ON THE IMPACT OF APPROXIMATION ERRORS ON EXTREME QUANTILE ESTIMATION WITH APPLICATIONS TO FUNCTIONAL DATA ANALYSIS

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Agenda of the presentation

Extreme Value Theory with Approximations

Impact of Approximation Errors

Extreme Quantile Estimation for L^p-Norms

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What is Extreme Value Theory?

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- Extreme quantile estimation
- Tail probability estimation
- Estimation of the endpoint of a given distribution

Maximum Domain of Attraction

Definition

Let $Y_1, \ldots, Y_n$ be i.i.d. observations of a random variable Y. If there exist sequences $a_n > 0$ and $b_n \in \mathbb{R}$, and a random variable G with a nondegenerate distribution such that

$$\frac{\max\left(Y_1, \ldots, Y_n\right) - b_n}{a_n} \xrightarrow{\mathcal{D}} G, \quad n \to \infty,$$

we say that Y belongs to the maximum domain of attraction of G, and denote $Y \in \mathrm{MDA}\left(G\right)$.

Extreme Value Index

Theorem (Fisher and Tippett 1928; Gnedenko 1943)

Up to location and scale, the distribution of $G = G_\gamma$ is characterized by the parameter γ, called the extreme value index. That is, the distribution of G_γ is of the type

$$F_{G_\gamma}(x) = \begin{cases} \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \quad 1+\gamma x > 0 & \text{if} \quad \gamma \neq 0, \\ \exp\left(-e^{-x}\right), \quad x \in \mathbb{R} & \text{if} \quad \gamma = 0. \end{cases}$$

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In the case $\gamma > 0$ the type of G_γ is Fréchet,

$$\Phi_\gamma(x) = \begin{cases} 0, & x \leq 0 \\ \exp\left(-x^{-1/\gamma}\right), & x > 0. \end{cases}$$

Tail Quantile Function

Define tail quantile function corresponding to distribution F by

$$U(t) = F^{\leftarrow}\left(1 - \frac{1}{t}\right), \quad t > 1,$$

where we denote left-continuous inverse of a nondecreasing function by $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\}$.

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That is, $U(1/p)$ is the $(1-p)$-quantile.

Definition (Regular variation)

A Lebesgue measurable function $f : \mathbb{R}^+ \to \mathbb{R}$ that is eventually positive is regularly varying with index $\alpha \in \mathbb{R}$ if for all $x > 0$,

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^{\alpha}.$$

Then we denote $f \in RV_{\alpha}$. Furthermore, we say that a function f is slowly varying if $f \in RV_0$.

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Intuition:

$$f \in \mathrm{RV}_\alpha \iff f(x) = L(x)x^\alpha, \quad L \in \mathrm{RV}_0\,.$$

We also have

$$\lim_{x\to\infty} x^{-\varepsilon} L(x) = 0, \quad \forall\, \varepsilon > 0.$$

Construction of an Extreme Quantile Estimator

Theorem ((Gnedenko 1943; de Haan 1970))

Let $\gamma > 0$. *We have*

$$Y \in \mathrm{MDA}\left(G_\gamma\right) \iff 1 - F \in RV_{-1/\gamma} \iff U \in RV_\gamma.$$

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Choose $t = n/k$ and $x = k/(np)$ to get the approximation

$$U\left(\frac{1}{p}\right) \approx U\left(\frac{n}{k}\right)\left(\frac{k}{np}\right)^\gamma .$$

Extreme Quantile Estimation

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$, $\gamma > 0$. Denote order statistics corresponding to the sample $\boldsymbol{Y}$ by $\boldsymbol{Y}_{1,n} \leq \cdots \leq \boldsymbol{Y}_{n,n}$. Then an estimator for the extreme $(1-p)$-quantile $x_p = U\,(1/p)$ can be given as

$$\hat{x}_p\,(\boldsymbol{Y}) = \boldsymbol{Y}_{n-k,n} \left(\frac{k}{np}\right)^{\hat{\gamma}(\boldsymbol{Y})},$$

where $\hat{\gamma}$ is an estimator for the extreme value index γ.

The Hill Estimator (Hill 1975; Mason 1982)

Suppose $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ is an i.i.d. sample of $Y \in \mathrm{MDA}\,(G_\gamma)$, $\gamma > 0$. The Hill estimator is defined as

$$\hat{\gamma}_H\,(\boldsymbol{Y}) = \frac{1}{k}\sum_{i=0}^{k-1} \ln\left(\frac{\boldsymbol{Y}_{n-i,n}}{\boldsymbol{Y}_{n-k,n}}\right).$$

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If additionally as $n \to \infty$, $k = k_n \to \infty$, $k/n \to 0$, then

$$\hat{\gamma}_H\,(\boldsymbol{Y}) \xrightarrow{\mathbb{P}} \gamma, \quad n \to \infty.$$

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