Recitation 1

Gustavo Pereira

September 7, 2019

In this recitation, I review the material presented in lectures 1 and 2. I also cover some things that might be challenging in the first problem sets.

1 Review: lectures 1 and 2

- Definition of probability space: $(\Omega, \mathcal{F}, \mathbf{P} : \mathcal{F} \to [0,1])$
  - The point of $(\Omega, \mathcal{F})$ is to provide a model for the *randomness of some outcome*.
  - Remember: we don't observe randomness. We observe some outcome. Then, we use a model to infer what are more or less likely "states of the world", because that allows us to predict things
  - The reason we keep Ω abstract (instead of focusing on say $\Omega = [0,1]$) is that it allows us to deal with a variety of possible structures for the outcome space!
- Random variables: *measurable* functions $X : \Omega \to S$ where S is some space of outcomes.
- Probability space induced by a random variable
  - Original space: $(\Omega, \mathcal{F}, \mathbf{P})$
  - RV 'measurably' maps original space to $(S, \mathcal{S})$
  - Induced measure: $\mathbf{P}_X(F) = \mathbf{P}\{\omega : X(\omega) \in F\}$ for $F \in \mathcal{S}$
    - ∗ Curiosity: this is called a push-forward measure in mesasure theory
  - Probability space $(S, \mathcal{S}, \mathbf{P}_X)$ is typically some Euclidean space (though it can be more complicated)

- Let's now focus on the case when $X : \Omega \to S$ is real valued, ie, $S = \mathbf{R}$.
- CDF of a random variable: $F_X(x) = \mathbf{P}\left\{\omega : X(\omega) \leq x\right\} = \mathbf{P}_X((-\infty, x])$
  - Result: all information in $\mathbf{P}_X$ is in F_X and vice-versa.
  - Properties of CDF
    1. F_X is non-decreasing
    2. $\lim_{x\to\infty} F_X(x) = 1$
    3. $\lim_{x\to-\infty} F_X(x) = 0$
    4. F_X is right continuous
  - **First main result**: every function F satisfying all four properties above is the CDF of some random variable.
- Absolutely continuous random variable: $\exists f_X$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(z)dz$$

  - Weirdly enough, the non-obvious thing about the statement above is not the $\exists f_X$ but the dz.
  - Measure theoretic details aside, the important thing is that dz is never a jump.
    * If X has a mass at some point x_0 in the real line – meaning that the $\mathbf{P}_X(\{x_0\}) > 0$, there will be a jump in F_X at x_0.
    * We can't have that becasuse $F_X(x_0) - F_X(x_0 - \epsilon) \approx f_X(x_0)\epsilon$
    * For $\epsilon > 0$ small enough, mass at x_0 would imply the LHS is $\mathbf{P}\{x_0\}$ while the RHS should be zero
  - Optional comment: in fact every F_X has an associated f_X with respect to *some* (generally non-uniform) measure. This is the consequence of a more general result called the *Radon-Nikodym theorem*.
- Expectation of absolutely continuous RV:

$$\mathbf{E}[g(X)] = \int_{\mathbf{R}} g(z)f_X(z)dz$$

  - "Law of the unconscious statistician"
- Moment generating function

$$m_X(t) = \mathbf{E}\left[e^{tX}\right] = \int_{\mathbf{R}} e^{tx} f_X(x)dx$$

  - The i-th moment of X can be found by taking the $i-th$ derivative of $m_X(t)$ and evaluating it at zero.
    * For this to be meaningful, the MGF must be well defined in $(-\epsilon, \epsilon)$ for some ϵ
    * Then for example $m'_X(t) = \mathbf{E}[Xe^{tX}]$

- **Second main result.** Let X_1 and X_2 be st

$$m_{X_1}(t) = m_{X_2}(t)$$

  for all t. Then $F_{X_1} = F_{X_2}$.

  - This essentially means that all information contained in F_X is also contained in $m_X(t)$

- Note: take the Taylor series of exponential around 0 and take expectations,

$$m_X(t) = \sum_{n=0}^{\infty} \frac{t^n \mathbf{E}(X^n)}{n!}$$

  - It is tempting to that knowledge of moments determines the distribution of X. This is not the case, however, because sometimes the series above doesn't converge even when all moments exist.

2 Problem 4 is not as easy as it might seem

Consider the proof, for example, that $F_X \to 1$ as $x \to \infty$. (The case of $x \to 0$ is similar.)

We know that:

1. $F(x) = \mathbf{P}\{\omega : X(\omega) \leq x\}$
2. $\{\omega : X(\omega) \leq x\} \uparrow \Omega$
3. $\mathbf{P}(\Omega) = 1$

So it must be the case that $F(x) = P\{\omega : X(\omega) \leq x\} \uparrow \mathbf{P}(\Omega) = 1$, isn't that right? Well, **no**. While that reasoning is in some sense in the right direction, at the very least it's an incomplete argument for two reasons.

- We haven't defined convergence of sets as in (2). Unless you can make that statement rigorous somehow, using it is not fair game.

- More importantly, when we took the statements together, we missed an important step: proving that (whatever the first arrow means)

$$A_x \uparrow \Omega \implies \mathbf{P}(A_x) \uparrow \mathbf{P}(\Omega)$$

The second step above is essentially the point of the exercise. Hint for actually solving the problem:

- Use the fact that
$$\lim_{x \to \infty} F(x) = L$$
if, and only if $F(x_n) \to L$ for all increasing sequences $x_n \to \infty$

- Show that for any probability measure, if $x_n \uparrow \infty$
$$\mathbf{P}\{\omega : X(\omega) \leq x_n\} \to \mathbf{P}(\Omega) = 1$$
You will need to use *countable* additivity for this.

For the right-continuity part, one useful way of checking your proof is to make sure you understand why your proof doesn't apply to the left limit.