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Weighted finite element method for elasticity problem with a crack

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ABSTRACT

We consider the Lamé system posed in a domain with the reentrant corner of 2π as a mathematical model for the crack problem. We construct a version of the weighted finite-element method (FEM) on the base of a novel definition of the R_ν-generalized solution. This allows us to suppress the influence of the singularity caused by the presence of the reentrant corner on the accuracy of computation of the approximate solution. Comparative numerical analysis of the presented approach with classical FEM and the method with mesh refinement has shown its advantages in the computational accuracy and stability as well as in the use of high-dimensional meshes. The results of investigation of the accuracy of solution to the model problem are presented in the Sobolev and energy norms. An absolute error in the mesh nodes is also analyzed.

1. Introduction

We consider a crack problem when a mathematical model is presented in the form of the Lamé system posed in a 2D domain Ω with the reentrant corner of 2π on the boundary. It is known that the weak solution of this boundary value problem belongs to the Sobolev space $W_2^{1+\beta-\varepsilon}(\Omega)$, where $\beta = 1/2$ when the Dirichlet or Neumann boundary conditions are specified on both sides of the crack, and $\beta = 1/4$ for the mixed Dirichlet-Neumann boundary conditions, where ε is an arbitrary positive number. In according to the principle of coordinated estimates (see, for example, [1]), the approximate solution by FEM converges to the exact one with the rate $O(h^\beta)$ ($\beta = 1/2$ or $\beta = 1/4$) in the norm of the space $W_2^1(\Omega)$.

In recent years different FEM versions were developed for solving 2D problems in elasticity posed in cracked domains; among them are smoothed FEM (see, for example, [2–7]), extended FEM (XFEM) [8–12], and meshless/meshfree methods [13–19]. Each of these methods has some advantages in comparison with classical FEM; however, one has to use adaptive meshes with refinement towards the crack tip in order to achieve the convergence rate $O(h)$ of the approximate solution to the exact one. Such approach leads to ill-conditioned systems of linear equations that makes computational process difficult and affects computational accuracy.

For this problem we have constructed and investigated weighted FEM [20–25] which provides the convergence rate $O(h)$. The solution of the boundary value problems for the Lamé systems was defined as R_ν-generalized one in the weighted set $W_{2,\nu}^1(\Omega, \delta)$ [26–30]. It gives a possibility, e.g., at a given accuracy 10^{-3}, to calculate the approximate solution of the mentioned problems by the weighted FEM 10^6–10^{12} times faster as compared with classical FEM. At the same time, one needs 10^6–10^{12} times less computational resources and energy consumption. Weighted FEM allows one to perform high-accuracy computations both inside the domain and in a neighborhood of the crack.

In addition to this, we have established, for the elasticity problem posed in domains with the reentrant corner, that for other computational conditions being equal, FEM with mesh refinement fails on high-dimensional meshes, whereas weighted FEM allows one to perform stable computations of approximate solution achieving theoretical accuracy [25]. We explain the reason of this failure by smallness of refined mesh steps in the neighborhood of the singular point. As a result, weighted FEM that has the same convergence rate $O(h)$ as FEM on meshes with refinement allows one to compute the solution on meshes with dimension by one or two orders higher [25].

This paper is organized as follows. In Section 2.1 we present the crack problem as a boundary value problem for the Lamé system posed in the domain with the reentrant corner of 2π. In Section 2.2 a new definition of the solution for the addressed problem as R_ν-generalized one is introduced and purposefulness of such approach is explained. In Section 2.3 the weighted FEM for the determination of approximate R_ν-generalized solution is elaborated and some

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comments about the theoretical convergence rate are made. In Section 3 a model problem is formulated. For this problem, a comparative numerical analysis of the convergence rate for the approximate solution obtained by classical FEM and weighted FEM in the energy and Sobolev norms is performed. In addition, comparative results for absolute errors in the mesh nodes for both methods are presented. Finally, some conclusions for these studies are summarized in Section 4.

2. Weighted finite element method

2.1. Problem setting

Let Ω be a bounded two-dimensional domain with a single straight edge crack (see Fig. 1). The boundary of Ω is denoted Γ. Let $\Gamma_C \subset \Gamma$ be a crack with tip located in the origin $(0,0)$ and Γ_C^+, Γ_C^- are crack sides. Let Γ_D and Γ_N be a disjoint parts of the boundary Γ with given displacements and traction.

Under small strains assumption, in Ω we consider the following boundary value problem of linear elasticity posed in displacements for isotropic homogeneous media:

$$-(2\mathbf{div}(\mu\varepsilon(\mathbf{u})) + \mathbf{grad}(\lambda \mathrm{div}\mathbf{u})) = \mathbf{f}, \quad x \in \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{q},\ x \in \Gamma_D, \quad \sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{p},\ x \in \Gamma_N. \tag{2}$$

Here $\mathbf{u} = (u_1, u_2)$ denotes a displacement field, $\varepsilon(\mathbf{u})$ is a strain tensor, $\sigma(\mathbf{u})$ is a stress tensor, $\mathbf{f}$ is a distributed body force, λ and μ are constant Lamé coefficients, $\mathbf{n} = (n_1, n_2)$ is the outward unit normal vector.

2.2. R_ν-generalized solution

We denote by $\Omega' = \{x \in \Omega : (x_1^2 + x_2^2)^{1/2} \leqslant \delta \ll 1\}$ the δ-neighbourhood of the point $(0,0)$ in Ω. We introduce a weight function $\rho(x)$ that coincides in Ω' with distance to the point $(0,0)$ and equals δ for $x \in \bar{\Omega} \setminus \bar{\Omega}'$.

We define the weighted spaces $L_{2,\alpha}(\Omega), L_{2,\alpha-1/2}(\Gamma_N)$ of functions with norms:

$$\|u\|_{L_{2,\alpha}(\Omega)} = \sqrt{\int_\Omega \rho^{2\alpha} u^2 dx}, \quad \|u\|_{L_{2,\alpha-1/2}(\Gamma_N)} = \sqrt{\int_{\Gamma_N} \rho^{2\alpha-1} u^2 ds}.$$

Let $W_{2,\alpha}^l(\Omega, \delta)$ $(l = 0, 1, \alpha \in R, \alpha \geqslant 0)$ be the set of functions satisfying the following conditions:

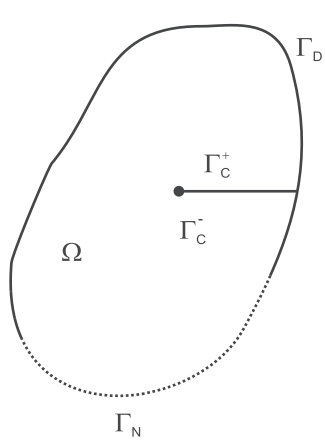

Fig. 1. Domain with a crack.

(a) $\left|D^k u(x)\right| \leqslant C_1(\delta/\rho(x))^{\alpha+|k|}, x \in \Omega'$,
$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2}}, k = (k_1, k_2), |k| = k_1 + k_2, k_1, k_2$ are non-negative integer numbers, $C_1 > 0$ is a constant independent of k;
(a) $\|u\|_{L_{2,\alpha}(\Omega \setminus \Omega')} \geqslant C_2, C_2 = \mathrm{const}$

with norm

$$\|u\|_{W_{2,\alpha}^l(\Omega)} = \sqrt{\sum_{|k| \leqslant l} \int_\Omega \rho^{2\alpha} (D^k u)^2 dx}. \tag{3}$$

For $l = 0$ we use notation $L_{2,\alpha}(\Omega, \delta)$. We also define the set $L_{2,\alpha-1/2}(\Gamma_N, \delta)$ of functions satisfying conditions (a) and (b) on the Γ_N with norm $\|u\|_{L_{2,\alpha-1/2}(\Gamma_N)}$.

We introduce the functional set $W_{2,\alpha}^{1/2}(\Gamma_D, \delta)$ consisting of traces on Γ_D of functions $\Phi \in W_{2,\alpha}^1(\Omega, \delta)$:

$$W_{2,\alpha}^{1/2}(\Gamma_D, \delta) = \{\varphi : \varphi = \Phi|_{\Gamma_D}, \Phi \in W_{2,\alpha}^1(\Omega, \delta)\}$$

with norm

$$\|\varphi\|_{W_{2,\alpha}^{1/2}(\partial\Omega,\delta)} = \inf_{\Phi|_\Gamma = \varphi} \|\Phi\|_{W_{2,\alpha}^1(\Omega)}.$$

Corresponding spaces and sets of vector-functions are designated with bold letters: $\mathbf{L}_{2,\alpha}(\Omega), \mathbf{W}_{2,\alpha}^1(\Omega, \delta), \mathbf{L}_{2,\alpha}(\Omega, \delta)$, $\mathbf{L}_{2,\alpha-1/2}(\Gamma_N, \delta), \mathbf{W}_{2,\alpha}^{1/2}(\Gamma_D, \delta)$.

We denote by $\mathbf{W}_{2,\alpha}^{1,D}(\Omega, \delta)$ the subset of $\mathbf{W}_{2,\alpha}^1(\Omega, \delta)$ that contains vector-functions with zero components on Γ_D:

$$\mathbf{W}_{2,\alpha}^{1,D}(\Omega, \delta) = \left\{\mathbf{v} = (v_1, v_2) \in \mathbf{W}_{2,\alpha}^1(\Omega, \delta) : v_i(x)|_{\Gamma_D} = 0, \quad i = 1, 2\right\}.$$

Assume that right hands of Eqs. (1) and boundary conditions (2) satisfy the requirements:

$$\mathbf{f} \in \mathbf{L}_{2,\beta}(\Omega, \delta), \quad \mathbf{q} \in \mathbf{W}_{2,\beta}^{1/2}(\Gamma_D, \delta), \quad \mathbf{p} \in \mathbf{L}_{2,\beta-1/2}(\Gamma_N, \delta), \tag{4}$$

where $\beta \geqslant 0$.

We define ***R***$_\nu$**-generalized solution** of problem (1), (2), (4) as a function $\mathbf{u}_\nu \in \mathbf{W}_{2,\nu}^1(\Omega, \delta)$ which meet the boundary conditions on Γ_D and satisfy the integral identity:

$$\int_\Omega 2\mu\varepsilon(\mathbf{u}_\nu) : \varepsilon(\rho^{2\nu}\mathbf{v}) + \lambda \mathrm{div}\mathbf{u}_\nu \mathrm{div}(\rho^{2\nu}\mathbf{v}) dx = \int_\Omega \rho^{2\nu} \mathbf{f} \cdot \mathbf{v}\mathbf{dx} + \int_{\Gamma_N} \rho^{2\nu} \mathbf{p} \cdot \mathbf{v}\mathbf{ds} \tag{5}$$

for all $\mathbf{v} \in \mathbf{W}_{2,\nu}^{1,D}(\Omega, \delta)$ and for any fixed value of ν such that

$\nu \geqslant \beta$.

The left hand side of identity (5) is a bilinear form $a(\mathbf{u}_\nu, \mathbf{v})$ and the right hand side is a linear form $l(\mathbf{v})$:

$$a(\mathbf{u}_\nu, \mathbf{v}) = \int_\Omega 2\mu\varepsilon(\mathbf{u}_\nu) : \varepsilon(\rho^{2\nu}\mathbf{v}) + \lambda \mathrm{div}\mathbf{u}_\nu \mathrm{div}(\rho^{2\nu}\mathbf{v}) dx,$$

$$l(\mathbf{v}) = \int_\Omega \rho^{2\nu} \mathbf{f} \cdot \mathbf{v}\mathbf{dx} + \int_{\Gamma_N} \rho^{2\nu} \mathbf{p} \cdot \mathbf{v}\mathbf{ds}.$$

Comment 1. In the beginning of the past century the Bubnov-Galerkin method was created to find an approximate solution on the base of introduction of the generalized (weak) solution to the boundary value problems. Such approach allowed one to develop the finite element method for determination of the approximate solution with accuracy $O(h)$ for the boundary value problems with discontinued initial data (coefficients and right hands of equations and boundary conditions). Presence of the reentrant corner on the

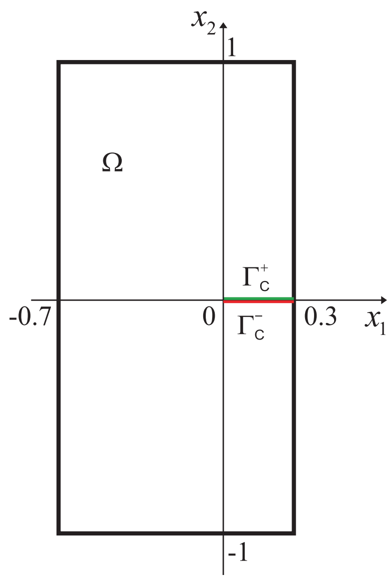

Fig. 2. Rectangle domain Ω with a crack.

boundary significantly decreases the accuracy of computation of the approximate solution. For boundary value problems posed in cracked domains the accuracy of the FEM becomes unacceptable. To overcome the singularity caused by the reentrant corner of 2π on the boundary, we introduce the weight function to the definition of the weak solution. This function is defined as a distance to the singularity point raised to some power. Such approach allowed one to decrease the influence of singularity to the accuracy of calculation of approximate solution and to archive the convergence rate $O(h)$ in the weighted norm without mesh refinement

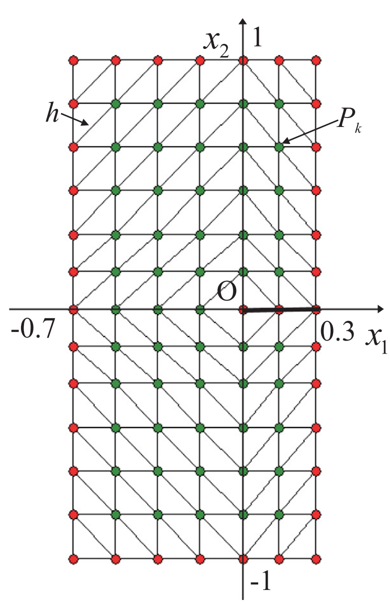

Fig. 3. Sample triangulation for model problem.

towards singularity point. And at the same time, numerical analysis showed that in our approach absolute difference between the approximate and the exact solution in the mesh nodes is in one or two times less than in the case of using traditional methods.

We proved the existence and uniqueness of the R_ν-generalized solution for the Lamé system with homogeneous Dirichlet boundary conditions posed on the whole boundary (see [31]).

2.3. Scheme of the weighted finite element method

In this section we construct the scheme of the weighted finite element method for determination of approximate R_ν-generalized solution of problem (1), (2), (4). For that we perform a quasiniform triangulation of domain Ω and introduce a special weighted vector basis functions. For concreteness all constructions will be realized when Ω is a rectangle as depicted on Fig. 2.

We decompose $\bar{\Omega}$ into the finite set of triangles K called finite elements. Their vertices $P_k, k = 1, \ldots, N$ are mesh nodes, the maximum length of sides of finite elements is designated h and called mesh step. We assume that the partition satisfies all conventional constrains imposed on quasiuniform triangulations (see [1]) and the mesh is coordinated with the crack Γ_C, i.e. Γ_C is composed of elements' sides and the crack tip is a mesh node.

The set of all triangulation nodes we designate $\mathrm{P} = \{P_k\}_{k=1}^N$. In P we single out the subset $\tilde{\mathrm{P}} = \{P_k\}_{k=1}^n$ of internal nodes and boundary nodes with Neumann boundary conditions and $\mathrm{P}^D = \{P_k\}_{k=n+1}^N$ is the subset of nodes where Dirichlet boundary conditions are imposed, $\mathrm{P} = \tilde{\mathrm{P}} \cup \mathrm{P}^D$.

Now we introduce weighted functions $\hat{\psi}_k(x)$ associated with nodes $P_k \in \mathrm{P}$:

$$\hat{\psi}_k(x) = \rho^{\nu^*}(x)\varphi_k(x),$$

here $\varphi_k(x)$ is a linear function on each finite element K and $\varphi_k(P_j) = \delta_{kj}, k, j = 1, \ldots, N, \delta_{kj}$ is a Kronecker delta and ν^* is a real number.

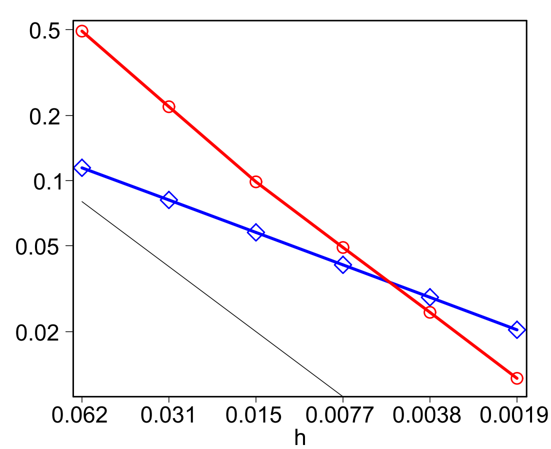

Fig. 4. Convergence rates of the relative errors η, η_ν for the approximate generalized (line with rhombs) and R_ν-generalized (line with circles) solutions in log scales respectively.

Table 1
Dependence of the relative errors η, η_ν for the approximate generalized and R_ν-generalized solutions from the mesh step h.

h	0.062	***k***	0.031	***k***	0.015	***k***	0.0077	***k***	0.0038	***k***	0.0019
η_ν	4.921e−1	***2.23***	2.198e−1	***2.23***	9.869e−2	***2.01***	4.908e−2	***2.00***	2.458e−2	***2.02***	1.216e−2
η	1.147e−1	***1.41***	8.135e−2	***1.41***	5.760e−2	***1.41***	4.077e−2	***1.41***	2.884e−2	***1.41***	2.040e−2

Using introduced functions $\hat{\psi}_i(x), i = 1, \ldots, N,$ we build weighted vector basis $\boldsymbol{\psi}_k(x), k = 1, \ldots, 2N$, in the following way:

$$\boldsymbol{\psi}_k(x) = \begin{cases} (\hat{\psi}_i(x), 0), & k = 2i - 1, \\ (0, \hat{\psi}_i(x)), & k = 2i, \end{cases} \quad i = 1, \ldots, N.$$

Let $\mathbf{V}^h$ be a vector linear span of basis functions $\{\boldsymbol{\psi}_k(x)\}_{k=1}^{2N}$. In $\mathbf{V}^h$ we single out the subset $\mathbf{V}^h_D$ containing vector functions equal to zero in nodes $P_k \in P^D$: $\mathbf{V}^h_D = \{\mathbf{v} \in \mathbf{V}^h : \mathbf{v}(P_k) = 0, P_k \in \mathrm{P}^D\}$.

A function $\mathbf{u}^h_\nu(x) \in \mathbf{V}^h$ we call an **approximate R_ν-generalized solution** of the problem (1), (2),(4) by the weighted FEM if it satisfy the boundary condition (2) for the mesh nodes $P_i \in \mathrm{P}^D$ and for any $\mathbf{v}^h_D(x) \in \mathbf{V}^h_D$ and $\nu > \beta$ the following integral identity holds

$$a(\mathbf{u}^h_\nu(x), \mathbf{v}^h_D(x)) = l(\mathbf{v}^h_D(x)).$$

Now we can write a finite-element approximation of the solution associated with the built triangulation:

$$\mathbf{u}^h_\nu = \sum_{k=1}^{2N} d_k \boldsymbol{\psi}_k(x), \tag{6}$$

here $d_k = \rho^{-\nu^*}(P_{[(k+1)/2]})c_k, c_k = \begin{cases} u^h_{\nu,1}(P_{[(k+1)/2]}), & k = 2i - 1 \\ u^h_{\nu,2}(P_{[(k+1)/2]}), & k = 2i \end{cases}, i = 1, \ldots, N$, $[(k+1)/2]$ is an integer part of number $(k+1)/2$. In (6) unknown coefficients $d_k, k = 1, \ldots 2n$, corresponding to the nodes $P_i \in \tilde{\mathrm{P}}$ can be found from the system of linear algebraic equations

$$a(\mathbf{u}^h_\nu, \boldsymbol{\psi}_k) = l(\boldsymbol{\psi}_k), \quad k = 1, \ldots, 2n, \tag{7}$$

or

Table 2
The values of the parameter δ for determination of the approximate R_ν-generalized solution in dependence of the mesh step h.

h	0.062	0.031	0.015	0.0077	0.0038	0.0019
δ	0.091	0.075	0.061	0.054	0.051	0.05

Table 3
Dependence of the relative errors η^E, η^E_ν for the approximate generalized and R_ν-generalized solutions from the mesh step h.

h	0.062	***k***	0.031	***k***	0.015	***k***	0.0077	***k***	0.0038	***k***	0.0019
η^E_ν	8.747e−1	***2.15***	4.068e−1	***2.25***	1.814e−2	***1.98***	9.135e−2	***1.98***	4.583e−2	***2.00***	2.291e−2
η^E	2.519e−1	***1.41***	1.784e−2	***1.41***	1.262e−2	***1.41***	8.928e−2	***1.41***	6.314e−2	***1.41***	4.465e−2

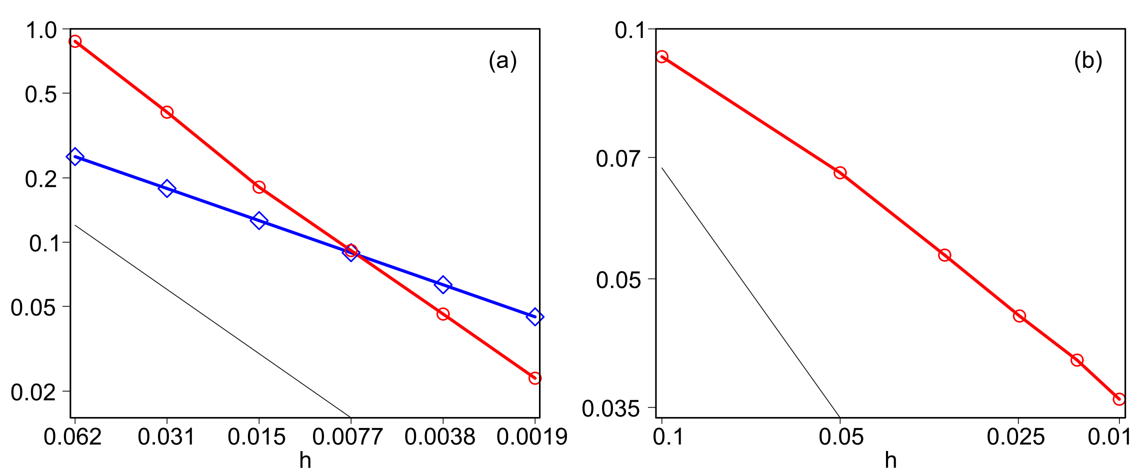

Fig. 5. (a) Convergence rates of the relative errors η^E, η^E_ν for the approximate generalized (line with rhombs) and R_ν-generalized (line with circles) solutions in log scales respectively. (b) Convergence rate of the relative error η^E for the approximate solution obtained by the sES-FEM-T3-5 method on the uniform meshes (line with circles) in log scale (see [6, Fig. 15a]).

Table 4
Percentage of nodes P_i where the absolute errors δ_{i1} (n_1, generalized solution) and δ^ν_{i1} (n^ν_1, R_ν-generalized solution) are less than the limit value $\bar{\Delta} = 10^{-7}$.

h	0.062	0.031	0.015	0.0077	0.0038	0.0019
n_1	4.194%	9.463%	20.400%	38.620%	60.382%	81.261%
n^ν_1	4.194%	10.362%	13.101%	46.144%	73.635%	96.860%

Table 5
Percentage of nodes P_i where the absolute errors δ_{i2} (n_2, generalized solution) and δ^ν_{i2} (n^ν_2, R_ν-generalized solution) are less than the limit value $\bar{\Delta} = 10^{-7}$.

h	0.062	0.031	0.015	0.0077	0.0038	0.0019
n_2	0.0%	1.169%	6.139%	12.853%	23.058%	38.078%
n^ν_2	0.0%	1.304%	3.749%	15.699%	30.588%	69.860%

$$A\mathbf{d} = \mathbf{F},$$

where

$$\mathbf{d} = (d_1, \dots, d_{2n})^T, \quad \mathbf{F} = (F_1, \dots, F_{2n})^T,$$

$$A_{ij} = a(\psi_i, \psi_j), \; F_i = l(\psi_i), \quad i, j = 1, \dots, 2n.$$

Comment 2. Singular functions introduced into the basis take into account the asymptotics of the solution near crack tip. This allowed one to create FEM without loss of accuracy for the Lamé system posed in the cracked domain. At that we use quasiuniform meshes without refinement toward the singularity point.

In [32, Theorem 8] we proved that the approximate solution (6) by the weighted FEM converges to the exact one for the problem (1), (2) with the first rate with respect to the mesh step h.

3. Numerical experiments

In this section we describe numerical experiments for the model problem using weighted FEM described in Section 2.3 and present derived results of comparative numerical analysis. The model problem is described in Section 3.1. In Section 3.2 we investigate the convergence rate of the approximate R_ν-generalized and generalized solutions in different norms. In Section 3.3 the absolute error of approximate R_ν-generalized and generalized solutions in mesh nodes P_k is analyzed.

3.1. Model problem

We consider a model problem (1), (2) in domain $\Omega = (-0.7, 0.3) \times (-1, 1)$ with crack $\Gamma_C = [0, 0.3) \times \{0\}$. We assume that Dirichlet boundary conditions are applied on all boundary Γ. The exact solution is a vector-function $\mathbf{u} = (u_1, u_2)$ with components

$$u_1 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\cos\left(\frac{\theta}{2}\right)\left(1 - \frac{\lambda}{\lambda+\mu} + \sin^2\left(\frac{\theta}{2}\right)\right),$$

$$u_2 = \frac{K_I}{\mu}\sqrt{\frac{r}{2\pi}}\sin\left(\frac{\theta}{2}\right)\left(2 - \frac{\lambda}{\lambda+\mu} - \cos^2\left(\frac{\theta}{2}\right)\right).$$

Lamé coefficients are $\lambda = 576.923, \mu = 384.615$ Pa and stress intensity factor $K_I = 1.611$.

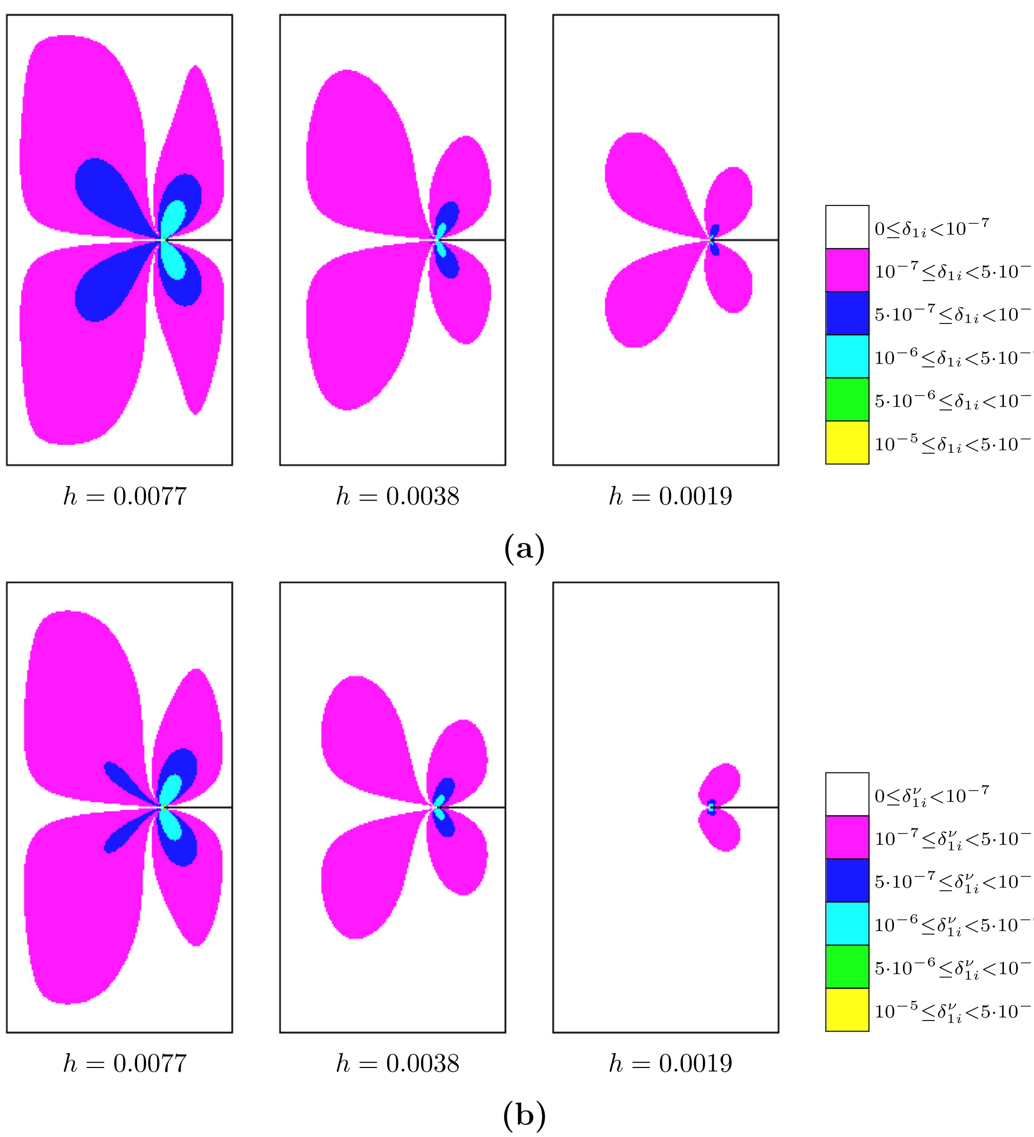

Fig. 6. Distribution of the absolute errors δ_{1i} for the approximate generalized (a) and δ^{ν}_{1i} for the approximate R_ν-generalized (b) solutions.

In Ω we construct quasiuniform mesh with step h and nodes P_k (see Fig. 3). The approximate R_ν-generalized $\mathbf{u}_\nu^h$ and generalized $\mathbf{u}^h$ ($\nu = 0, \rho(x) \equiv 1$) solutions were obtained using weighted FEM described in Section 2.3. Optimal parameters δ, ν, ν^* of the weighted FEM were established by software complex [33]. Results of numerical analysis are presented on graphs, pictures and in tables.

3.2. Convergence in Sobolev and energy norms

Relative errors for the approximate generalized and R_ν-generalized solutions were computed in Sobolev norm and weighted Sobolev norm (with $\delta = 0.005$) respectively:

$$\eta = \frac{\|\mathbf{u}-\mathbf{u}^h\|_{W_2^1(\Omega)}}{\|\mathbf{u}\|_{W_2^1(\Omega)}}, \eta_\nu = \frac{\|\mathbf{u}-\mathbf{u}_\nu^h\|_{W_{2,\nu}^1(\Omega)}}{\|\mathbf{u}\|_{W_{2,\nu}^1(\Omega)}}.$$

Also we computed relative errors in energy norm and weighted energy norm:

$$\eta^E = \frac{\|\mathbf{u}-\mathbf{u}^h\|_{E(\Omega)}}{\|\mathbf{u}\|_{E(\Omega)}}, \; \eta_\nu^E = \frac{\|\mathbf{u}-\mathbf{u}_\nu^h\|_{E_\nu(\Omega)}}{\|\mathbf{u}\|_{E_\nu(\Omega)}},$$

where

$$\|\mathbf{u}-\mathbf{u}^h\|_{E(\Omega)} = \sqrt{\frac{1}{2}\int_\Omega \left(\lambda(\mathrm{div}(\mathbf{u}-\mathbf{u}^h))^2 + 2\mu\sum_{i,j=1}^{2}(\varepsilon_{ij}(\mathbf{u}-\mathbf{u}^h))^2\right)dx},$$

$$\|\mathbf{u}-\mathbf{u}_\nu^h\|_{E_\nu(\Omega)} = \sqrt{\frac{1}{2}\int_\Omega \rho^{2\nu}\left(\lambda(\mathrm{div}(\mathbf{u}-\mathbf{u}^h))^2 + 2\mu\sum_{i,j=1}^{2}(\varepsilon_{ij}(\mathbf{u}-\mathbf{u}^h))^2\right)dx}.$$

In Table 1 we present the values of η, η_ν for meshes with different steps h, and k is a ratio between previous and subsequent values of norms. On Fig. 4 we depict the convergence rates graphs for the approximate generalized (line with rhombs) and R_ν-generalized (line with circles) solutions in log scales. Solid line designates the convergence rate $O(h)$.

Comment 3. In the present there is no universal mathematic algorithm which allows us to calculate by some formula the optimal values of the parameters ν, ν^*, δ for determination of approximate solutions to the boundary value problems for the elliptic equations, Lamé and Maxwell systems and the Stokes problem [21,22,24].

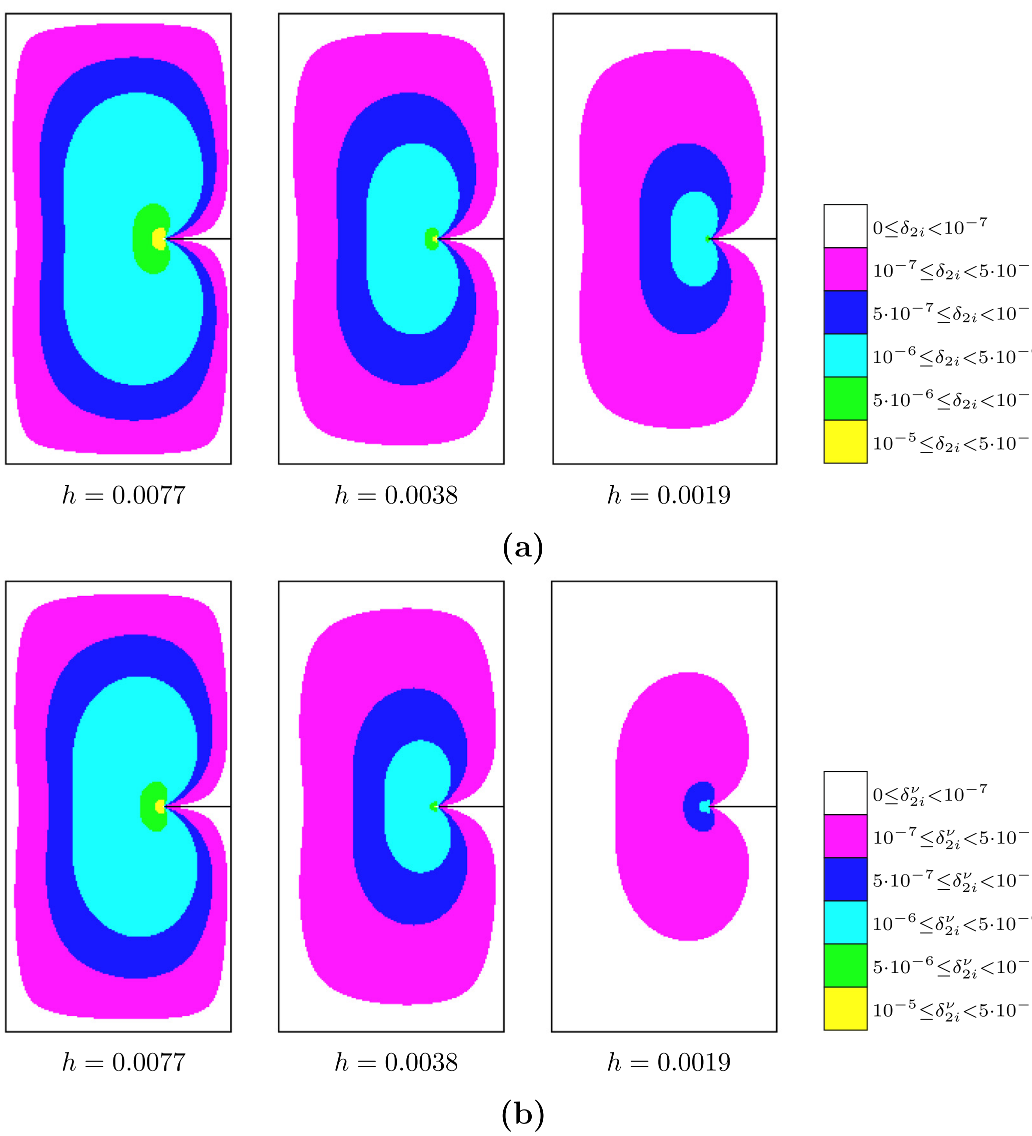

Fig. 7. Distribution of the absolute errors δ_{2i} for the approximate generalized (a) and δ^ν_{2i} for the approximate R_ν-generalized (b) solutions.

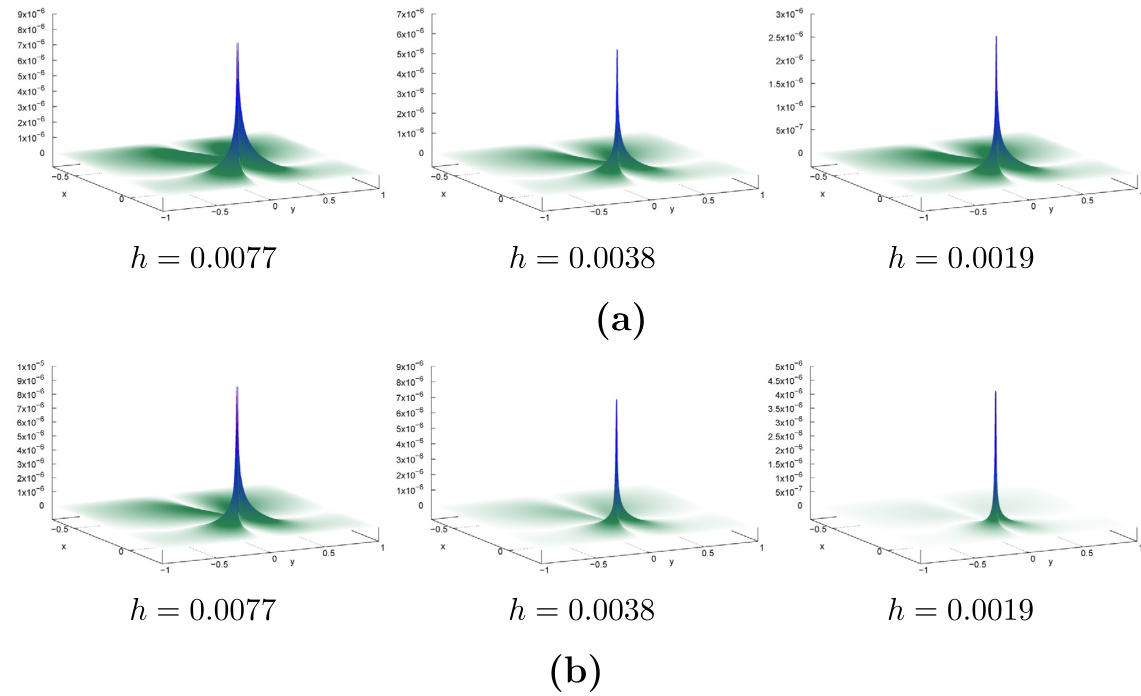

Fig. 8. Graph of the absolute errors δ_{1i} for the approximate generalized (a) and δ_{1i}^{ν} for the approximate R_ν-generalized (b) solutions.

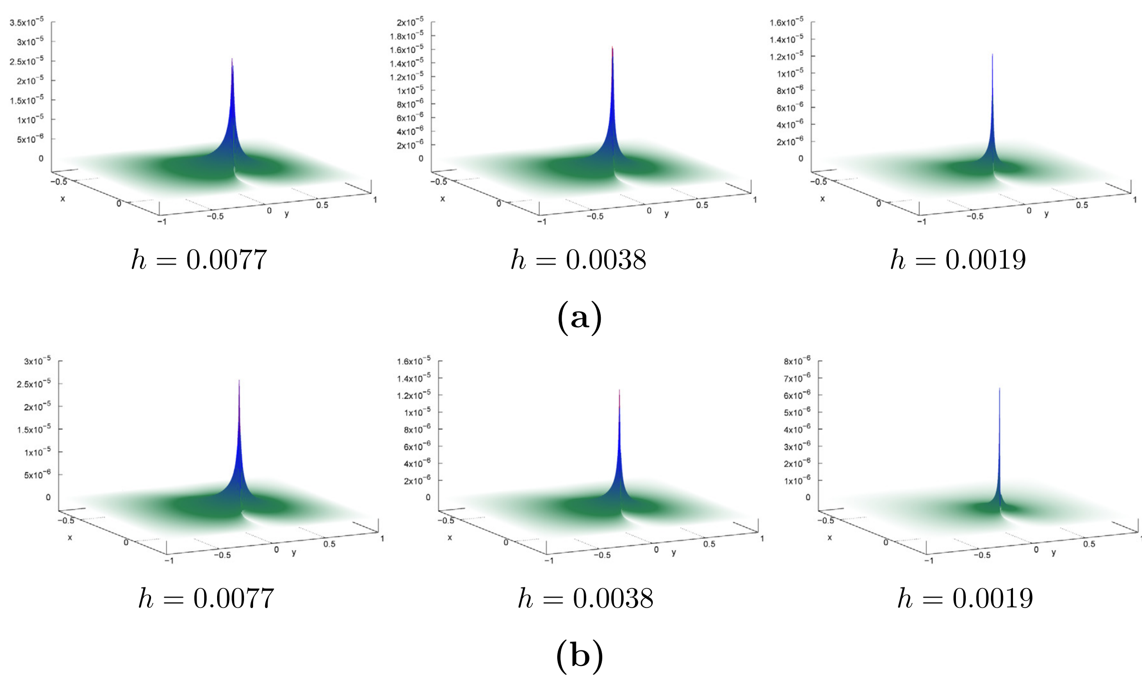

Fig. 9. Graph of the absolute errors δ_{2i} for the approximate generalized (a) and δ_{2i}^{ν} for the approximate R_ν-generalized (b) solutions.

We call as optimal values such intervals of the parameters ν, ν^*, δ for which changing speed of the approximate solution relative error calculated in the weighted norm in not less than theoretical convergence rate $O(h)$ when the mesh in refined.

It is known that the values of the parameters ν, ν^* depend on the set $W^1_{2,\nu'}(\Omega, \delta)$ where the R_ν-generalized solution is contained. The value of ν' in definition of the set $W^1_{2,\nu'}(\Omega, \delta)$ depends on the regularity of the coefficients, right hands of equations and boundary conditions, change of type of the boundary conditions and the geometry of the domain boundary. For the Dirichlet problem for the Lamé system posed in the domain with reentrant corner on the boundary, the lower bound of the parameter ν' is established in [31]. In this work it is also proved that the R_ν-generalized solution is the same for all values of ν satisfying the inequality $\nu \geqslant \nu'$.

We created the code for numerical analysis of model problems with known exact solution [33]. This code allows us to define by the trial-and-error method the intervals of optimal parameters ν, ν^*, δ. On the base of a large amount of numerical experiments we noted a relationship of value of the parameter ν^* with asymptotic of the exact solution in the neighborhood of the singularity point, and value of the parameter δ with the mesh step. In computations the domain Ω' (the δ-neighborhood of the singularity point $O(0,0)$) should contain not large amount of the mesh nodes (not more of several dozens of nodes).

For the considered problem we defined the following intervals of optimal parameters: $\nu* \in [0, 0.5]$, $\nu \in [1.0, 2.9]$, and the universal values of the parameter δ for each mesh (see Table 2).

In the present paper for numerical analysis of the mathematical model (1), (2) we used the values $\nu^* = 0.0, \nu = 1.8$. □

In Table 3 and on Fig. 5a we present analogous data for η^E and η^E_ν. The graph of the convergence rate of relative error in energy norm $O(h^{0.57})$ for the approximate solution obtained by the sES-FEM-T3-5 method on the uniform meshes is presented on Fig. 5b in log scale (we used data from [6], Fig. 15a). The convergence rate of the sES-FEM-T3-5 is higher than for other methods on the uniform meshes (see [6], Fig. 15a).

3.3. Absolute error distribution

We calculated the absolute differences between the exact solution and the approximate generalized and R_ν-generalized one in the mesh nodes $P_i : \delta_{ij} = \left|u_j(P_i) - u^h_j(P_i)\right|, \delta^\nu_{ij} = |u_j(P_i) - u^h_{\nu,j}(P_i)|, j = 1, 2$. Let n_j, n^ν_j be a mesh nodes P_i where the absolute errors δ_{ij} and δ^ν_{ij}, respectively, are less than the limit value $\bar{\Delta} = 10^{-7}$. We adduce numbers n_j, n^ν_j for different values of h in Table 4 and Table 5. We present distribution of the absolute errors $\delta_{i1}, \delta^\nu_{i1}$ and $\delta_{i2}, \delta^\nu_{i2}$ on Figs. 6 and 7. Three dimensional graphs of $\delta_{i1}, \delta^\nu_{i1}$ and $\delta_{i2}, \delta^\nu_{i2}$ are depicted on Figs. 8 and 9 respectively.

4. Discussion of the results. Conclusion

In this paper we suggested a new weighted FEM for numerical solution of the for elasticity problem with a crack. The following conclusions and discussions can be drawn:

1. A novel definition of the R_ν-generalized solution for this problem allowed one to suppress the influence of the reentrant corner of 2π on the boundary on the accuracy of computation of the solution. We note that the R_ν-generalized solution identically coincides with the weak solution for the problem posed. These solutions differ only in the form of integral identity (5) and in the spaces in which they are defined.
2. The approximate R_ν-generalized solution of the problem posed in the cracked domain (1), (2) converges to the exact one with the rate $O(h)$ in the norm of the space $W^1_{2,\nu}(\Omega)$ and weighted energy norm $\|\cdot\|_{E_\nu(\Omega)}$ unlike the generalized solution which has the convergence rate $O(h^\beta)$, ($\beta = 1/2$ or $\beta = 1/4$) in the norm of the space $W^1_2(\Omega)$ and energy norm $\|\cdot\|_{E(\Omega)}$. Suggested approach allows one to compute approximate solution by the weighted FEM for given accuracy 10^{-3} in 10^6–10^{12} times faster than in the case of classical FEM. We also note that for realization of the weighted FEM we use in times less of computing resources and energy consumption.
3. The absolute value of the error of the approximate R_ν-generalized solution is by one or two orders of magnitude better than for the generalized solution in the overwhelming majority of nodes.
4. The reported results can be achieved only due to the proper introduction the of the solution of problem. The proposed approach does not require complicated schemes with mesh adaptation and does not lead to the ill-conditioned systems of linear equations. As discussed in [25], the weighted FEM, compared with the FEM with mesh refinement, allows one to compute the solution on meshes with dimensions by one to two orders of magnitude higher.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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