Intro to Econometrics: Recitation 2

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Review

Random variables - *univariate* case

$$(\Omega, \mathcal{F}, \mathbf{P})$$

- $X : \Omega \to \mathbf{R}$
- CDF:
$$F_X(x) = \mathbf{P}(\{\omega : X(\omega) \leq x\})$$
  - Completely characterizes $\mathbf{P}\{X \in B\}$ for $B \subset \mathbf{R}$
- Absolutely continuous:
$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

Review

Random variables - *multivariate* case

$$(\Omega, \mathcal{F}, \mathbf{P})$$

- $X : \Omega \to \mathbf{R}^S$ where $X(\omega) = (X_1(\omega), \ldots, X_S(\omega))'$
- CDF:

$$F_X(x_1, \ldots, x_S) = \mathbf{P}(\{\omega : X_1(\omega) \leq x_1, \ldots, X_S(\omega) \leq x_S\})$$

  - Completely characterizes $\mathbf{P}\{X \in B\}$ for $B \subset \mathbf{R}^S$
- Absolutely continuous:

$$F_X(x_1, \ldots, x_S) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_S} f_X(x_1, \ldots, x_S) dx_S \cdots dx_1$$

Review

Random variables - *multivariate* case

- Result: if $F : \mathbf{R} \to [0, 1]$ is
  1. Increasing
  2. Right-continuous
  3. Satisfies $\lim_{x\to\infty} F(x) = 1 - \lim_{x\to-\infty} F(x) = 1$

  Then it is the CDF of some random variable $X : \Omega \to \mathbf{R}$

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  Then it is the CDF of some random variable $X : \Omega \to \mathbf{R}$
- Can you think of (or prove?) an S-dimensional analog of the statement above?

Review

Random variables - *multivariate* case

- If $F : \mathbf{R}^2 \to [0, 1]$ is
  1. Increasing
  2. "Continuous from above"
  3. Has the following limits:
     - 3.1 $\lim_{x_1 \to -\infty} F(x_1, x_2) = 0$ for all x_2
     - 3.2 $\lim_{x_2 \to -\infty} F(x_1, x_2) = 0$ for all x_1
     - 3.3 $\lim_{x_1 \to \infty} \lim_{x_2 \to \infty} F(x_1, x_2) = 1$

Then F is the CDF of a random variable $X : \Omega \to \mathbf{R}^2$

(Durrett, sec 2.9)

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  4. Satisfies, for $x_1^* \geq x_1$ and $x_2^* \geq x_2$,

$$F(x_1^*, x_2^*) - F(x_1^*, x_2) - F(x_1, x_2^*) + F(x_1, x_2) \geq 0$$

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Review

Marginals

- Marginal with respect to coordinate s, $F_s : \mathbf{R} \to [0,1]$

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- Marginal with respect to coordinate s, $F_s : \mathbf{R} \to [0, 1]$

$$F_s(x) = \mathbf{P}(\{\omega : X_s(\omega) \leq x\})$$

- How do you obtain it?
- Just take limits. Suppose $S = 2$ and we want to recover first coordinate:

$$F_1(x_1) = \lim_{x_2 \to \infty} F(x_1, x_2)$$

  Proof?

- How do you recover a marginal pdf? Suppose $X : \Omega \to \mathbf{R}^2$ has pdf $f(x_1, x_2)$:

$$f_1(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2) dx_2$$

- Proof?

Review

Digression: marginals don't determine joints

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$$\mathbf{P}(W = 1) = \mathbf{P}(W = -1) = \tfrac{1}{2}$$

  - Define $Y = WX$. Claim: (X, Y) has normal marginals, but (X, Y) is not jointly normal.

$$\begin{aligned} F_Y(y) = \mathbf{P}(WX \leq y) &= \frac{1}{2}\mathbf{P}(X \leq y) + \frac{1}{2}\mathbf{P}(-X \leq y) \\ &= F_X(y) \end{aligned}$$

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    So marginals of (X, Y) are the same
  - (X, Y) is not multivariate normal. Why?
  - $X + Y$ has a mass at zero, with probability $\frac{1}{2}$!

Review

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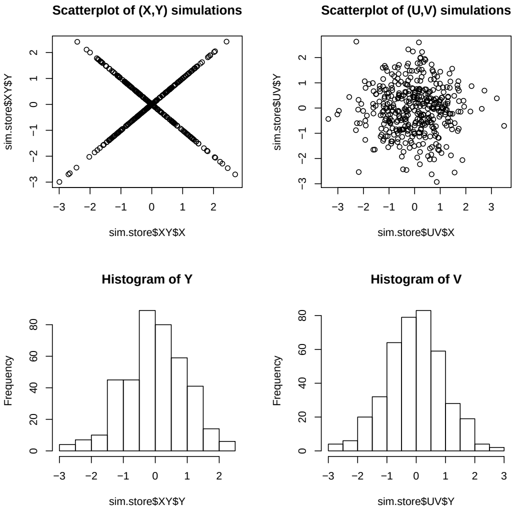

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Moments of multivariate RVs

- Focus on the case when there is a pdf
- "Definition"

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$$V(X) = \mathbf{E}\left[(X - \mu_X)(X - \mu_X)'\right]$$

  - When is $V(X)$ finite?

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- Second moment:

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  - When is $V(X)$ finite?
- Covariance btw X and Y:

$$\text{Cov}(X, Y) = \mathbf{E}\left[(X - \mu_X)(Y - \mu_Y)'\right]$$

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Moment generating functions of multivariate RVs

- MGF:
$$m_X(\mathbf{t}) = \mathbf{E}\left[e^{\mathbf{t}'X}\right] = \mathbf{E}\left[e^{\sum_{i=1}^{S} t_i X_i}\right]$$

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- Result: suppose X and Y have a moment generating function, and
$$m_X(\mathbf{t}) = m_Y(\mathbf{t})$$
for all $\mathbf{t}$. Then $F_X(\mathbf{t}) = F_Y(\mathbf{t})$ for all $\mathbf{t}$.

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- Result: suppose X and Y have a moment generating function, and
$$m_X(\mathbf{t}) = m_Y(\mathbf{t})$$
for all $\mathbf{t}$. Then $F_X(\mathbf{t}) = F_Y(\mathbf{t})$ for all $\mathbf{t}$.
- Result (stronger): suppose that, for all $\mathbf{t} \in \mathbf{R}^S$, $\alpha \in \mathbf{R}$,
$$\mathbf{P}\{\mathbf{t}'X \leq \alpha\} = \mathbf{P}\{\mathbf{t}'Y \leq \alpha\}$$
then $F_X(z) = F_Y(z)$ for all $z \in \mathbf{R}^S$

PS2: Projections, conditioning, linear predictors

Projections

Let $(V, \langle \cdot, \cdot \rangle)$ be a vector space with an inner product.

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- Orthogonal projection of v into (closed) $W \subseteq V$:

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Theorem (Projection in a Hilbert Space)

Let $W \subset V$ be a closed vector subspace of V.
For any $v \in V$, the distance minimization problem

$$\min_{w \in W} \|v - w\|$$

has a unique solution $w^* \in W$. Moreover, $w^* = Proj_W(v)$.

What if W has a finite basis?

$$W = \mathrm{span}\{w_1, \ldots, w_K\}$$

- Orthogonal projection of v into W is

$$\mathrm{Proj}_W(v) = \sum_{i=1}^{K} \frac{\langle w_i, v \rangle}{\langle w_i, w_i \rangle} w_i$$

Using this result in the pset is fair game

Space $V = \{X : \Omega \to \mathbf{R}^S : \mathbf{E}\|X\|^2 < \infty\}$ is a Hilbert space with

$$\langle X, Y \rangle = \mathbf{E}XY$$

PS2: Projections, conditioning, linear predictors

Projections

Space $V = \{X : \Omega \to \mathbf{R}^S : \mathbf{E}\|X\|^2 < \infty\}$ is a Hilbert space with

$$\langle X, Y \rangle = \mathbf{E}XY$$

- Fix variables X, Y in V and consider the subspace

$$W = \{Z : \Omega \to \mathbf{R} : Z = \alpha + \beta(X - \mu_X)\}$$

(Is there a finite basis for W?)

The problem

$$\min_{(\alpha,\beta)} [Y - \alpha - \beta(X - \mu_X)]^2$$

is equivalent to some norm minimization problem involving Y, X and W.
What is it?