r + \delta$. If $k^{unc} > \lambda a$, the entrepreneur is borrowing-constrained and sets $k^* = \lambda a$.

3 The Recursive Household Problem

The agent's state is (a, z). The total income is $y(a, z) = \max\{\pi(a, z, k^*), w\} + (1 + r)a$. The Bellman equation is:

$$V(a, z) = \max_{a'} \left\{ \frac{(y(a, z) - a')^{1-\sigma} - 1}{1 - \sigma} + \beta \text{Cont}(a', z) \right\} \tag{6}$$

where the continuation value is:

$$\text{Cont}(a', z) = \psi V(a', z) + (1 - \psi) \int V(a', z')\pi(z')dz' \tag{7}$$

3.1 The Euler Equation

The first-order condition for the optimal savings a' (assuming an interior solution) is given by the Euler equation:

$$c_t^{-\sigma} = \beta(1 + r)\mathbb{E}_t[c_{t+1}^{-\sigma}] \tag{8}$$

Defining marginal utility as $MU(a, z) = (y(a, z) - a'(a, z))^{-\sigma}$, the equation becomes:

$$MU(a, z) = \beta(1 + r) \left[\psi MU(a', z) + (1 - \psi) \int MU(a', z')\pi(z')dz' \right] \tag{9}$$

4 Spectral Time Iteration Logic

The policy function $a'(a, z)$ is approximated by a bivariate Chebyshev polynomial $\hat{a}(a, z; \boldsymbol{C})$.

4.1 Detailed First-Order Condition in Spectral Form

Let $\boldsymbol{\Phi}(a, z)$ be the vector of bivariate Chebyshev basis functions, such that the savings policy is $\hat{a}(a, z; \boldsymbol{C}) = \boldsymbol{\Phi}(a, z)^\top \boldsymbol{C}$. The household's optimal choice a' satisfies the following first-order condition:

$$F(a', a, z; \boldsymbol{C}) \equiv [y(a, z) - a']^{-\sigma} - \beta(1 + r) \left[\psi \text{MU}' + (1 - \psi)\mathbb{E}_{z'}\text{MU}' \right] = 0 \tag{10}$$

where the next-period marginal utility MU' is evaluated using the **known** coefficients $\boldsymbol{C}$ from the previous iteration:

$$\text{MU}'(a', z'; \boldsymbol{C}) = \left[y(a', z') - \boldsymbol{\Phi}(a', z')^\top \boldsymbol{C} \right]^{-\sigma} \tag{11}$$

In this formulation, a' is the target savings level for the current period (the variable we solve for), while $\boldsymbol{C}$ defines the behavior of the agent from $t+1$ onwards. The spectral method is powerful because $\boldsymbol{\Phi}(a', z')$ and its derivatives are defined for any $a' \in [a_{min}, a_{max}]$, allowing us to solve $F(a') = 0$ using continuous root-finding methods (e.g., Brent's method) or by direct inversion.

4.2 Grid and Mapping

Chebyshev polynomials are defined on $x \in [-1, 1]$. We map the state space:

1. **Assets**: $x_a = 2\frac{\ln(a+\text{shift}) - \ln(a_{min}+\text{shift})}{\ln(a_{max}+\text{shift}) - \ln(a_{min}+\text{shift})} - 1$
2. **Ability**: $x_z = 2\frac{z - z_{min}}{z_{max} - z_{min}} - 1$

We choose N_a and N_z nodes in each dimension, resulting in $N = N_a \times N_z$ collocation points.

4.3 The Iterative Step (Time Iteration)

Given a current set of coefficients $\boldsymbol{C}^{(n)}$:

1. **Expectation Integration**: For each Chebyshev node (a_i, z_j), compute the next-period asset $a' = \hat{a}(a_i, z_j; \boldsymbol{C}^{(n)})$.
2. **Marginal Utility at** $t+1$: Evaluate $MU(a', z_k; \boldsymbol{C}^{(n)})$ for all z_k in a quadrature grid. The integral is computed as:

$$\mathbb{E}[MU] = (1 - \psi) \sum_k w_k MU(a', z_k; \boldsymbol{C}^{(n)}) + \psi MU(a', z_j; \boldsymbol{C}^{(n)}) \tag{12}$$

3. **Invert Euler Equation**: Find the target consumption $c^* = [\beta(1+r)\mathbb{E}[MU]]^{-1/\sigma}$.
4. **Compute New Policy**: The target savings is $a^*_{target} = y(a_i, z_j) - c^*$. We apply the constraint $a' = \max\{a_{min}, a^*_{target}\}$.
5. **Update Coefficients**: Solve the linear system $\boldsymbol{TC}^{(n+1)} = \boldsymbol{a}^*$, where $\boldsymbol{T}$ is the matrix of Chebyshev polynomials evaluated at the nodes.

4.4 Coefficient Transformation

The coefficients $\boldsymbol{C}$ are related to the values at the collocation nodes $\boldsymbol{A}^*$ via the basis matrix $\boldsymbol{T}$. Since the bivariate basis is a Kronecker product of the univariate bases $\boldsymbol{T} = \boldsymbol{T}_z \otimes \boldsymbol{T}_a$, the update can be written as:

$$\boldsymbol{C}^{(n+1)} = (\boldsymbol{T}_z^{-1} \otimes \boldsymbol{T}_a^{-1})\boldsymbol{A}^* \tag{13}$$

In practice, this is solved efficiently using matrix inversions of the smaller univariate matrices.

5 Coupled Spectral Solver for Distorted Economies

In the pre-reform state, agents face idiosyncratic distortions $\tau \in \{\tau^+, \tau^-\}$. The implementation maintains two distinct policy approximations $\hat{a}^+(a,z)$ and $\hat{a}^-(a,z)$.

5.1 Coupled Euler Equations

The Euler equations for the two types are coupled through the ability reset process:

$$MU^\tau(a,z) = \beta(1+r)\left[\psi MU^\tau(\hat{a}^\tau, z) + (1-\psi)\mathbb{E}_{z'}[\hat{p}(z')MU^+(\hat{a}^\tau, z') + (1-\hat{p}(z'))MU^-(\hat{a}^\tau, z')]\right] \tag{14}$$

where $\hat{p}(z')$ is the probability of drawing the τ^+ distortion conditional on new ability z'. This requires a joint iteration on both sets of coefficients until $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$ stabilize.

6 Distributional and Transition Dynamics

The distribution is tracked on a dense grid using a Markov transition matrix Q derived from the continuous policy $\hat{a}$.

$$\mu_{t+1}(a', z') = \sum_{a,z} \mathbb{P}(a'|a,z)\mathbb{P}(z'|z)\mu_t(a,z) \tag{15}$$

Aggregate labor L_t and capital K_t are then calculated by summing firm decisions over the distribution μ_t.

7 Convergence and General Equilibrium

The general equilibrium wage w and interest rate r are found by iterating on the price paths using Algorithm B.2. The spectral method's continuity ensures that the excess demand functions $ED_L(w,r)$ and $ED_K(w,r)$ are smooth, allowing for robust bisection even with a relatively small number of spectral nodes.