Polytropic Model of a Non-Relativistic White Dwarf

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Numerical analysis of the Lane-Emden equation applied to white dwarfs is presented as an assignment for the subject *Estructura y Evolución Estelar* within the Astrophysics Master's program at Universidad de La Laguna. In this work we provide numerical solutions of the Lane-Emden equation for a variety of polytropic indexes, the computation of their structural constants, and a comparison of numerical solutions with theoretical expectations. Results are applied to infer mass and radii of polytopic white dwarfs of different central densities.

1 Introduction

The Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{1}$$

was introduced by Lane [1870], and further studied by Emden [1907]. It was originally used to describe the mass density distribution inside a spherical polytropic star in hydrostatic equilibrium. The ξ and θ adimensional variables are related to the physical radius, r, and density, ρ, of the star by

$$\rho = \rho_c\theta^n \ , \tag{2}$$

$$r = \alpha\xi \ , \tag{3}$$

where ρ_c is the density at the center of the star and α is a scale constant given by

$$\alpha^2 = \frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}} \tag{4}$$

where G is the gravitational constant, K is the polytropic constant, and n is the polytropic index. The polytropic constant K relates the pressure of the star, P with its density via the equation of state of a polytropic star

$$P = K\rho^{1+1/n} \ . \tag{5}$$

The Lane-Emden equation must be solved only in the domain where $\theta_n(\xi) \geq 0$, as only in this range the solutions are physically meaningful (otherwise would imply negative density values). We denote by R_n the radial value such that $\theta_n(R_n) = 0$. R_n has physical significance: it denotes the radius of the polytropic star in units of α. Similarly for the density gradient we define

$$\Lambda_n = -\left.\frac{d\theta_n}{d\xi}\right|_{\xi=R_n} \ . \tag{6}$$

It is possible to express some important properties of a polytropic star in terms of combinations of R_n and Λ_n. The total mass of the star can be expressed as

$$M = 4\pi\alpha^3\rho_c M_n \ . \tag{7}$$

where

$$M_n \equiv R_n^2\Lambda_n \ . \tag{8}$$

The central density of the star can be expressed as

$$\rho_c = \frac{3M}{4\pi R}D_n \ . \tag{9}$$

where R is the radius of the star and

$$D_n \equiv \frac{R_n}{3\Lambda_n} \ , \tag{10}$$

And the pressure at the center of the star is

$$P_c = \frac{K}{4\pi G}\rho_c^2 B_n \ , \tag{11}$$

with B_n usually defined as

$$B_n \equiv \frac{(3D_n)^{\frac{3-n}{3n}}}{(n+1)M_n^{\frac{n-1}{n}}R_n^{\frac{3-n}{n}}} \ . \tag{12}$$

We see that B_n is undefined for $n = 0$. However, when writting B_n solely in terms of R_n and Λ_n we find the following equivalent expression

$$B_n = \frac{1}{n+1}\left(R_n^2\Lambda_n\right)^{-2/3} \ , \tag{13}$$

which is well behaved for $n = 0$.

2 Results

2.1 Numerical Solutions of the Lane-Emden Equation

It is a well established result that the Lane-Emden equation only has analytic solutions expressable in

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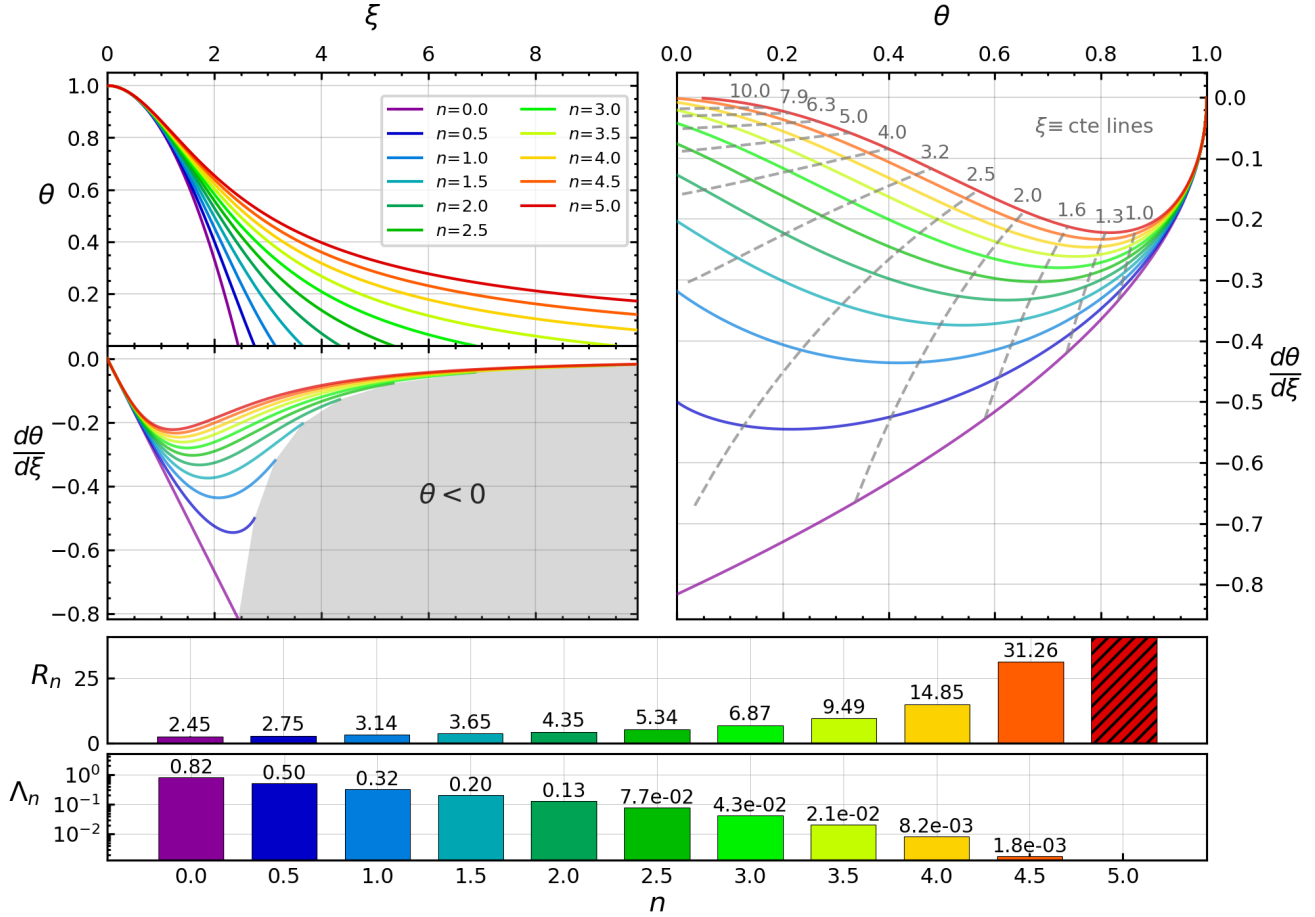Figure 1: Numerical solutions of the Lane-Emden equation for $n = 0$ to 5 in half-integer steps. Top Left (up): Radial solutions $\theta(\xi)$. Top Left (down): Radial solutions of the density gradient $d\theta/d\xi$. Top Right: $d\theta/d\xi$ vs. θ with equiradial lines of constant ξ overlaid. Bottom (bar plots): Dimensionless radius R_n and the density gradient at the star's boundary Λ_n for each polytropic index n. $\lim_{n \to 5} R_n$ tends to infinity.

terms of elementary functions for indexes $n = 0, 1$ and 5. Finding other solutions it is thought to be "complicated and involves elliptic integrals" [Chandrasekhar, 1957]. For $n = 0$ and $n = 1$ a general solution can be found. However, a general solution for $n = 5$ is not known. Instead, a collection of solutions satisfying particular boundary conditions has been found for $n = 5$ [Mach, 2012]. The known general solutions and a particular solution for $n = 5$ satisfying the boundary conditions $\theta_n(0) = 1$ and $d\theta_n/d\xi|_{\xi=0} = 0$ are

$$\theta_0(\xi) = 1 - \xi^2/6 \,, \qquad (n = 0) \qquad (14)$$
$$\theta_1(\xi) = \sin\xi/\xi \,, \qquad (n = 1) \qquad (15)$$
$$\theta_5(\xi) = \left(1 + \xi^2/3\right)^{-1/2} \,, \qquad (n = 5) \qquad (16)$$

where we use θ_n to denote the solution to the Lane-Emden equation for a specific n. The first boundary condition translates to $\rho(r=0) = \rho_c$, and the second implies that the density gradient vanishes at the center of the star.

Other solutions besides $n = 0, 1, 5$ must be found by numerical integration. Numerical solutions of the Lane-Emden equation for $n = 0$ to 5 in half-integer steps are shown in Figure 1. We used a 4th order Runge-Kutta integrator algorithm with a ξ-resolution of 10^{-4} in the domain $\xi \in (10^{-2}, 35)$. In Figure 2 we see that the numerical error is negligible when compared to the known analytical solutions, being of the order of 10^{-5} in all cases for a ξ-step $\Delta\xi = 10^{-4}$. Interestingly, we find that the error does not monotonically decrease as the step size $\Delta\xi$ decreases (see Table 1). Instead, it converges to specific values (of the order of 10^{-5}) for different n's.

2.2 Polytropic Coefficients

We now compute the polytropic coefficients M_n, D_n and B_n for our numerical solutions. Results can be found in Table 2. An n-continuum for the coefficients
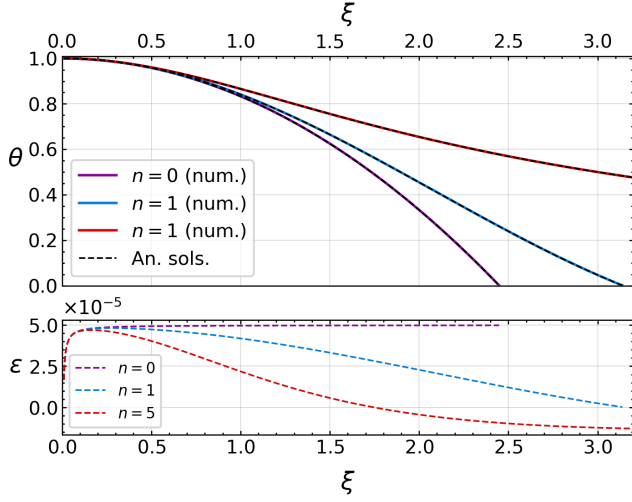
Figure 2: Comparison of numerical and analytic solutions of the Lane-Emden equation. Top: Radial profiles of numerical solutions for the polytropic Lane-Emden equation with indices $n = 0, 1$, and 5 (colored) compared to the respective analytic solutions (grey-dashed). Bottom: Corresponding error plots quantifying the discrepancy between numerical and analytical approaches.

Table 1: Integration errors ε for polytropic Lane-Emden equation solutions with indices $n = 0, 1$, and 5 for various step sizes $\Delta\xi$.

$\Delta\xi$	$\varepsilon_{n=0}$	$\varepsilon_{n=1}$	$\varepsilon_{n=5}$
10^{-1}	3.36×10^{-4}	2.41×10^{-4}	1.23×10^{-4}
10^{-2}	4.90×10^{-5}	2.89×10^{-5}	1.33×10^{-5}
10^{-3}	4.92×10^{-5}	2.90×10^{-5}	1.33×10^{-5}
10^{-4}	4.92×10^{-5}	2.90×10^{-5}	1.33×10^{-5}
10^{-5}	4.92×10^{-5}	2.90×10^{-5}	1.33×10^{-5}

is shown in Figure 3. From the numerical behaviour of the coefficients we can infer the following limits:

$$\lim_{n \to 5} D_n = \infty \, , \tag{17}$$

$$\lim_{n \to 5} R_n = \infty \, , \tag{18}$$

$$\lim_{n \to 5} \Lambda_n = 0 \, . \tag{19}$$

These limits are consistent with what one would obtain with the particular analytical solution for $n = 5$ using eq. (16). It is rellevant to mention that the numerical coefficients as $n \to 5$ can only be computed up to a value $n_{\max}$ for a given range of integration $\xi \in [0, \xi_{\max}]$, such that $R_{n_{\max}}$ is the last value of R_n that is smaller than $\xi_{\max}$.

n	R_n	Λ_n	D_n	M_n	B_n
0.0	2.450	0.815	1.002	4.894	0.347
0.5	2.755	0.501	1.834	3.802	0.274
1.0	3.140	0.320	3.276	3.152	0.233
1.5	3.651	0.204	5.958	2.722	0.205
2.0	4.346	0.128	11.320	2.417	0.185
2.5	5.346	0.077	23.239	2.191	0.169
3.0	6.876	0.043	53.616	2.021	0.156
3.5	9.491	0.021	150.605	1.893	0.145
4.0	14.847	0.008	606.649	1.798	0.135
4.5	31.265	0.002	5860.395	1.738	0.126
5.0	∞^*	0^*	∞^*	1.725	0.116

Table 2: Computed polytropic coefficients for indices n ranging from 0 to 5 in half-integer steps. Asterisks represent values that require special consideration in their computation due to the singular nature of the solution at $n = 5$.
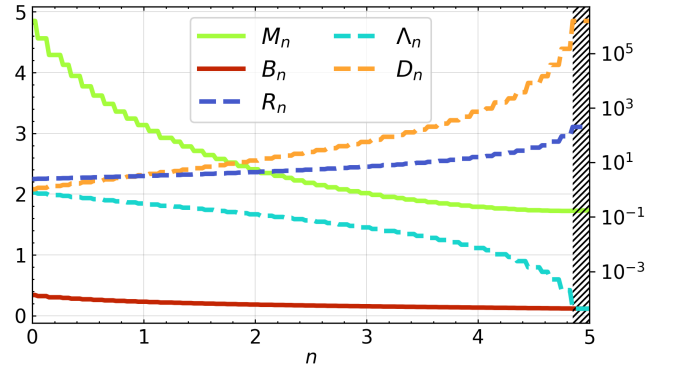

Figure 3: Variation of polytropic coefficients as functions of the polytropic index n, computed for a n-continuum with a step size of 0.025 over the domain $\xi \in (10^{-2}, 200)$. Solid lines are represented in linear scale (left ticks) while dashed lines are represented in log scale (right ticks). Hatches represent regions where divergent coefficients cannot be computed accurately (see main text.)

2.3 Polytropic White Dwarfs

The equation of state of a non-relativistic White Dwarf (WD) is

$$P_{e,\mathrm{deg}} = K_1 \left(\frac{\rho}{\mu_e} \right)^{5/3} \, , \tag{20}$$

where $P_{e,\mathrm{deg}}$ is the degenerate electron pressure, μ_e is the mean molecular weight per electron, and K_1 is a constant given by

$$K_1 = \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{3}{\pi} \right)^{2/3} \, , \tag{21}$$

where h is the Planck constant, m_{e} is the electron mass, and m_{p} is the proton mass. Comparing the equation of state of a non-relativistic WD and the

equation of state of a generic polytropic model (eq. 20) we see that the WD matches a polytropic model with $n = 3/2$. Armed with the formalism developed in the previous sections we can now compute the properties of a WD using the polytropic coefficients. We will do so for a set WD central densities as well as for a WD representative of the Sun in its latest stages of evolution.

Astrophysical models predict that the Sun will ultimately transition into a white dwarf state, with a projected mass of $M_{WD} = 0.5405$ $M_\odot$ [Horedt, 1986]. By combining eqns. (4) and (7), and considering $n = 3/2$ we can get the following expression for the central density of a WD as a function of its mass, M_{WD}:

$$\rho_c = 4\pi \left(\frac{2G}{5K} \right)^3 \left(\frac{M_{WD}}{M_n} \right)^2 . \tag{22}$$

For the Sun's WD mass the central density is $\rho_{c,\odot} = 1.23 \times 10^9$ kg m^{-3}.

In Table 3 we show the mass and radius of a WD with central densities ranging from 5×10^8 to 5×10^9 kg m^{-3} and $\rho_{c,\odot}$. Radial density and mass profiles for the example WDs in table 3 are shown in Figure 4.
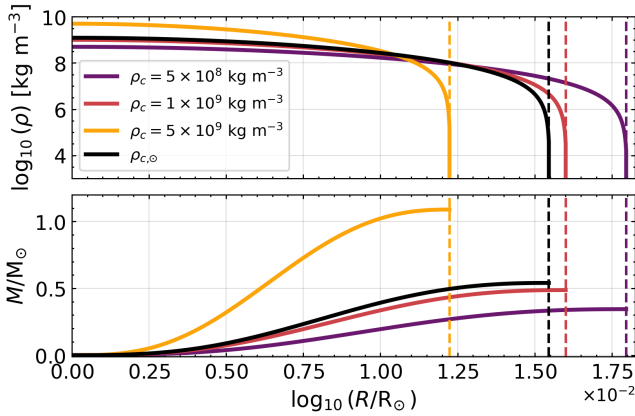

Figure 4: Top: Logarithmic radial density profiles as a function of solar radii $R/R_\odot$ for three non-relativistic WDs polytropic models at different central densities, compared with the Sun's estimated WD central density $\rho_{c,\odot}$ at its final evolutionary stage. Bottom: Corresponding mass profiles as a function of normalized radius, expressed in solar masses $M/M_\odot$.

Table 3: Relationship between central density ρ_c, radius R, and mass M, normalized to solar values R$_\odot$ and M$_\odot$ for a polytropic non-relativistic WD model. $\rho_{c,\odot}$ represents the Sun's expected white dwarf density.

ρ_c	$R/R_\odot$	$M/M_\odot$
5×10^8	1.80×10^{-2}	0.344
1×10^9	1.60×10^{-2}	0.487
5×10^9	1.22×10^{-2}	1.089
$\rho_{c,\odot}$	1.55×10^{-2}	0.541

and radius of non-relativistic WD across various central densities. The derived coefficients also allowed for a comparison with a WD model representative of the Sun in its final evolutionary stages. The numerical approach achieved significant accuracy, aligning closely with analytic benchmarks.

4 Data Availability Statement

The results supporting the findings of this report, along with the code and methodologies employed, are publicly available on GitHub. For complete details on our methods and to fully reproduce our results, interested parties can refer to the following repository: `https://github.com/pererossello/polytropic_WD_model`.

3 Conclusions

We numerically solved the Lane-Emden equation using a 4th order Runge-Kutta integrator for a range of polytropic indices n from 0 to 5 in half-integer steps. Our solutions facilitated the computation of polytropic coefficients, which allowed for determination of the mass

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