Recitation 4

Log-linearization & Dynare

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1 Log-linearizing

The first order Taylor approximation of a (differentiable) function of two variables is

$$f(x,y) \approx f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0)$$

where f_i denotes the partial derivative of f with respect to the i-th coordinate.

We can get the log-linear approximation by defining the following auxiliary function:

$$\phi(w,z) \equiv f(e^w, e^z)$$

so that $\phi_1(w,z) = f_1(e^w,e^z)e^w$ and $\phi_2(w,z) = f_2(e^w,e^z)e^z$. Applying a linear Taylor approximation of ϕ around $\bar{w} = \log(\bar{x})$ and $\bar{z} = \log(\bar{y})$ yields

$$\begin{aligned}\phi(w,z) &\approx \phi(\bar{w},\bar{z}) + \phi_1(\bar{w},\bar{z})(w-\bar{w}) + \phi_2(\bar{w},\bar{z})(z-\bar{z}) \\ &= f(\bar{x},\bar{y}) + \phi_1(\bar{x},\bar{y})\,(w-\bar{w}) + \phi_2(\bar{x},\bar{y})(z-\bar{z})\end{aligned}$$

Applying that to $w = \log(x)$ and $z = \log(y)$, and substituting the partial derivatives, the approximation we get is

$$f(x,y) \approx f(\bar{x},\bar{y}) + \bar{x}f_1(\bar{x},\bar{y})\log\left(\frac{x}{\bar{x}}\right) + \bar{y}f_2(\bar{x},\bar{y})\log\left(\frac{y}{\bar{y}}\right)$$

or, with hat notation,

$$f(x,y) \approx f(\bar{x},\bar{y}) + \bar{x}f_1(\bar{x},\bar{y})\hat{x} + \bar{y}f_2(\bar{x},\bar{y})\hat{y} \tag{1}$$

Finally, suppose we are interested in approximating the log-deviations of a function $h(x,y)$ from the steady state value $h(\bar{x},\bar{y})$. We can again define $f(x,y) = \log h(x,y)$ and apply the approximation above to f, obtaining

$$\widehat{h(x,y)} = \bar{x}\frac{h_1(\bar{x},\bar{y})}{h(\bar{x},\bar{y})}\hat{x} + \bar{y}\frac{h_2(\bar{x},\bar{y})}{h(\bar{x},\bar{y})}\hat{y} \tag{2}$$

1.1 Mnemonics

You can think of the log-linear approximation through the following mnemonic:

$$\widehat{h(x,y)} \approx \left.\frac{\partial \log h}{\partial \log x}\right|_{\bar{x},\bar{y}}\hat{x} + \left.\frac{\partial \log h}{\partial \log y}\right|_{\bar{x},\bar{y}}\hat{y}$$

1.2 Simple example

Suppose a system is characterized by the equation

$$x_t^\alpha y_t = 1 + z_t$$

and that we know $(\bar{x}, \bar{y}, \bar{z})$ such that

$$\bar{x}^\alpha \bar{y} = 1 + \bar{z}$$

We can apply the technique in the previous parts to log-linearize the system around the point $(\bar{x}, \bar{y}, \bar{z})$. There are equivalent ways of achieving that. I'll mention a few and you can apply whatever feels more natural.

1. *Apply formula* (1) *to both sides.* We can write the system as $g(x_t, y_t) = f(z_t)$. With that notation, we know that the point around the system is being approximated satisfies $g(\bar{x}, \bar{y}) = f(\bar{z})$. LHS is approximated by:

$$\begin{aligned} g(x_t, y_t) &\approx g(\bar{x}, \bar{y}) + \bar{x} g_1(\bar{x}, \bar{y}) \hat{x}_t + \bar{y} g_1(\bar{x}, \bar{y}) \hat{y}_t \\ &= g(\bar{x}, \bar{y}) + \bar{x} \left[\alpha \bar{x}^{\alpha - 1} \bar{y}\right] \hat{x}_t + \bar{y} \left[\bar{x}^\alpha\right] \hat{y}_t \\ &= g(\bar{x}, \bar{y}) + \left[\bar{x}^\alpha \bar{y}\right] \left[\alpha \hat{x}_t + \hat{y}_t\right] \end{aligned}$$

   and the RHS:

$$f(z_t) \approx f(\bar{z}) + \bar{z} f'(\bar{z}) \hat{z}_t = f(\bar{z}) + \bar{z} \hat{z}_t$$

   Therefore the log-linear approximation to the system is

$$\cancel{g(\bar{x}, \bar{y})} + \left[\bar{x}^\alpha \bar{y}\right] \left[\alpha \hat{x}_t + \hat{y}_t\right] = \cancel{f(\bar{z})} + \bar{z} \hat{z}_t$$

   or simply

$$\left[\bar{x}^\alpha \bar{y}\right] \left[\alpha \hat{x}_t + \hat{y}_t\right] = \bar{z} \hat{z}_t \tag{3}$$

2. *Apply formula* (1) *to the log of both sides.* Taking logs, we have the equivalent system

$$\alpha \log x_t + \log y_t = \log(1 + z_t)$$

   Let $\tilde{g}(x_t, y_t) = \alpha \log x_t + \log y_t$ and $\tilde{f}(z_t) = \log(1 + z_t)$. The log-linearization formula (1) applied to the LHS yields

$$\begin{aligned} \tilde{g}(x_t, y_t) &\approx \tilde{g}(\bar{x}, \bar{y}) + \bar{x} \tilde{g}_1(\bar{x}, \bar{y}) \hat{x}_t + \bar{y} \tilde{g}_1(\bar{x}, \bar{y}) \hat{y}_t \\ &= \tilde{g}(\bar{x}, \bar{y}) + \bar{x} \frac{\alpha}{\bar{x}} \hat{x}_t + \bar{y} \frac{1}{\bar{y}} \hat{y}_t \\ &= \tilde{g}(\bar{x}, \bar{y}) + \alpha \hat{x}_t + \hat{y}_t \end{aligned}$$

   and to the RHS,

$$\tilde{f}(z_t) \approx \tilde{f}(\bar{z}) + \bar{z} \frac{1}{1 + \bar{z}} \hat{z}_t$$

and bringing the two together, the approximation is

$$\alpha \hat{x}_t + \hat{y}_t = \frac{\bar{z}}{1+\bar{z}} \hat{z}_t \tag{4}$$

At first glance, approximations (3) and (4) are not the same, but in fact they are: just note that $1 + \bar{z} = \bar{x}^\alpha \bar{y}$.

3. *Apply formula* (2) *to both sides.* On the LHS, we get

$$\widehat{x_t^\alpha y_t} = \bar{x} \frac{\alpha \bar{x}^{\alpha-1} \bar{y}}{\bar{x}^\alpha \bar{y}} \hat{x}_t + \bar{y} \frac{\bar{x}^\alpha}{\bar{x}^\alpha \bar{y}} \hat{y}_t = \alpha \hat{x}_t + \hat{y}_t$$

RHS:

$$\widehat{1 + z_t} = \bar{z} \frac{1}{1+\bar{z}} \hat{z}_t$$

Again, equating both sides we get the same result:

$$\alpha \hat{x}_t + \hat{y}_t = \frac{\bar{z}}{1+\bar{z}} \hat{z}_t$$

4. *Use the mnemonic technique.* Taking the log of the LHS,

$$\log LHS = \alpha \log x_t + \log y_t$$

therefore

$$\begin{aligned}\widehat{LHS} &= \left.\frac{\partial \log LHS}{\partial \log x_t}\right|_{\bar{x},\bar{y}} \hat{x}_t + \left.\frac{\partial \log LHS}{\partial \log y_t}\right|_{\bar{x},\bar{y}} \hat{y}_t \\ &= \alpha \hat{x}_t + \hat{y}_t\end{aligned}$$

The log of RHS can be writen as

$$\log RHS = \log \left(1 + e^{\log z_t}\right)$$

therefore

$$\begin{aligned}\widehat{RHS} &= \left.\frac{\partial \log RHS}{\partial \log z_t}\right|_{\bar{z}} \hat{z}_t \\ &= \frac{1}{1+e^{\log \bar{z}}} e^{\log \bar{z}} \hat{z}_t \\ &= \frac{\bar{z}}{1+\bar{z}} \hat{z}_t\end{aligned}$$

equating $\widehat{LHS} = \widehat{RHS}$, we get the same formula:

$$\alpha \hat{x}_t + \hat{y}_t = \frac{\bar{z}}{1+\bar{z}} \hat{z}_t$$

2 Dynare

Consider the following setting:

- Flow utility:

$$u(c,l) = \log(c) + \theta_n \log(l)$$

- Production function:

$$F(k,n,z) = k^\alpha (zn)^{1-\alpha}$$

- First order conditions:

$$\begin{aligned}
\frac{1}{c_t} &= \beta \mathbf{E}_t \left\{ \frac{1}{c_{t+1}} \left[1 - \delta + \alpha k_{t+1}^{\alpha-1} \left(z_{t+1} n_{t+1} \right)^{1-\alpha} \right] \right\} \\
\frac{\theta_n}{1-n} &= \frac{1}{c}(1-\alpha) z_t^{1-\alpha} k_t^\alpha n_t^{-\alpha} \\
c_t + i_t &= y_t \\
y_t &= k_t^\alpha \left(z_t n_t \right)^{1-\alpha}
\end{aligned}$$

The dynare mod file `example.mod` in the github repo generates IRFs and simulations for this example.