ON THE IMPACT OF APPROXIMATION ERRORS ON EXTREME VALUE INFERENCE: APPLICATIONS TO MULTIDIMENSIONAL EXTREMES

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Agenda of the presentation

Table of Contents

What is Extreme Value Theory?

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- Extreme quantile estimation
- Tail probability estimation
- Estimation of the endpoint of a given distribution

Definition (Maximum domain of attraction)

Let $Y_1, \ldots, Y_n$ be i.i.d. observations of a random variable Y. If there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$, and a random variable G with a nondegenerate distribution such that

$$\frac{\max(Y_1, \ldots, Y_n) - b_n}{a_n} \xrightarrow{\mathcal{D}} G, \quad n \to \infty,$$

we say that Y belongs to the maximum domain of attraction of G, and denote $Y \in \mathrm{MDA}(G)$.

Extreme Value Index

Theorem (Fisher and Tippett, 1928; Gnedenko, 1943)

Up to location and scale, the distribution of $G = G_\gamma$ is characterized by the parameter γ, called the extreme value index. That is, the distribution of G_γ is of the type

$$F_{G_\gamma}(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \quad 1+\gamma x > 0 & \text{if} \quad \gamma \neq 0, \\ \exp\left(-e^{-x}\right), \quad x \in \mathbb{R} & \text{if} \quad \gamma = 0. \end{cases}$$

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In the case $\gamma > 0$ the type of G_γ is Fréchet,

$$\Phi_\gamma(x) = \begin{cases} 0, & x \leq 0 \\ \exp\left(-x^{-1/\gamma}\right), & x > 0. \end{cases}$$

Tail Quantile Function

Define the tail quantile function corresponding to a distribution F by

$$U(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right), \quad t > 1,$$

where we denote the left-continuous inverse of a nondecreasing function by $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\}$.

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That is, $U(1/p)$ is the $(1-p)$-quantile.

Definition (Regular variation)

A Lebesgue measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ that is eventually positive is regularly varying with index $\alpha \in \mathbb{R}$ if for all $x > 0$,

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha.$$

Then we denote $f \in \mathrm{RV}_\alpha$. Furthermore, we say that a function f is slowly varying if $f \in \mathrm{RV}_0$.

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Intuition:

$$f \in \mathrm{RV}_\alpha \iff f(x) = L(x)x^\alpha, \quad L \in \mathrm{RV}_0 .$$

We also have

$$\lim_{x \to \infty} x^{-\varepsilon} L(x) = 0, \quad \forall \varepsilon > 0.$$

Construction of an Extreme Quantile Estimator

Theorem (de Haan, 1970; Gnedenko, 1943)

Let $\gamma > 0$. *We have*

$$Y \in \mathrm{MDA}\,(G_\gamma) \iff 1 - F \in \mathrm{RV}_{-1/\gamma} \iff U \in \mathrm{RV}_\gamma\,.$$

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Choose $t = n/k$ and $x = k/(np)$ to get the approximation

$$U\left(\frac{1}{p}\right) \approx U\left(\frac{n}{k}\right)\left(\frac{k}{np}\right)^\gamma.$$

Extreme Quantile Estimation

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}(G_\gamma)$, $\gamma > 0$. Denote order statistics corresponding to the sample $\boldsymbol{Y}$ by $\boldsymbol{Y}_{1,n} \leq \cdots \leq \boldsymbol{Y}_{n,n}$. Then an estimator for the extreme $(1-p)$-quantile $x_p = U(1/p)$ can be given as

$$\hat{x}_p(\boldsymbol{Y}) = \boldsymbol{Y}_{n-k,n} \left(\frac{k}{np} \right)^{\hat{\gamma}(\boldsymbol{Y})},$$

where $\hat{\gamma}$ is an estimator for the extreme value index γ.

The Hill Estimator (Hill, 1975; Mason, 1982)

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}(G_\gamma)$, $\gamma > 0$. The Hill estimator is defined as

$$\hat{\gamma}_H(\boldsymbol{Y}) = \frac{1}{k} \sum_{i=0}^{k-1} \ln\left(\frac{\boldsymbol{Y}_{n-i,n}}{\boldsymbol{Y}_{n-k,n}}\right).$$

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$$\hat{\gamma}_H(\boldsymbol{Y}) = \frac{1}{k}\sum_{i=0}^{k-1} \ln\left(\frac{\boldsymbol{Y}_{n-i,n}}{\boldsymbol{Y}_{n-k,n}}\right).$$

If additionally as $n \to \infty$, $k = k_n \to \infty$, $k/n \to 0$, then

$$\hat{\gamma}_H(\boldsymbol{Y}) \xrightarrow{\mathbb{P}} \gamma, \quad n \to \infty.$$

Table of Contents

A Traditional Approach to Extremes

Definition (Multivariate regular variation)

Let Θ be a probability measure on the unit sphere $\mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}$. A d-dimensional random vector X is multivariate regularly varying with the extreme value index $\gamma > 0$ and the probability measure Θ if

$$\lim_{t \to \infty} \frac{\mathbb{P}\left(\|X\|_2 \geq tx, X/\|X\|_2 \in A\right)}{\mathbb{P}\left(\|X\|_2 \geq t\right)} = x^{-1/\gamma}\Theta\left(A\right),$$

for every $x > 0$ and for every Borel set A in $\mathbb{S}^{d-1}$ with $\Theta\left(\partial A\right) = 0$.

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for every $x > 0$ and for every Borel set A in $\mathbb{S}^{d-1}$ with $\Theta\left(\partial A\right) = 0$.

- Estimation of multivariate extreme quantile regions under multivariate regular variation based on
  - density (Cai et al., 2011), and
  - half-space depth (He & Einmahl, 2016).

An Alternative Framework

1. Approach in multidimensional extremes:
   - Let $X \in \mathbb{S}$ be a random object, where, e.g., $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \mathsf{L}^p([0, 1]^d)$.

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2. Often instead of the sample $\boldsymbol{Y}$, only approximations $\hat{\boldsymbol{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_n)$ are available.
   - How the approximation error affects the asymptotics?

Applications

- Elliptical extreme quantile region estimation (Pere, Ilmonen, & Viitasaari, 2024).
- Extreme value index estimation for latent model (Virta et al., 2024).
- Estimation of the extreme value index corresponding to functional PCA scores (Kim & Kokoszka, 2019).
- Extreme quantile estimation under approximated L^p-norms (Pere, Avelin, et al., 2024).

Table of Contents

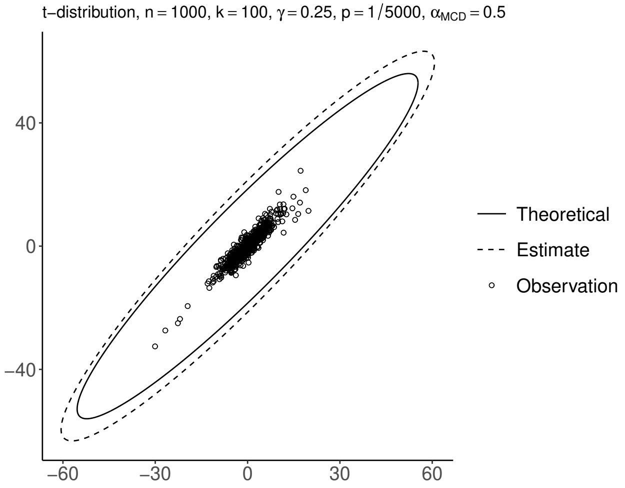
J. Pere, P. Ilmonen, and L. Viitasaari (2024). On extreme quantile region estimation under heavy-tailed elliptical distributions. *Journal of Multivariate Analysis*.
doi: https://doi.org/10.1016/j.jmva.2024.105314

Extreme Quantile Region Estimation

Let $X \in \mathbb{R}^d$ be a random vector with density f. We wish to estimate

$$Q_p = \left\{ x \in \mathbb{R}^d : f(x) \leq \beta_p \right\},$$

where β_p is chosen such that $\mathbb{P}\left(X \in Q_p\right) = p$, and p is small.



t–distribution, n = 1000, k = 100, γ = 0.25, p = 1/5000, α_MCD = 0.5
40
0
–40
–60
–30
0
30
60
Theoretical
Estimate
Observation

Elliptical Distributions (Cambanis et al., 1981)

We say that a d-variate random vector X is elliptically distributed if

$$X \stackrel{\mathcal{D}}{=} \mu + \mathcal{R}\Lambda S,$$

where

- $\mu \in \mathbb{R}^d$ is called the *location vector*,
- $\mathcal{R}$ is a nonnegative random variable called the *generating variate*,
- $\Lambda \in \mathbb{R}^{d \times d}$ is a matrix such that $\Sigma = \Lambda\Lambda^\intercal$ is a symmetric positive definite matrix (matrix Σ is called *scatter matrix*),
- S is an d-dimensional random vector uniformly distributed over the unit-sphere $\{x \in \mathbb{R}^d : x^\intercal x = 1\}$ and
- Random variables $\mathcal{R}$ and S are independent.

Role of the Generating Variate $\mathcal{R}$

- $\mathcal{R}$ is regularly varying $\iff$ X is multivariate regularly varying (Hult & Lindskog, 2002)

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- $\mathcal{R} \overset{\mathcal{D}}{=} \sqrt{(X-\mu)^\intercal \, \Sigma^{-1} \, (X-\mu)} =: \|X-\mu\|_\Sigma$

Role of the Generating Variate $\mathcal{R}$

- $\mathcal{R}$ is regularly varying $\iff$ X is multivariate regularly varying (2002)
- $\mathcal{R} \overset{\mathcal{D}}{=} \sqrt{(X-\mu)^\intercal \Sigma^{-1} (X-\mu)} =: \|X-\mu\|_\Sigma$
- $\hat{R}_i = \|X_i - \hat{\mu}(\boldsymbol{X})\|_{\hat{\Sigma}(\boldsymbol{X})}$
- $\hat{\boldsymbol{R}} = (\hat{R}_1, \ldots, \hat{R}_n)$

The Estimator

- We wish to estimate

$$Q_p = \left\{ x \in \mathbb{R}^d : \|x - \mu\|_\Sigma \geq r_p \right\},$$

for a very small p, where r_p is the $(1 - p)$-quantile of the generating variate $\mathcal{R}$.

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  for a very small p, where r_p is the $(1-p)$-quantile of the generating variate $\mathcal{R}$.
- Define estimator $\hat{Q}_p$ as

$$\hat{Q}_p = \left\{ x \in \mathbb{R}^d : \|x - \hat{\mu}(\boldsymbol{X})\|_{\hat{\Sigma}(\boldsymbol{X})} \geq \hat{x}_p(\hat{\boldsymbol{R}}) \right\},$$

  where

$$\hat{x}_p(\hat{\boldsymbol{R}}) = \hat{\boldsymbol{R}}_{n-k,n} \left(\frac{k}{np} \right)^{\hat{\gamma}_H(\hat{\boldsymbol{R}})},$$

  and $\hat{\gamma}_H(\hat{\boldsymbol{R}})$ is the separating Hill estimator (2019).

Draft of the Consistency Result

Suppose that $\mathcal{R} \in \mathrm{MDA}\,(G_\gamma)$ for $\gamma > 0$ and $p_n \to 0$ with a suitable rate, as $n \to \infty$ (and other technical conditions). Then

$$\frac{\mathbb{P}\left(X \in \hat{Q}_{p_n} \triangle Q_{p_n}\right)}{p_n} \to 0, \quad \text{as } n \to \infty,$$

where $B_1 \triangle B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1), \quad B_1, B_2 \in \mathbb{R}^d$.

Affine Equivariance

Let $A \in \mathbb{R}^{d \times d}$ be an invertible matrix and let $a \in \mathbb{R}^d$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from an elliptical distribution, $Z_i = AX_i + a$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$. Let $\hat{Q}_p$ and $\hat{Q}'_p$ be extreme quantile region estimators computed with $\boldsymbol{X}$ and $\boldsymbol{Z}$, respectively. If estimators of the location and the scatter are affine equivariant in the sense that

$$\hat{\mu}(\boldsymbol{Z}) = A\hat{\mu}(\boldsymbol{X}) + a \quad \text{and} \quad \hat{\Sigma}(\boldsymbol{Z}) = A\hat{\Sigma}(\boldsymbol{X}) A^{\intercal},$$

then

$$\hat{Q}'_p = \{Ax + a : x \in Q_p\}.$$

Table of Contents

J. Pere, B. Avelin, V. Garino, P. Ilmonen, and L. Viitasaari (2024). On the Impact of Approximation Errors on Extreme Quantile Estimation with Applications to Functional Data Analysis. *Submitted*.
doi: https://doi.org/10.48550/arXiv.2307.03581

Draft of a Result: Controlling the Approximation Errors

Let $\gamma > 0$. Let $Y_1, \ldots, Y_n$ be i.i.d. copies of $Y \in \mathrm{MDA}\left(G_\gamma\right)$ and $\hat{\boldsymbol{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_n)$ the corresponding approximations. Denote errors by $E_i = \left|\hat{Y}_i - Y_i\right|$. If

$$\sqrt{k}\frac{\boldsymbol{E}_{n,n}}{U_Y\left(n/k\right)} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty,$$

then

$$\sqrt{k}\left(\hat{\gamma}(\hat{\boldsymbol{Y}}) - \gamma\right) \quad \text{and} \quad \frac{\sqrt{k}}{\ln\left(k/(np)\right)}\left(\frac{\hat{x}_p(\hat{\boldsymbol{Y}})}{U(1/p)} - 1\right)$$

are asymptotically normally distributed under the standard assumptions (second-order condition, rate for $p = p_n$, $k = k_n \to \infty$, $k/n \to 0$, as $n \to \infty$).

Approximated L^p-Norms

- Let $X \in \mathrm{L}^p\left([0,1]^d\right)$, and let $X_1, \ldots, X_n$ be i.i.d. copies of X.
- We wish to estimate extreme value index and extreme quantiles corresponding to $\|X\|_p \in \mathrm{MDA}\left(G_\gamma\right)$, $\gamma > 0$.

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- Approximate norms with Riemann sums or Monte Carlo integration.
- Use approximated norms $\hat{Y}_i$ in the estimation.

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- In practice we never observe $X_1, \ldots, X_n$.
- Approximate norms with Riemann sums or Monte Carlo integration.
- Use approximated norms $\hat{Y}_i$ in the estimation.
- As the estimator of the extreme value index we choose the Hill estimator

$$\hat{\gamma}(\hat{\boldsymbol{Y}}) = \frac{1}{k} \sum_{i=0}^{k-1} \ln\left(\frac{\hat{\boldsymbol{Y}}_{n-i,n}}{\hat{\boldsymbol{Y}}_{n-k,n}}\right).$$

Riemann Sum Approximated Norms

Let $\gamma > 0$. Let X_i be i.i.d. copies of $X \in \mathrm{L}^p([0,1])$, $p \in [1,\infty]$, s.t. $Y = \|X\|_p \in \mathrm{MDA}(G_\gamma)$. Let $\hat{Y}_i$ be the Riemann sum approximated norms (based on discretizations with m equidistant observed points). Suppose for all $s, t \in [0,1]$, X satisfies

$$|X(t) - X(s)| \leq V\phi(|t-s|) \quad a.s.,$$

for some random variable $V \in \mathrm{MDA}\left(G_{\gamma'}\right)$, $\gamma' > 0$, and for some continuous decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$.

Riemann Sum Approximated Norms

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for some random variable $V \in \mathrm{MDA}\left(G_{\gamma'}\right)$, $\gamma' > 0$, and for some continuous decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Then the condition

$$\sqrt{k}\frac{\boldsymbol{E}_{n,n}}{U_Y(n/k)} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty,$$

translates into

$$\sqrt{k}\phi\left(\frac{1}{m}\right)k^{\gamma}n^{\gamma'-\gamma} \to 0, \quad n \to \infty.$$

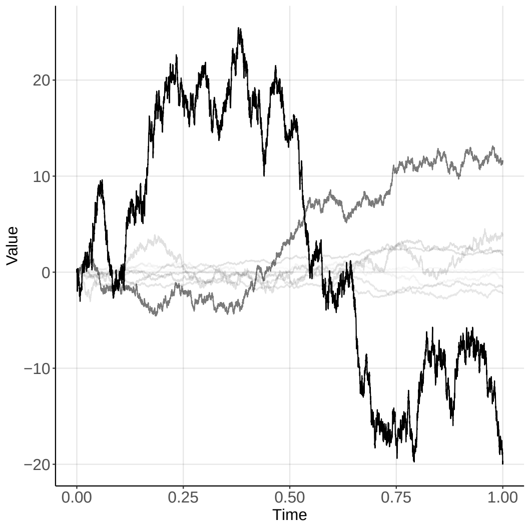

Figure: Independent and identically distributed observations from a stochastic process $X(t) = \mathcal{R}Z(t)$, where $\mathcal{R} \in \text{MDA}\,(G_\gamma)$, $\gamma > 0$, and Z is a Brownian motion.

Concentration for $\hat{\gamma}(\hat{\boldsymbol{Y}})$

In order to give concentration inequality for $\mathbb{P}\left(\left|\hat{\gamma}(\hat{\boldsymbol{Y}}) - \hat{\gamma}(\boldsymbol{Y})\right| > x\right)$ one needs to control the errors

$$\mathbb{P}\left(\frac{\boldsymbol{E}_{n,n}}{U_Y\left(n/k\right)} > x\right)$$

and the convergence

$$\mathbb{P}\left(\left|\frac{\boldsymbol{Y}_{n-k,n}}{U(n/k)} - 1\right| > x\right).$$

Chernoff-Type Bound for Intermediate Order Statistics

Let $\gamma > 0$. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$ and assume that, as $n \to \infty$, $k = k_n \to \infty$, and $k/n \to 0$. Then for sufficiently large n

$$\mathbb{P}\left(\left|\frac{\boldsymbol{Y}_{n-k,n}}{U(n/k)} - 1\right| > x\right) \leq C_1 e^{-C_2 k},$$

where the constants $C_1 > 0$ and $C_2 > 0$ depend on x and γ.

Thank you for your attention!

- Link to slides (Github): https://github.com/perej1/ics-and-related

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