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Methods with high accuracy for finite element probability computing

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Abstract

This paper introduces methods with high accuracy for finite element probability computing, by which the function value on one or a few nodes can be calculated without forming the total stiffness matrix.

Keywords: Finite element; Mathematical expectation; Random walk; Multi-grid; Probability computing method

The finite element probability computing method is the method which uses the Monte Carlo method to get the approximate solution of finite element approximation of the partial differential equation. Using this method, the approximate value on one or a few nodes of the finite element solution can be directly calculated without forming the total stiffness matrix. Thus, not only the computer's inner memory unit can be saved, but also the procedure is easy to be achieved. If the dividing step length h is very small, then the time that moves about to the boundary will be very long and therefore finding a method with high accuracy is significant. This paper presents two kinds of methods with high accuracy using the first boundary value problem in two dimensions, viz. the Laplace equation. The method also can be generalized to a general elliptic equation.

1. Brief introduction of the finite element probability computing method

We consider the first boundary value problem in two dimensions i.e. Laplace equation:

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$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega, \\ &u = f \qquad \text{on } \partial\Omega = \Gamma. \end{aligned} \tag{1}$$

The corresponding variational form is:

$$\begin{aligned} &\text{Find} \qquad u \in K = \{u \in H^1(\Omega) \text{ and } u|_\Gamma = f\} \\ &\text{satisfying} \quad a(u,v) = 0 \quad \text{for any } v \in K, \end{aligned} \tag{2}$$

where $a(u,v) = \int_\Omega \nabla u \nabla v \, dx$, $H^1(\Omega)$ is the usual Sobolev space.

We use the uniform triangulation $£^h$ for area Ω and denote the inner nodes set as T^h, boundary as Γ^h, and the total number of nodes as N. Let

$$S^{h,k}(\Omega) = \{v \in C(\bar{\Omega});\, v|_e = P_k(e) \text{ for any } e \in £^h\},$$

$$K^{h,k} = \{v \in S^{h,k};\, v|\Gamma^h = f^{\text{I}}\},$$

where $P_k(e)$ is a kth order polynomial on e. Then the finite element equation corresponding to (2) is

$$\begin{aligned} &\text{Find} \qquad u^h \in K^{h,k} \\ &\text{satisfying} \quad a(u^h, v^k) = 0 \quad \text{for any } v^h \in K^{h,k}. \end{aligned} \tag{3}$$

Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be the basis functions of $K^{h,k}$ and take v^h as $\varphi_1, \varphi_2, \ldots, \varphi_N$ independently, we get the following algebraic equation:

$$\sum_{i=1}^{N} a_{ij} u_i = 0, \quad j = 1, 2, \ldots, N, \tag{4}$$

where $a_{ij} = a(\varphi_i, \varphi_j)$, $u_i = u^h(x_i)$, $x_i \in T^h \cup \Gamma^h$, and $u_i = u^h(x_i) = f^{\text{I}}(x_i)$ and $x_i \in \Gamma^h$.

Using the properties of the basis function, for any $x_i \in T^h$, we have

$$u_i = \sum_{j=1}^{i_s} P_{ij} u_j, \tag{5}$$

where $P_{ij} = -(a_{ij}/a_{ii})$, $u_{i1}, u_{i2}, \ldots, u_{is}$ are the function values of the nodes $x_{i1}, x_{i2}, \ldots, x_{is}$ which are adjacent to x_i.

Next using (5) we structure a probability model to calculate the function value $u^h(x_i)$ of node x_i.

First let $P_{ij} = g_{ij} \cdot P'_{ij}$ such that $0 \leqslant P'_{ij} < 1$, $\sum_{j=1}^{i_s} P'_{ij} \leqslant 1$, and let

$$P'_i \equiv 1 - \sum_{j=1}^{i_s} P'_{ij} \geqslant 0.$$

Let the particle leave from $i(x_i)$. Let the probability that the particle moves about to the adjacent node x_{ij} be P'_{ij}, and with the probability of stopping P'_i. (In this time, let the value of the random variable be zero.) If the moving route is $i \to i_1 \to i_2 \to \cdots \to i_r \in \Gamma^h$, then the value of random variable ξ is

$$\xi(x_i) = g_{ii_1} g_{i_1 i_2} g_{i_{r-1} i_r} f(x_{ir}), \quad x_{ir} \in \Gamma^h.$$

Lemma 1. *From* [2, 3], *we have that the conditions that the above random variable ξ exists and is finite are* $\max|\lambda_i(A)| < 1$, *and* $E(\xi(x_i)) \equiv u^h(x_i) = u^i$, *where λ_i are eigenvalues of the finite element total stiffness matrix* $A = (P_{ij})$.

Lemma 2. *The conditions that the variance value $D(\xi)$ of the random variable ξ is finite are* $\max|\lambda_i(\bar{A})| < 1$ *where* $\bar{A} = (a_{ij})$, $a_{ij} = g_{ij}^2 \cdot P_{ij}$.

From Lemmas 1 and 2, using the strong law of large numbers in the theory of probability, we have $E(\xi(x_i)) \approx \sum_{i=1}^{M} \xi_i/M$; $\xi_1, \xi_2, \ldots, \xi_M$ are M sample points of ξ. Therefore we can use the approximate value of $E(\xi)$ to replace the function value of this point and this method can be easily implemented using a computer [3].

Lemma 3. *Let u^h be the solution of problem* (3) *with $k = 1$, for any $x_i \in T^h$ let the corresponding adjacent nodes be $x_{i1}, x_{i2}, \ldots, x_{is}$. Then there exist real numbers P_{ij} ($j = 1, 2, \ldots, s$) such that*

(i) $P_{ij} \geqslant 0$, $j = 1, 2, \ldots, s$,

(ii) $\sum_{j=1}^{s} P_{ij} = 1$,

(iii) $u^h(x_i) = \sum_{j=1}^{s} P_{ij} u^h(x_{ij})$.

Particularly, if we use the uniform triangulation of Ω as in Fig. 1, *then we have $P_{ij} = \frac{1}{4}$ and this method is consistent with the difference method given in* [3].

Lemma 4. *For a linear triangular element, the expectation value of the random variable, according to the above probability model, exists and is finite;* $E(\xi) = u^h(x_i)$.

Proof. Since in time $g_{ij} = 1$ and the transition probability that the particle moves about m steps is A^m, where A is the transition probability matrix. From [2], the probability that the particle moves

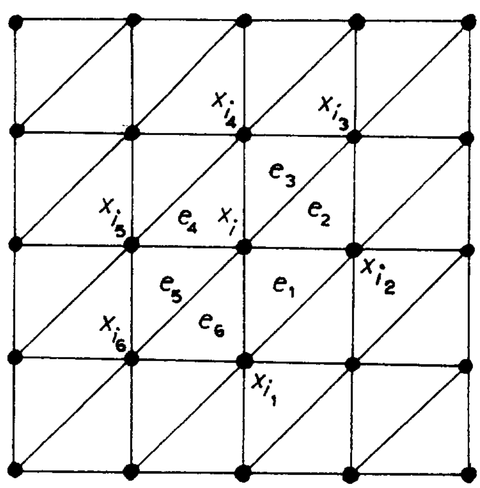

Fig 1. Uniform triangulation.

about infinite steps is zero, which means that $\lim_{m\to\infty} A^{\infty} = 0$. Therefore $\max_i\{\lambda_i(A)\} < 1$. The rest of the proof follows from Lemma 1. $\square$

Lemma 5. *The variance of the random variable of the linear triangular element probability computing method is finite.*

Proof. Since $g_{ij} \equiv 1$, $A \equiv \bar{A}$. Then $\max_i\{\lambda_i(\bar{A})\} < 1$. From Lemma 2 the proof is complete. $\square$

From Lemmas 3–5, we know that the finite element probability computing method is valid.

2. A high accuracy method

For the finite element probability computing method holds, if the dividing step length h is very small, then the time that a particle moves about to the boundary will be very long. Therefore finding a method with high accuracy is significant. Next we introduce two kinds of methods with high accuracy: the probability multigrid method and the boundary thickening method.

2.1. *The probability multigrid method*

Using the idea of the multigrid method, let the particle move in variable grid with some fixed probability in the different scale grid. Thus the mathematical expectation value $E(\xi)$ of the random variable ξ is consistent with finite element with 2-degree element approximation of u. Next we introduce the concrete method.

Let Ω be the polygonal area. We use the uniform triangulation of Ω as shown in Fig. 2, where the solid lines denote the thick grid dissection and the dashed lines denote the thickening grid. Let

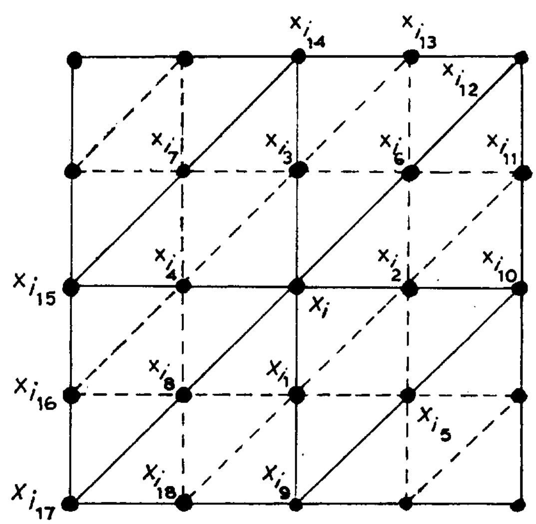

Fig. 2. Uniform triangulation.

$A_1 = \{x_{i0}, x_{i9}, x_{i12}, x_{i14}, x_{i15}, x_{i17}, \dots\}$ denote the thick grid nodes set, $A_2 = \{x_{i5}, x_{i6}, x_{i7}, x_{i8}, \dots\}$ the first thin grid nodes set and $A_3 = \{x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i11}, x_{i13}, x_{i18}, \dots\}$ the second thin grid nodes set.

The particle moves randomly from any $x_i = x_{i0} \in T^h$ to the boundary with many nets: it moves to the first thin grid nodes $x_{ij} \in A_2$ $(j = 5, 6, 7, 8)$ adjacent to x_i with the probability $\frac{1}{4}$, there is no harm in letting the particle move about to x_{i5}, next it moves to the second thin grid nodes $x_{il} \in A_3$ $(l = 2, 3, 11, 13)$ adjacent to x_{i5} with the probability $\frac{1}{4}$. Furthermore it moves to the first thin grid nodes $x_{ij} \in A_2$ $(j = 5, 6)$ or $x_{ik} \in A_1$ $(k = 0, 10)$ and so on and so forth, at last the particle reaches the boundary $x' \in \Gamma^h$ and be absorbed. This random moving procedure can be denoted as follows:

$$A_1 \to A_2 \to A_3 \left\{ \begin{array}{l} \xrightarrow{1/2} A_1 \\[2ex] \xrightarrow[1/2]{} A_2 \end{array} \right.$$

From A_1 to A_2 the moving step is $\sqrt{2}\,h$ and, from $A_2 \to A_3$ and $A_3 \to A_2$ the moving steps are h. So the moving law is similar to solving the process of multiple net. The differences are that the grid's thickness is a random variable and the grid's dimensions are also different.

Next we will prove that the mathematical expectation value $E(\xi)$ of the random variable ξ obtained with the above moving procedure is consistent with that obtained from a solution problem (2) corresponding to a finite element with degree 2.

Theorem 6. *Let u be the solution of problem* (2) *and u^h be the solution of problem* (3). *When $K^{h,k}$ is the 2-degree triangular element space, then $E(\xi(x_i)) \equiv u^h(x_i)$ for any $x_i \in T^h$.*

Proof. We use the uniform triangulation of Ω as in Fig. 3, $x_i = x_{i0} \in T^h$, the basis functions in adjacent unit nodes are φ_{ij} $(i = 0, 1, 2, \dots, 18)$. From (5)

$$u_i = \sum_{j=1}^{18} P_{ij} U_{ij} \quad (u_i = u^h(x_i)),$$

where

$$P_{ij} = \frac{a(\varphi_{i0}, \varphi_{ij})}{a(\varphi_{i0}, \varphi_{i0})} \quad (j = 1, 2, \dots, 18). \tag{6}$$

Calculating (6) we get

$$\begin{aligned} &P_{i1} = P_{i2} = P_{i3} = P_{i4} = \tfrac{1}{3}, \\ &P_{i9} = P_{i10} = P_{i11} = P_{i15} = -\tfrac{1}{12}, \\ &P_{ij} = 0, \quad \text{otherwise}. \end{aligned} \tag{7}$$

Substituting (7) into (5), we have the uniform triangulation

$$u_i = \tfrac{1}{3}(u_{i1} + u_{i2} + u_{i3} + u_{i4}) - \tfrac{1}{12}(u_{i9} + u_{i10} + u_{i14} + u_{i15}). \tag{8}$$

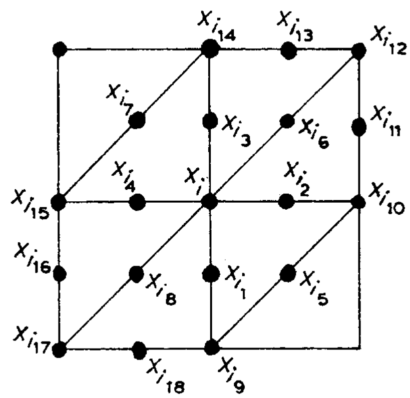

Fig. 3. Uniform triangulation.

Similarly, we can get

$$
\begin{aligned}
&u_{i1} = \tfrac{1}{4}(u_i + u_{i5} + u_{i6} + u_{i9}), \qquad u_{i2} = \tfrac{1}{4}(u_i + u_{i5} + u_{i6} + u_{i10}),\\
&u_{i3} = \tfrac{1}{4}(u_i + u_{i6} + u_{i7} + u_{i14}), \qquad u_{i4} = \tfrac{1}{4}(u_i + u_{i7} + u_{i8} + u_{i15}).
\end{aligned} \tag{9}
$$

Substituting (9) into (8), we have

$$u_i = u_{i5} + u_{i6} + u_{i7} + u_{i8}. \tag{10}$$

Since Eqs. (10) + (9) and (8) + (9) have the same solution, Eqs. (9) and (10) can be regarded as the local algebraic structure of the finite element with 2-degree. Because of the arbitrariness of x_i, we only need to prove that the mathematical expectation value $E(\xi)$ of the random variable ξ varies with many-nets satisfying Eqs. (9) and (10), i.e. $E(\xi(x_i)) \equiv u^h(x_i)$.

The procedure followed by the moving particle is the following: the particle leaves from x_i to reach the first four thin grid nodes $x_{ij} \in A_2$ $(j = 5, 6, 7, 8)$ with probability $\frac{1}{4}$, next it leaves from the first thin grid nodes to reach the boundary Γ^h and gets absorbed. If $P(a, b)$ denotes the transition probability with which the particle leaves from node a to reach $b \in \Gamma^h$ and be absorbed, then

$$P(a, b) = \frac{1}{4}\sum_{i=1}^{4} P(a_i, b),$$

where a_i $(i = 1, 2, 3, 4)$ are the adjacent nodes of node a. Then we have

$$
\begin{aligned}
E(\xi(x_i)) &= \sum_{x^i \in \Gamma^h} P(x_i, x')\, f(x')\\
&= \frac{1}{4}\sum_{j=5}^{8} P(x_{ij}, x')\, f(x')\\
&= \tfrac{1}{4}[\xi(x_{i5}) + E(\xi(x_{i5})) + E(\xi(x_{i7})) + E(\xi(x_{i8}))].
\end{aligned} \tag{11}
$$

For the same reason, we have

$$\begin{aligned}
E(\xi(x_{i1})) &= \tfrac{1}{4}[E(\xi(x_i) + E(\xi(x_{i5})) + E(\xi(x_{i8})) + E(\xi(x_{i9}))],\\
E(\xi(x_{i2})) &= \tfrac{1}{4}[E(\xi(x_i) + E(\xi(x_{i5})) + E(\xi(x_{i6})) + E(\xi(x_{i10}))],\\
E(\xi(x_{i3})) &= \tfrac{1}{4}[E(\xi(x_i) + E(\xi(x_{i6})) + E(\xi(x_{i7})) + E(\xi(x_{i14}))],\\
E(\xi(x_{i4})) &= \tfrac{1}{4}[E(\xi(x_i) + E(\xi(x_{i7})) + E(\xi(x_{i8})) + E(\xi(x_{i15}))].
\end{aligned} \tag{12}$$

If $x' \in \Gamma^h$, then $E(\xi(x')) = f^1(x')$. Therefore $E(\xi(x_i))$ satisfies Eqs. (9) and (10), so

$$E(\xi(x_i)) \equiv u^h(x_i). \qquad \square$$

Theorem 7. *The expectation value $E(\xi)$ and the variance $D(\xi)$ of the random variable ξ which moves with many-nets exist and are finite.*

Proof. Similar to Lemma 5. $\square$

Theorem 8. *If u is the solution of problem* (2) *and $u \in H^3(\Omega)$, u^h is the finite element with 2-degree solution of problem* (3), *$E(\xi)$ is the expectation value of ξ, then we have the error estimate*:

$$\|u - E(\xi)\|_{0.2} = \|u - u^h\|_{0.2} \leqslant Ch^3\|u\|_{3.2},$$

$$\|u - E(\xi)\|_{1.2} = \|u - u^h\|_{1.2} \leqslant Ch^3\|u\|_{3.2}.$$

Proof. From Theorem 6, for any $x_i \in T^h U \Gamma^h$, we have $E(\xi(x_i)) = u^h(x_i)$. From [4] if $u \in H^3$, then

$$\|u - u^h\|_{0.2} \leqslant Ch^3\|u\|_{3.2}, \qquad \|u - u^h\|_{1.2} \leqslant Ch^2\|u\|_{3.2}.$$

Therefore the proof is complete. $\square$

From Theorems 6–8, using the strong law of large numbers, let the particle move M times. Then we get M sample points $\xi^1(x_i), \xi^2(x_i), \dots, \xi^M(x_i)$. Let

$$\bar{E}(\xi) = \sum_{j=1}^{M} \xi^j(x_i)/M,$$

then $E(\xi) \cong \bar{E}(\xi) \cong u^h(x_i)$. Therefore we have a finite element with 2-degree precision approximation.

2.2. The boundary thickening method

In order to increase precision, we can use the boundary thickening method as shown in Fig. 4. Let Γ^h denote the boundary area and $\bar{\Gamma}^h$ denote the grid points near the boundary. The boundary thickening method can be described as follows. The particle moves about from $x_i \in T^h$ to T^h, we fix

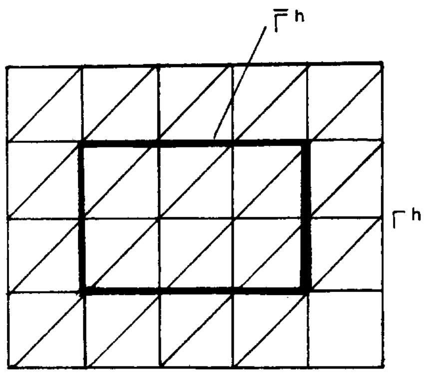

Fig. 4. The boundary thickening method.

the thin grid to be $L^{h/2}$ and the particle continues to move until it gets absorbed at the boundary. Examples show that this method has better approximation precision when we add a little computational work, particularly, this method is more valid for some complicated boundary.

3. Example

Consider the following problem:

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= \sin y\, \mathrm{e}^z \quad \text{on } \Omega, \end{aligned} \tag{13}$$

where $\Omega = \{(x, y);\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}$.

The strict solution of (13) is $u = \sin y\, \mathrm{e}^x$, for any $(x, y) \in \Omega$, we use the uniform triangulation $£^h$ as shown in Fig. 2 for two values of h, $h = \frac{1}{4}, \frac{1}{8}$. The results are shown in Tables 1–3.

Remarks. (1) Table 1 gives the error in the approximate solution $E(x_0)$ on the point $x_0 = (\frac{1}{4}, \frac{1}{4})$. The accuracy is approximately equal to $o(h^2)$.

Table 1
Linear triangular element probability computing method

h	M	$\lvert u - \bar{E}(\xi(\frac{1}{4}, \frac{1}{4}))\rvert$
$\frac{1}{4}$	100	$4.8 \cdot 10^{-2}$
	200	$3.4 \cdot 10^{-2}$
$\frac{1}{8}$	100	$1.4 \cdot 10^{-2}$
	200	$1.5 \cdot 10^{-2}$

Table 2
The probability multigrid method

h	M	$\lvert u - \bar{E}(\xi(\frac{1}{4},\frac{1}{4}))\rvert$
$\frac{1}{4} \sim \frac{1}{4}\sqrt{2}$	100	$3.3 \cdot 10^{-2}$
	200	$1.7 \cdot 10^{-2}$
$\frac{1}{8} \sim \frac{1}{8}\sqrt{2}$	100	$1.1 \cdot 10^{-2}$
	200	$3.2 \cdot 10^{-2}$

Table 3
The boundary thickening method

h	M	$\lvert u - \bar{E}(\xi(\frac{1}{4},\frac{1}{4}))\rvert$
	200	$1.4 \cdot 10^{-2}$
$\frac{1}{4} \sim \frac{1}{8}$	300	$3.6 \cdot 10^{-3}$
	400	$1.8 \cdot 10^{-3}$

(2) In Table 2, $h = \frac{1}{4} \sim \frac{1}{4}\sqrt{2}$ denotes the result of the walk for the digrid with step length $h = \frac{1}{4}$ and $h = \frac{1}{4}\sqrt{2}$. Its error is smaller than the error in Table 1. The error approximately equals the accuracy of the finite element with 2-degree: $o(h^3)$ between 10^{-2} and 10^{-3}.

(3) Table 3 shows that, in order to enhance the accuracy, we do not thicken triangulation at first, we can let the particle walk with thickening triangulation when it reaches the node adjoining the boundary.

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