A generalization of intransitive dice

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Abstract

In this paper, a generalization of intransitive dice is addressed. A general expression for a set of instransitive dice of any given size is provided, and the the probability of one die rolling higher than other in the set is explicitly computed. Using this expression, the non-transitivity property of these sets is demonstrated, along with some symmetries of these sets. Additionally, some relevant questions about intransitive dice are addressed providing some insight about this mathematical curiosity.

1 Introduction

Commonly speaking, we will say that a set of dice is intransitive if contains dice with the property that given a particular die in the set, some dice roll higher more than half the time, but also others roll lower half the time. This means that the relation between the dice "die A beats die B" is not transitive. Obtaining such a set of dice is difficult in general, the structure of these sets is not very intuitive. Intransitive dice in general do not need to have 6 faces, and the number of dice in the set may vary but note that there should be at least 3 dice to have a non-degenerate set. We will write $A > B$ when "die A beats die B".

Some particular cases have been produced over time, for several number of players and several number of faces in the dice. Next some examples are depicted, in each row of the matrices the different faces of each dice are displayed.

Efron's dice This set of dice consist of 4 6-faced dice. Observe that faces are repeated, and also a trivial die D_2 is in the set, but no face value appears in two different dice. It can be verified that $\mathbb{P}[D_1 < D_2] > \frac{1}{2}, \mathbb{P}[D_2 < D_3] > \frac{1}{2}, \mathbb{P}[D_3 < D_4] > \frac{1}{2}$ and $\mathbb{P}[D_1 < D_2] > \frac{1}{2}$, so $D_1 > D_4, D_4 > D_3, D_3 > D_2$ and $D_2 > D_1$.

$$\begin{array}{ll} D_1: & [\begin{array}{cccccc} 2 & 2 & 2 & 2 & 6 & 6 \end{array}] \\ D_2: & [\begin{array}{cccccc} 3 & 3 & 3 & 3 & 3 & 3 \end{array}] \\ D_3: & [\begin{array}{cccccc} 4 & 4 & 4 & 4 & 0 & 0 \end{array}] \\ D_4: & [\begin{array}{cccccc} 5 & 5 & 5 & 1 & 1 & 1 \end{array}]. \end{array} \tag{1}$$

Miwin's dice This set of dice consist of 3 6-faced dice. Here we have a nice property, which is the symmetry in the probabilities of one die beating each other ($\mathbb{P}[D_1 < D_2] = \mathbb{P}[D_2 < D_3] = \mathbb{P}[D_3 < D_1]$). Also notice that in this case no face is repeated inside a dice, but some faces appear in different dice. It can also be verified that $D_1 > D_3, D_3 > D_2$ and $D_2 > D_1$.

$$\begin{array}{ll} D_1: & [\begin{array}{cccccc} 1 & 2 & 5 & 6 & 7 & 9 \end{array}] \\ D_2: & [\begin{array}{cccccc} 1 & 3 & 4 & 5 & 8 & 9 \end{array}] \\ D_3: & [\begin{array}{cccccc} 2 & 3 & 4 & 6 & 7 & 8 \end{array}] \end{array} \tag{2}$$

Oskar dice Oskar van Deventer introduced a set of seven intransitive dice. The original version consist of a set of common dice with 6 faces with each of the next faces repeated twice per dice.

$$\begin{array}{ll} D_1: & [\begin{array}{ccc} 2 & 14 & 17 \end{array}] \\ D_2: & [\begin{array}{ccc} 7 & 10 & 16 \end{array}] \\ D_3: & [\begin{array}{ccc} 5 & 13 & 15 \end{array}] \\ D_4: & [\begin{array}{ccc} 3 & 9 & 21 \end{array}] \\ D_5: & [\begin{array}{ccc} 1 & 12 & 20 \end{array}] \\ D_6: & [\begin{array}{ccc} 6 & 8 & 19 \end{array}] \\ D_7: & [\begin{array}{ccc} 4 & 11 & 18 \end{array}] \end{array} \tag{3}$$

It can be verified that $D_1 > \{D_2, D_3, D_4\}$, $D_2 > \{D_3, D_4, D_6\}$, $D_3 > \{D_4, D_5, D_7\}$, $D_4 > \{D_1, D_5, D_6\}$, $D_5 > \{D_2, D_6, D_7\}$, $D_6 > \{D_1, D_3, D_7\}$ and $D_7 > \{D_1, D_2, D_4\}$.

Grime dice Dr. James Grime discovered a set of five intransitive dice. Again this set of dice has repated faces inside each die, but dice do not share faces.

$$\begin{array}{ll} D_4: & [\begin{array}{cccccc} 4 & 4 & 4 & 4 & 4 & 9 \end{array}] \\ D_5: & [\begin{array}{cccccc} 3 & 3 & 3 & 3 & 8 & 8 \end{array}] \\ D_1: & [\begin{array}{cccccc} 2 & 2 & 2 & 7 & 7 & 7 \end{array}] \\ D_2: & [\begin{array}{cccccc} 1 & 1 & 6 & 6 & 6 & 6 \end{array}] \\ D_3: & [\begin{array}{cccccc} 0 & 5 & 5 & 5 & 5 & 5 \end{array}] \end{array} \tag{4}$$

Tetrahedra Also intransitive dice of other number of faces have been discovered, for instance the next 4-faced dice.

$$\begin{array}{rl} D_1: & [\ 1\ \ 4\ \ 7\ \ 7\ ] \\ D_2: & [\ 2\ \ 6\ \ 6\ \ 6\ ] \\ D_3: & [\ 3\ \ 5\ \ 5\ \ 8\ ] \end{array} \tag{5}$$

A very common application of these dice is to "cheat" in the game of initially choosing a die from the set and betting which player rolls a bigger number most often. In this game, the last player to choose die (the "cheater") can always find a die that beats all previously selected dice. Efron's and Miwin's dice allow games of two players, while Oskar's and Grime's dice allow games of three players. We will call this game of betting on who rolls a higher die more often the *High Roller*.

In this report, for a given size N of the intransitive set of dice, a expression for a generalized set of dice with N faces is provided. These set of dice allow cheating for two players *High Roller* games.

2 Definitions

In this section, the notation to be used throughout the paper is described. Additionally, some concepts and hypotheses that will be necessary for the subsequent descriptions and proofs are formalized.

2.1 Set of dice

Given $N \geq 3$, we define a set of dice indexed by the index $n \in \{1, 2, \ldots, N\}$ as the set of uniform random variables $\{D_n\}_{n=1}^{N}$ each taking values in the set $V_n = \{v_{n,1}, v_{n,2}, \ldots, v_{n,M}\} \subset \mathbb{N}$. We consider fair dice, so $\mathbb{P}[D_n = v_{n,j}] = \frac{1}{M}, j = 1, 2, \ldots, M$. It is, we have a set of N fair dice, each with M faces.

2.2 Intransitive property of a set of dice

We define two types of intransitive property.

2.2.1 Weak intransitive property

For a set of dice $\{D_n\}_{n=1}^{N}$, we say that the set of dice is weak intransitive if $\forall m = 1, 2, \ldots, N$ there exists an $n = 1, 2, \ldots, N$ with

$$\mathbb{P}[D_m < D_n] > \frac{1}{2}. \tag{6}$$

This means that for any die in the set, there is another die that beats it.

2.2.2 Strong intransitive property

For a set of dice $\{D_n\}_{n=1}^{N}$, we say that the set of dice is strong intransitive if $\forall m, n = 1, 2, \ldots, N$ holds

$$(n - m) \,\%\, (N) < N/2 \Rightarrow \mathbb{P}\left[D_m < D_n\right] > \frac{1}{2}. \tag{7}$$

This means that each die is beaten by $N/2 - 1$ dice.

Visually, if we arrange the dice in a cycle, then each dice beats the ones at one side and is beaten by the dice to the other side. A visualization of this concept is depicted in Figure 1.

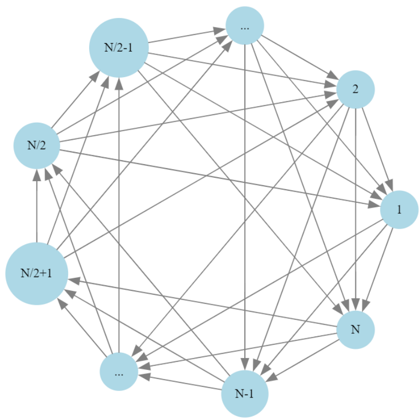

Figure 1: Diagram of strong intransitive dice.

3 Generalized set of intransitive dice

Given $N \geq 3$, a generalized formula for the face values of intransitive dice is

$$v_{n,j} = (j - 1)N + (n - j) \,\%\, (N) + 1. \tag{8}$$

The generated matrix of faces has next shape

$$\begin{array}{lllllllll}
D_1 & : & [& 1 & N+N & 2N+(N-1) & \cdots & (N-2)N+3 & (N-1)N+2 &] \\
D_2 & : & [& 2 & N+1 & 2N+N & \cdots & (N-2)N+4 & (N-1)N+3 &] \\
D_3 & : & [& 3 & N+2 & 2N+1 & \cdots & (N-2)N+5 & (N-1)N+4 &] \\
\cdots & & & & & & \cdots & & \cdots & \\
D_{N-2} & : & [& N-2 & N+(N-3) & 2N+(N-4) & \cdots & (N-2)N+N & (N-1)N+(N-1) &] \\
D_{N-1} & : & [& N-1 & N+(N-2) & 3N+(N-3) & \cdots & (N-2)N+1 & (N-1)N+N &] \\
D_N & : & [& N & N+(N-1) & 3N+(N-2) & \cdots & (N-2)N+2 & (N-1)N+1 &].
\end{array} \tag{9}$$

A more visual description is that the value of each face is computed by adding as a column a permutation of the vector $[1, 2, \ldots, N]$

$$
\begin{array}{lclllllll}
D_1 & : & \bar{v} + [& 1^* & N & N-1 & \cdots & 3 & 2 &] \\
D_2 & : & \bar{v} + [& 2 & 1^* & N & \cdots & 4 & 3 &] \\
D_3 & : & \bar{v} + [& 3 & 2 & 1^* & \cdots & 5 & 4 &] \\
\cdots & & & & & & \cdots & & \cdots & \\
D_{N-2} & : & \bar{v} + [& N-2 & N-3 & N-4 & \cdots & N & N-1 &] \\
D_{N-1} & : & \bar{v} + [& N-1 & N-2 & N-3 & \cdots & 1^* & N &] \\
D_N & : & \bar{v} + [& N & N-1 & N-2 & \cdots & 2 & 1^* &],
\end{array}
\tag{10}
$$

where $\bar{v} = [0, N, 2N, \ldots, (N-1)N]$.

In this particular case, a shift of a single position is choosen, but any generator of the cyclic group can be choosen. In other words, a more general expression could be considered

$$
v_{n,j} = (j-1)N + (p(n-j))\,\%\,(N) + 1, \tag{11}
$$

for any $p \in \mathbb{N}$ with $\mathrm{mcd}(p, N) = 1$. For the sake of simplicity we consider $p = 1$.

3.1 Properties

Some properties to mention is that no face is repeated, inside a die or among dice in the intransitive set. This guarantees that $\forall m, n, \mathbb{P}\left[D_m = D_n\right] = 0$, so no tie in roll can occur.

3.1.1 Probability of one die rolling higher

In Appendix B it is computed the analytic expression of the probability of one die in the set being higher than other die

$$
\mathbb{P}\left[D_m < D_n\right] = \frac{1}{2} + \frac{1}{2N} - \frac{(n-m)\,\%\,(N)}{N^2}. \tag{12}
$$

As a consequence a symmetric property is derived, it is $\mathbb{P}\left[D_{m+k} < D_{n+k}\right] = \mathbb{P}\left[D_m < D_n\right], \forall k = 1, 2, \ldots, N$. This means that the probability of a die rolling higher than other depends on how far are the indices of the dice. Note also that if N is even, then for any n, m with $n - m = N/2$, then $\mathbb{P}\left[D_m < D_n\right] = \mathbb{P}\left[D_m > D_n\right] = 1/2$. In other words, if N is even, for any die ther is another die in the set that does not beat it and is not beaten by it.

3.1.2 Intransitive property

We have a (even stronger) version of the strong intransitive property of the proposed dice, as there is an equivalence statement instead of the implication in the general statement of the property.

Statement For a set of dice $\{D_n\}_{n=1}^{N}$, with $\mathbb{P}[D_n = v_{n,j}] = \frac{1}{N}$, where

$$v_{n,j} = (j-1)N + (n-j)\,\%\,(N) + 1, \tag{13}$$

then

$$(n-m)\,\%\,(N) < N/2 \iff \mathbb{P}[D_m < D_n] > \frac{1}{2}. \tag{14}$$

Proof Using the hypothesis $(n-m)\,\%\,(N) < N/2$, the strong intransitive property is proof just by using that

$$\frac{(n-m)\,\%\,(N)}{N^2} < \frac{1}{2N}, \tag{15}$$

into equation 12 to obtain the lower bound

$$\mathbb{P}[D_m < D_n] > \frac{1}{2}. \tag{16}$$

Also assuming $\mathbb{P}[D_m < D_n] > \frac{1}{2}$, in expression in equation 12 we obtain the inequality

$$\frac{1}{2} + \frac{1}{2N} - \frac{(n-m)\,\%\,(N)}{N^2} > \frac{1}{2}, \tag{17}$$

thus

$$\frac{N}{2} > (n-m)\,\%\,(N)\,. \tag{18}$$

3.1.3 Cheating on ***High Roller***

To cheat at *High Roller* game with two players, if the first player chooses dice m, then the second player can choose the $(m+1)\,\%\,(N)$-th die to have the advantage. In general the second player could choose any dice n with $(n-m)\,\%\,(N) < N/2$, so it can be verified that there are $\lfloor N/2 - 1 \rfloor$ candidates.

These remaining dice that beat the die chosed by first player intuitively might provide somehow room for allowing cheating in the *High Roller* game for more players. However, it is enough to see that if first player chooses dice 1 and second player chooses $N/2$ dice, then there is no die to be choosen by last player that beats both (For instance die 2 beats die 1, but does not beat $N/2$ as $(2 - N/2)\,\%\,(N) = N + 2 - N/2 = N/2 + 2 \not< N/2$, and die $N/2 + 1$ beats $N/2$ but does not beat 1 since $(N/2 + 1 - 1)\,\%\,(N) = N/2 \not< N/2$, and dice 2 and $N/2 + 1$ are the best (closer to both dice) candidates). Thus the generalization of set of intransitive dice allowing cheating in *High Roller* games for three or more players remains as future work.

3.2 Examples

In this section, some particular cases of generalized intransitive dice are depicted to illustrate the structure described in this paper.

$N = 3$

$$\begin{array}{lcllll} D_1 & : & [& 1 & 6 & 8 &] \\ D_2 & : & [& 2 & 4 & 9 &] \\ D_3 & : & [& 3 & 5 & 7 &] \end{array} \tag{19}$$

If one uses these values in a 3-sided dice set, then the simplest set of transitive dice is obtained. To craft this set one can also glue two regular tetrahedra by a face or a common cubic dice and repeat twice each value. Using common cubic dice is obtianed this Wikipedia's example.

$N = 4$ **(Muñoz's dice)**

$$\begin{array}{lclllll} D_1 & : & [& 1 & 8 & 11 & 14 &] \\ D_2 & : & [& 2 & 5 & 12 & 15 &] \\ D_3 & : & [& 3 & 6 & 9 & 16 &] \\ D_4 & : & [& 4 & 7 & 10 & 13 &] \end{array} \tag{20}$$

This set can be crafted used regular tetrahedra as dice. Very aesthetic in my opinion.

$N = 5$

$$\begin{array}{lcllllll} D_1 & : & [& 1 & 10 & 14 & 18 & 22 &] \\ D_2 & : & [& 2 & 6 & 15 & 19 & 23 &] \\ D_3 & : & [& 3 & 7 & 11 & 20 & 24 &] \\ D_4 & : & [& 4 & 8 & 12 & 16 & 25 &] \\ D_5 & : & [& 5 & 9 & 13 & 17 & 21 &] \end{array} \tag{21}$$

$N = 6$ **(Perera's dice)**

$$\begin{array}{lcl} D_1 & : & [\ 1\ \ 12\ \ 17\ \ 22\ \ 27\ \ 32\ ] \\ D_2 & : & [\ 2\ \ 7\ \ 18\ \ 23\ \ 28\ \ 33\ ] \\ D_3 & : & [\ 3\ \ 8\ \ 13\ \ 24\ \ 29\ \ 34\ ] \\ D_4 & : & [\ 4\ \ 9\ \ 14\ \ 19\ \ 30\ \ 35\ ] \\ D_5 & : & [\ 5\ \ 10\ \ 15\ \ 20\ \ 25\ \ 36\ ] \\ D_6 & : & [\ 6\ \ 11\ \ 16\ \ 21\ \ 26\ \ 31\ ] \end{array} \tag{22}$$

This set can be crafted using commun cubic as dice. Very aesthetic also in my opinion.

4 Conclusions

In this report, a general expression for sets of given size of intransitive fair dice is provided, with the restriction taht each die has the same number of faces as the number of dice in the set. The expression of the probability of one die in the set rolling higher than other is computed. This analytic expression is used to proof the intransitive properties of the sets described. These set of dice can be used to "cheat" in the *High Roller* game for two players, allowing the second player to always choose a winning die.

adeep knowledge in the limitations on the relation between the number of dice in the set and the number of faces in each die remain as future work. Also remains as future work the generalization of sets of intransitive dice that allow cheating in *High Roller* games for three or more players. To generate such sets, it might be required to repeat faces in some dice and/or among dice in the set, as other existing examples employ.

As a final practical note, a die with N faces can be crafted by gluing together two pyramids whose base is a regular polygon with N faces. The drawbacks of this die are that it is not a regular polyhedron and that each face will be repeated twice. Another alternative with no repeated faces is to use a "lemon-shape" form, like a beach ball, in which the cross-section is a regular polygon with N faces. In this case the die has curved faces, which in my opinion is a drawback both in terms of the practical aspect of manufacturing but most important in terms of aesthetics.

A Probability for one die rolling higher than other

Given two dice namely D_1 and D_2, lets compute the probability of one rolling higher than the other in terms of the faces V_1 and V_2 respectively. Using the law of total probability

$$\mathbb{P}\left[D_1 < D_2\right] = \sum_{j=1}^{M} \mathbb{P}\left[v_{1,j} < D_2 | D_1 = v_{1,j}\right] \mathbb{P}\left[D_1 = v_{1,j}\right]. \tag{23}$$

Then using the definition of conditional probability and independence of D_1, D_2,

$$\mathbb{P}\left[v_{1,j} < D_2 | D_1 = v_{1,j}\right] = \frac{\mathbb{P}\left[v_{1,j} < D_2, D_1 = v_{1,j}\right]}{\mathbb{P}\left[D_1 = v_{1,j}\right]} = \mathbb{P}\left[v_{1,j} < D_2\right] \tag{24}$$

thus as $\mathbb{P}\left[D_1 = v_{1,j}\right] = \frac{1}{M}$

$$\mathbb{P}\left[D_1 < D_2\right] = \frac{1}{M} \sum_{j=1}^{M} \mathbb{P}\left[v_{1,j} < D_2\right]. \tag{25}$$

Now we compute $\mathbb{P}\left[v_{1,j} < D_2\right]$, just by counting how many values $v_{2,k}$ are higher than $v_{1,j}$.

$$\mathbb{P}\left[v_{1,j} < D_2\right] = \sum_{k=1}^{M} \mathbb{I}\left[v_{1,j} < v_{2,k}\right] \mathbb{P}\left[D_2 = v_{2,k}\right] = \frac{1}{M} \sum_{k=1}^{M} \mathbb{I}\left[v_{1,j} < v_{2,k}\right]. \tag{26}$$

Thus in summary

$$\mathbb{P}\left[D_1 < D_2\right] = \frac{1}{M^2} \sum_{j=1}^{M} \sum_{k=1}^{M} \mathbb{I}\left[v_{1,j} < v_{2,k}\right], \tag{27}$$

and that means that we have to count for each value v_j in D_1 how many faces in D_2 are higher than this value, sum all and divide by M^2.

A.1 Example

Lets see a example with $N = 3$, $M = 4$.

$$\begin{array}{rl} D_1: & \begin{pmatrix} 2 & 2 & 5 & 5 \end{pmatrix} \\ D_2: & \begin{pmatrix} 3 & 3 & 3 & 6 \end{pmatrix} \\ D_3: & \begin{pmatrix} 1 & 4 & 4 & 4 \end{pmatrix} \end{array} \tag{28}$$

We have next computation of probabilities

$$\begin{array}{c}\mathbb{P}\left[D_1 < D_2\right] : (2+2+2+4)/(4^2) = 10/16\\ \mathbb{P}\left[D_2 < D_3\right] : (0+3+3+3)/(4^2) = 9/16\\ \mathbb{P}\left[D_3 < D_1\right] : (1+1+4+4)/(4^2) = 10/16\end{array} \tag{29}$$

B Probability for one die rolling higher than other in a set of instransitive dice

Lets take the formula of the probability derived in Appendix A

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\mathbb{I}\left[v_{m,j} < v_{n,k}\right]. \tag{30}$$

Note that

$$(j-1)N < (j-1)N + (m-j)\,\%\,(N) + 1 < jN, \tag{31}$$

thus for any m, n,

$$i < j \Rightarrow v_{m,i} < v_{n,j}. \tag{32}$$

If we use this property in the expression of the probability we can split the sum into parts

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{N^2}\sum_{k=1}^{N}\left[\sum_{j=1}^{k-1}\mathbb{I}\left[v_{m,j} < v_{n,k}\right] + \mathbb{I}\left[v_{m,k} < v_{n,k}\right] + \sum_{j=k+1}^{N}\mathbb{I}\left[v_{m,j} < v_{n,k}\right]\right]. \tag{33}$$

Then $\mathbb{I}\left[v_{m,j} < v_{n,k}\right] = 1$, for $j < k$ and $\mathbb{I}\left[v_{m,j} < v_{n,k}\right] = 0$ for $j > k$, so

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{N^2}\sum_{k=1}^{N}\left[(k-1) + \mathbb{I}\left[v_{m,k} < v_{n,k}\right]\right] = \tag{34}$$

$$= \frac{1}{N^2}\left[\frac{N(N-1)}{2} + \sum_{k=1}^{N}\mathbb{I}\left[v_{m,k} < v_{n,k}\right]\right]. \tag{35}$$

$$\tag{36}$$

The right part of the sum can be computed explicitly. Lets plug in these expressions of $v_{m,k}, v_{n,k}$ into the formula and simplify

$$\sum_{k=1}^{N}\mathbb{I}\left[v_{m,k} < v_{n,k}\right] = \sum_{k=1}^{N}\mathbb{I}\left[(m-k)\,\%\,(N) < (n-k)\,\%\,(N)\right]. \tag{37}$$

To continue, we will split the sum again and consider two cases. Also, we will use that if $a, b < N$, then

$$(a-b)\,\%\,(N) = \begin{cases} a-b & \text{if } b \le a \\ N+a-b & \text{if } b > a \end{cases} \tag{38}$$

B.1 Case $m < n$

We split the sum and apply the expression of the modulus of the substractions in each case to obtain

$$\sum_{k=1}^{N} \mathbb{I}\left[(m-k)\,\%\,(N) < (n-k)\,\%\,(N)\right] = \tag{39}$$

$$\sum_{k=1}^{m} \mathbb{I}\left[m < n\right] + \sum_{k=m+1}^{n} \mathbb{I}\left[N + m < n\right] + \sum_{k=n+1}^{N} \mathbb{I}\left[m < n\right] = \tag{40}$$

$$m + N - n = N - (n - m) = N + m - n. \tag{41}$$

B.2 Case $n < m$

We again split the sum

$$\sum_{k=1}^{N} \mathbb{I}\left[(m-k)\,\%\,(N) < (n-k)\,\%\,(N)\right] = \tag{42}$$

$$\sum_{k=1}^{n} \mathbb{I}\left[m < n\right] + \sum_{k=n+1}^{m} \mathbb{I}\left[m < N + n\right] + \sum_{k=m+1}^{N} \mathbb{I}\left[m < n\right] = \tag{43}$$

$$m - n. \tag{44}$$

And we have obtained that in general

$$\sum_{k=1}^{N} \mathbb{I}\left[(m-k)\,\%\,(N) < (n-k)\,\%\,(N)\right] = \begin{cases} m - n & \text{if } n < m \\ N + m - n & \text{if } n > m, \end{cases} \tag{45}$$

and this is by definition, as $n \neq m$

$$\sum_{k=1}^{N} \mathbb{I}\left[(m-k)\,\%\,(N) < (n-k)\,\%\,(N)\right] = (m-n)\,\%\,(N)\,. \tag{46}$$

If we plug in the computed expression into the formula of the probability we obtain

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{N^2}\left[\frac{N(N-1)}{2} + (m-n)\,\%\,(N)\right]. \tag{47}$$

And if we observe that $(m-n)\,\%\,(N) = N - (n-m)\,\%\,(N)$ then we obtain

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{N^2}\left[\frac{N(N+1)}{2} - (n-m)\,\%\,(N)\right], \tag{48}$$

So we have an analytic expression for the probability of the die n rollin higher than the die m

$$\mathbb{P}\left[D_m < D_n\right] = \frac{1}{2} + \frac{1}{2N} - \frac{(n - m)\,\%\,(N)}{N^2}. \tag{49}$$