Review: optimization and equilibrium

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1 Setup

- Two periods, one good (e.g., apples)
- Production structure:
  1. In the first period, a fixed endowment (ω) materializes
  2. No endowment in the second period. All available output must be produced by a technology $f(k)$, whose input is capital, measured in terms of the good.
     - If k units of capital are used in production, δk units become unavailable for consumption (depreciation).
- Consumption side:
  1. Single consumer with preference $u(c_1) + \beta u(c_2)$ wrt its consumption profile over time
  2. The consumer is the owner of the endowment available in the first period

The market environment in this problem is the following. There is a financial market that intermediates the savings of the consumer and the capital use of the firm. The firm promises to pay r goods in period 2 for every good invested through the financial market in period 1. The consumer is a passive owner of the firm. We assume that the firm operates so as to maximize profits, and all profits accrued by the firm are distributed at the end of period 2 in the form of dividends.

2 Competitive equilibrium

A competitive equilibrium begins by specifying an *optimization problem given prices* to each relevant entity of the model. In our case:

- consumers maximize lifetime utility subject to budget constraints

- firms maximize profit

Problem of the consumer.

$$\begin{aligned}\max_{c_1,c_2,a}\ & u(c_1)+\beta u(c_2)\\ \text{s.t } & c_1+a\le\omega\\ & c_2\le(1+r)a+\pi^e\end{aligned}$$

Note the term π^e above. It represents a guess consumers make about how much dividends they're going to get from the firm. It is going to be a parameter of the consumer problem, but as we will see, rational expectations will imply that in equilibrium, π^e corresponds to the actual value of profits accrued by the firm.

Moreover, there is a single price in this problem: the interest rate r, which governs the supply of savings of the consumer and, as we will see, the demand for capital by the firm.

A solution to the problem above, given prices and expected dividends, is

$$(c_1^*(r,\pi^e), c_2^*(r,\pi^e), a^*(r,\pi^e))$$

Problem of the firm.

$$\max_k \pi(k;r)=f(k)-(r+\delta)k$$

The term δ shows up there because we are assuming the (accounting) incidence of depreciation to fall over the firm. In terms of actual allocations, in equilibrium nothing would change if the depreciation was instead paid by the consumer.

In any case, a solution is simply $k^*(r)$, with associated profits $\pi^*(r)=\pi(k^*(r);r)$.

Competitive equilibrium.

A *competitive* (or *decentralized*) equilibrium consists of allocations and prices satisfying a few requirements:

- at the specified prices, the allocation specifies to each agent a choice that is optimal for him
- all markets clear
- (rational expectations:) whenever an agent bases her decision on a guess, the guess is correct in equilibrium

In our case, a competitive equilibrium is a list $(r^{ce}, k^{ce}, c_1^{ce}, c_2^{ce}, a^{ce}, \pi^{e,ce})$ such that

1. People/firms optimize given prices:

$$\begin{aligned}c_t^{ce}&=c_t^*(r^{ce},\pi^{e,ce}) \qquad t\in\{1,2\}\\ a^{ce}&=a^*(r^{ce},\pi^{e,ce})\\ k^{ce}&=k^*(r^{ce})\end{aligned}$$

2. Asset and goods markets clear:

$$
\begin{aligned}
a^{ce} &= k^{ce} \\
c_1^{ce} + a^{ce} &= \omega_1 \\
c_2^{ce} &= f(k^{ce}) + (1-\delta)k^{ce}
\end{aligned}
$$

3. Rational expectations: guess about profit is correct;

$$
\pi^{e,ce} = \pi(k^{ce}; r^{ce})
$$

Computing the competitive equilibrium. The task of computing a decentralized equilibrium is hard, because you have to solve for demand and supply functions and then solve a large non-linear (in macro, often infinite-dimensional) system.

In the current, very simple setting, we can solve it with Lagrangian techniques which will be useful throughout the course. Therefore, for the sake of reviewing optimization, I'll solve the model.

Assumption: $u(c) = \log c$, $f(k) = k^\alpha$, $\delta = 1$.

1. *Solving the consumer problem.* Let's set up the Lagrangian.

$$
\mathcal{L}(c_1, c_2, a) = u(c_1) + \beta u(c_2) + \lambda_1[\omega - c_1 - a] + \lambda_2[\pi^e + (1+r)a - c_2]
$$

The first order conditions (MPP rule of thumb) are:

$$
\begin{aligned}
u'(c_1) - \lambda_1 &= 0 \\
\beta u'(c_2) - \lambda_2 &= 0 \\
-\lambda_1 - (1+r)\lambda_2 &= 0
\end{aligned}
$$

along with the budget constraints and complementary slackness. Because the budget constraints will hold with equality (at least under our assumption), we get the system:

$$
\begin{aligned}
u'(c_1^*(r,\pi^e)) &= \beta(1+r)u'(c_2^*(r,\pi^e)) \\
c_1^*(r,\pi^e) + a^*(r,\pi^e) &= \omega \\
c_2^*(r,\pi^e) &= (1+r)a^*(r,\pi^e) + \pi^e
\end{aligned}
$$

Under the log functional form for u we get

$$
\begin{aligned}
a^*(r,\pi^e) &= \frac{1}{1+\beta}\left[\beta\omega - \frac{\pi^e}{1+r}\right] \\
c_2^*(r,\pi^e) &= \frac{\beta}{1+\beta}[(1+r)\omega + \pi^e] \\
c_1^*(r,\pi^e) &= \frac{1}{1+\beta}\left[w + \frac{\pi^e}{1+r}\right]
\end{aligned}
$$

2. *Solving the firm problem.* The firm maximizes

$$f(k) - (r+\delta)k$$

which has the simple foc $f'(k^*(r)) = r + \delta$.

3. *Finding the equilibrium.* We could continue to write all the equilibrium conditions and say "the solution is whatever solves this system". Let's try to find some characterization. First, asset market equilibrium and firm optimization imply

$$f'(k^{ce}) = f'(a^{ce}) = r^{ce} + \delta$$

By rational expectations,

$$\pi^e = \pi(k^{ce}, r^{ce}) = f(k^{ce}) - r^{ce}k^{ce}$$

Consumer optimization then implies

$$\begin{aligned} u'(c_1^{ce}) &= u'(c_2^{ce})\beta(1 + f'(k^{ce})) \\ c_1^{ce} + k^{ce} &= \omega \\ c_2^{ce} &= (1 + \cancel{f'(k^{ce})} - \delta)k^{ce} + f(k^{ce}) - \cancel{f'(k^{ce})k^{ce}} \end{aligned}$$

This is a system of three equations in three unknowns.

3 The social planner problem

We consider the case of a fictitious benevolent planner who wants to maximize some notion of "collective welfare", subject only to the physical constraints of the economy. That is, the planner completely bypasses the market process, and instead just chooses an allocation that is productively feasible and maximizes welfare.

In our economy, the social planner problem is thus to find an allocation that solves

$$\begin{aligned} &\max_{c_1, c_2, a} u(c_1) + \beta u(c_2) \\ &\text{s.t } c_1 + k \le \omega \\ &\quad c_2 \le f(k) + (1-\delta)k \end{aligned}$$

Solution: exactly the system of equations that you get for the competitive problem!

The crucial thing to note is that **the above problem does not involve any prices**. In general, computing the solution to a social planner problem is easier than a competitive equilibrium. Under certain regularity conditions, the solutions will coincide. Using that can spare you substantial time in computing the solution to a more complicated problem.