ON THE IMPACT OF APPROXIMATION ERRORS ON EXTREME VALUE INFERENCE: APPLICATIONS TO MULTIDIMENSIONAL EXTREMES

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27th of May, 2025

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Agenda of the presentation

Univariate Extreme Value Theory

Multidimensional Extremes and Impact of Approximation Errors

Extreme Quantile Region Estimation Under Ellipticity

Extreme Quantile Estimation for L^p-Norms

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What is Extreme Value Theory?

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- Extreme quantile estimation
- Tail probability estimation
- Estimation of the endpoint of a given distribution

Definition (Maximum domain of attraction)

Let $Y_1, \ldots, Y_n$ be i.i.d. observations of a random variable Y. If there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$, and a random variable G with a nondegenerate distribution such that

$$\frac{\max(Y_1, \ldots, Y_n) - b_n}{a_n} \xrightarrow{\mathcal{D}} G, \quad n \to \infty,$$

we say that Y belongs to the maximum domain of attraction of G, and denote $Y \in \mathrm{MDA}(G)$.

Extreme Value Index

Theorem (Fisher and Tippett, 1928; Gnedenko, 1943)

Up to location and scale, the distribution of $G = G_\gamma$ is characterized by the parameter γ, called the extreme value index. That is, the distribution of G_γ is of the type

$$F_{G_\gamma}(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \quad 1+\gamma x > 0 & \text{if} \quad \gamma \neq 0, \\ \exp\left(-e^{-x}\right), \quad x \in \mathbb{R} & \text{if} \quad \gamma = 0. \end{cases}$$

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In the case $\gamma > 0$ the type of G_γ is Fréchet,

$$\Phi_\gamma(x) = \begin{cases} 0, & x \leq 0 \\ \exp\left(-x^{-1/\gamma}\right), & x > 0. \end{cases}$$

Tail Quantile Function

Define the tail quantile function corresponding to a distribution F by

$$U(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right), \quad t > 1,$$

where we denote the left-continuous inverse of a nondecreasing function by $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\}$.

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That is, $U(1/p)$ is the $(1-p)$-quantile.

Definition (Regular variation)

A Lebesgue measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ that is eventually positive is regularly varying with index $\alpha \in \mathbb{R}$ if for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\alpha.$$

Then we denote $f \in \mathrm{RV}_\alpha$. Furthermore, we say that a function f is slowly varying if $f \in \mathrm{RV}_0$.

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Intuition:

$$f \in \mathrm{RV}_\alpha \iff f(x) = L(x)x^\alpha, \quad L \in \mathrm{RV}_0 .$$

We also have

$$\lim_{x \to \infty} x^{-\varepsilon} L(x) = 0, \quad \forall\, \varepsilon > 0.$$

Construction of an Extreme Quantile Estimator

Theorem (de Haan, 1970; Gnedenko, 1943)

Let $\gamma > 0$. *We have*

$$Y \in \mathrm{MDA}\left(G_\gamma\right) \iff 1 - F \in \mathrm{RV}_{-1/\gamma} \iff U \in \mathrm{RV}_\gamma \,.$$

Construction of an Extreme Quantile Estimator

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Choose $t = n/k$ and $x = k/(np)$ to get the approximation

$$U\left(\frac{1}{p}\right) \approx U\left(\frac{n}{k}\right)\left(\frac{k}{np}\right)^{\gamma}.$$

Extreme Quantile Estimation

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$, $\gamma > 0$. Denote order statistics corresponding to the sample $\boldsymbol{Y}$ by $\boldsymbol{Y}_{1,n} \leq \cdots \leq \boldsymbol{Y}_{n,n}$. Then an estimator for the extreme $(1-p)$-quantile $x_p = U\,(1/p)$ can be given as

$$\hat{x}_p\,(\boldsymbol{Y}) = \boldsymbol{Y}_{n-k,n}\left(\frac{k}{np}\right)^{\hat{\gamma}(\boldsymbol{Y})},$$

where $\hat{\gamma}$ is an estimator for the extreme value index γ.

The Hill Estimator (Hill, 1975; Mason, 1982)

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$, $\gamma > 0$. The Hill estimator is defined as

$$\hat{\gamma}_H\,(\boldsymbol{Y}) = \frac{1}{k}\sum_{i=0}^{k-1} \ln\left(\frac{\boldsymbol{Y}_{n-i,n}}{\boldsymbol{Y}_{n-k,n}}\right).$$

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$$\hat{\gamma}_H(\boldsymbol{Y}) = \frac{1}{k} \sum_{i=0}^{k-1} \ln\left(\frac{\boldsymbol{Y}_{n-i,n}}{\boldsymbol{Y}_{n-k,n}}\right).$$

If additionally as $n \to \infty$, $k = k_n \to \infty$, $k/n \to 0$, then

$$\hat{\gamma}_H(\boldsymbol{Y}) \xrightarrow{\mathbb{P}} \gamma, \quad n \to \infty.$$

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A Traditional Approach to Extremes

Definition (Multivariate regular variation)

Let Θ be a probability measure on the unit sphere $\mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 = 1\}$. A d-dimensional random vector X is multivariate regularly varying with the extreme value index $\gamma > 0$ and the probability measure Θ if

$$\lim_{t\to\infty} \frac{\mathbb{P}\left(\|X\|_2 \geq tx, X/\|X\|_2 \in A\right)}{\mathbb{P}\left(\|X\|_2 \geq t\right)} = x^{-1/\gamma}\Theta(A),$$

for every $x > 0$ and for every Borel set A in $\mathbb{S}^{d-1}$ with $\Theta(\partial A) = 0$.

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for every $x > 0$ and for every Borel set A in $\mathbb{S}^{d-1}$ with $\Theta\left(\partial A\right) = 0$.

- Estimation of multivariate extreme quantile regions under multivariate regular variation based on
  - density (Cai et al., 2011), and
  - half-space depth (He & Einmahl, 2016).

An Alternative Framework

1. Approach in multidimensional extremes:
    - Let $X \in \mathbb{S}$ be a random object, where, e.g., $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \mathsf{L}^p([0,1]^d)$.

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2. Often instead of the sample $\boldsymbol{Y}$, only approximations $\hat{\boldsymbol{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_n)$ are available.
    - How the approximation error affects the asymptotics?

Applications

- Elliptical extreme quantile region estimation (Pere, Ilmonen, & Viitasaari, 2024).
- Extreme value index estimation for latent model (Virta et al., 2024).
- Estimation of the extreme value index corresponding to functional PCA scores (Kim & Kokoszka, 2019).
- Extreme quantile estimation under approximated L^p-norms (Pere, Avelin, et al., 2024).

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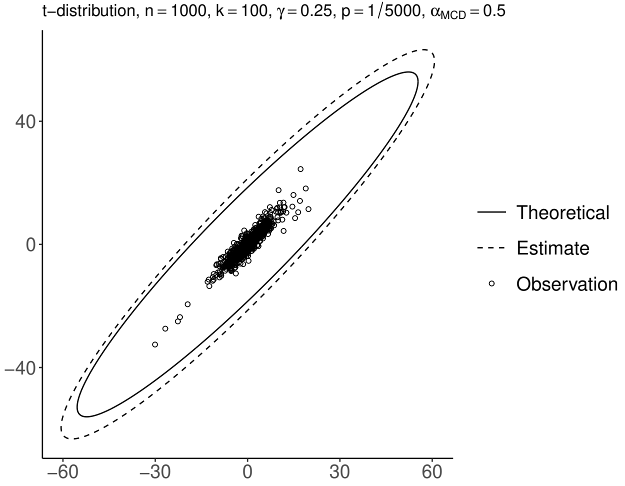
J. Pere, P. Ilmonen, and L. Viitasaari (2024). On extreme quantile region estimation under heavy-tailed elliptical distributions. *Journal of Multivariate Analysis*.
doi: https://doi.org/10.1016/j.jmva.2024.105314

Extreme Quantile Region Estimation

Let $X \in \mathbb{R}^d$ be a random vector with density f. We wish to estimate

$$Q_p = \left\{ x \in \mathbb{R}^d : f(x) \leq \beta_p \right\},$$

where β_p is chosen such that $\mathbb{P}\left(X \in Q_p\right) = p$, and p is small.



t−distribution, n = 1000, k = 100, γ = 0.25, p = 1/5000, α_MCD = 0.5
40
0
−40
−60
−30
0
30
60
Theoretical
Estimate
Observation

Elliptical Distributions (Cambanis et al., 1981)

We say that a d-variate random vector X is elliptically distributed if

$$X \stackrel{\mathcal{D}}{=} \mu + \mathcal{R}\Lambda S,$$

where

- $\mu \in \mathbb{R}^d$ is called the *location vector*,
- $\mathcal{R}$ is a nonnegative random variable called the *generating variate*,
- $\Lambda \in \mathbb{R}^{d\times d}$ is a matrix such that $\Sigma = \Lambda\Lambda^\intercal$ is a symmetric positive definite matrix (matrix Σ is called *scatter matrix*),
- S is an d-dimensional random vector uniformly distributed over the unit-sphere $\{x \in \mathbb{R}^d : x^\intercal x = 1\}$ and
- Random variables $\mathcal{R}$ and S are independent.

Role of the Generating Variate $\mathcal{R}$

- $\mathcal{R}$ is regularly varying $\iff$ X is multivariate regularly varying (Hult & Lindskog, 2002)
- $\mathcal{R} \overset{\mathcal{D}}{=} \sqrt{(X-\mu)^\intercal \Sigma^{-1} (X-\mu)} =: \|X-\mu\|_\Sigma$
- $\hat{R}_i = \|X_i - \hat{\mu}(\boldsymbol{X})\|_{\hat{\Sigma}(\boldsymbol{X})}$
- $\hat{\boldsymbol{R}} = (\hat{R}_1, \ldots, \hat{R}_n)$

The Estimator

- We wish to estimate

$$Q_p = \left\{ x \in \mathbb{R}^d : \|x - \mu\|_\Sigma \geq r_p \right\},$$

for a very small p, where r_p is the $(1 - p)$-quantile of the generating variate $\mathcal{R}$.

The Estimator

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for a very small p, where r_p is the $(1 - p)$-quantile of the generating variate $\mathcal{R}$.

- Define estimator $\hat{Q}_p$ as

$$\hat{Q}_p = \left\{ x \in \mathbb{R}^d : \|x - \hat{\mu}\,(\boldsymbol{X})\|_{\hat{\Sigma}(\boldsymbol{X})} \geq \hat{x}_p(\hat{\boldsymbol{R}}) \right\},$$

where

$$\hat{x}_p(\hat{\boldsymbol{R}}) = \hat{\boldsymbol{R}}_{n-k,n} \left(\frac{k}{np} \right)^{\hat{\gamma}_H(\hat{\boldsymbol{R}})}.$$

Draft of the Consistency Result

Suppose that $\mathcal{R} \in \mathrm{MDA}\,(G_\gamma)$ for $\gamma > 0$ and $p_n \to 0$ with a suitable rate, as $n \to \infty$ (and other technical conditions). Then

$$\frac{\mathbb{P}\left(X \in \hat{Q}_{p_n} \triangle Q_{p_n}\right)}{p_n} \to 0, \quad \text{as } n \to \infty,$$

where $B_1 \triangle B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1), \quad B_1, B_2 \in \mathbb{R}^d$.

Affine Equivariance

Let $A \in \mathbb{R}^{d \times d}$ be an invertible matrix and let $a \in \mathbb{R}^d$. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a sample from an elliptical distribution, $Z_i = AX_i + a$ and $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$. Let $\hat{Q}_p$ and $\hat{Q}'_p$ be extreme quantile region estimators computed with $\boldsymbol{X}$ and $\boldsymbol{Z}$, respectively. If estimators of the location and the scatter are affine equivariant in the sense that

$$\hat{\mu}(\boldsymbol{Z}) = A\hat{\mu}(\boldsymbol{X}) + a \quad \text{and} \quad \hat{\Sigma}(\boldsymbol{Z}) = A\hat{\Sigma}(\boldsymbol{X}) A^\intercal,$$

then

$$\hat{Q}'_p = \{Ax + a : x \in Q_p\}.$$

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J. Pere, B. Avelin, V. Garino, P. Ilmonen, and L. Viitasaari (2024). On the Impact of Approximation Errors on Extreme Quantile Estimation with Applications to Functional Data Analysis. *Submitted*.
doi: https://doi.org/10.48550/arXiv.2307.03581

Draft of a Result: Controlling the Approximation Errors

Let $\gamma > 0$. Let $Y_1, \ldots, Y_n$ be i.i.d. copies of $Y \in \mathrm{MDA}\left(G_\gamma\right)$ and $\hat{\boldsymbol{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_n)$ the corresponding approximations. Denote errors by $E_i = \left|\hat{Y}_i - Y_i\right|$. If

$$\sqrt{k}\frac{\boldsymbol{E}_{n,n}}{U_Y\left(n/k\right)} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty,$$

then

$$\sqrt{k}\left(\hat{\gamma}(\hat{\boldsymbol{Y}}) - \gamma\right) \quad \text{and} \quad \frac{\sqrt{k}}{\ln\left(k/(np)\right)}\left(\frac{\hat{x}_p(\hat{\boldsymbol{Y}})}{U(1/p)} - 1\right)$$

are asymptotically normally distributed under the standard assumptions (second-order condition, rate for $p = p_n$, $k = k_n \to \infty$, $k/n \to 0$, as $n \to \infty$).

Approximated L^p-Norms

- Let $X \in \mathrm{L}^p\left([0,1]^d\right)$, and let $X_1, \ldots, X_n$ be i.i.d. copies of X.
- We wish to estimate extreme value index and extreme quantiles corresponding to $\|X\|_p \in \mathrm{MDA}\,(G_\gamma)$, $\gamma > 0$.

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- Approximate norms with Riemann sums or Monte Carlo integration.
- Use approximated norms $\hat{Y}_i$ in the estimation.

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- In practice we never observe $X_1, \dots, X_n$.
- Approximate norms with Riemann sums or Monte Carlo integration.
- Use approximated norms $\hat{Y}_i$ in the estimation.
- As the estimator of the extreme value index we choose the Hill estimator

$$\hat{\gamma}(\hat{\boldsymbol{Y}}) = \frac{1}{k}\sum_{i=0}^{k-1} \ln\left(\frac{\hat{\boldsymbol{Y}}_{n-i,n}}{\hat{\boldsymbol{Y}}_{n-k,n}}\right).$$

Riemann Sum Approximated Norms

Let $\gamma > 0$. Let X_i be i.i.d. copies of $X \in \mathrm{L}^p([0,1])$, $p \in [1,\infty]$, s.t. $Y = \|X\|_p \in \mathrm{MDA}(G_\gamma)$. Let $\hat{Y}_i$ be the Riemann sum approximated norms (based on discretizations with m equidistant observed points). Suppose for all $s, t \in [0,1]$, X satisfies

$$|X(t) - X(s)| \leq V\phi(|t-s|) \quad a.s.,$$

for some random variable $V \in \mathrm{MDA}\left(G_{\gamma'}\right)$, $\gamma' > 0$, and for some continuous decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$.

Riemann Sum Approximated Norms

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for some random variable $V \in \mathrm{MDA}\left(G_{\gamma'}\right)$, $\gamma' > 0$, and for some continuous decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Then the condition

$$\sqrt{k}\frac{\boldsymbol{E}_{n,n}}{U_Y(n/k)} \xrightarrow{\mathbb{P}} 0, \quad n \to \infty,$$

translates into

$$\sqrt{k}\phi\left(\frac{1}{m}\right)k^\gamma n^{\gamma'-\gamma} \to 0, \quad n \to \infty.$$

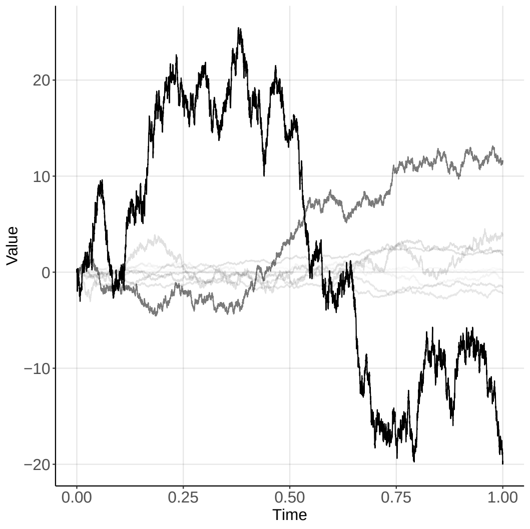

Figure: Independent and identically distributed observations from a stochastic process $X(t) = \mathcal{R}Z(t)$, where $\mathcal{R} \in \text{MDA}\,(G_\gamma)$, $\gamma > 0$, and Z is a Brownian motion.

Concentration for $\hat{\gamma}(\hat{\boldsymbol{Y}})$

In order to give concentration inequality for $\mathbb{P}\left(\left|\hat{\gamma}(\hat{\boldsymbol{Y}}) - \hat{\gamma}(\boldsymbol{Y})\right| > x\right)$ one needs to control the errors

$$\mathbb{P}\left(\frac{\boldsymbol{E}_{n,n}}{U_Y\,(n/k)} > x\right)$$

and the convergence

$$\mathbb{P}\left(\left|\frac{\boldsymbol{Y}_{n-k,n}}{U(n/k)} - 1\right| > x\right).$$

Chernoff-Type Bound for Intermediate Order Statistics

Let $\gamma > 0$. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ be an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$ and assume that, as $n \to \infty$, $k = k_n \to \infty$, and $k/n \to 0$. Then for sufficiently large n

$$\mathbb{P}\left(\left|\frac{\boldsymbol{Y}_{n-k,n}}{U(n/k)} - 1\right| > x\right) \leq C_1 e^{-C_2 k},$$

where the constants $C_1 > 0$ and $C_2 > 0$ depend on x and γ.

Thank you for your attention!

- Link to slides (Github): https://github.com/perej1/hel-seminar

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