ECE 278C Imaging Systems

4. Fresnel and Fraunhofer approximation

Department of Electrical and Computer Engineering
University of California, Santa Barbara

plane-to-plane case

$$g(x,y,z_0) = s(x,y,0) \ast h(x,y,z_0)$$

$z = z_0$ $\qquad$ $z = 0$ $\qquad$ z_0: propagation distance

$$= \iint s(x',y',0)\,\frac{1}{j\lambda r}\exp\left(j2\pi \frac{r}{\lambda}\right)dx'dy'$$

$$r = \sqrt{(x-x')^2 + (y-y')^2 + z_0^2}$$

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when z_0 is large

$$r \approx z_0\sqrt{1 + \frac{(x-x')^2}{z_0^2} + \frac{(y-y')^2}{z_0^2}}$$

$$\approx z_0\left(1 + \frac{(x-x')^2}{2z_0^2} + \frac{(y-y')^2}{2z_0^2}\right)$$

$$= z_0 + \frac{(x-x')^3}{2z_0} + \frac{(y-y')^2}{2z_0}$$

zero-order approximation

$$r \approx \longrightarrow z_0$$

1st-order approximation

$$z_0 + \frac{x^2 + x'^2 - 2xx'}{2z_0} + \frac{y^2 + y'^2 - 2yy'}{2z_0}$$

$$= z_0 + \left(\frac{x^2+y^2}{2z_0} + \frac{x'^2+y'^2}{2z_0}\right)$$

$$\left(-\frac{2xx'}{2z_0} - \frac{2yy'}{2z_0}\right)$$

Approximation:

1. $$\frac{1}{j\lambda r} \approx \frac{1}{j\lambda z_0}$$

2. $$\exp(j2\pi r/\lambda) = \exp(j2\pi z_0/\lambda)$$

$$\cdot \exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right) \cdot \exp\left(j\pi \frac{x'^2+y'^2}{\lambda z_0}\right)$$

$$\cdot \exp\left(-j2\pi\left(\frac{x}{\lambda z_0}\right)x'\right)$$

$$\cdot \exp\left(-j2\pi\left(\frac{y}{\lambda z_0}\right)y'\right)$$

$$g(x,y,z_0) = s(x,y,0) * h(x,y,z_0)$$

$$\cong \iint s(x',y',0) \cdot \frac{1}{j\lambda z_0} \cdot \exp\left(j2\pi \frac{z_0}{\lambda}\right)$$

$$\cdot \exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right) \cdot \exp\left(j\pi \left(\frac{x'^2+y'^2}{\lambda z_0}\right)\right)$$

$$\cdot \exp\left(-j2\pi \left(\frac{x}{\lambda z_0}\right) x'\right)$$

$$\cdot \exp\left(-j2\pi \left(\frac{y}{\lambda z_0}\right) y'\right) \quad dx'dy'$$

$$= \frac{1}{j\lambda z_0} \cdot \exp\left(j2\pi \frac{z_0}{\lambda_0}\right) \cdot \exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right) \cdot$$

$$\cdot \iint \left[s(x',y',0) \cdot \exp\left(j\pi \frac{x'^2+y'^2}{\lambda z_0}\right) \right]$$

$$f_x = \frac{x}{\lambda z_0}$$

$$\cdot \exp\left(-j2\pi \left(\frac{x}{\lambda z_0}\right) x'\right) \quad dx'dy'$$

$$\cdot \exp\left(-j2\pi \left(\frac{y}{\lambda z_0}\right) y'\right)$$

$$f_y = \frac{y}{\lambda z_0}$$

Fresnel approximation

Source $s(x, y, 0)$

$\downarrow$

Mask $\exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right)$

$\downarrow$

Fourier Transform

$\downarrow$

Scaling $f_x = \frac{x}{\lambda z_0}$

$f_y = \frac{y}{\lambda z_0}$

$\downarrow$

Mask $\exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right)$

$\downarrow$

Constant term $\frac{1}{j\lambda z_0} \cdot \exp\left(j 2\pi \frac{z_0}{\lambda}\right)$

$\downarrow$

$g(x, y, z_0)$

Fraunhofer approximation

$$\exp\left(j\pi \frac{x^2+y^2}{\lambda z_0}\right) = \exp\left(j\pi \frac{R^2}{\lambda z_0}\right)$$

when $\dfrac{R^2}{\lambda z_0}$ is small

large

$$\exp\left(j\pi \frac{R^2}{\lambda z_0}\right) \approx 1$$

Source

$\downarrow$

Fourier Transform

$\downarrow$

wavefield pattern