



**Master of Science
Sustainable Management and Technology**

SUSTAINABLE AND ENTREPRENEURIAL FINANCE

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SUSTAINABLE FINANCE

Lecture 2: Fundamentals of Asset Management

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Objectives of the lecture

This lecture is an introduction to **Investment** (or Asset Management or Portfolio Management).

- Presentation of the major steps in **asset management**:
 - 1- Optimal portfolio allocation
 - 2- Capital Asset Pricing Model (CAPM)
 - 3- Market efficiency
- How to measure **inputs** of the allocation process?

Objectives of the lecture

Readings

Fama, E.F. (1965), The Behavior of Stock-Market Prices, *Journal of Business*, 38(1), 34-105.

Lintner, J. (1965), “The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets”, *Review of Economics and Statistics*, 47(1), 13-37.

Markowitz, H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), 77-91.

Sharpe, W.F. (1964), “Capital asset prices: A Theory of Market Equilibrium under Conditions of Risk”, *Journal of Finance*, 19(3), 425-442

Tobin, J. (1958), “Liquidity Preference as Behavior Towards Risk”, *Review of Economic Studies*, 25.1(2), 65-86

Objectives of the lecture

➔ **Optimal Allocation**

- **Capital Asset Pricing Model**
- **Market Efficiency**
- **Predicting Risk Premia and the Covariance Matrix**

Major Steps in Asset Management

1- **Optimal portfolio allocation** (Markowitz, 1952): The optimization problem reduces to a simple mean-variance criterion.

2- **Capital Asset Pricing Model (CAPM)** (Sharpe, 1964, Lintner, 1965): At equilibrium, there is a unique factor explaining cross-section differences between firms' expected return, i.e., their **beta**

What if everyone does the same thing?

3- **Market efficiency** (Fama, 1965): If markets are efficient, abnormal returns should be unpredictable.

What does it mean if markets are efficient?

We now discuss these 3 steps.

Notations with Many Risky Securities

Assume that we have N risky securities.

Securities returns are denoted by: $R = (R_1, \dots, R_N)'$.

Expected returns are denoted by: $E[R] = \overset{\text{expected return}}{\mu} = (\mu_1, \dots, \mu_N)'$.

The covariance matrix is denoted by: $\Sigma = E[(R - \mu)(R - \mu)'] = \begin{pmatrix} \sigma_1^2 & \dots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{1N} & \dots & \sigma_N^2 \end{pmatrix}$

Portfolio weights are denoted by: $\alpha = (\alpha_1, \dots, \alpha_N)'$, with $\alpha'e = \sum_{i=1}^N \alpha_i = 1$
alpha = portefolio

Portfolio return is denoted by: $R_P = \alpha'R = \sum_{i=1}^N \alpha_i R_i$

Portfolio expected return is denoted by: $E[R_P] = \mu_P = \alpha'\mu = \sum_{i=1}^N \alpha_i \mu_i$
exante: expected return
expost: realized return
(end of period)

v=variance
Portfolio variance is denoted by: $V[R_P] = \sigma_P^2 = \alpha'\Sigma\alpha = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \sigma_{ij}$

Mean-Variance case (Markowitz, 1952)

On the efficient frontier, there are many portfolios. Are we indifferent between all these portfolios? It depends on our **risk aversion** and therefore on how we trade-off risk and return.

We denote by λ the risk aversion parameter

This allows us to define the mean-variance criterion (Markowitz, 1952):

The optimal portfolio weight α^* is the solution of the problem

$$\begin{cases} \max_{\alpha} \mu_P - \frac{\lambda}{2} \sigma_P^2 = \alpha' \mu - \frac{\lambda}{2} \alpha' \Sigma \alpha & \text{subtract the risk} \\ \text{s.t. } \alpha' e = 1 & (\gamma: \text{Lagrange multiplier}) \end{cases}$$

additional restriction

The Lagrangian is: $\mathcal{L} = \alpha' \mu - \frac{\lambda}{2} \alpha' \Sigma \alpha - \gamma(\alpha' e - 1)$

e=vector of 1

The derivatives are:

gamma=unknown
lambda=known
(need to compute)

alpha prime = transposed

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \mu - \lambda \Sigma \alpha - \gamma e = 0 \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \gamma} = \alpha' e - 1 = 0$$

Mean-Variance case (Markowitz)

Proposition: When there is no risk-free asset and no restriction on portfolio weights, the weights of the **optimal portfolio** are:

First part: only depends on risk

"speculative portefolio"

$$\alpha^* = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}e\right) = \alpha_{\text{gmV}}^* + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}e\right)$$

global min variance port

where α_{gmV} is independent from expected returns μ . Rearranging terms, we obtain:

$$\alpha^* = \left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha_{\text{gmV}}^* + \left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha_{\text{spec}}^*$$

We obtain the well-known **Mutual Fund Separation Theorem**. Investors invest in: y=return

- **the minimum-variance portfolio:**

$$\alpha_{\text{gmV}}^* = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} \text{ with weight } \left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)$$

the two portefolios does
not depend on us

lambda drives the wheights

- **a speculative portfolio:**

$$\alpha_{\text{spec}}^* = \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu} \text{ with weight } \left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)$$

x=volatility

Mean-Variance case (Markowitz)

Alternative derivations

An investor minimizing the portfolio variance subject to a return constraint or maximizing the portfolio return subject to a volatility constraint will find the same solution (with no particular value for the risk aversion parameter):

$$\left\{ \begin{array}{l} \min_{\alpha} \sigma_P^2 = \alpha' \Sigma \alpha \\ \text{s.t. } \mu_P \geq \tilde{\mu}_P \text{ and } \alpha' e = 1 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \max_{\alpha} \mu_P = \alpha' \mu \\ \text{s.t. } \sigma_P \leq \tilde{\sigma}_P \text{ and } \alpha' e = 1 \end{array} \right.$$

Proposition: When there are no restrictions on the portfolio weights, the weights of the **minimum variance portfolio (MVP)** for a required return $\tilde{\mu}_P$ are:

$$\alpha^* = \Lambda_1 + \Lambda_2 \tilde{\mu}_P$$

$$\text{where } \Lambda_1 = \frac{\Sigma^{-1}}{D} [Be - A\mu] \quad \text{and} \quad \Lambda_2 = \frac{\Sigma^{-1}}{D} [C\mu - Ae]$$

A,B,C,D=combinations of expected returns and covariance matrix

$$\text{with } A = e' \Sigma^{-1} \mu, \quad B = \mu' \Sigma^{-1} \mu, \quad C = e' \Sigma^{-1} e \quad \text{and} \quad D = BC - A^2.$$

Remark: The collection of MVPs for various $\tilde{\mu}_P$ gives the **mean-variance efficient frontier**

Mean-Variance case (Markowitz)

The efficient frontier

The efficient frontier is given by the relation:

$$\tilde{\mu}_P = \frac{A}{C} + \sqrt{\frac{D}{C} \left(\tilde{\sigma}_P^2 - \frac{1}{C} \right)}$$

Remarks:

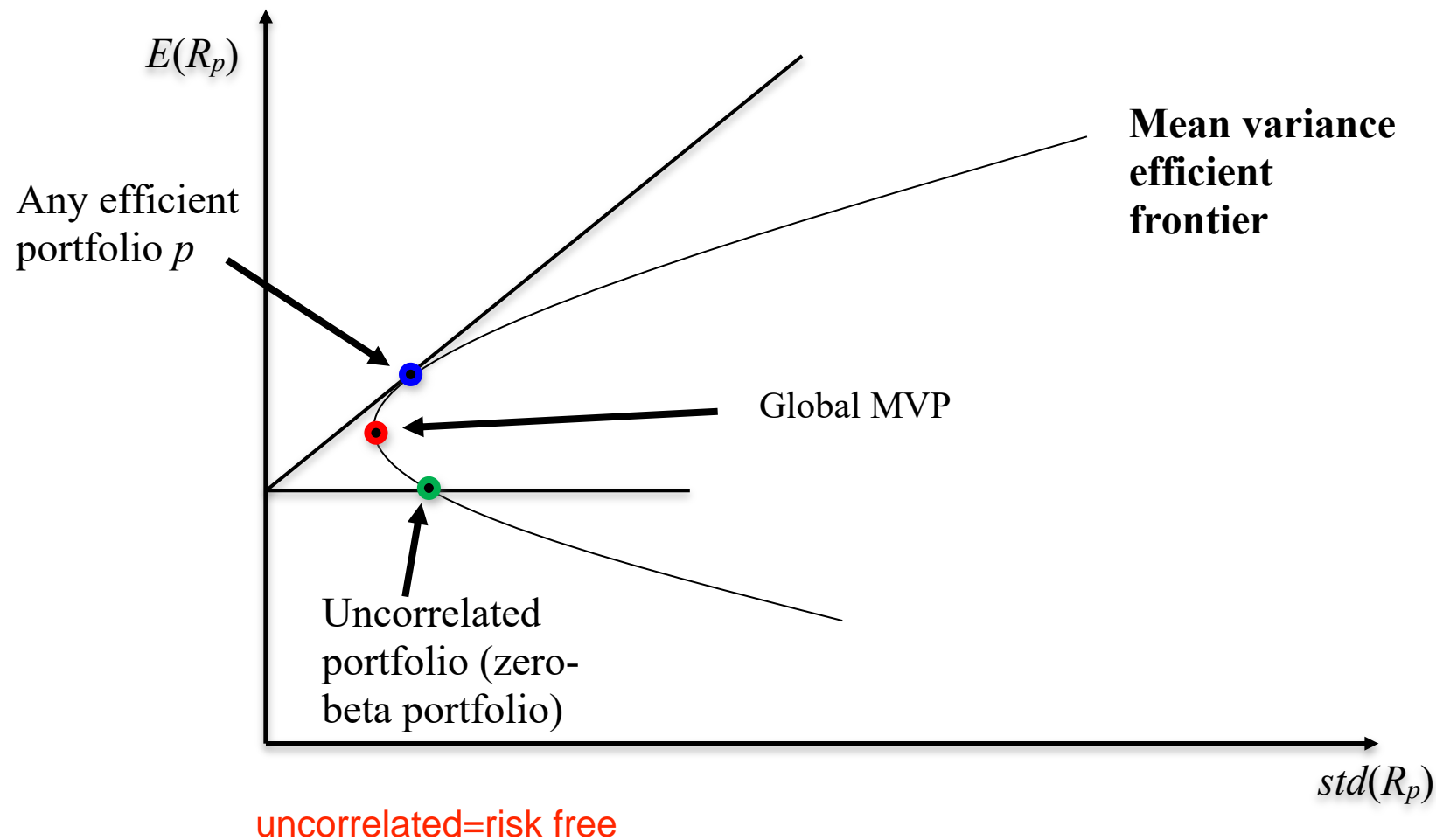
- Any portfolio of MVPs is also an MVP.
- The **Global Minimum Variance Portfolio** (Global MVP) is given by

$$\alpha_{\text{gmv}} = \frac{\Sigma^{-1}e}{C} \quad \text{with} \quad \mu_{\text{gmv}} = \frac{A}{C} \quad \text{and} \quad \sigma_{\text{gmv}}^2 = \frac{1}{C}$$

- The covariance of any asset or portfolio return R_P with the Global MVP is $\text{Cov}[R_{\text{gmv}}, R_P] = \frac{1}{C}$.

Mean-Variance case (Markowitz)

Mean-variance efficient frontier



Mean-Variance case (Tobin, 1958)

no restrictions on weights

The solution with a risk-free asset

There is a risk-free asset, which the investor can borrow or lend with unlimited amount. The solution is the portfolio weight α^* that maximizes:

$$\mu_P - \frac{\lambda}{2} \sigma_P^2 = [\alpha' \mu + (1 - \alpha' e) R_f] - \frac{\lambda}{2} \alpha' \Sigma \alpha$$

Proposition: If there is a risk-free asset, the weights of the **optimal portfolio** are:

$$\alpha^* = \frac{1}{\lambda} \Sigma^{-1} (\mu - R_f e) \quad \text{in risky assets}$$

$$1 - e' \alpha^* \quad \text{in the risk-free asset}$$

over 1 means that we borrow
(leverage)

The sum of the α^* does not necessarily sum to 1, because the investor can hold some amount of risk-free asset.

Remark: If there is one risky asset only (μ_M, σ_M^2) , then $\alpha^* = \frac{\mu_M - R_f}{\lambda \sigma_M^2}$

Mean-Variance case (Tobin)

The solution with a risk-free asset – Alternative derivation

An investor minimizing the portfolio variance subject to a return constraint or maximizing the portfolio return subject to a volatility constraint will find the same solution:

$$\left\{ \begin{array}{l} \min_{\alpha} \sigma_P^2 = \alpha' \Sigma \alpha \\ \text{s.t. } \mu_P = \alpha' \mu + (1 - e' \alpha) R_f \geq \tilde{\mu}_P \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \max_{\alpha} \mu_P = \alpha' \mu + (1 - e' \alpha) R_f \\ \text{s.t. } \sigma_P = \sqrt{\alpha' \Sigma \alpha} \leq \tilde{\sigma}_P \end{array} \right.$$

Proposition: If there is a risk-free asset, the weights of the risky assets in the **optimal portfolio** for a required return $\tilde{\mu}_P$ are:

$$\alpha^* = \frac{\tilde{\mu}_P - R_f}{\mu^e' \Sigma^{-1} \mu^e} \Sigma^{-1} \mu^e = \gamma \Sigma^{-1} \mu^e$$

where $\mu^e = \mu - R_f e$ is the vector of excess returns and γ is the Lagrange multiplier.

Mean-Variance case (Tobin)

The solution with a risk-free asset – Alternative derivation

Tobin's Two-fund Separation Theorem: Every mean-variance efficient portfolio is a combination of the risk-free asset and the **tangency portfolio** with weight:

weights you are investing in risky portfolio
normalized with sum to 1

$$\alpha_T = \frac{\alpha^*}{e' \alpha^*} = \frac{\Sigma^{-1} \mu^e}{e' \Sigma^{-1} \mu^e}$$

The tangency portfolio is also characterized by:

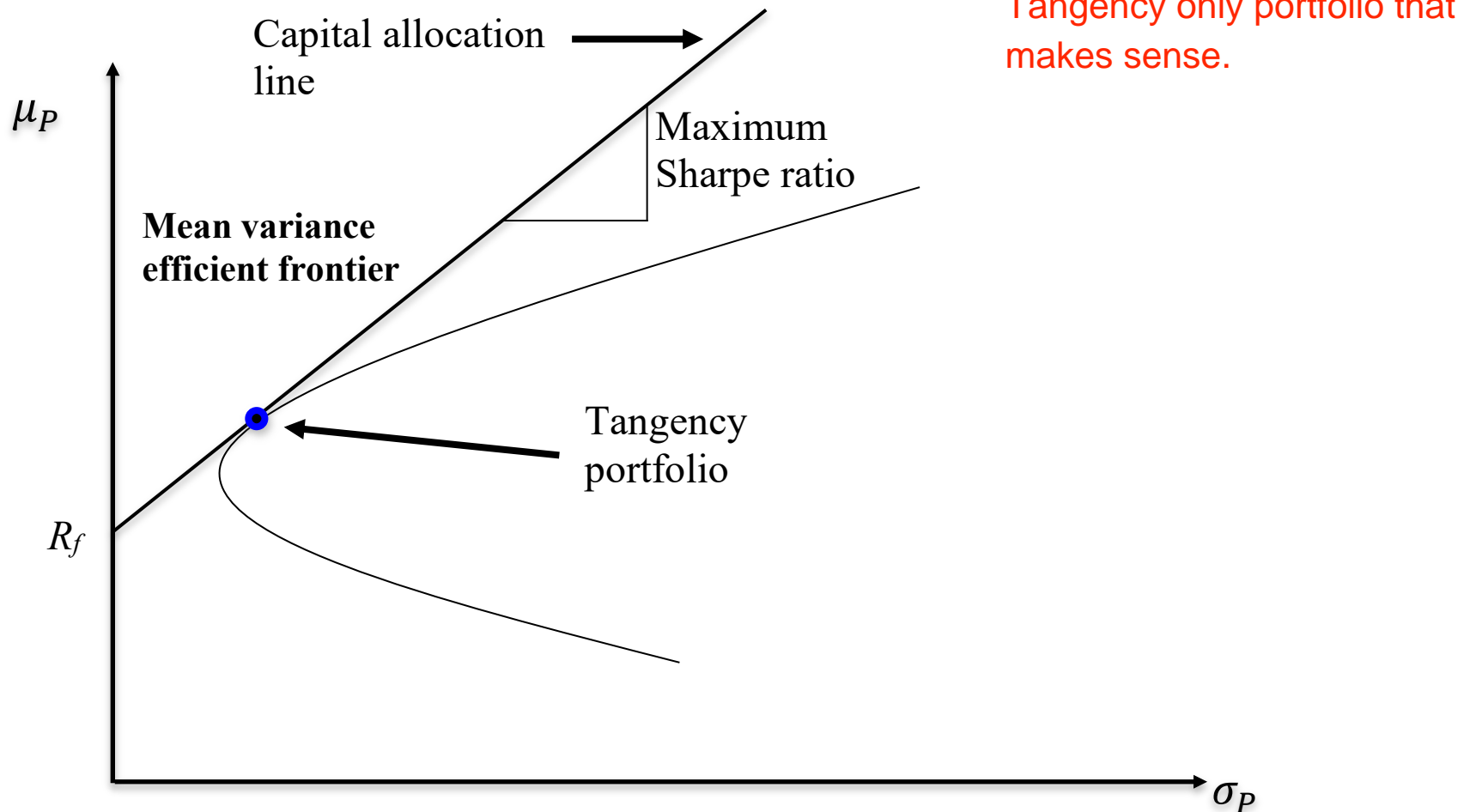
$$\mu_T - R_f = \alpha'_T \mu^e = \frac{\mu^{e' \Sigma^{-1} \mu^e}}{e' \Sigma^{-1} \mu^e} \quad \text{and} \quad \sigma_T^2 = \alpha'_T \Sigma^{-1} \alpha_T = \frac{\mu^{e' \Sigma^{-1} \mu^e}}{(e' \Sigma^{-1} \mu^e)^2}$$

so that its Sharpe ratio is

$$\frac{\mu_T - R_f}{\sigma_T} = \sqrt{(\mu - R_f e)' \Sigma^{-1} (\mu - R_f e)}$$

Mean-Variance case (Tobin)

Mean-variance efficient frontier



Remark: The tangency portfolio is the risky portfolio with the maximum Sharpe ratio. It is often referred to as the **market portfolio** (true at equilibrium).

Objectives of the lecture

- **Optimal Allocation**

➔ **Capital Asset Pricing Model**

- **Market Efficiency**

- **Predicting Risk Premia and the Covariance Matrix**

Market Equilibrium and CAPM

Assumptions

A1. Investors are **price-takers** (believe that security prices are unaffected by their own trades)

A2. All investors plan for one **identical holding period**. This behavior is myopic in that it ignores everything that might happen after the end of the single-period horizon

A3. Investments are limited to a **universe of publicly traded financial assets**, such as stocks and bonds, and to risk-free borrowing or lending arrangement. Investors may borrow or lend any amount at the risk-free rate

A4. Investors pay **no taxes** on returns and **no transaction costs** on trades in securities

A5. All investors are **rational mean-variance optimizers**, meaning that they all use the Markowitz/Tobin portfolio selection model.

A6. **Homogeneous expectations**: All investors use the same expected returns and covariance matrix of security returns to generate the efficient frontier and the unique optimal risky portfolio. They may have different aversion to risk.

Killing finance industry. Do we need HF, banks etc?

Market Equilibrium and CAPM

Results

The only portfolio we can invest in at the same time

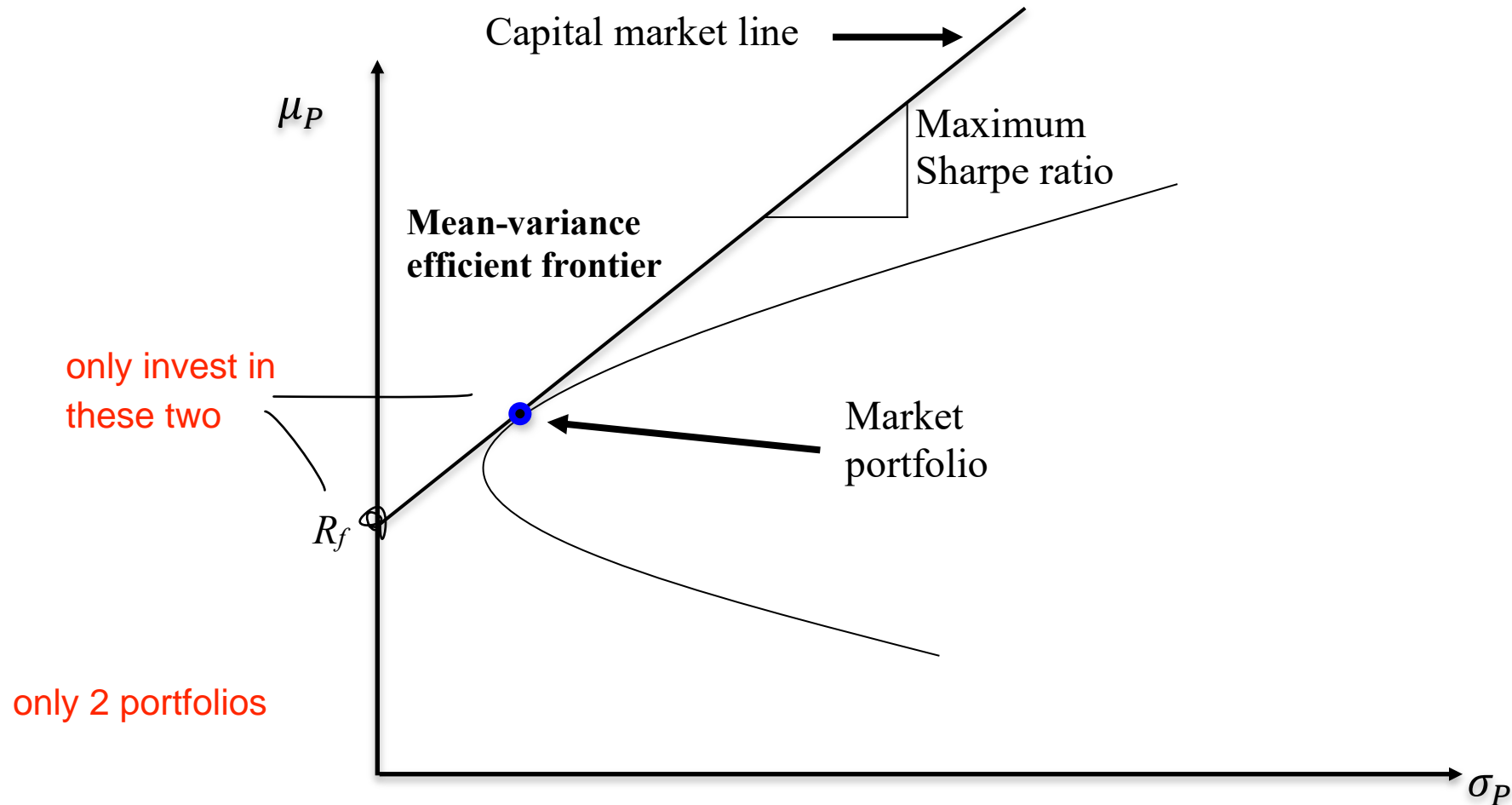
R1. All investors will choose to hold the same **market portfolio** (M), which is a market-value-weighted portfolio of all existing securities

R2. Not only will the market portfolio be on the efficient frontier, but it also will be the tangency portfolio to the optimal **capital allocation line** derived by each investor. As a result, the **capital market line (CML)**, the line from the risk-free rate through the market portfolio, M, is also the best attainable capital allocation line.

All investors hold M as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.

R3. The **risk premium on the market portfolio** will be proportional to its risk and the degree of risk aversion of the representative investor: $E[R_M] - R_f = \lambda \sigma_M^2$ ($\alpha^* = 1$)

Market Equilibrium and CAPM



Market Equilibrium and CAPM

Results (cont'd)

R4. The **risk premium on individual assets** is proportional the risk premium on the market portfolio, M , and the beta coefficient of the security relative to the market portfolio. Beta measures the extent to which returns on the stock and the market move together. Beta is defined as $\beta_i = \frac{cov[R_i, R_M]}{V[R_M]} = \frac{e_i' \Sigma \alpha}{\alpha' \Sigma \alpha}$

The risk premium on individual securities is (price times quantity of risk)

How much you should be compensated for the risk you take.

$$\underbrace{E[R_i] - R_f}_{\text{Expected excess return of asset } i} = \underbrace{\frac{E[R_M - R_f]}{\sigma_M}}_{\text{Price of risk}} \times \underbrace{\frac{cov[R_i, R_M]}{\sigma_M}}_{\text{Systematic risk of asset } i}$$

not depend on asset depend on asset

or

Risk premium of the asst = $E[R_i] - R_f = \frac{cov[R_i, R_M]}{V[R_M]} E[R_M - R_f] = \beta_i E[R_M - R_f]$

sensitivity of asset x market expectation of return

CAPM: The risk premium of the asset i is equal to its beta times the excess return of the market portfolio. Assets with higher systematic risk (higher beta) are priced to offer higher expected return.

1-factor model (Beta)

CAPM and Multi-factor Models

CAPM was first published by Sharpe (1964) and Lintner (1965)

Limitations to CAPM

- Market Portfolio is not directly observable
- Research shows that other factors affect returns

Fama French Three-Factor Model:

- Market beta
- Size
- Book value relative to market value

Fund managers are often evaluated based on running the adjusted CAPM regression:

$$R_{i,t} - R_f = \alpha_i + \beta_i(R_{b,t} - R_f) + \varepsilon_{i,t}$$

where $R_{i,t}$ is the fund return and $R_{b,t}$ is the return of the benchmark adopted by the fund

Passive management = beta

versus

Active management = alpha

Objectives of the lecture

- **Optimal Allocation**
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➔ **Market Efficiency**

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Efficient Market Hypothesis (EMH)

In an **efficient** capital market, stock prices fully reflect available information (Fama, 1965)

= we cannot predict returns.

Efficient Market Hypothesis (EMH) has implications for investors and firms:

- Since information is reflected in security prices quickly, knowing information *when it is released* does an investor no good.
- Firms should expect to receive the fair value for securities that they sell. Firms cannot profit from fooling investors in an efficient market.

Definitions of Market Efficiency

Three forms of efficiency:

1. Weak-form Efficiency: The information set includes only the history of prices and volumes. Using past prices or returns (*technical analysis*) does not yield a positive return. Weak-form efficiency is related to the hypothesis of random walk in prices or non-predictability of returns. *No predictions.*

2. Semistrong-form Efficiency: The information set includes all information known to all market participants (*publicly available information*). Using public information (fundamental analysis) does not yield a positive abnormal return. *Public information is useless. The price has already moved when you think you know something. Already reflected by the market.*

3. Strong-form Efficiency: The information set includes all information known to any market participants (*private information*). Even private information does not yield a positive abnormal return. *Even if you have private information it's not beneficial. Musk's tweets are a trial to manipulate the market. Market efficiency will be rejected. Very hard, due to robots.*

Definitions of Market Efficiency

Market efficiency does not imply that the return should be zero. Rather, it means that there is no profit beyond the **normally required return**.

Usually, normal returns are supposed to be constant over time, but they may be time varying as well. This is important, as returns may be predictable for a given information set, just because the normal return is moving as a function of this information.

Then, **abnormal returns** are computed as the difference between the return on a security and its normal return.

Forecasts of the abnormal return are constructed using the chosen information set. If the abnormal return is forecastable, then the EMH is rejected.

Tests of the EMH are in fact joint tests because they test (1) the EMH and (2) the model for normal returns. Therefore, if the joint hypothesis is rejected, it may be due to the rejection of the equilibrium model or to the rejection of the EMH.

Implications of Efficient Market Hypothesis

Time-series implications

- A **stock price** (including dividends) is a **random walk** with a positive trend (compensation for risk taking)
- **Expected returns** (risk premia) are given by an equilibrium model (e.g., CAPM)
- As the flow of information is random, **abnormal returns** are **unpredictable** conditional on a given information set (depending on the version of the EMH)
- There is no free lunch: in efficient markets, underpriced or overpriced securities do not exist, so it is not possible to design a strategy that will generate an abnormal return on a regular basis.
- Timing the market does not work.

Implications of Efficient Market Hypothesis

Investment implications

- **Fundamental analysis** does not work because the quality of firms is already public. The only form that can work is security analysis: find firms that are better than everyone else's estimated
- Under EMH, active management is wasted effort and costly
- The only worthy strategy is **passive management**:
 - Investors should invest in a buy-and-hold strategy
 - Investors should invest in a highly diversified portfolio (market portfolio)
 - Investment firms should create an index fund and the fund manager should only tailor the portfolio to the needs of the investors

Efficient Market Hypothesis (EMH)

Evidence

1. The weak form of EMH is not rejected by the data
2. The performance of professional managers is broadly consistent with market efficiency (semi-strong form). The amounts by which professional managers as a group beat or are beaten by the market fall within the margin of statistical uncertainty.
3. Markets are very efficient, but especially diligent, intelligent, or creative investors may probably make more money than the average investor.
4. Having access to private information is valuable. In general, it is prohibited (insider trading). What about high frequency trading?

Record of U.S. Mutual Funds

**Average annual return performance of U.S. equity mutual funds
relative to the S&P 500, since inception**

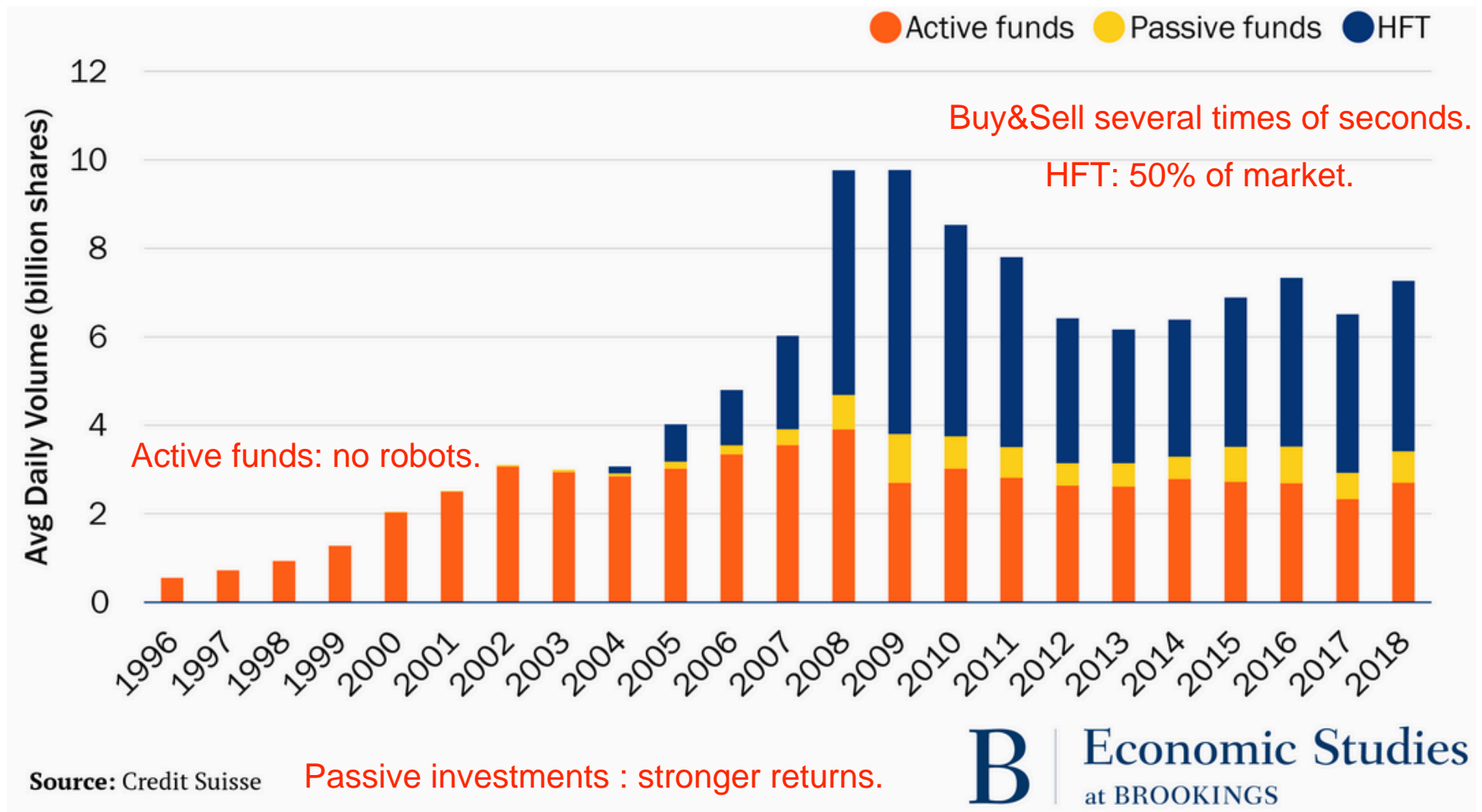
Equity Funds - Life Time Performance						
Inception Years	Age	No. of Funds	Beat SP500		Beat T-Bill	
1924 - 1962	50+	59	26	44%	57	97%
1963 - 1972	40+	58	31	53%	57	98%
1973 - 1982	30+	75	23	31%	74	99%
1983 - 1987	25+	195	91	47%	191	98%
1988 - 1992	20+	277	115	42%	267	96%
1993 - 1997	15+	739	407	55%	715	97%
1998 - 2000	12+	606	411	68%	432	71%
2001 - 2002	10+	324	204	63%	290	90%
2003 - 2005	7+	475	247	52%	420	88%
2006 - 2007	5+	675	240	36%	349	52%
2008 - 2009	3+	592	158	27%	376	64%
2010 - 2011	1+	824	124	15%	467	57%
2012	New	349	99	28%	208	60%
	Total	5248	2176	41%	3903	74%

*Based on monthly returns since inception or Jan '62

<http://www.mutualfundobserver.com/2012/12/mutual-funds-that-beat-the-market/>

Efficient Market Hypothesis (EMH) ETF: fund reflecting the market.

High frequency trading (average daily volume of U.S. equity)



Objectives of the lecture

- **Optimal Allocation**
- **Capital Asset Pricing Model**
- **Market Efficiency**

➔ Predicting Risk Premia and the Covariance Matrix

Estimating Risk Premia: Unconditional Mean

Given historical prices (including dividends) $\{P_0, P_1, \dots, P_T\}$, we compute

- simple returns: $R_t = (P_t - P_{t-1})/P_{t-1}$
- log-returns: $r_t = \log(P_t) - \log(P_{t-1})$

Using the log-risk-free rate, r_f , we obtain excess log-returns, $\{r_1^e, \dots, r_T^e\}$.

We then estimate unconditional risk premia as:

$$\hat{\mu}^e = \frac{1}{T} \sum_{t=1}^T r_t^e$$

Properties:

average return

"hard to beat the market prediction"

High cost with low return
to try to improve the predictions.

- easy to compute and update
- assumes constant risk premia
- with iid normal returns, the central limit theorem states that $\sqrt{T}(\hat{\mu}^e - \mu^e) \sim N(0, \Sigma)$

Estimating Risk Premia: Unconditional Mean

Example:

Assume $\hat{\mu}^e$ = 6%, σ = 15%, and T = 120 (10 years of monthly data).

The standard error of the sample mean is $std[\hat{\mu}^e] = 15\%/\sqrt{120} = 1.4\%$

This implies a 95% confidence interval of [3.25%; 8.75%]

Risk premia are difficult to estimate precisely.

Remark: Most models are designed for daily returns. However, the asset allocation is often performed at the weekly or monthly frequency (\rightarrow temporal aggregation).

Estimating Risk Premia: Linear Conditional Mean

Conditional risk premia can be defined as linear functions of macro-economic variables or firm-specific variables:

- Time series:

$$\mu_{i,t}^e = a_i + b_i' Z_t \quad t = 1, \dots, T$$

where Z_t may include business cycle indicators, such as interest rate, inflation rate, or dividend yield.

- Cross section:

$$\mu_{i,t}^e = a_i + b_i' Z_{i,t} \quad t = 1, \dots, T, i = 1, \dots, N$$

where $Z_{i,t}$ may be the market beta $\beta_{i,t}$.

Factor Fishing

- **Theory**

- Market portfolio (CAPM) or Intertemporal hedge portfolios: portfolios maximally correlated with changes in investment opportunities (I-CAPM)

- **Fundamental factors (accounting-based)**

- Size
- Book to market
- Earnings to market
- Cash-flow to market
- Dividend yield
- Industry factors

- **Macroeconomic factors**

- Default premium
- Term premium
- Industrial production
- Inflation

- **Statistical procedure**

- Factor analysis or Principal components

Estimating Covariance Matrix: Unconditional Moment

Given excess returns, $\{r_1^e, \dots, r_T^e\}$, sample covariance matrix can be estimated as:

$$V[r_i^e] = \frac{1}{T-1} \sum_{t=1}^T (r_{i,t}^e - \hat{\mu}_i^e)^2 \quad \text{and} \quad \text{Cov}[r_i^e, r_j^e] = \frac{1}{T-1} \sum_{t=1}^T (r_{i,t}^e - \hat{\mu}_i^e)(r_{j,t}^e - \hat{\mu}_j^e)$$

Properties:

- easy to compute and update
- assumes constant return distribution

However:

- the horizon of the allocation is not necessarily the same as the frequency of the data
- volatilities and correlations are time varying

Where does this come from?

Remark: For N assets, the number of parameters to estimate is $N(N+1)/2$.

- For 500 stocks and 600 months of data (50 years): 125'250 parameters and 300'000 observations, so that each parameter is estimated with 2.4 observations on average.
- The sample covariance matrix is singular when $N > T - 1$.

Estimating Time-Varying Moments: Rolling Window

An alternative approach that accounts for time-variability is the use of **Rolling Windows**

- Consider the first τ observations ($\tau \ll T$)
- Estimate risk premia (μ_τ^e) and the covariance matrix (Σ_τ) using $t = 1, \dots, \tau$

$$\hat{\mu}_\tau^e = \frac{1}{\tau} \sum_{t=1}^{\tau} r_t^e \quad \Sigma_\tau = \frac{1}{\tau - 1} \sum_{t=1}^{\tau} (r_t^e - \hat{\mu}_\tau^e)(r_t^e - \hat{\mu}_\tau^e)'$$

- Roll the sample by 1 observation
- Estimate risk premia ($\mu_{\tau+1}^e$) and the covariance matrix ($\Sigma_{\tau+1}$) using $t = 2, \dots, \tau + 1$
- ...
- Estimate risk premia (μ_T^e) and the covariance matrix (Σ_T) using $t = T - \tau + 1, \dots, T$

Estimating Time-Varying Moments: EWMA

A slightly better estimator is the **Exponentially Weighted Moving Average (EWMA)**

- Consider a memory parameter ϕ ($0 \leq \phi \leq 1$), an initial risk premium (μ_0^e), and an initial covariance matrix Σ_0 . μ_0^e and Σ_0 can be based on a pre-sample.
- At date τ , compute the risk premium and the covariance matrix as:

$$\hat{\mu}_\tau^e = \phi \hat{\mu}_{\tau-1}^e + (1 - \phi)r_\tau^e$$

$$\Sigma_\tau = \phi \Sigma_{\tau-1} + (1 - \phi)(r_\tau^e - \hat{\mu}_\tau^e)(r_\tau^e - \hat{\mu}_\tau^e)'$$

Remark: Problems related to the number of observations are even more severe.