Comments

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1 Finite additivity

Let's define some notation. I can define the following for any indexed collection of sets A_i:

$$A_1 + A_2 := A_1 \cup A_2$$

or, more generally

$$\sum_i A_i := \bigcup_i A_i$$

whenever the collection A_i is pairwise disjoint.

The idea of assuming additivity – without any further qualification – is that set-function $\mathbf{P}$ satisfies some form of linearity, that is

$$\mathbf{P}\left(\sum A_i\right) = \sum_i \mathbf{P}\left(A_i\right)$$

It turns out that the set of indices over which this assumption is made is consequential.

We call $\mathbf{P}$ *finitely additive* if the above is required to hold for all finite sets of indices. Similarly, if the relationship holds for countably many indices, $\mathbf{P}$ is called *countably additive*.

Let's investigate an example of finitely, but not countably, additive measure. Here, we are working with a triple $(X, \mathcal{A}, \mathbf{P})$. $\mathcal{A}$ is an *algebra* of sets. Very similar to the usual $\sigma -$ algebra couterpart, but we don't require the assumptions of closedness under unions and intersections to hold for infinitely many set, only finitely many.

We will work with the following algebra, which is not a σ-algebra. Let X be the set of all natural numbers, $\mathbf{N}$. Define also

$$\mathcal{A} = \{A \subset \mathbf{N} : A \text{ is finite or } A^c \text{ is finite}\}$$

Example of sets in $\mathcal{A}$: $\{1, 2, 3\}$ and $\{5001, 5002, \ldots\}$. Example of a sets *not* in $\mathcal{A}$: the set of all odd/even/prime numbers.[1]

It's not hard to see that this is satisfies: $\emptyset \in \mathcal{A}$ (since $\emptyset$ is finite) and closedness under intersections/unions. The reason why $\mathcal{A}$ is not a σ-algebra is that each $A_i = \{1, 3, \ldots, 2i+1\}$ is in $\mathcal{A}$, but its infinite union, the set of all odd numbers, is not.

Now consider the probability measure: $\mathbf{P} : \mathcal{A} \to [0,1]$:

$$\mathbf{P}(A) = \begin{cases} 1 & \text{if } A \text{ is infinite} \\ 0 & \text{otherwise} \end{cases}$$

Thus, for example, $\mathbf{P}(1,2,3) = 0$ and $P(\{1023, 1024, \ldots\}) = 1$.

Such $\mathbf{P}$ trivially satisfies $\mathbf{P}(A + A') = \mathbf{P}(A) + \mathbf{P}(A')$ because the finite union of finite sets is finite.

This probability measure is interesting because it provides a counter-example to continuity when $\mathbf{P}$ is only finitely, but not countably, additive.

For example, it holds that $\{1, 2, \ldots, n\} \uparrow \mathbf{N}$, but

$$1 = \mathbf{P}(\mathbf{N}) = \mathbf{P}\left(\bigcup_n \{1, 2, \ldots, n\}\right) \neq \lim_n \mathbf{P}\left(\{1, 2, \ldots, n\}\right) = 0$$

Moreover, $\{n+1, n+2, \ldots\} \downarrow \emptyset$, but

$$0 = \mathbf{P}(\emptyset) = \mathbf{P}\left(\bigcap_n \{n+1, n+2, \ldots\}\right) \neq \lim_n \mathbf{P}\left(\{n+1, n+2, \ldots\}\right) = 1$$

The CDF of the random variable $X : \mathbf{N} \to \mathbf{N}$, $X(n) = n$ according to $\mathbf{P}$ will satisfy:

$$F_X(k) = \mathbf{P}\{n : X(n) \leq k\} = 0$$

for all n, so $\lim F_X(k) = 0$ for $k \to \infty$.

[1] The sets whose complement is finite are called co-finite sets.