Graphing Functions

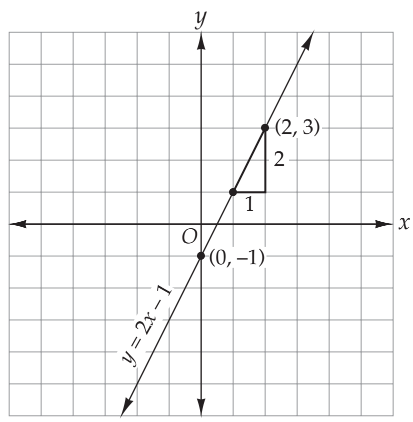
LESSON 1.1

Graphing and Writing Linear Equations Using the Slope-Intercept and Point-Slope Forms

Certain forms of a linear equation can provide enough information to easily graph the corresponding line. Consider the table of values for each of the two given graphs and their linear equations. The slopes and y-intercepts are easily found from looking at the linear equation. There is a pattern.

1. Equation: $y = 2x - 1$

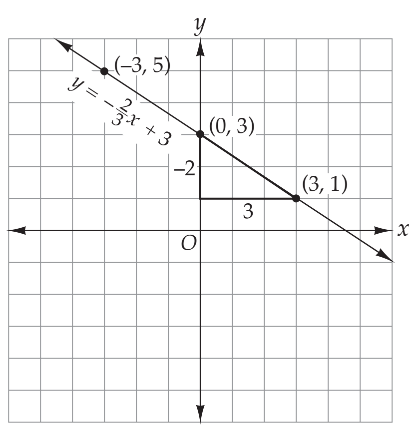
x	0	1	2
y	-1	1	3



The slope is 2 or $\frac{2}{1}$ and the y-intercept is -1.

2. Equation: $y = -\frac{2}{3}x + 3$

x	-3	0	3
y	5	3	1



The slope is $-\frac{2}{3}$ and the y-intercept is 3.

Slope-Intercept Form

In several of the examples discussed in this and previous sections, the first step in graphing the equation was to solve the equation for y. You may have noticed that in each of these examples, the value of the slope equaled the coefficient of x, and the y-intercept equaled the numeric constant. These statements are always true when y is by itself.

> If a linear equation is expressed in the form $y = mx + b$, then m represents the slope of the line and b represents the y-intercept. $y = mx + b$ is called the **slope-intercept form** of the equation.

Direct variation is a special case of the slope-intercept form when b = 0.

We can identify the slope and intercept in the equation:

$$y = \frac{-2}{3}x + 4:$$

The slope is $\frac{-2}{3}$ and the y-intercept is 4.

A negative slope can be written as

$$-\frac{2}{3} \text{ or } \frac{-2}{3} \text{ or } \frac{2}{-3}$$

but not $\frac{-2}{-3}$.

If we are told that the slope of a line is $\frac{3}{4}$ and the y-intercept is -2,

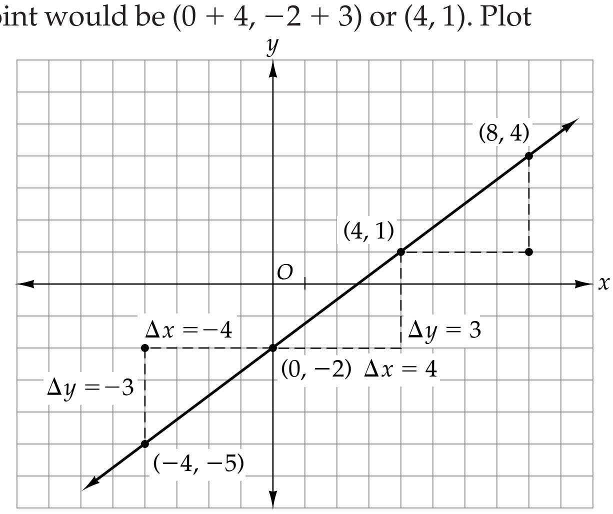
then the equation of the line is $y = \frac{3}{4}x - 2$.

The slope-intercept form is an easy form to use when a graph of the line is required. We graph $y = \frac{3}{4}x - 2$:

(1) *Remember that b is the value of y where the graph crosses the y-axis.* The y-intercept (-2), is at the point $(0, -2)$ when $y = \frac{3}{4}x - 2$.

(2) To find a second point, use the slope, $m = \frac{3}{4}$. Remember that $\frac{3}{4} = \frac{\Delta y}{\Delta x}$. This means that when y changes $+ 3$, x changes $+ 4$. So starting at the y-intercept $(0, -2)$, the next point would be $(0 + 4, -2 + 3)$ or $(4, 1)$. Plot that point.

(3) We can repeat the procedure by adding 3 to y and 4 to x. The next point would then be $(8, 4)$. Plot that point and draw a line connecting the three points.



MODEL PROBLEMS

1. Find the slope and y-intercept of $4x + 3y = 3$.

SOLUTION

Solve for y.

$$4x + 3y = 3$$
$$3y = 3 - 4x$$
$$y = 1 - \frac{4}{3}x$$

Rewrite in slope-intercept form: $y = mx + b$.

$$y = -\frac{4}{3}x + 1$$

Answer: Slope $m = -\frac{4}{3}$ and y-intercept $b = 1$

2. **MP 7** Write the equation of a line in standard form $Ax + By = C$, if the slope is $-\frac{4}{5}$ and the y-intercept is 5.

SOLUTION

Begin with the slope-intercept form $y = mx + b$ and replace m with $-\frac{4}{5}$ and b with 5. Thus, $y = -\frac{4}{5}x + 5$. To change into standard form, add $\frac{4}{5}x$ to both sides.

Rewrite as $\qquad \frac{4}{5}x + y = 5$

Multiply every term by 5, so $\qquad 4x + 5y = 25$

Answer: $4x + 5y = 25$

3. **MP 5** Predict what feature of the following graphs is the same and what feature is different. Use your graphing calculator to check your prediction.

$y = 3x - 1$

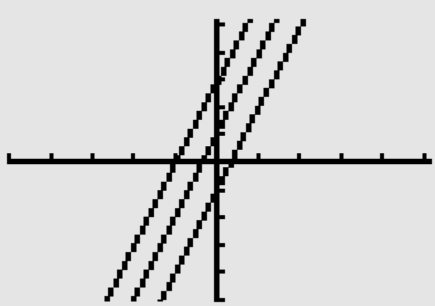
$y = 3x + 1$

$y = 3x + 3$

SOLUTION

The graphs have the same slope. The slope is 3. However, each has a different y-intercept. The graphs will be parallel and cross the y-axis at -1, 1, and 3, respectively.

The graphing window on the right verifies the prediction.



Answer: The equations are parallel lines. The y-intercepts are $(0, -1)$, $(0, 1)$, and $(0, 3)$.

Point-Slope Form

Given the slope m of line l and the coordinates (x_1, y_1) of any point on the line, it is possible to write an equation for the line. Since the slope is the same for any two points on the line, we can let (x, y) represent another point on the line and write the slope as $m = \frac{y - y_1}{x - x_1}$. By cross multiplying we get $y - y_1 = m(x - x_1)$.

If a line passes through the given point (x_1, y_1) and has slope m, the **point-slope form** of the equation of the line is $y - y_1 = m(x - x_1)$.

MODEL PROBLEMS

1. **MP 1, 2, 7** Write the equation of the line parallel to $2y = x + 3$ and passing through the point $(6, -2)$.

SOLUTION

First, solve for y. $y = \frac{1}{2}x + \frac{3}{2}$.

Since the new line is parallel, it has the *same slope*, $m = \frac{1}{2}$.

Using the point-slope method, $y - (-2) = \frac{1}{2}(x - 6)$ and $y + 2 = \frac{1}{2}x - 3$.

Thus, $y = \frac{1}{2}x - 5$.

2. **MP 1, 2, 7** Write the equation of the line perpendicular to $3y = -2x - 3$ and passing through the point $(2, -1)$.

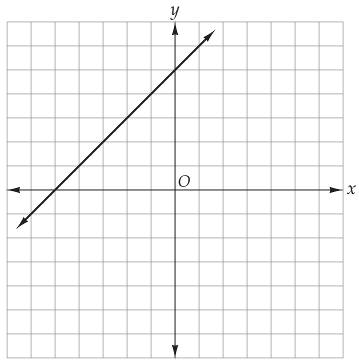
SOLUTION

First, solve for y. $y = -\frac{2}{3}x - 1$. Since the new line is perpendicular, the slope is the *negative reciprocal* of $-\frac{2}{3}$, which is $\frac{3}{2}$. Again, using the point-slope method, $y - (-1) = \frac{3}{2}(x - 2)$ and $y + 1 = \frac{3}{2}x - 3$. Thus, $y = \frac{3}{2}x - 4$.

PRACTICE

1. Which is the graph of $x + y = 5$?

A.



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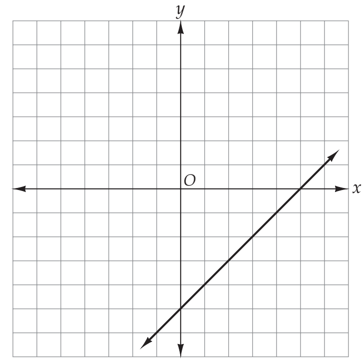

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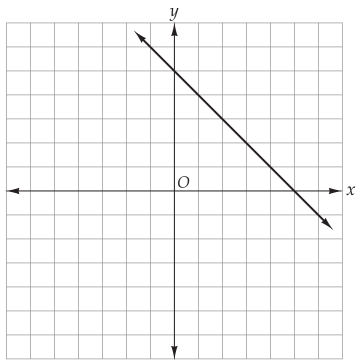

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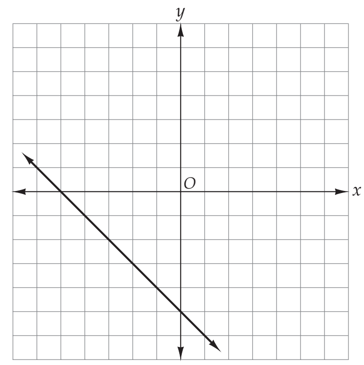


Exercises 2–5: Fill in the table.

Equation	Solve for y	Slope	y-intercept
2. $2x + 2y = 6$			
3. $3x - 3y = 12$			
4. $3x - y + 7 = 0$			
5. $3y - 7 = 0$			

Exercises 6–9: Write an equation of the line whose slope and y-intercept are given, and graph the line.

6. Slope is 4 and y-intercept is 1.

7. Slope is $-\frac{1}{2}$ and y-intercept is 0.

8. Slope is -1 and y-intercept is 1.

9. Slope is 0 and y-intercept is 6.

Exercises 10–11: Using the point-slope form, write an equation of the line with the given slope that passes through the given point. Transform the equation into the $y = mx + b$ form and graph.

10. $m = -2, (4, 2)$

11. $m = -\frac{1}{2}, (-4, -2)$

Finding the Slope-Intercept Form Given Two Points

To Find the Slope-Intercept Form of a Line Given Two Points

Method 1

- Use the two points to find slope m of the line.
- Use slope m and either one of the given points to substitute for x and y in the slope-intercept formula: $y = mx + b$.
- Solve for b, the y-intercept.
- Substitute the values for m and b in $y = mx + b$.

Method 2

- Use the two points to find slope m of the line.
- Then use slope m and either one of the given points to substitute in the point-slope formula: $y - y_1 = m(x - x_1)$
- Simplify and change into $y = mx + b$ form.

MODEL PROBLEMS

1. MP 1 Find the equation of the line that joins the points (1, 3) and (2, 5).

SOLUTION

METHOD 1 (Slope-Intercept)

Substitute and find slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{2 - 1} = \frac{2}{1} \text{ or } 2$$

Now substitute 2 for slope m, and point (1, 3) for (x, y) in the slope-intercept formula $y = mx + b$.

Solve for b:

$3 = 2(1) + b$, and $b = 1$

Once again, substitute 2 for m, and 1 for b in $y = mx + b$:

The equation of the line is $y = 2x + 1$.

Check. Check by substituting the coordinates of the other point (2, 5) for (x, y) in the equation of the line.

$5 = (2)(2) + 1$

$5 = 5$ ✓

METHOD 2 (Point-Slope)

Substitute and find the slope m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{2 - 1} = \frac{2}{1} \text{ or } 2$$

Now substitute 2 for slope m and point (1, 3) for (x_1, y_1) in the point-slope formula:

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - 1)$$

If we had chosen the point (2, 5), then

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 2(x - 2)$$

In fact, these are dependent lines; that is, they have the same graph. If we convert each equation to slope-intercept form, the result is the same equation.

$$y - 3 = 2(x - 1) \qquad y - 5 = 2(x - 2)$$
$$y - 3 = 2x - 2 \qquad y - 5 = 2x - 4$$
$$y = 2x + 1 \qquad y = 2x + 1$$

PRACTICE

1. Which equation passes through $(3, -5)$ and $(-1, 3)$?

A. $y + 5 = x - 3$
B. $y = 2x + 1$
C. $y - 3 = -2(x + 1)$
D. $y = x + 2$

Exercises 2–6: Write the linear equation determined by the following pairs of points. Put your answer in $y = mx + b$ form and then $Ax + By = C$ form, where A, B, and C are integers.

2. $(1, 3)$ and $(-2, -6)$

3. $(3, 5)$ and $(4, 7)$

4. $(-2, 3)$ and $(1, 5)$

5. $(2, 3)$ and $(2, 5)$

6. $(-1, -2)$ and $(2, 2)$

7. If the vertices of triangle ABC are located at $(2, -2)$, $(6, 4)$, and $(-4, 2)$, what are the slopes of the three lines that determine the triangle?

Exercises 8–11: Write the linear equation determined by the following information.

8. y-intercept $(0, 3)$ and point $(1, 2)$

9. y-intercept $(0, -2)$ and point $(2, 5)$

10. $b = 5$ and point $(3, -2)$

11. $b = -4$ and point $(3, 1)$

Exercises 12–17: Find the x- and y-intercepts of the following equations.

12. $x + 2y = 2$

13. $2x - 5y = 20$

14. $y = \frac{3}{5}x + 5$

15. $y = 10$

16. $y = -\frac{2}{3}x + 3$

17. $3x + y - 3 = 0$

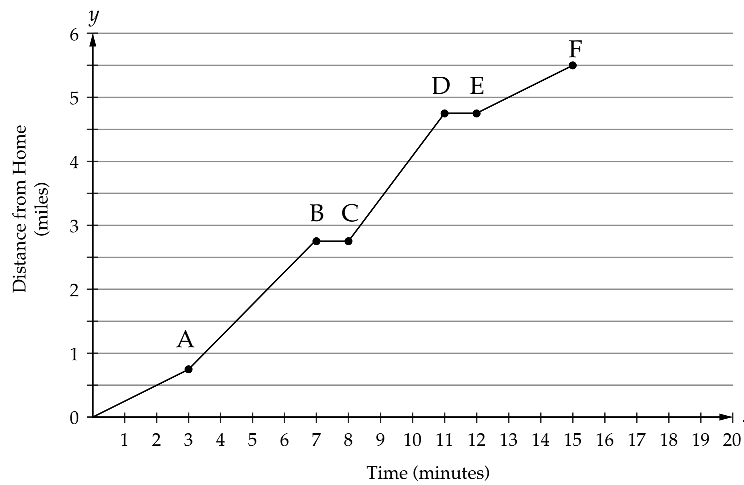
Exercises 18–20: Given the x- and y-intercepts, write the linear equation.

18. $(2, 0)$ and $(0, -6)$

19. $(4, 0)$ and $(0, 1)$

20. $(-5, 0)$ and $(0, -3)$

MP 1, 4, 5, 7 The graph below shows Michelle driving to work.



21. Which of the following could describe what happened between B and C?

A. Michelle drove away from home.
B. Michelle drove toward her home.
C. Michelle increased her speed.
D. Michelle stopped at a stoplight.

22. Which of the following describes what happened at A?

A. Michelle reduced her speed.
B. Michelle drove toward her home.
C. Michelle increased her speed.
D. Michelle stopped at a stoplight.

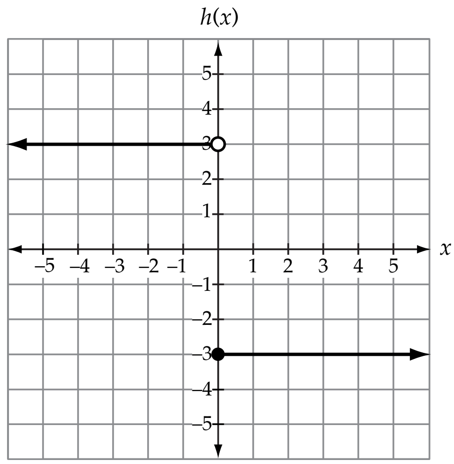
LESSON 1.2

Graphing Piecewise and Step Functions

Piecewise Functions

A **piecewise function** is defined by more than one condition for distinct intervals in its domain. Here are three examples of piecewise functions and their corresponding graphs.

(1) $h(x) = \begin{cases} 3 & \text{if } x < 0 \\ -3 & \text{if } x \geq 0 \end{cases}$

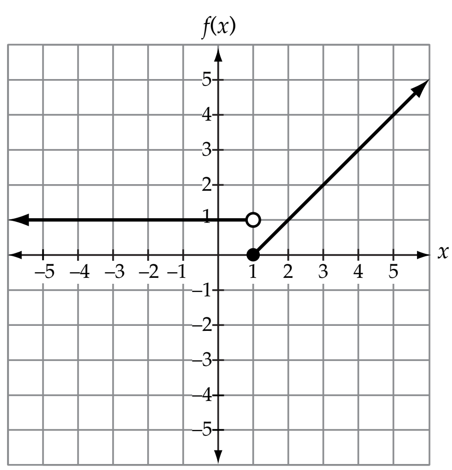


One can think of piecewise functions as disjunctions. In example 1, $h(x)$ is 3 if $x < 0$ or $h(x)$ is -3 if $x \geq 0$.

From the graph and the defining function $h(x)$, we can see that there is a constant for each condition.

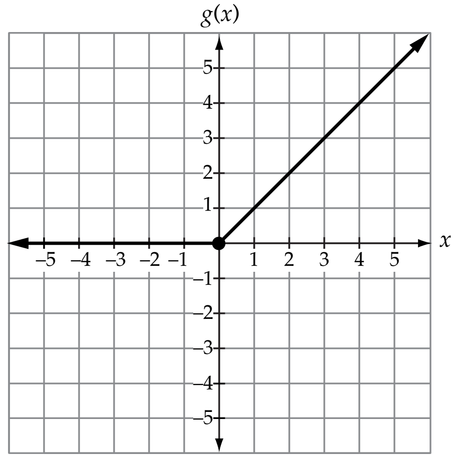
- For the first condition, we graph $h(x) = 3$. The first condition tells us that for all negative x, where $x < 0$, we graph the constant $y = 3$. This expression is not defined for 0, so the empty circle means at $x = 0$, $h(x)$ does not equal 3.
- For the second condition, we graph $h(x) = -3$ for $x \geq 0$. Since $h(0) = -3$, there is a closed circle at the point $(0, -3)$.

(2) $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$



- For the first condition, $f(x) = 1$, for all x less than 1. Since $x \neq 1$, there is an open circle at 1.
- For the second condition, $f(x) = x - 1$, for all x greater than or equal to 1. Start at the closed circle $(1, 0)$ and graph the linear function $f(x) = x - 1$ for values of x greater than one.

(3) $g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$



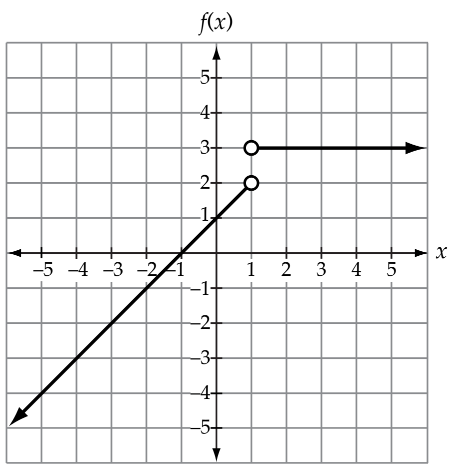
Although the graph is continuous, it is piecewise because it has more than one condition.

- For the first condition, for all non-positive x, $g(x) = 0$ with a closed circle at 0.
- For the second condition, for all positive numbers, we graph the ray $g(x) = x$. The endpoint is already filled in by the first rule.

MODEL PROBLEMS

1. Use the graph below to:

a Determine the x-values where the rules change.
b Write the piecewise function that describes the given graph.
c Within the piecewise function, what types of functions are there?



SOLUTION

a The rules change for the piecewise function at $x = 1$.

b $f(x) = \begin{cases} 3 & \text{if } x > 1 \\ x + 1 & \text{if } x < 1 \end{cases}$

c Both piecewise functions are linear functions.

Model Problems continue . . .

Model Problems *continued*

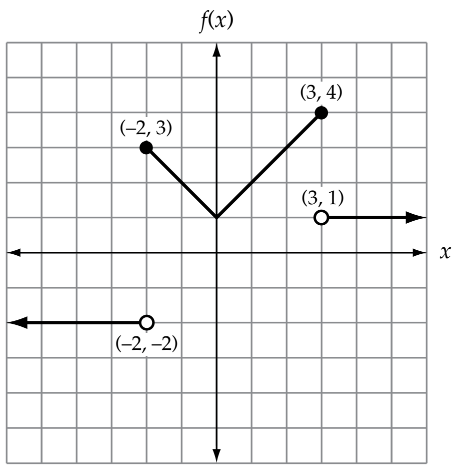
2. For the following piecewise function: $f(x) = \begin{cases} -2 & \text{if } x < -2 \\ -x + 1 & \text{if } -2 \le x \le 0 \\ x + 1 & \text{if } 0 \le x \le 3 \\ 1 & \text{if } 3 < x \end{cases}$

a Graph the piecewise function.

b Determine the x-values where the rules change.

SOLUTION

a



The absolute value function we will study later can also be defined algebraically as a piecewise function:

$f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$

b The rules change when $x = -2$, $x = 0$, and $x = 3$.

3. If $f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x + 1 & \text{if } x \ge 0 \end{cases}$, find

a $f(-4)$ **c** $f(1.4)$

b $f(-0.5)$ **d** $f(3)$

SOLUTION

a Since $-4 < 0$, then $-x = -(-4) = 4$

b Since $-0.5 < 0$, then $-x = -(-0.5) = 0.5$

c Since $1.4 > 0$, then $x + 1 = 1.4 + 1 = 2.4$

d Since $3 > 0$, then $x + 1 = 3 + 1 = 4$

Model Problems continue . . .

Model Problems *continued*

4. **MP 2** At Eagle Nest Golf Club, golf clubs are rented by the day, and you pay for the whole day even if you return your clubs early. The first day is $20, and then $20 more for a second day. For every day after the second day, the rental is only $10. Construct a graph that shows the cost of renting golf clubs for different periods of time.

a What should we label the x- and y-axes?

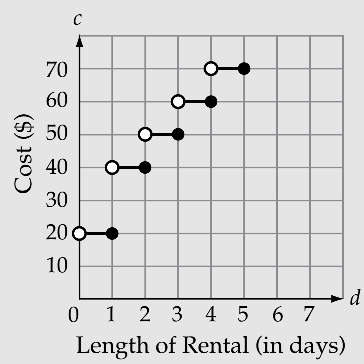
b How will the ordered pairs appear, and what will they mean?

c What are the independent and dependent variables?

d What is the cost of renting clubs for 2 days? for 3.5 days? How long can you rent the clubs if you have only $75?

e Define the rental as a piecewise function.

SOLUTION



a The horizontal axis is labeled with the variable d. This reinforces the fact that it is associated with the number of days the clubs will be rented. The vertical axis is labeled with the letter c (for cost).

b In this case, the ordered pairs on the graph will be of the form (d, c). For example, the point (3, 50) would tell us that for 3 days the rental cost is $50.

c Since the days are unknown, the d variable is independent. The cost, c, is dependent on the number of days rented.

d 2-day rental is $40; 3.5 days is $60; for $75 you can only rent for 5 days.

e $$c(d) = \begin{cases} 20 & \text{if } 0 < d \leq 1 \\ 40 & \text{if } 1 < d \leq 2 \\ 50 & \text{if } 2 < d \leq 3 \\ 60 & \text{if } 3 < d \leq 4 \\ 70 & \text{if } 4 < d \leq 5 \end{cases}$$

PRACTICE

1. Consider the function defined:

$$g(x) = \begin{cases} x - 2 & \text{if } -4 \leq x \leq 1 \\ -x + 1 & \text{if } x > 1 \end{cases}$$

What is the value of $g(-4) + g(1)$?

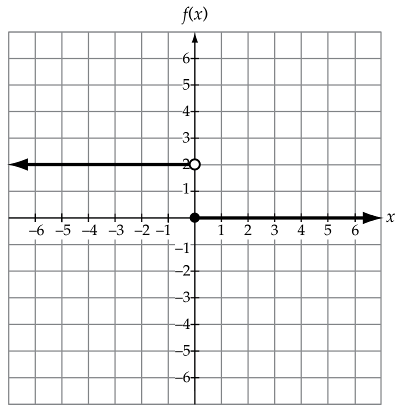
A. -7 C. -5

B. -6 D. 5

Practice Problems continue . . .

2. **MP 3** Which of the following is the piecewise graph of the given function? Explain your reasoning. $f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$

A.



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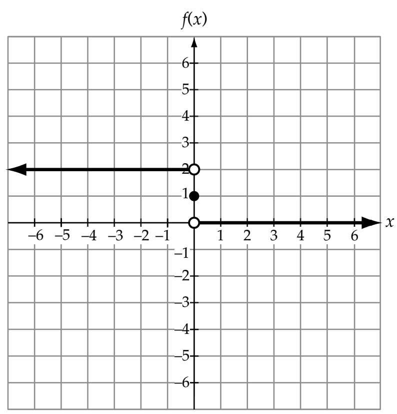
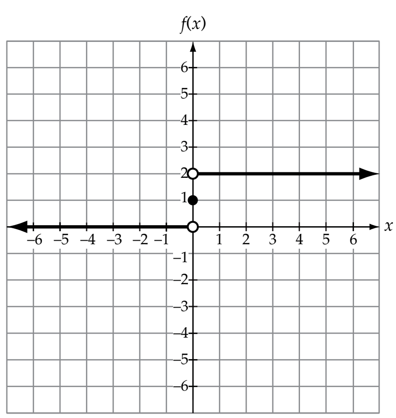


C.



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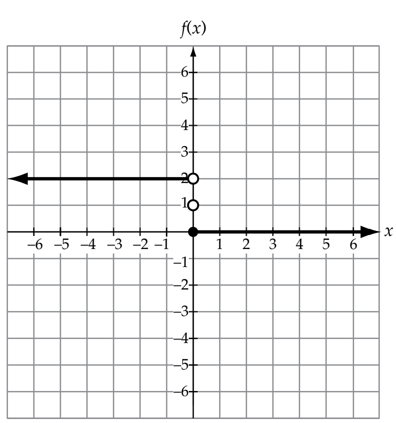
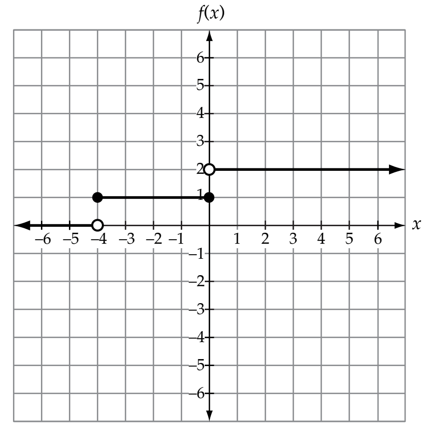
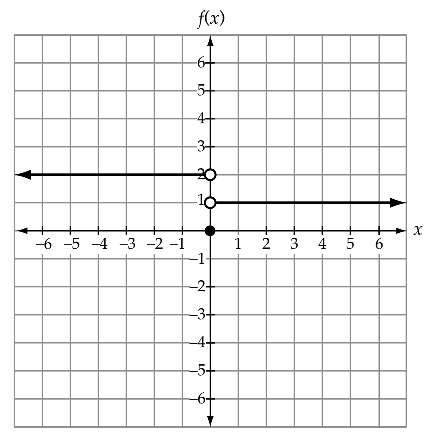


3. Write the piecewise function that describes the given graphs.

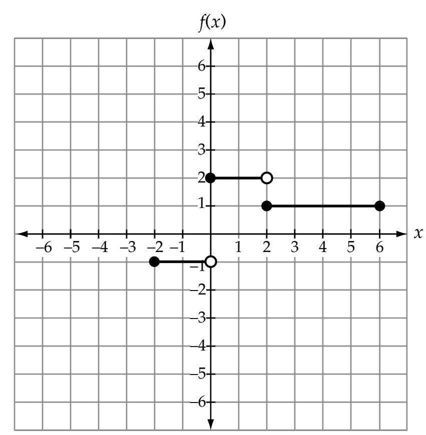
a



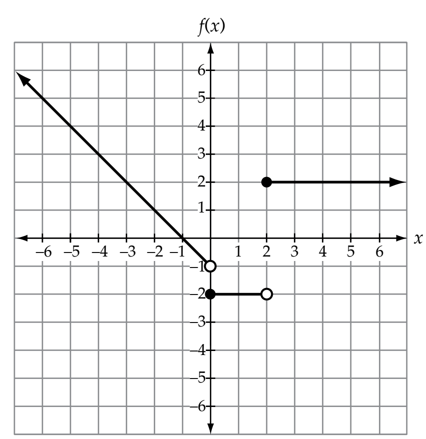
b



c

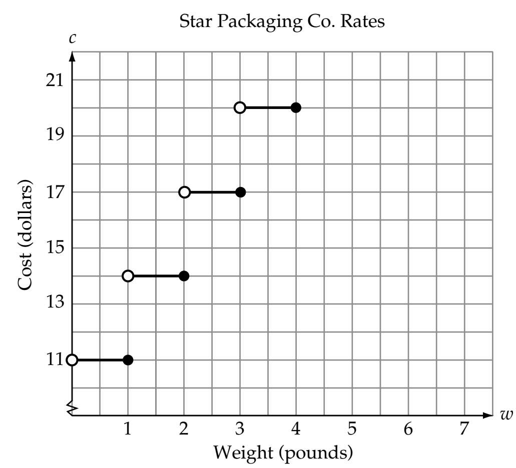


d



Step Functions

Another type of piecewise function is called a **step function**. The graphs of these functions appear to "step" from one change to another, usually horizontally because they appear to be a constant function at each interval. The following problem and graph show a typical example of a step function:

<table>
<tr><th>Example of a Real-World Step Function</th><th>Graph of a Step Function</th></tr>
<tr><td>The Star Packaging Co. charges $11 for shipping and handling any package weighing one pound or less. Each additional pound or fraction of a pound will cost $3 more, as indicated by the graph.</td><td>
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Among step functions, one of the most important is the **greatest integer function**.

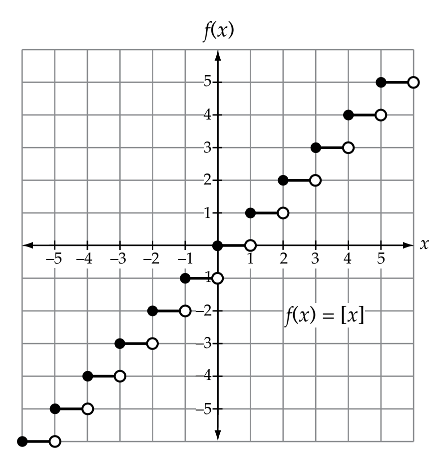
The greatest integer function is a step function f such that $f(x) = [x]$, where $[x]$ means the greatest integer not greater than x (or the greatest integer less than or equal to x).

$f(x) = [x]$	
x	$f(x)$
-2.5	-3
-1.1	-2
-0.8	-1
3	3
3.7	3
4.8	4
5.9	5

If x is a positive number in a greatest integer function, such as $x = 3.7$ and $f(x) = [x]$, then $f(3.7) = [3.7] = 3$, because 3 is the greatest integer not bigger than 3.7. The coordinates $(x, f(x))$ would be $(x, [x])$ or, in this case, $(3.7, 3)$.

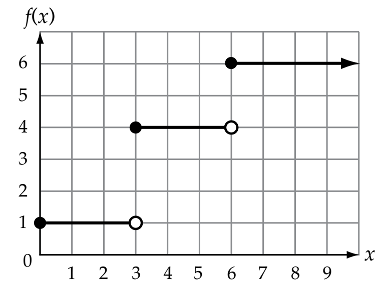
However, in the case of negative numbers, the numbers for $[x]$ are to the left. For example, if $x = -1.1$, then $f(-1.1) = [x]$ would mean $(-1.1, -2)$, since the greatest integer not greater than -1.1 would have to be -2.

We show a graph of the greatest integer function $f(x) = [x]$, where the domain is the real numbers and the range is integers only. Note the locations of the closed and open dots. They show that $f(3) = 3$, for instance, and that $f(3.999) = 3$, but $f(4) = 4$.



MODEL PROBLEMS

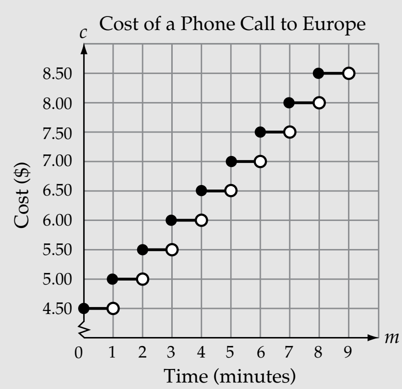
1. The following graph of $f(x)$ is a typical illustration of a step function. Describe the set of rules for this graph.



SOLUTION

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 3 \\ 4 & \text{if } 3 \leq x < 6 \\ 6 & \text{if } x \geq 6 \end{cases}$$

2. **MP 2, 4** A long-distance phone call to a city in Eastern Europe initially costs \$4.50 up to the first minute. For one full minute up to two minutes, the cost increases by \$0.50; it increases by another \$0.50 for two full minutes up to three minutes, and so on. The greatest integer step function graphed below illustrates this pricing.



a What is the cost of a 5-minute-and-40-second call?

b What is the difference between the cost of the call in part **a** and the cost of an 8-minute call?

c About how many minutes could you talk for on a phone call that cost \$7.50? Give two possible answers.

d Write the function $c(m)$ to represent the cost for any given time.

SOLUTION

a \$7.00

b 8-minute call = \$8.50. The difference is \$1.50.

c time, m, spent talking: 6 min $\leq m <$ 7 min

d $c(m) = \$4.50 + 0.50\,[m]$

PRACTICE

1. **MP 3, 4** Which of the following applications could *not* be represented by the graph of a typical step function? Explain your reasoning.

a Postage rates per ounce

b Charges at a parking lot per hour

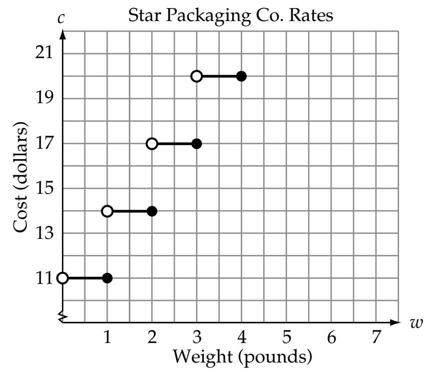
c Taxicab fares determined by the distance traveled

d Temperatures over the course of an afternoon

2. For the greatest integer function $f(x) = [x]$, complete the following table of values.

x	$[x]$	$f(x)$
−3.1	[−3.1]	
−3	[−3]	
−2.8	[−2.8]	
−2.3	[−2.3]	
−1.8	[−1.8]	
−1	[−1]	
−0.8	[−0.8]	
0	[0]	
0.7	[0.7]	
0.9	[0.9]	
1	[1]	
1.1	[1.1]	
1.9	[1.9]	
2	[2]	
2.4	[2.4]	
3.7	[3.7]	
4	[4]	
4.5	[4.5]	

3. The Star Packaging Co. charges $11 for shipping and handling any package weighing one pound or less. Each additional pound or fraction of a pound will cost $3 more, as indicated by the graph.

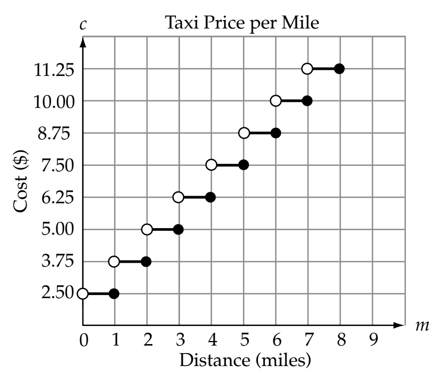


a What is the total cost of sending three separate packages weighing 1.8 pounds, 3.7 pounds, and 4 pounds?

b If it costs $17 to send a particular package, what could have been the weight of that package?

c Mary reasons that since an additional pound costs $3, an additional half-pound will cost $1.50. Is this true? Justify your response.

4. On a city taxi meter, the cost begins at $2.50 per person for the first mile plus $1.25 for every additional mile or fraction of a mile thereafter. The following step function graph illustrates this distance/cost relationship. Using the graph, answer the following:



a If Mark takes the city taxi and travels a distance of 3.7 miles, what is his cost for the taxi ride?

b If the cost to travel from the airport to his office is $8.75, how many miles did he travel? If he also gave a 15% tip to the driver, what did it cost him (rounded to the nearest dollar)?

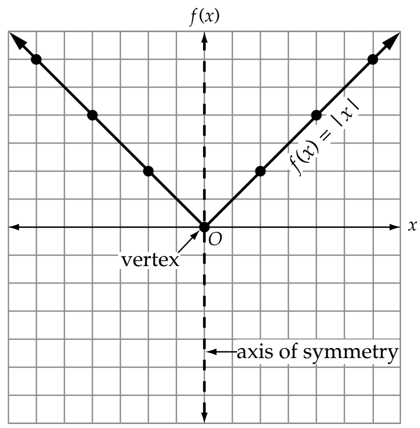
LESSON 1.3

Graphing Absolute Value Functions

Key Features of the Graph

We will look at the *key features* of graphs of absolute value functions of the form $f(x) = |x|$.

x	$f(x)$
−6	6
−4	4
−2	2
0	0
2	2
4	4
6	6



When graphing on an *xy*-coordinate system, the *domain* is the *x*-axis and the *range* is the *y*-axis. And the graph of the function $y = f(x)$ is the set of all points $(x, f(x))$.

The axis of symmetry is an equation of a line, not just a number.

1. **axis of symmetry:** The graph of $f(x) = |x|$ is symmetric with respect to the y-axis, also called the **axis of symmetry**.

2. **vertex**, or **turning point:** The graph of $f(x) = |x|$ is a V-shaped union of two rays with a common endpoint called the **vertex**. The vertex is the *lowest point* (minimum) if the graph opens upward, and it is the *highest point* (maximum) if the graph opens downward.

3. **end behavior of the graph:** Absolute value functions open up or down. The graph of $f(x) = |x|$ opens up.

4. ***y*-intercept:** The y-intercept is where $x = 0$. The y-intercept in $f(x) = |x|$ is the origin.

5. ***x*-intercept** or ***x*-intercepts:** The x-coordinates are the x-values when $y = 0$. The x-intercept in $f(x) = |x|$ is the origin.

6. **average rate of change:** To find the average rate of change, we calculate the slope of a line between those points.

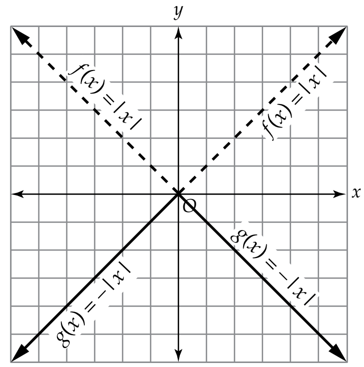
   - When $x < 0$ in the absolute value function $f(x) = |x|$, we see that function has a negative slope. We confirm this by calculating the average rate of change over an interval using $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. The average rate of change from the point $(-6, 6)$ to the point $(-4, 4)$ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 4}{-6 - (-4)} = \frac{2}{-2} = -1$.
   - When $x > 0$ in the absolute value function $f(x) = |x|$, we see that function has a positive slope. The average rate of change from the point $(4, 4)$ to the point $(6, 6)$ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - 4}{6 - 4} = \frac{2}{2} = 1$.

Transformations of Absolute Value Functions

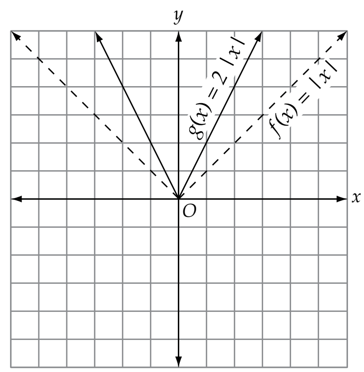
As we continue to observe patterns in functions, we will examine **transformations** of the most basic absolute value parent function, $f(x) = |x|$, compared to the general form $f(x) = a|x - h| + k$.

A **parent function** is the simplest form of a function. In this case, $a = 1$, $h = 0$, and $k = 0$ in the parent function.

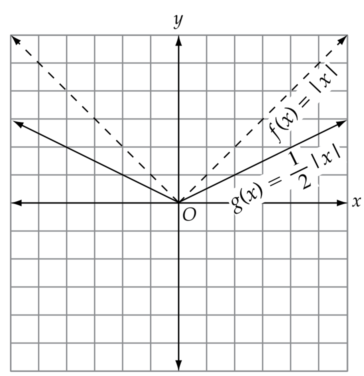
1. **Reflection about the *x*-axis:** If $a < 0$, the coefficient is negative, and the graph opens downward, as in the graph of $g(x) = -|x|$. The vertex remains the same but is now the maximum.



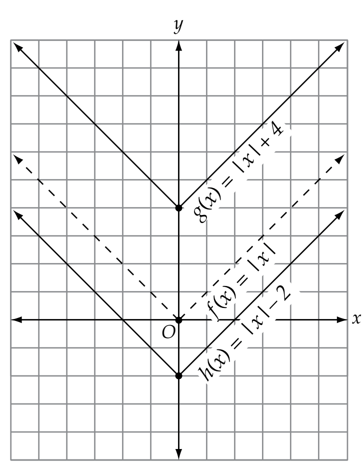
2. **Vertical stretching:** If $a > 1$, the coefficient is greater than 1, the graph is steeper (or narrower) than the parent function, as is the case in $g(x) = 2|x|$, and the slopes of the two rays are -2 and 2.



3. **Vertical shrinking:** If $0 < a < 1$, the coefficient is between 0 and 1, and the graph is wider (less steep) than the parent function, as is the case in $g(x) = \frac{1}{2}|x|$. The slopes of the rays are $-\frac{1}{2}$ and $\frac{1}{2}$.

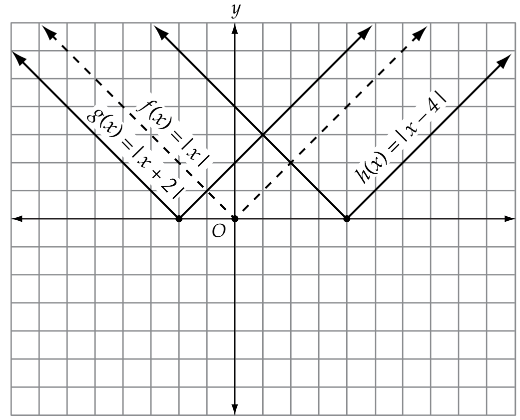


4. **Vertical translation:** If $f(x) = |x| + k$, as in the graph of $g(x) = |x| + 4$, then the parent function shifts up k units. If $f(x) = |x| - k$, as in the graph of $h(x) = |x| - 2$, then the parent function shifts down k units. This is true for all functions, including quadratic functions and exponential functions. The vertex of $g(x) = |x| + 4$ is the y-intercept (0, 4), and the vertex of $h(x) = |x| - 2$ is (0, –2).



Similar to the b in the linear function $f(x) = mx + b$, if we add or subtract constants as in $f(x) = |x| + k$, the graph moves vertically.

5. **Horizontal translation:** If $f(x) = |x + h|$, as in the graph of $g(x) = |x + 2|$, then the parent function shifts the graph to the left h units. If $f(x) = |x - h|$, as in the graph of $h(x) = |x - 4|$, then the parent function shifts the graph to the right h units. This is true for all functions. The vertex of $g(x) = |x + 2|$ is the x-intercept $(-2, 0)$, and the vertex of $h(x) = |x - 4|$ is $(4, 0)$.

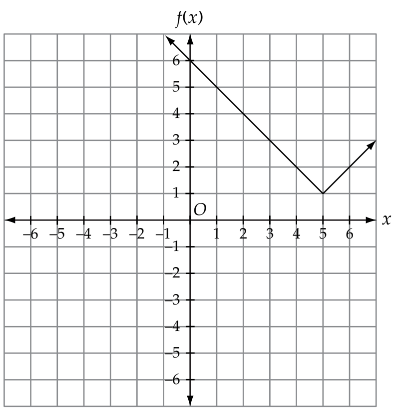


Changing the values of a, h, and k in a function has the effect of translating the graph *left* or *right* (that's the h value), *up* or *down* (that's the k value), as well as changing whether the graph *vertically stretches* or *vertically shrinks* (that's the a value). Lastly, the *sign* of the coefficient of a determines whether the function opens up or down.

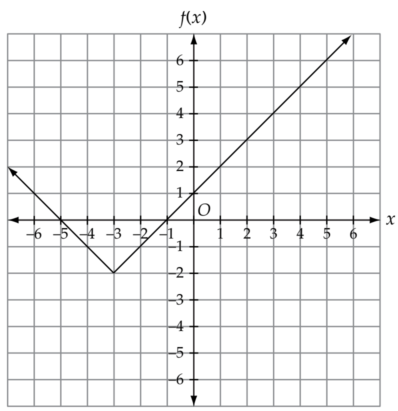
MODEL PROBLEMS

1. Write the corresponding absolute value functions that describe each of the three graphs below.

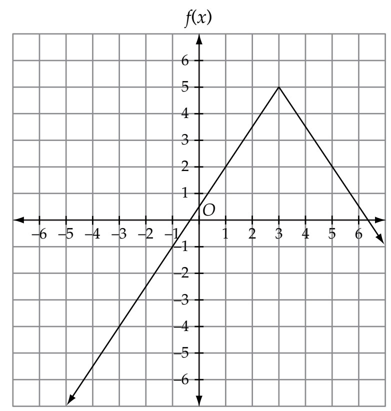
a



b



c



SOLUTION

a $f(x) = |x - 5| + 1$ **b** $f(x) = |x + 3| - 2$ **c** $f(x) = -\frac{3}{2}|x - 3| + 5$

2. **MP 7** If $f(x) = |x|$, what graphic transformation has been applied to create each of the functions $g(x)$?

a $g(x) = |x - 2|$

b $h(x) = |x - 2| - 1$

c $z(x) = |x + 4| + 3$

SOLUTION

a Move the vertex (0, 0) for $f(x) = |x|$ 2 units to the right to the new vertex (2, 0).

b Move the vertex (0, 0) for $f(x) = |x|$ 2 units to the right and 1 unit down to the new vertex $(2, -1)$.

c Move the vertex (0, 0) for $f(x) = |x|$ 4 units left and 3 units up to the new vertex $(-4, 3)$.

PRACTICE

1. **MP 5, 7** Graph the following functions. How are these functions the same? How are they different?

   $f(x) = x + 3$ and $g(x) = |x| + 3$

2. **MP 7, 8** In each comparison, explain clearly what changes occur.

   a Compare $f(x) = |x|$ with the following three functions: $g(x) = |x| + 1$, $h(x) = |x| + 3$, and $k(x) = |x| - 2$

   b Compare $f(x) = |x|$ with the following three functions: $p(x) = |x + 1|$, $q(x) = |x + 3|$, and $s(x) = |x - 2|$

3. **MP 7** Determine the equation from the description.

   a Given the graph of $f(x) = |x - 2| + 3$, what would be the equation of this same graph if it was shifted left one unit and down two units?

   b Given the graph of $f(x) = |x + 3| + 1$, what would be the equation of this same graph if it was shifted right five units and down two units?

   c Given the graph of $f(x) = |x + 1| + 3$, what would be the equation of this same graph if it was shifted right one unit and down three units?

   d Given the graph of $f(x) = |x - 5| - 1$, what would be the equation of this same graph if it was shifted left six units and up three units?

4. **MP 8** What graphic transformation has been applied to $f(x) = |x|$ to create each of the four functions $g(x)$ given below?

   a $g(x) = |x + 2| + 3$

   b $g(x) = |x - 4| - 5$

   c $g(x) = 2|x - 1| - 4$

   d $g(x) = -3|x + 2| + 6$

5. Graph each of the following functions.

   a $f(x) = |x + 3| + 2$

   b $f(x) = |x - 2| - 3$

   c $f(x) = |x - 3| + 2$

   d $f(x) = |x + 1| + 1$

   e $f(x) = -2|x| + 1$

   f $f(x) = |x + 1| - 2$

   g $f(x) = |x - 2| + 3$

   h $f(x) = |x - 1| + 1$

6. Graph each piecewise function for the given domain.

   a $f(x) = \dfrac{x}{|x|}$ for all $x \neq 0$

   b $h(x) = \begin{cases} -1 & \text{if } x < -1 \\ |x| & \text{if } -1 \leq x \leq 2 \\ -1 & \text{if } x > 2 \end{cases}$

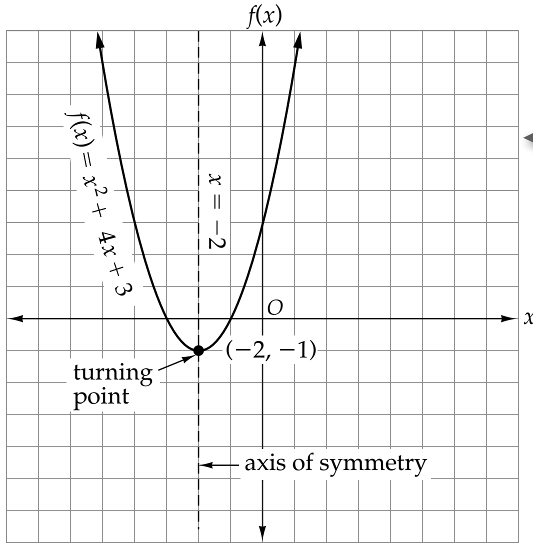
   c $g(x) = \begin{cases} x^2 & \text{if } -2 \leq x < 2 \\ x & \text{if } 2 \leq x \leq 5 \end{cases}$

LESSON 1.4

Graphing Quadratic Functions

Key Features of the Graph

We will look at the key features of graphs of parabolic functions of the form, $f(x) = ax^2 + bx + c$.



$f(x) = ax^2 + bx + c$ is the standard form of a quadratic function.

1. **axis of symmetry:** The axis of symmetry is a line that passes through the vertex and is parallel to the y-axis. It divides the parabola into mirror images.

   The axis of symmetry can be found algebraically by using the formula $x = \frac{-b}{2a}$. In the graph of $f(x) = x^2 + 4x + 3$, we substitute for a and b to get the axis of symmetry, $x = \frac{-4}{2(1)} = -2$, or the vertical line $x = -2$.

The axis of symmetry is an equation of a line, not just a number.

2. **vertex**, or **turning point:** The vertex is either a minimum point (the graph curves up) or a maximum point (the graph curves down). The x-coordinate of the turning point is $\frac{-b}{2a}$. In the graph of $f(x) = x^2 + 4x + 3$, the x-coordinate is $\frac{-b}{2a} = \frac{-(4)}{2(1)} = -2$. The y-coordinate of this point is found by substituting the x-value at the turning point into the original equation, so $y = x^2 + 4x + 3 = (-2)^2 + 4(-2) + 3 = 4 - 8 + 3 = -1$. The turning point is $(-2, -1)$.

3. **end behavior of the graph:** Quadratic functions either open up or open down. The coefficient of the x^2 term, a, determines the direction.
   - If a is positive, the graph opens up. The vertex is the minimum and the y-values increase as x changes in either direction. Both ends of the function extend to positive infinity. In the graph of $f(x) = x^2 + 4x + 3$, a is positive, so the graph opens up.
   - If a is negative, the graph opens down. The vertex is the maximum and the y-values decrease as x changes in either direction. Both ends of the function extend to negative infinity.

4. ***y*-intercept:** The y-intercept is where $x = 0$. Simply substitute 0 for x into the original equation, so $f(x) = (0)^2 + 4(0) + 3 = 3$. The y-intercept is (0, 3).

5. ***x*-intercept** or ***x*-intercepts:** The roots or zeros (since $f(x) = 0$ at these points) are the x-values or x-intercepts of the equation. The graph of $f(x) = x^2 + 4x + 3$ crosses the x-axis twice at $x = -3$ and $x = -1$. The roots are -3 and -1, so we can say that the x-intercepts are $(-3, 0)$ and $(-1, 0)$.

    - If the quadratic can be factored, we can use the zero product property to find the zeros. In this case, the quadratic function $f(x) = x^2 + 4x + 3 = (x + 3)(x + 1)$. Using the zero product property, we find that $x = -3$ and $x = -1$. This quadratic has two zeros, but that need not be the case. If the quadratic were shifted vertically, it might have no x-intercepts, and no zeros.
    - If the quadratic cannot be factored, we will use the quadratic formula, which we explain later.

6. **average rate of change:** We analyze the average rate of change between $-2 \leq x \leq 2$. To find the average rate of change, we calculate the slope of a line between those points. For example, the average rate of change over the interval is $\frac{y_2 - y_1}{x_2 - x_1}$ or $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. In the example graph, the average rate of change from the vertex $(-2, -1)$ to the point $(-1, 0)$ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - (-1)}{-1 - (-2)} = \frac{1}{1} = 1$.

x	$f(x)$
−2	−1
−1	0
0	3
1	8
2	15

Average Rate of Change
1
3
5
7

The average rate of change is increasing incrementally (1, 3, 5, 7). This is a pattern that can be used to graph the rest of the parabola. We can confirm this increase by looking at the graph; you can see it is becoming steeper as we move farther from the vertex.

The slope is constantly changing on any curve. With a curve, such as a quadratic, we are limited to finding the average rate of change between two points.

Graphing Quadratic Functions of the Form $f(x) = ax^2 + bx + c$

To Graph a Quadratic Function of the Form $f(x) = ax^2 + bx + c$

- Plot the vertex.
- Determine the direction of the parabola (up or down).
- Calculate and plot the y-intercept, and then use the axis of symmetry to plot a symmetrical point.
- Plot the zeros, or x-intercepts, of the function. If the quadratic equation can be factored, use the zero product property to find the zeros. If the quadratic equation cannot be factored, use the quadratic formula (which we discuss later) to find the zeros.
- Use the pattern found from analyzing the average rate of change to graph additional points on the parabola.

To Graph a Quadratic Function Using a Graphing Calculator

- Enter the function on the [Y=] screen of your calculator.
- Choose an appropriate window.
- Graph the function.
- Press [2nd] [CALC] [2] to find the zeros or roots.
- Enter left and right bounds for each zero.

To Find the Turning Point, or Vertex, Using a Graphing Calculator

- Graph the function.
- Look at the graph to determine whether the turning point is a maximum or a minimum.
- Press [2nd] [CALC] [3] to find the minimum or [2nd] [CALC] [4] to find the maximum.
- Enter left and right bounds for the minimum or maximum.

MODEL PROBLEM

MP 6 Graph the function $f(x) = x^2 - 2x - 8$. Label the vertex, axis of symmetry, and zeros of the function. Find the average rate of change from the vertex to the y-intercept.

SOLUTION

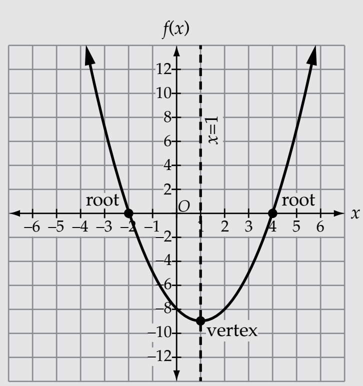
Axis of symmetry: The axis of symmetry is the line $x = 1$.

Plot the vertex: The x-coordinate of the vertex can be found using $\frac{-b}{2a}$, so that $\frac{-(-2)}{2(1)} = 1$. To find the y-value of the vertex, we substitute 1 into the expression $x^2 - 2x - 8$ and find the y-coordinate of the vertex is $1^2 - 2(1) - 8 = -9$. The vertex is $(1, -9)$.

Determine the direction of the parabola: a is positive, since $a = 1$, so the graph opens up.

Plot the y-intercept and symmetrical point: The y-intercept is found by setting x equal to 0 in the expression, $0^2 - 2(0) - 8 = -8$. The y-intercept is $(0,-8)$. The point across the axis of symmetry from the y-intercept is $(2,-8)$.

Plot the zeros of the function: The function, in factored form, is $f(x) = (x + 2)(x - 4)$. From the zero product property, we can tell that the zeros of this function are -2 and 4. The x-intercepts are $(-2, 0)$ and $(4, 0)$.



Average rate of change: The average rate of change between the vertex $(1,-9)$ and the y-intercept $(0, -8)$ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-8-(-9)}{0-1} = \frac{1}{-1} = -1$.

PRACTICE

1. What is the equation of the axis of symmetry for $f(x) = x^2 - 4x$?

A. $x = 2$

B. $x = -\frac{1}{8}$

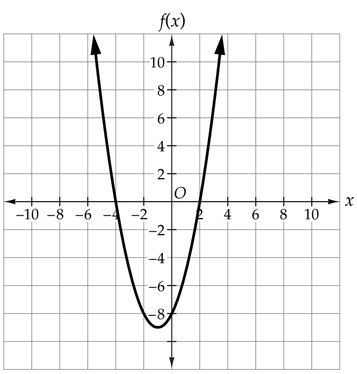
C. $y = \frac{1}{8}$

D. $y = -2$

2. Identify the turning point of $f(x) = -x^2 - 2x$.

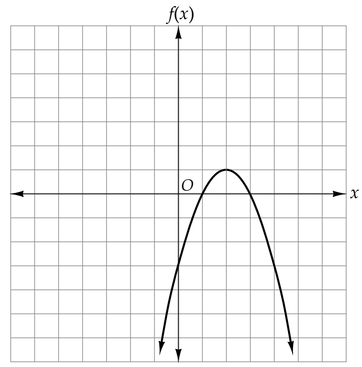
A. $(4, 8)$
B. $(1, -3)$
C. $(-1, 1)$
D. $(-4, -8)$

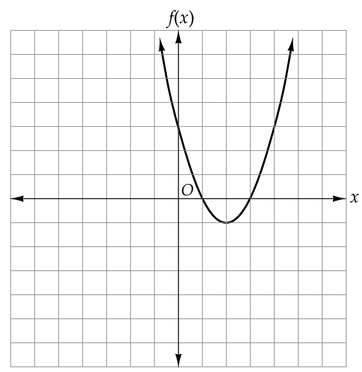
3. Which equation does the parabola in the graph represent?

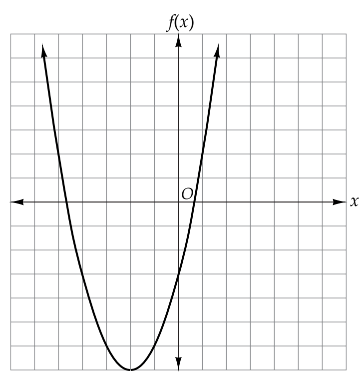


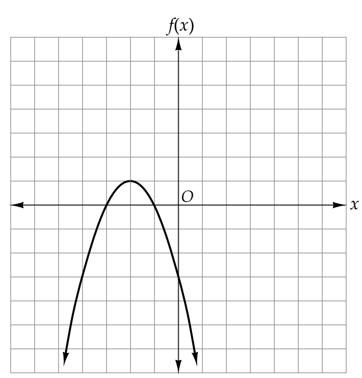
A. $f(x) = x^2 + 2x + 8$
B. $f(x) = x^2 + 2x - 8$
C. $f(x) = x^2 - 2x - 8$
D. $f(x) = x^2 - 2x + 8$

4. Which parabola is the graph of the equation $f(x) = -x^2 + 4x - 3$?

A.


B.


C.


D.


Practice Problems continue . . .

Exercises 5–9: What are the coordinates of the vertex of each of the following? Identify the vertex as a maximum or a minimum.

5. $f(x) = -x^2 + 6x + 10$

6. $f(x) = x^2 - 10x - 14$

7. $f(x) = 4x^2 + 8x - 11$

8. $f(x) = -x^2 - 6$

9. $f(x) = 2x^2 - 7x$

Exercises 10–15: Using the domain indicated in the parentheses, prepare a table of values and graph the quadratic function. Find the equation of the axis of symmetry and the turning point. Identify the turning point as a maximum or a minimum.

10. $f(x) = x^2 - 6x - 7$ $(0 \leq x \leq 6)$

11. $f(x) = -2x^2$ $(-3 \leq x \leq 3)$

12. $f(x) = x^2 + 2$ $(-3 \leq x \leq 3)$

13. $f(x) = x^2 - 3x - 4$ $(-1 \leq x \leq 5)$

14. $f(x) = 2(x^2 - 1)$ $(-3 \leq x \leq 3)$

15. $f(x) = -x^2 + 4x - 1$ $(1 \leq x \leq 5)$

Exercises 16–20: For each of the functions below,

a Find the vertex and axis of symmetry.

b Determine the direction of the parabola.

c Find the y-intercept and use the y-intercept to find another point across the axis of symmetry.

d Find the zero(s).

e Sketch the graph.

f Describe the end behavior of the graph.

g State increasing and decreasing intervals graphed.

h Find the average rate of change from the vertex to the smallest x-intercept.

i Find the average rate of change from the vertex to the largest x-intercept.

16. $f(x) = x^2 + 4x + 3$

17. $f(x) = x^2 - 4x + 3$

18. $f(x) = 2x^2 - 8x$

19. $f(x) = -(x^2 - 2x - 8)$

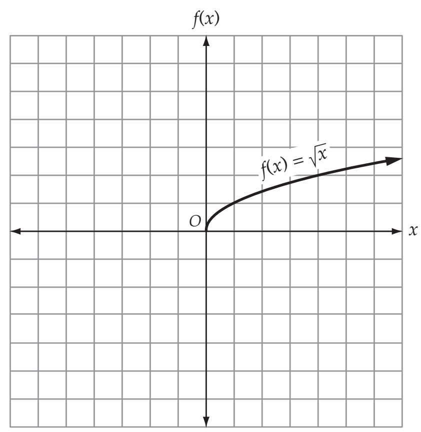
20. $f(x) = -x^2 + 4x + 21$

LESSON 1.5

Graphing Radical Functions

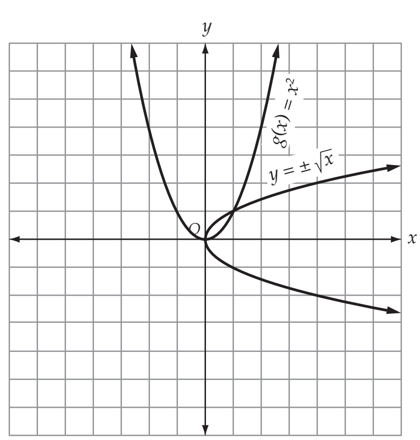
Graphing Square Root Functions

The parent **square root function** is $f(x) = \sqrt{x}$, the principal square root of x. The domain of this function is all non-negative real numbers, since the square root of a negative number does not exist in the set of real numbers. The range of a square root function is also all non-negative real numbers.



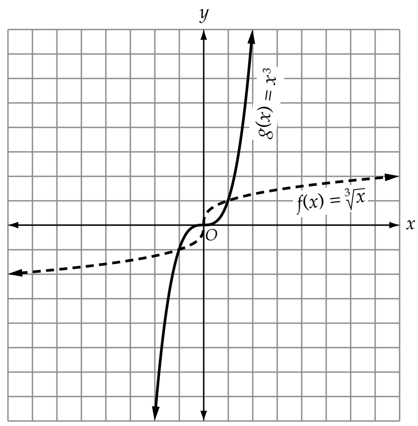
Looking at the graph of $f(x) = \sqrt{x}$, we see that the graph begins at the origin and that the values of $f(x)$ increase as x increases. However, as x increases, the rate of change of $f(x)$ decreases.

The relation $y = \pm\sqrt{x}$ is the inverse of the quadratic function $g(x) = x^2$. However, $y = \pm\sqrt{x}$ fails the vertical line test and is not a function. The inverse of the quadratic function $g(x) = x^2$, $x \geq 0$, is the function $f(x) = \sqrt{x}$.



Graphing Cube Root Functions

The parent **cube root function** is $f(x) = \sqrt[3]{x}$. Since we can find the cube root of any real number, positive or negative, the domain of this function is all real numbers.



The graph is strictly increasing without an upper or lower boundary. Therefore, the range is also the set of all real numbers. As the graph shows, the closer the domain value is to zero, the greater the rate of change. If we shift the graph of $f(x) = \sqrt[3]{x}$ left, right, up, or down, or if we change the amplitude or scale, the domain and range remain the set of all real numbers. While the rate of change increases as domain values move farther from the origin for $g(x) = x^3$, the rate of change decreases as domain values move farther from the origin for $f(x) = \sqrt[3]{x}$.

MODEL PROBLEMS

1. Find the domain and range of the function $f(x) = \sqrt{x} + 2$.

SOLUTION

The domain is determined by the radicand and is therefore all non-negative real numbers.

Either plot points or use your calculator to graph $y = \sqrt{x} + 2$ to find the range. The graph remains strictly increasing. The minimum value occurs when $x = 0$. That value is 2. The range is all real numbers greater than or equal to 2.

2. Find the domain and range of the function $f(x) = \sqrt{x - 4} + 1$.

SOLUTION

The radicand expression is non-negative when $x \geq 4$. Therefore, the domain is the set of all real numbers greater than or equal to 4. The graph remains strictly increasing, with a minimum value when the radical expression is zero. When $\sqrt{x - 4} = 0$, the value of $f(x) = \sqrt{x - 4} + 1$ is 1. Thus, the range of the function is all real numbers greater than or equal to 1.

PRACTICE

1. Describe the relationship between the graphs of $g(x) = x^2$ and $f(x) = \sqrt{x}$.
2. Find the domain and the range of the function $f(x) = \sqrt{x} - 1$.
3. Find the domain and the range of the function $f(x) = \sqrt{3x}$.
4. Find the domain and range of the function $f(x) = \sqrt{\dfrac{x}{3}}$.
5. Find the domain and range of the function $f(x) = 3\sqrt{x}$.
6. Find the domain and range of the function $f(x) = \sqrt{x + 1}$.
7. Is $y = -\sqrt{x}$ a function? Explain your answer.
8. Describe the relationship between the graphs of $y = x^3$ and $y = \sqrt[3]{x}$. How are they alike? How are they different? Specifically, compare the average rates of change in three intervals: before zero, near zero, and after zero.
9. Is the domain of $y = \dfrac{1}{\sqrt[3]{x}}$ the same as the domain of $y = \sqrt[3]{x}$? Explain your answer.
10. For what value(s) of x is $\sqrt{x} = \sqrt[3]{x}$ true?
11. For what values of x is $\sqrt[3]{x} > \sqrt{x}$ true?
12. For all positive odd values of n, what are the domain and range of $\sqrt[n]{x}$?

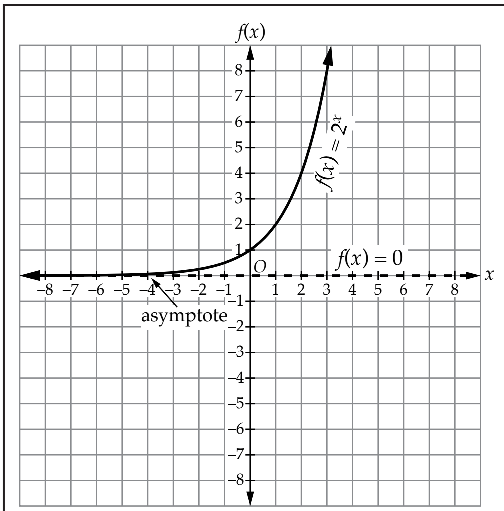
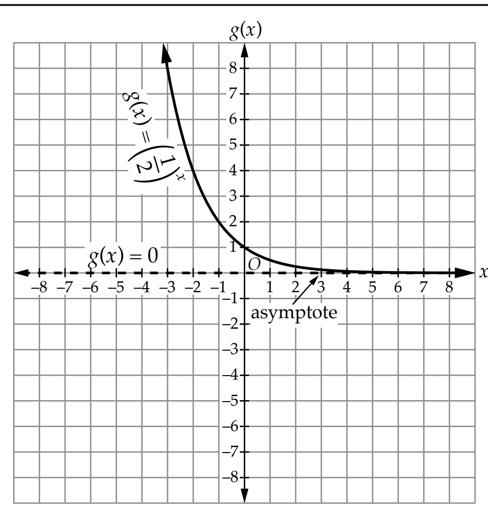
LESSON 1.6

Graphing Exponential Functions

Key Features of the Graph

$g(x) = \left(\dfrac{1}{2}\right)^x$ is the same as saying $g(x) = 2^{-x}$.

A function that contains a variable in the place of a numerical exponent is called an **exponential function**. The form of that function can be described by the equation $f(x) = b^x$ where $b > 0$, $b \neq 1$, and the variable x is a real number.

	
An exponential function can increase, as with $f(x) = 2^x$.	An exponential function can also decrease, as with $g(x) = \left(\dfrac{1}{2}\right)^x$.

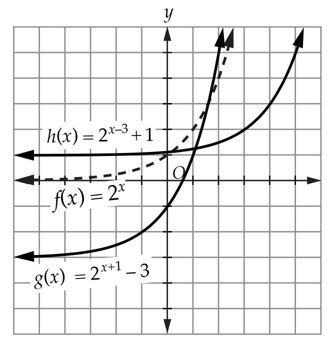
MODEL PROBLEMS

1. Compare and contrast the graphs of two functions $g(x) = y = 2^{x+1} - 3$ and $h(x) = y = 2^{x-3} + 1$ against the parent function $f(x) = y = 2^x$ using a, h, and k.

SOLUTION

The graphs indicate that $g(x) = 2^{x+1} - 3$ has been horizontally shifted left 1 unit and vertically shifted down 3 units, while $h(x) = 2^{x-3} + 1$ indicates a horizontal shift 3 units to the right and vertical shift 1 unit up.

We can specify the asymptotes and the y-intercepts: for $f(x)$ the asymptote is $y = 0$ and y-intercept is (0, 1); for $g(x)$ the asymptote is $y = -3$ and y-intercept (0, −1); for $h(x)$ the asymptote is $y = 1$ and y-intercept (0, 1.125).



2. For each of the given functions, $f(x) = 2^x$, $g(x) = 2^x + 3$, $h(x) = 2^{x+3}$, and $m(x) = 2^{x-1} - 2$

a State the transformation that applies to the parent function $f(x)$ in order to generate the functions $g(x)$, $h(x)$, and $m(x)$.

b Identify the asymptote.

c Identify the y-intercept.

SOLUTION

a Transformations: for $g(x)$, the function is vertically shifted up 3 units; for $h(x)$, the function is horizontally shifted to the left 3 units; for $m(x)$, the function is horizontally shifted to the right 1 unit and vertically shifted down 2 units.

b Asymptotes: for $f(x)$, $y = 0$; for $g(x)$, $y = 3$; for $h(x)$, $y = 0$; for $m(x)$, $y = -2$

c y-intercepts: for $f(x)$, (0, 1); for $g(x)$, (0, 4); for $h(x)$, (0, 8); for $m(x)$, (0, −1.5)

3. Graph $y = \left(\frac{1}{2}\right)^x$ and its reflection about the x-axis. Provide a table of values. Show the algebra that identifies this function.

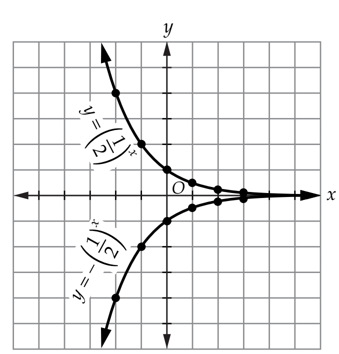
SOLUTION

The reflection of a function about the x-axis follows the rule of keeping the x and taking the opposite of y.

$y = \left(\frac{1}{2}\right)^x$ becomes $-y = \left(\frac{1}{2}\right)^x$ and (solving for y) we have $y = -\left(\frac{1}{2}\right)^x$.

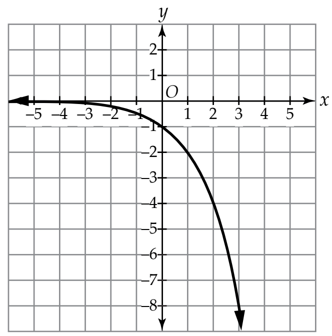
x	−3	−2	−1	0	1	2	3
$\left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

x	−3	−2	−1	0	1	2	3
$-\left(\frac{1}{2}\right)^x$	−8	−4	−2	−1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$



PRACTICE

1. Which is the equation of the graph?



A. $y = 2^{-x}$ C. $y = 2^{x}$

B. $y = -2x$ D. $y = -2^{x}$

2. The graph of $f(x) = b^x$, where $b > 0$, contains the point

A. $(-1, 0)$ C. $(0, 1)$

B. $(0, 0)$ D. $(1, 0)$

3. Which statement describes the graph of $y = 6^x$?

A. It is an increasing function that lies entirely in quadrants I and IV.

B. It is an increasing function that lies entirely in quadrants I and II.

C. It is a decreasing function that lies entirely in quadrants I and II.

D. It is a decreasing function that lies entirely in quadrants I and IV.

4. The graph of $y = \left(\frac{1}{3}\right)^x$ lies in which two quadrants?

A. I and II

B. II and III

C. III and IV

D. I and IV

5. The graph of the equation $y = 3^x$ intersects

A. the x-axis and the y-axis

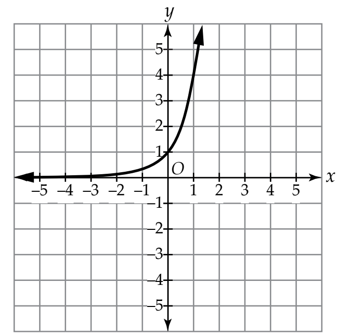
B. the y-axis only

C. the x-axis only

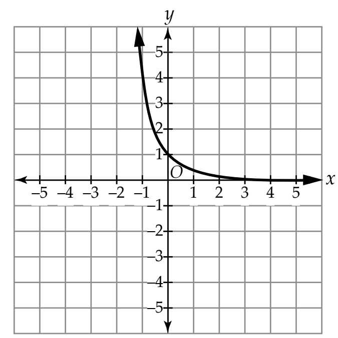
D. neither the x-axis nor the y-axis

6. Which of the following is the graph of the given equation: $y = 4^{-x}$?

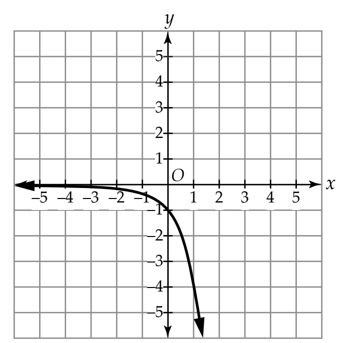
A.



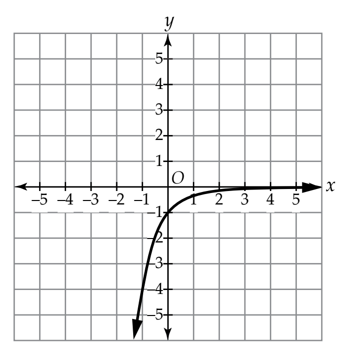
B.



C.

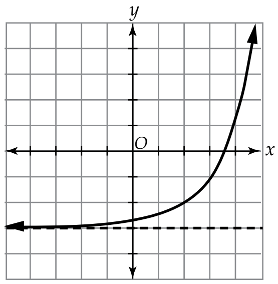


D.



Practice Problems continue . . .

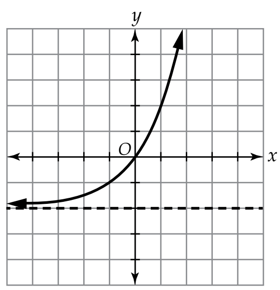
7. Given the following graph, which function represents that graph?



A. $y = 2^{x-2} - 2$ C. $y = 2^{x+2} - 2$

B. $y = 2^{x-2} - 3$ D. $y = 2^{x+2} - 3$

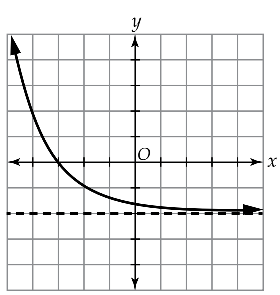
8. Given the following graph, which function represents that graph?



A. $y = 2^{x+1} - 1$ C. $y = 2^{x} + 2$

B. $y = 2^{x-1} + 1$ D. $y = 2^{x+1} - 2$

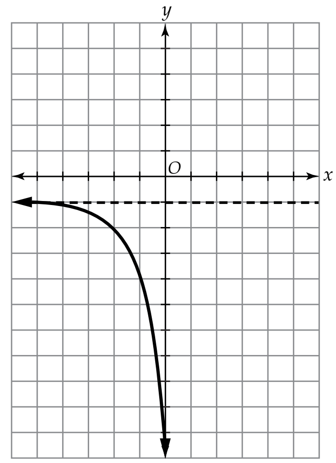
9. Given the following graph, which function represents that graph?



A. $y = \left(\frac{1}{2}\right)^{x+2} - 2$ C. $y = \left(\frac{1}{2}\right)^{x-2} - 1$

B. $y = \left(\frac{1}{2}\right)^{x-2} - 2$ D. $y = \left(\frac{1}{2}\right)^{x+2} - 1$

10. Given the following graph, which function represents that graph?



A. $y = -3^{x+2}$

B. $y = -3^{x-2} - 1$

C. $y = -3^{x+2} - 1$

D. $y = -3^{x+1} - 1$

11. Identify each of the following as functions of exponential growth or exponential decay. Justify your reasoning.

a $y = 4(0.75)^x$
b $y = 3(2.1)^x$
c $y = \frac{3}{4}(1.01)^x$
d $y = 0.25(0.25)^x$
e $f(n) = 0.05(1.50)^n$
f $f(t) = 0.1(0.1)^t$

Exercises 12–13: Complete the following tables.

12. $y = 10^x$

x	−3	−2	−1	0	0.5	1	1.5	2	3
y	0.001								

13. $f(x) = \left(\frac{1}{3}\right)^x$

x	−3	−2	−1	0	1	2	3	4	5
$f(x)$	27								

LESSON 1.7

Graphing Polynomial Functions

Finding Zeros of Polynomial Functions

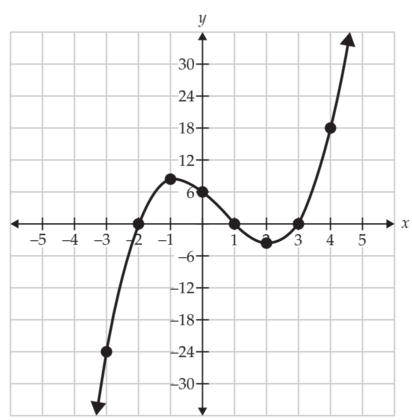
In this section, we focus on the graphs of cubic polynomials, such as $P(x) = x^3 - 2x^2 - 6x + 6$, and quartic polynomials, such as $P(x) = 5x^4 - 3x^2 + 1$. The domains of these functions are the set of real numbers. The graphs of these functions are continuous—they have no gaps—and like the graphs of all functions, they pass the vertical line test.

The graphs of cubic and quartic functions are curves, not straight lines. Polynomials of the first degree (linear equations), $P(x) = ax + b$, have straight-line graphs. Polynomials of the second degree (quadratic equations), $P(x) = ax^2 + bx + c$, have parabolic graphs. The graphs of higher-degree polynomials have more complicated shapes.

Graphing Polynomial Functions by Plotting Points

To graph and analyze a polynomial function by plotting points:

1. Graph by plotting points.



x	$P(x)$
−3	−24
−2	0
−1	8
0	6
1	0
2	−4
3	0
4	18

A polynomial function, such as $P(x) = x^3 - 2x^2 - 5x + 6$, can be graphed by plotting points. Calculate some points, plot them, and draw a curve through them.

2. Identify *x*-intercepts. The x-intercepts are at −2, 1, and 3. The x-intercepts of the graph are the real zeros of the polynomial, since at these locations the value of the function is 0. We can use graphs to approximate zeros. When the graph crosses the x-axis, the value of the function must change from positive to negative, or vice versa.

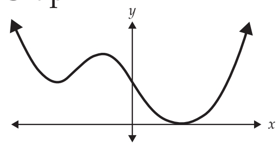
3. Describe end behavior. As x goes toward negative infinity, y also goes toward negative infinity. As x goes toward positive infinity, y also goes toward positive infinity.

MODEL PROBLEMS

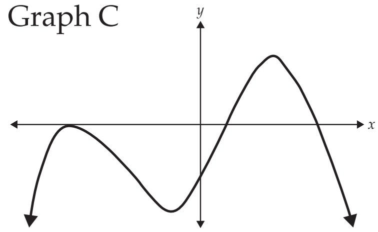
1. Determine the number of real zeros of each quartic poynomial.

> Quartic equations can have up to four real zeros.

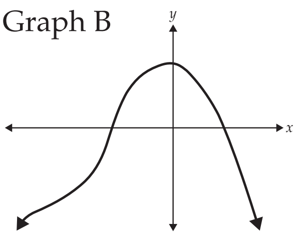
Graph A



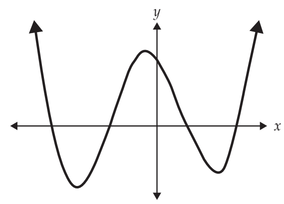
Graph C



Graph B



Graph D



SOLUTION

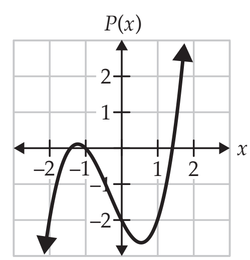
Graph A One real zero. The number of x-intercepts of the graph of a polynomial equals the number of distinct real zeros of the polynomial. Graph A has one x-intercept, so the polynomial has one real zero.

Graph B Two real zeros. Since Graph B has two x-intercepts, the polynomial has two distinct real zeros.

Graph C Three real zeros. Graph C intersects the x-axis three times, so it has three real zeros.

Graph D Four real zeros. Graph D crosses the x-axis four times, so it has four real zeros.

2. Find the positive real zeros of $P(x) = x^3 + x^2 - 2x - 2$.



SOLUTION

Iterate to find roots

x	$P(x)$
0	−2
1	−2
1.1	−1.659
1.2	−1.232
1.3	−0.713
1.4	−0.096
1.5	0.625
2	6

The function crosses the x-axis between integers. Iterate solutions: try values, and use them to determine the next trials.

> Trying values and noting when these values cause the value of the function to change sign is one way to estimate zeros.

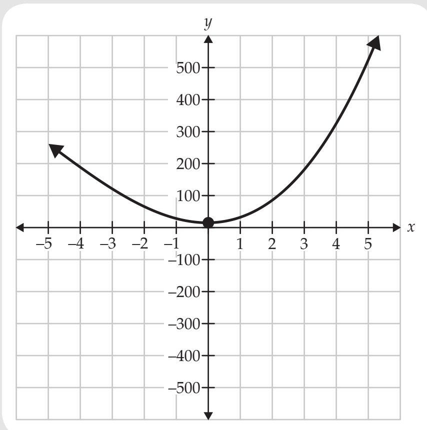
When it changes sign, we have found two points between which the graph must pass through the x-axis. The graph crosses the x-axis between 1 and 2.

Employ the same idea with the tenths, and find that the value of the function changes sign between 1.4 and 1.5, and closer to 1.4. The process above could be continued, with values between 1.4 and 1.5 evaluated to determine the root to the nearest hundredth. The estimate above is close, and shows both the benefits and limits of estimation.

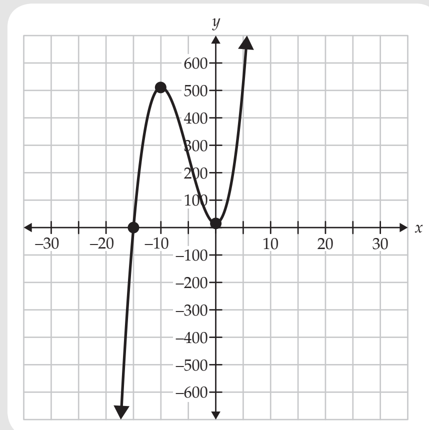
Model Problems continue . . .

3. **MP 3** Graph A and Graph B both represent the polynomial function $P(x) = x^3 + 15x^2 + x + 15$. They are viewed through different windows, showing different ranges and domains. Explain how the Graph B view changes one's view of the nature of the function's graph.

Graph A



Graph B



SOLUTION

Graph A displays the function from −5 to 5, while Graph B displays the function from −30 to 30. Graph A looks like a parabola and seems to have no zeros (since it does not cross the x-axis).

In Graph B, we show a larger domain of the same function and see a different shape for the graph. We see a zero where the graph crosses the x-axis at −15. It also seems like there might be a zero near $x = 0$. $x = 0$ is not a zero of the polynomial since $P(0) = 15$. It can be shown that the polynomial's relative minimum near $x = 0$ is at $x \approx -0.03345$, and the value of the polynomial there is $P(-0.03345) \approx 14.98$. Graph A shows the region near $x = 0$ in more detail and indicates that the relative minimum is above the x-axis.

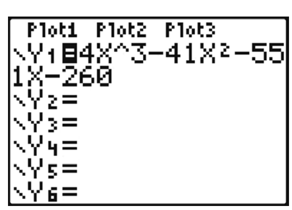
We also get a more accurate view of the end behavior of the graph to the left of the origin in Graph B; it will become increasingly negative as x becomes more negative, since the cube term will be negative, and it will dominate the output.

Graphing Calculator: Graphing Polynomial Functions

We use a calculator to analyze $y = 4x^3 - 41x^2 - 551x - 260$. In addition to finding its zeros and describing end behavior, we also analyze the function for its relative minimums and maximums. Relative minimums and maximums are low and high points on a section of the graph.

Graphing a polynomial function can be a useful way to determine its zeros, even when the zeros are irrational. This is particularly true in the era of graphing calculators, which can quickly produce a graph of a polynomial.

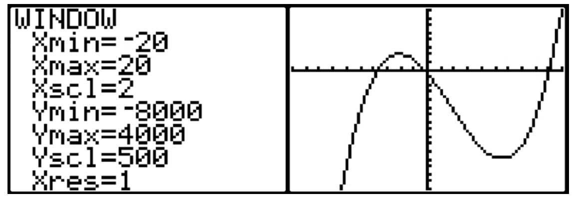
1. Enter function. Press $\boxed{\text{Y=}}$ to get to the equation entry screen, and type the above polynomial into a Y variable line.



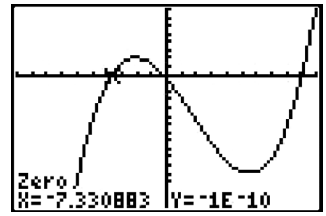
Directions continue . . .

Directions continued . . .

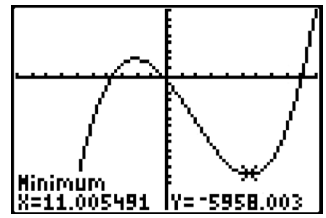
2. Graph the function. To graph, press ZOOM **6:ZStandard**, and graph the function using the graphing window of $-20 < x < 20$ and $-8000 < y < 4500$.



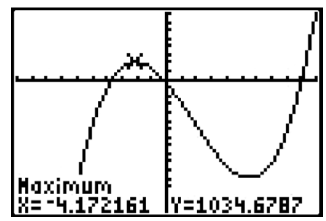
3. Find the intercepts. Press 2nd TRACE **2:Zero**. Use the ◀ and ▶ keys to move the cursor to the left and right of each x-intercept. Press ENTER. Move the cursor so it is close to the same x-intercept and press ENTER.



4. Finding the relative minimums. Press 2nd TRACE **3:Minimum** to find the relative minimum of the polynomial. Use the ◀ and ▶ keys to move the cursor so it is to the left and right of each relative minimum. Press ENTER. Use the ◀ and ▶ keys to move the cursor close to the relative minimum. Press ENTER.

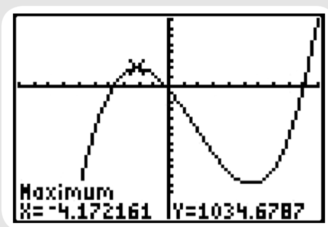


5. Finding the relative maximums. Press 2nd TRACE **4:Maximum** to find the relative maximum of the polynomial. Use the ◀ and ▶ keys to move the cursor to the left and right of each relative maximum. Press ENTER. Use the ◀ and ▶ keys to move the cursor close to the relative maximum. Press ENTER.



MODEL PROBLEM

MP 5, 6 Now that we have graphed the function $y = 4x^3 - 41x^2 - 551x - 260$ on a graphing calculator, complete an analysis based on the end behavior, x- and y-intercepts, domain and range, relative minimum and maximum, and the intervals the function increases and decreases. (We show one image of the graphed function to the right, but you should graph it on your own to complete your analysis or use the images above.)



SOLUTION

End behavior: As x goes toward negative infinity, y also goes toward negative infinity. As x goes toward positive infinity, y also goes toward positive infinity.

x-intercepts (zeros): $x \approx -7.33$, $x \approx -0.49$, $x \approx 18.07$

Domain and range: All real numbers

y-intercept: -260

Relative minimum: $x \approx 11$ and $y \approx -5958$

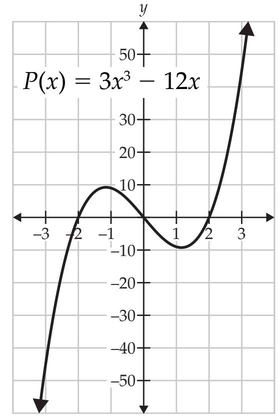
Relative maximum: $x \approx -4.17$ and $y \approx 1034.68$

Increases on about $x > 11$ and $x \leq -4.17$

Decreases on about $-4.17 \leq x \leq 11$

Graphing Polynomial Functions Using Zeros

Another way to get an accurate view of a polynomial exists if you can calculate its zeros. We use $P(x) = 3x^3 - 12x$ as an example. Factor the expression to get $(3x)(x^2 - 4)$. Recognize the second factor as a difference of squares, $(x + 2)(x - 2)$. Using the zero-product property, the roots are 0, 2, and -2. Plot those 3 points, some adjacent points and the points in between, to arrive at the graph to the right.



Considering the structure of the function, we would expect the function output to become increasingly negative as x becomes more negative, since the cubed term will dominate the output. For instance, if we substitute for $x = -1000$ into the two parts of the function, $3x^3 \rightarrow 3(-1000)^3$ is far greater than $12x \rightarrow 12(-1000)$. And as x becomes more positive, the output will become more positive, if we use similar analysis. So the directions of the endpoints in the graph will be maintained.

MODEL PROBLEM

MP 1, 6, 7 Graph $f(x) = -(x^2 + 2x + 1)(x^2 + 9)$ and analyze the end behavior of the function.

SOLUTION

Analyze structure of expression

$$
\begin{aligned}
f(x) &= -(x^2 + 2x + 1)(x^2 + 9) \\
&= -(x^4 + 9x^2 + 2x^3 + 18x + x^2 + 9) \\
&= -(x^4 + 2x^3 + 10x^2 + 18x + 9)
\end{aligned}
$$

Analyze the structure of the expression. It is a quartic expression (multiplying the two x^2 expressions will yield x^4). Then factor the expression.

Factor

$$= -(x + 1)(x + 1)(x^2 + 9)$$

Factor the expression. The expression $(x^2 + 2x + 1)$ is a perfect square.

Use zero-product property

$$
\begin{array}{lll}
-(x + 1) = 0 & x + 1 = 0 & x^2 + 9 = 0 \\
-x - 1 = 0 & x = -1 & x^2 = -9 \\
x = -1 & & x = -3i, 3i
\end{array}
$$

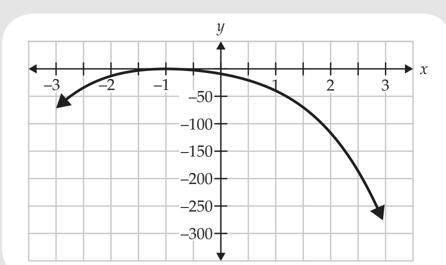
Using the zero-product property for the first two factors, we obtain from each that -1 is a zero. The third factor does not contribute real zeros. Using the zero-product property, $x^2 = -9$, and no real number when squared equals -9.

Analyze end behavior

x	y
−3	−72
−2	−13
−1	0
0	−9
1	−40
2	−117
3	−288

We consider the appearance of the graph at its endpoints: the leading term is $-x^4$, and as x becomes increasingly positive or negative, that term will become more negative. So we would expect the endpoints to point down as we move from the origin.

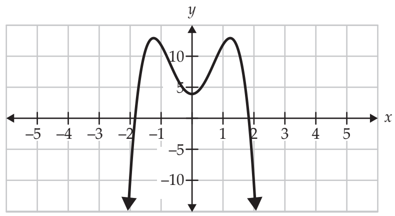
Graph



When we graph the function from $x = -3$ to $x = 3$, we get a graph that meets our description.

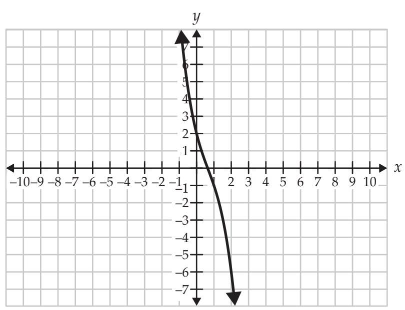
PRACTICE

1. Which of the following statements best describes the zeros of the quartic polynomial function graphed?



A. There are 4 real zeros.
B. There are 4 complex zeros.
C. There are 3 real zeros and 1 complex zero.
D. There are 2 real zeros and 2 complex zeros.

2. Which of the following polynomial functions does this graph best represent?



A. $f(x) = -x^3 + 2x^2 - 4x + 2$
B. $f(x) = x^3 + 3x^2 - 2x + 1$
C. $f(x) = -x^3 + 5x^2 - 2x$
D. $f(x) = -x^3 - x^2 + 3x - 4$

3. **MP 2** Mark analyzes a polynomial $P(x)$ and makes the following table of values.

x	3	4
$P(x)$	0.12	−1.6

Which of his statements is true about $P(x)$?

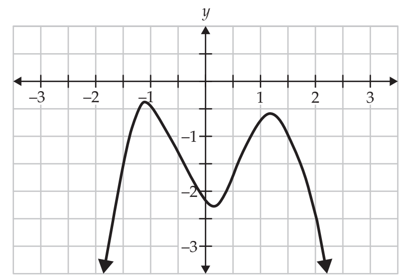
A. The polynomial has a zero at $x = 3$ and $x = 4$.
B. The polynomial has a zero between $x = 3$ and $x = 4$.
C. The polynomial has factors $x - 0.12$ and $x + 1.6$.
D. The polynomial has at least 2 x-intercepts.

4. The graph of a polynomial function never passes through the x-axis but passes through the y-axis once. How many real zeros does it have?

5. The graph of a polynomial function passes through the x-axis three times and the y-axis once. How many real zeros does it have?

Exercises 6–9: Use each graph to determine the number of real zeros of the function.

6.



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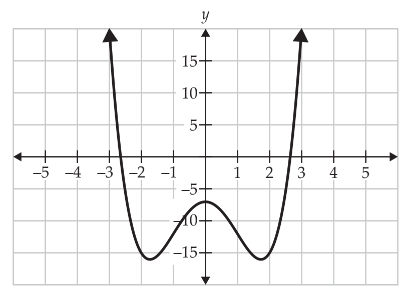

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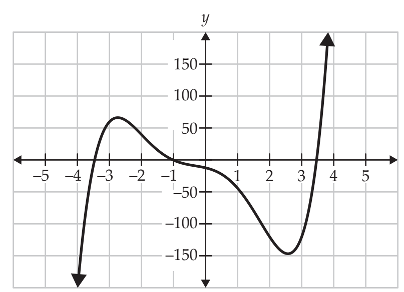

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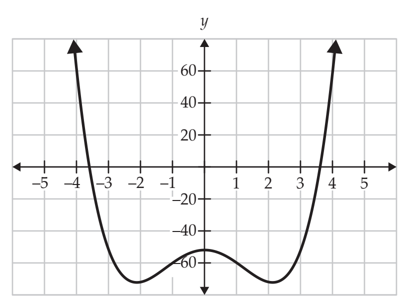
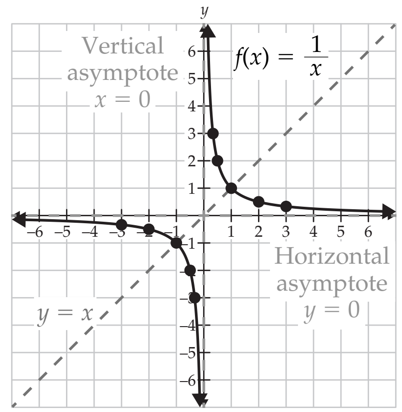


LESSON 1.8

Graphing Rational Functions

> Rational functions are also called **reciprocal functions.**

We show you how to graph rational functions of the form $f(x) = \frac{p(x)}{q(x)}$. We start with the properties of the graph of the parent function $f(x) = \frac{1}{x}$. It is the parent function of rational functions because it is the simplest rational function. Its graph helps you graph other rational functions. These graphs have asymptotes. Remember, asymptotes are lines that are approached but never reached.



x	$f(x)$
-3	$-\frac{1}{3}$
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
$-\frac{1}{3}$	-3
0	undefined
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

- Domain is real numbers <u>except</u> zero — Since division by zero is undefined, the domain of the function is all real numbers except zero.

- Range is real numbers except 0 — The range of $f(x) = \frac{1}{x}$ is all real numbers except zero. There is no value of x that will cause this function to equal 0.

- Asymptote at value excluded from domain — The graph has a vertical asymptote at $x = 0$, which is the y-axis. This asymptote occurs at the domain restriction, that x cannot equal 0.

- Asymptote also at value excluded from range — When x is positive, $\frac{1}{x}$ is also positive. As x gets larger, $\frac{1}{x}$ gets closer and closer to zero. Since $f(x) = \frac{1}{x}$ never equals zero, the graph has a horizontal asymptote at $y = 0$, which is the x-axis. When x gets close to 0, the absolute values of y get larger and larger.

- Two symmetric branches — The graph of $\frac{1}{x}$ has two symmetric branches. One branch is in the first quadrant, and the other is in the third. The graph is symmetric about the line $y = x$. It is also symmetric about the origin, which makes it an odd function. There are two symmetric branches that approach but never reach the asymptotes.

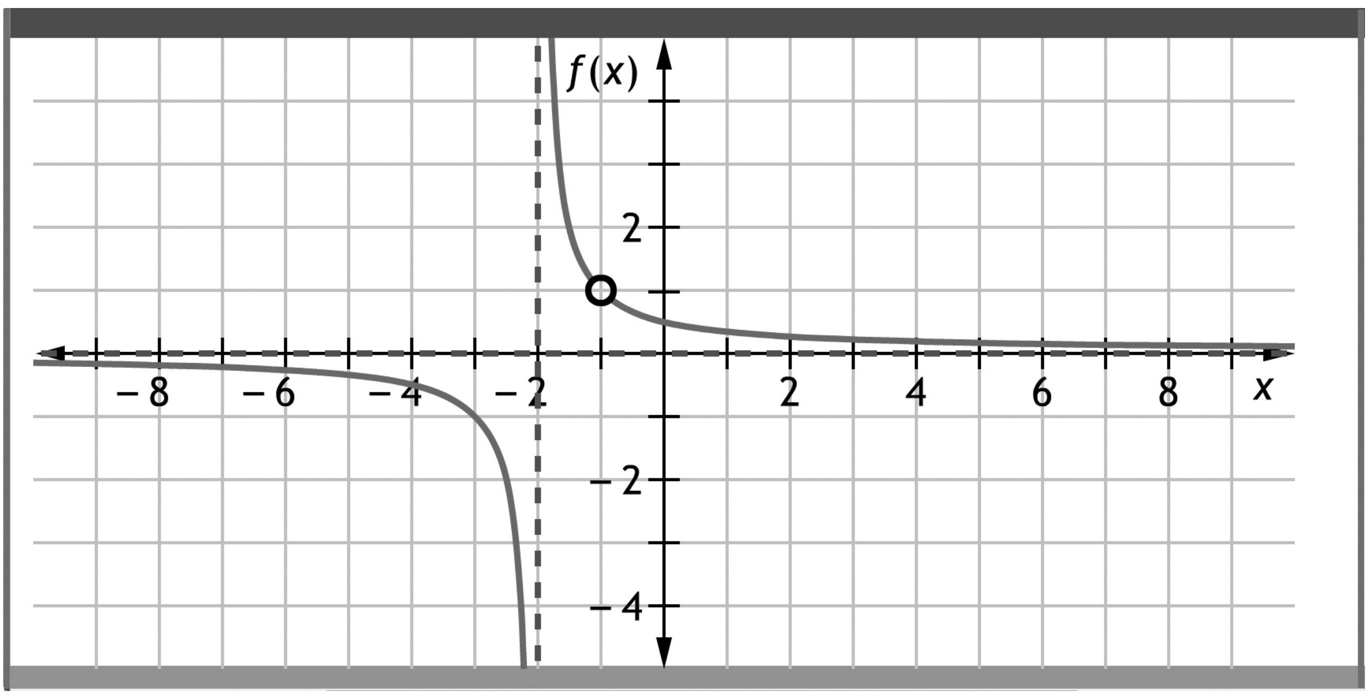
- End behaviors — The asymptotes define the end behaviors of the graph. As the absolute value of x increases, $f(x)$ approaches 0. And as the absolute value of x approaches 0, $|f(x)|$ becomes infinitely large.

Graphs with Discontinuities

This section covers rational functions of the form $f(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials that have zeros in common.

Look at the rational function graphed below. The open circle indicates a point of discontinuity. It is a gap in the curve where the function is not defined. In this case, the *point of discontinuity* occurs because $f(x)$ has common zeros with the same multiplicity in the numerator and denominator.

In the graph below, the point of discontinuity is at $x=-1$, so $p(x)$ and $q(x)$ must have a common factor of $(x+1)$. Points of discontinuity are excluded from a function's domain. The domain of the graphed function is all real numbers except $x=-1$ and $x=-2$.



1. For the reciprocal function $f(x) = \frac{a}{x}$, which statement is true about the graph of $a > 1$?

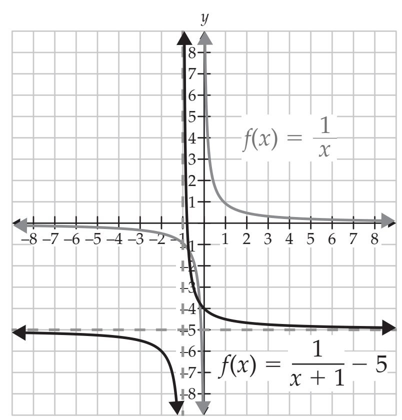
A. If a increases, the horizontal asymptote moves vertically upward.
B. If a increases, the domain and range remain the same.
C. If a decreases, the vertical asymptote moves horizontally to the left.
D. If a decreases, the domain and range become smaller.

SOLUTION

A. The value of a increasing stretches the graph vertically and does not change the asymptotes of $x = 0$ and $y = 0$.
B. **Correct answer.** The value of a increasing stretches the graph vertically and does not affect the domain and range of the function, which are all real numbers except 0.
C. Changing a does not move the vertical asymptote from $x = 0$.
D. Changing a does not change the domain and range of all real numbers except 0.

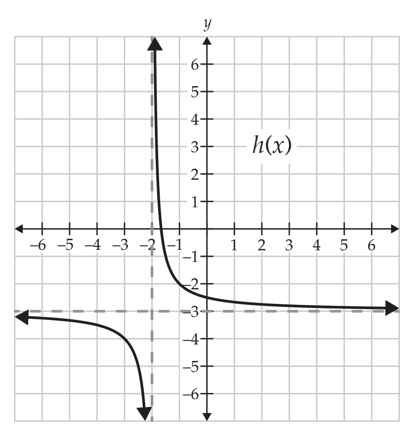
2. Graph $f(x) = \frac{1}{x+1} - 5$. Determine the domain and asymptotes. Describe the function compared to the parent function.

SOLUTION



Determine domain	For this function, the denominator is zero when $x = -1$, so the domain is all real numbers except -1.
Asymptotes	The graph of this function has a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = -5$.
Determine translations based on $f(x) = \frac{1}{x-h} + k$	Since the denominator is $x + 1$, the graph is translated to the left by 1 compared to the graph of $\frac{1}{x}$. To put it another way, $h = -1$.
Translate down by 5	Since the fraction has 5 subtracted from it, the graph is translated down by 5. In other words, $k = -5$. Note that the horizontal asymptote is also translated. It is $y = -5$.

3. Travis translated the graph of a reciprocal function, $g(x)$, vertically up by 2 units and graphed the new function, $h(x)$. Which could be $g(x)$?



A. $g(x) = \frac{1}{x} - 5$
B. $g(x) = \frac{1}{x-2} - 5$
C. $g(x) = \frac{1}{x+2} - 3$
D. $g(x) = \frac{1}{x+2} - 5$

SOLUTION

A. This does not have a horizontal shift as indicated in the graph.
B. The vertical asymptote of this function would be at $x = 2$, not $x = -2$.
C. This is the function of $h(x)$, not $g(x)$.
D. **Correct answer.** This function, when translated vertically 2 units, gives the graph shown.

Model Problems continue . . .

4. Find the domain, equations of any asymptote(s), and points of discontinuity for the function $f(x)\dfrac{x+1}{x^2+3x+2}$.

SOLUTION

Set the denominator equal to zero and solve. Those values will be excluded from the domain. The domain of this function is all real numbers except –1 and –2.

$$x^2+3x+2=0$$
$$(x+1)(x+2)=0$$
$$x=-1,-2$$

Rewrite $f(x)$ and cancel the common factors to simplify.

There is a vertical asymptote at $x=-2$.

$$\frac{\cancel{x+1}}{\cancel{(x+1)}(x+2)}=\frac{1}{x+2}$$

There is a common factor of $(x+1)$ in both the numerator and denominator of $f(x)$. This mean that there is a point of discontinuity at $x=-1$.

5. What is the domain of the function $f(x)\dfrac{x^2-6x-7}{x^2-9x+14}$?

A. All real numbers.
B. All real numbers except $x=-1$.
C. All real numbers except $x=2$.
D. All real numbers except $x=7$.
E. All real numbers except $x-2,7$.
F. All real numbers except $x=-1,2$ and 7.

SOLUTION

Factor the function.

$$\frac{x^2-6x-7}{x^2-9x+14}=\frac{(x+1)\cancel{(x-7)}}{(x-2)\cancel{(x-7)}}=\frac{x+1}{x-2}$$

Interpret the factorization.

The denominator will be equal to zero when $x=2$.

There is a point of discontinuity at $x=7$.

The domain of this function is all real numbers except $x-2,7$ which is answer choice E.

PRACTICE

1. What is the domain of $f(x) = \frac{3}{x - 13} - 11$?

A. All real numbers
B. All real numbers except 0
C. All real numbers except 11
D. All real numbers except −11
E. All real numbers except 13
F. All real numbers except −13

2. What is the domain of $f(x) = \frac{2}{x + 25} - 28$?

A. All real numbers
B. All real numbers except 0
C. All real numbers except 25
D. All real numbers except −25
E. All real numbers except 28
F. All real numbers except −28

3. What is the domain of $f(x) = \frac{1}{x - 32} + 15$?

A. All real numbers
B. All real numbers except 0
C. All real numbers except 15
D. All real numbers except −15
E. All real numbers except 32
F. All real numbers except −32

4. How does the graph of $f(x) = \frac{1}{x + 12}$ compare to the graph of $f(x) = \frac{1}{x}$?

A. $f(x) = \frac{1}{x + 12}$ is 12 units to the left of $f(x) = \frac{1}{x}$

B. $f(x) = \frac{1}{x + 12}$ is 12 units to the right of $f(x) = \frac{1}{x}$

C. $f(x) = \frac{1}{x + 12}$ is 12 units up from $f(x) = \frac{1}{x}$

D. $f(x) = \frac{1}{x + 12}$ is 12 units down from $f(x) = \frac{1}{x}$

5. The two branches of the parent reciprocal function $f(x) = \frac{1}{x}$ are symmetric about what line?

6. What is the domain of $f(x) = \frac{1}{x} - 8$?

7. What is the domain of $f(x) = \frac{1}{x - 2} + 3$?

8. What are the domain and range of $f(x) = \frac{1}{x}$?

9. What are the domain and range of the function $f(x) = \frac{1}{x + 2} - 10$?

10. How do the graphs of $f(x) = \frac{1}{x}$ and $f(x) = -\frac{1}{x}$ differ, and why?

11. The graph of a reciprocal function has a vertical asymptote at $x = -5$. What is the domain of this function?

Exercises 12–15: For each value of a, explain how the graph of the function $f(x) = \frac{a}{x}$ will differ from its parent function $f(x) = \frac{1}{x}$. Use terms like stretched, compressed, and reflected in your descriptions.

12. $a = 3$

13. $a = \frac{1}{2}$

14. $a = -1$

15. $a = -4$

16. What are the asymptotes of the graph $f(x) = \frac{1}{x - 3}$?

17. What are the asymptotes of the graph $f(x) = \frac{1}{x} + 12$?

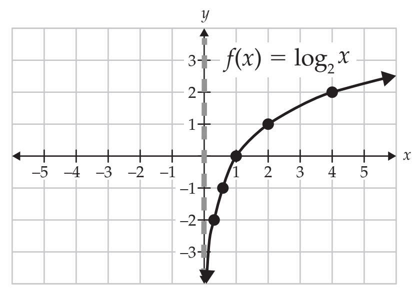
18. What are the asymptotes of the graph $f(x) = \frac{1}{x + 6} - 4$?

19. What are the asymptotes of the graph $f(x) = \frac{1}{x - 5} + 3$?

LESSON 1.9

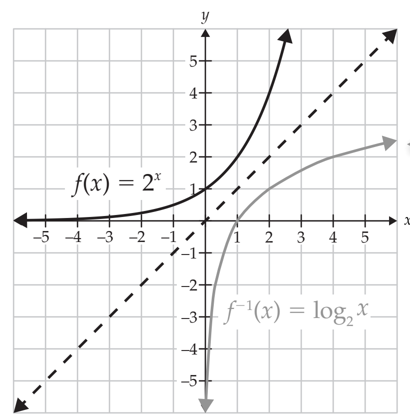
Logarithmic Function Graphs

The graph is a typical logarithmic function. As x approaches 0, $f(x)$ becomes increasingly negative. As x becomes larger, the graph "flattens out." The vertical axis is the vertical asymptote for the logarithmic function. The graph of the logarithmic function does not extend to the left of the y-axis because logarithms are not defined for 0 or for negative numbers.



x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

An exponential function and the appropriate logarithmic function are inverses of each other. Two functions that are inverses have graphs that are mirror images of each other and are symmetric about the line $y = x$. Since the exponential function $f(x) = 2^x$ and the logarithmic function $f^{-1}(x) = \log_2 x$ are inverses, their graphs are symmetric about $y = x$.



Just as with other inverse functions, the graphs of a function and its inverse are symmetric about the line $y = x$.

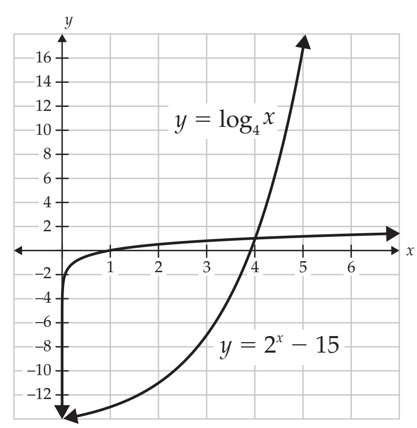
MODEL PROBLEMS

1. Solve the system of equations, $y = \log_4 x$ and $y = 2^x - 15$, by graphing.

SOLUTION

Create table and graph

x	$\log_4 x$	$2^x - 15$
1	0	-13
2	0.50	-11
3	0.79	-7
4	1	1
5	1.16	17



Solution: (4, 1) A solution to the system is where the graphs intersect. In this example, the graphs intersect twice. We can use x-y tables to get a sense of the number of solutions. A solution in the table is a row where the two functions have the same value of y for a given x. A logarithmic function is undefined for 0 and negative inputs, so there cannot be an intersection for values less than or equal to 0. Using the x-y table, we know one solution to this system of equations is (4, 1). The other solution is where $y = 2^x - 15$ crosses $y = \log_4 x$, very close to the y-axis. If you iterate a solution, you will find it between 10^{-8} and 10^{-9}.

Model Problems continue . . .

MODEL PROBLEMS *continued*

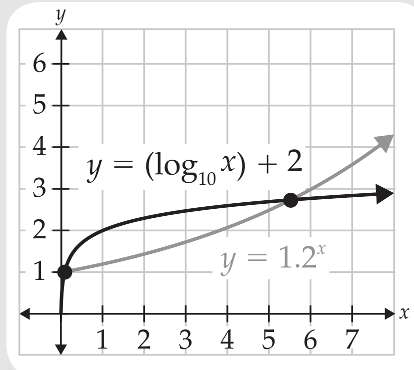
2. **MP 1, 5, 6** **a** Solve $y = (\log_{10} x) + 2$ and $y = 1.2^x$ to find approximate solutions.

b Continue to iterate to find a more precise solution for the greater value of x.

SOLUTION

a Create table and graph

x	$(\log_{10} x) + 2$	1.2^x
1	2	1.2
2	2.30	1.44
3	2.48	1.73
4	2.60	2.07
5	2.70	2.49
6	2.78	2.99
7	2.85	3.58

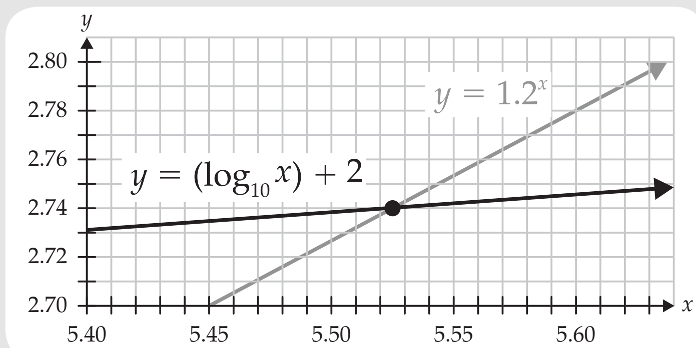


Two solutions | Approximately (0.1, 1.1) | Use the graph to find approximate solutions. One seems to be about (0.1, 1.1). The other seems to be about (5.5, 2.7).

Approximately (5.5, 2.7)

b

x	$(\log_{10} x) + 2$	1.2^x
5.45	2.74	2.70
5.475	2.74	2.71
5.5	2.74	2.73
5.525	2.74	2.74
5.55	2.74	2.75
5.575	2.75	2.76
5.6	2.75	2.78
5.625	2.75	2.79



Iterate values between 5.45 and 5.625 | We estimate that the solution for the greater value of x was about 5.5. Using a spreadsheet, generate values for x, starting with 5.45, and adding 0.025 to create each additional value.

Approximately (5.525, 2.74) | Using the graph, we can approximate the solution as (5.525, 2.74). We could also state that the x-value of the solution is between 5.525 and 5.55, and the y-value is between 2.74 and 2.75.

We could continue to iterate: We could start with 5.525 and add even smaller increments, such as 0.01, to calculate an even more precise solution.

MODEL PROBLEM

Using the parent function $f(x) = \log_4 x$, graph $g(x) = 5 + \log_4 (x - 3)$.

SOLUTION

Create table

$f(x) = \log_4 x$	(x, y)	(1, 0)	(4, 1)	(16, 2)
$g(x) = 5 + \log_4 (x - 3)$	$(x + 3, y + 5)$	(4, 5)	(7, 6)	(19, 7)

Create a table of values that satisfies the parent function, $f(x)$, and the new function, $g(x)$. Using properties of logarithms you can identify simple points for the graph of $f(x)$, (1, 0) and (4, 1), and the corresponding translated points, (4, 5) and (7, 6). You also know that $4^2 = 16$, so (16, 2) is a point on the $f(x)$ graph, with corresponding translated point (19, 7).

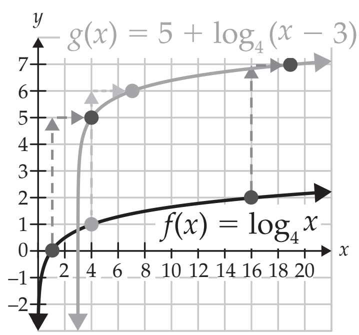
Compare functions

The new function $g(x) = 5 + \log_4 (x - 3)$ has the same shape as the parent function. It's simply translated. The $(x - 3)$ part of the function indicates that the graph of the function moves 3 units to the right. The constant of 5 moves the function 5 units vertically.

Consider domain

Before plotting the points on a graph, the domain of the translated function must be considered. $(x - 3)$ can only be a positive value. Setting $x - 3 > 0$ and solving finds that $x > 3$. If $x > 3$, then the graph of $g(x)$ approaches the vertical line $x = 3$.

Graph



Plot the points (4, 5), (7, 6), and (19, 7) and sketch a logarithmic curve approaching the vertical line $x = 3$. Both the translated function and the parent function are shown on the graph. Note the translation is shown by the arrows.

PRACTICE

1. Which is true about the graph of $y = \log_5 (x - 2)$? Choose all that apply.

A. The graph is 2 units left of $y = \log_5 x$.
B. The graph is 2 units right of $y = \log_5 x$.
C. The graph is 5 units left of $y = \log_2 x$.
D. The graph is 2 units down from $y = \log_5 x$.
E. The graph has the same shape as $y = \log_5 x$.
F. The graph is a steeper curve than $y = \log_5 x$.

Exercises 2–5: Graph.

2. $f(x) = \log_3 x$

3. $f(x) = \log_{1.5} x$

4. $f(x) = \log_4 x$ and $g(x) = 4^x$

5. $f(x) = \log_{0.5} x$ and $g(x) = 0.5^x$

6. Describe the difference between the graphs of $y = \log x$ and $y = 5 + \log x$ using a translation.

7. Describe the difference between $y = \log_2 x$ and $y = \log_2 (x + 5)$ using a translation.

8. Describe the translations that change the graph of $y = \log_3 x$ into the graph of $y = \log_3 (x + 1) - 9$.

9. Describe the translations that change the graph of $y = 2 \log_6 x$ into the graph of $y = 2 \log_6 (x - 3) + 2$.

10. How are the graphs of $y = \log_7 x$ and $y = \log_7 (x + 10) - 4$ similar?

LESSON 1.10

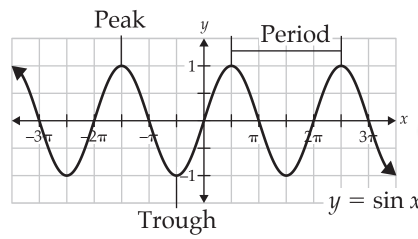
Trigonometric Function Graphs

Properties of Trigonometric Function Graphs

Functions that have a pattern that repeats over and over are called **periodic functions**. The graph of the function repeats its pattern indefinitely. Many functions are not periodic. For instance, linear and exponential functions do not have a repeating form, so they are not periodic. Trigonometric functions are periodic functions.

The period can be calculated as the distance from any *x*-value to the closest *x*-value for which the graph starts to repeat. It is often convenient to use two adjacent peaks or two adjacent troughs.

Period



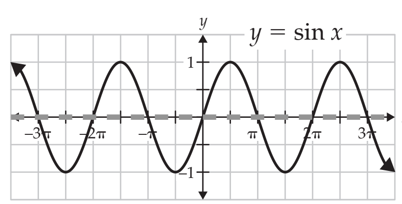
The **period** is the interval on which the graph repeats once. A cycle of a periodic function is the smallest repeating unit of its graph. A period is the interval of the independent variable (x) that contains a single cycle. The graph of a sine function has a period of 2π, as shown in the diagram. This means it repeats its pattern, or completes a cycle, every 2π units as you move from left to right or from right to left. In this type of graph, high points are called *peaks* and low points are called *troughs*.

Frequency

The **frequency** is the number of cycles contained in one unit interval of the independent variable (x). The frequency is the reciprocal of the period. For instance, if a function has a period of 4π, then the frequency is $\frac{1}{4\pi}$.

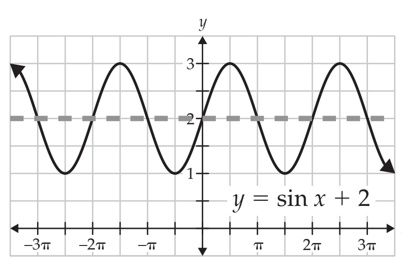
Midline

A **midline** is a horizontal line about which a periodic function *oscillates*. It is the graph's vertical midpoint. In the graph of $y = \sin x$, the midline is the line $y = 0$ (the horizontal axis of the graph).



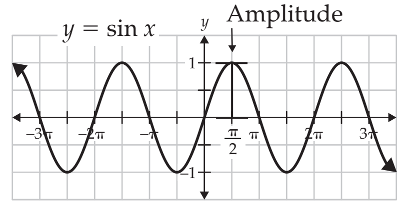
The midline is halfway between the peaks and the troughs.

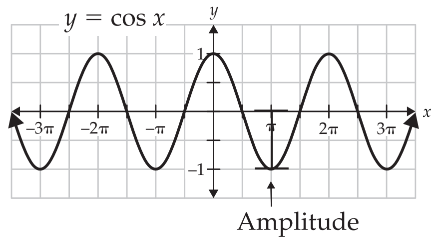
In the graph of $y = \sin x + 2$, the midline is $y = 2$. The graph has been shifted up 2, and the graph is oscillates about the line $y = 2$.



Amplitude

The **amplitude** is the distance from the midline of the graph to its highest or lowest point, or the distance to a peak or a trough. We show the amplitude of the parent functions sin x and cos x. The maximum value of both sin x and cos x is 1, and the minimum value is -1. These values occur at a point such as π radians, for cos x, and $\frac{\pi}{2}$ radians, for sin x. The amplitude of these functions is 1, or the distance between their midline (the x-axis) and a peak or trough.





One way to calculate the amplitude is to measure the distance from peak to trough and divide by two. As a distance, the amplitude is always positive.

End Behavior

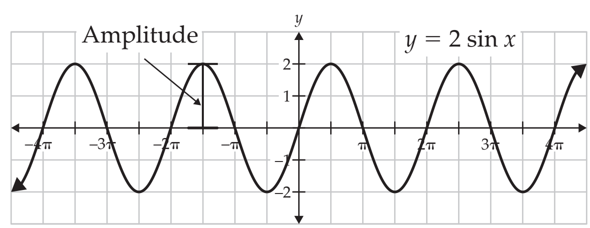
The end behavior of trigonometric functions is to extend infinitely along the x-axis, unless the domain of the function is restricted. The range of the function will be determined by the y-coordinates of a peak and a trough, which are functions of the amplitude and midline.

MODEL PROBLEMS

1. State the amplitude of $y = 2 \sin x$.

SOLUTION

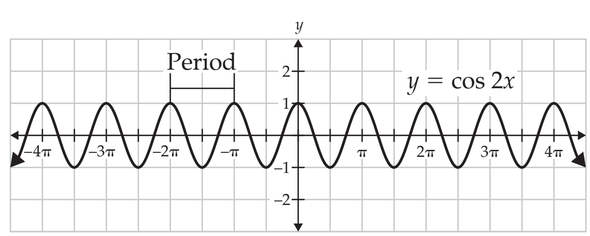
Calculate distance from midline	Amplitude = 2	The amplitude is the maximum distance from the midline, or x-axis in this case.



2. State the period of $y = \cos 2x$.

SOLUTION

Calculate period	Period = π	The distance from peak to peak is the period. From the graph, the period is π.



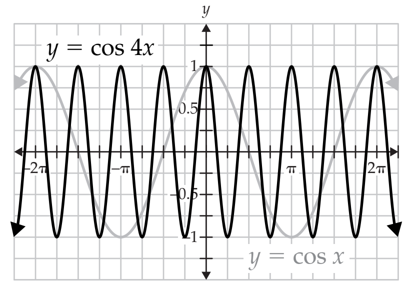
Scaling Trigonometric Function Graphs

Horizontal Scaling

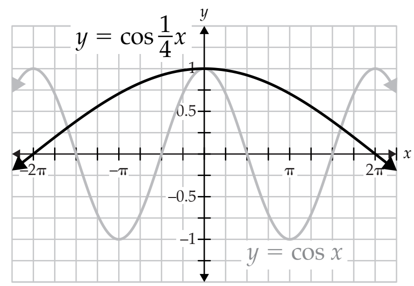
As with other functions, trigonometric functions can be scaled. We use the cosine function as the parent function to discuss the concept of **horizontal scaling**. Scaling horizontally means that you can squeeze points like peaks closer together, or stretch them farther apart. This changes the period of a function.

The period equals $\frac{2\pi}{b}$ in equations of the form $y = a \cos bx$ or $y = a \sin bx$.

We show $y = \cos x$ horizontally scaled by a factor of $\frac{1}{4}$. This pushes the peaks closer together. In other words, it reduces the period by a factor of 4, and since the frequency is the reciprocal of the period, it increases the frequency by a factor of 4.



We show $y = \cos x$ horizontally scaled by a factor of 4. This pushes the peaks farther apart. In other words, it increases the period by a factor of 4.

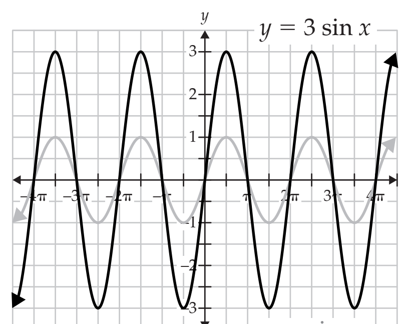


Multiplying the input by b in $y = a \cos bx$ causes *horizontal scaling*.

A periodic function of the form $f(kx)$ changes the period of $f(x)$, scaling its period, by a factor of $\frac{1}{|k|}$. If k's sign is changed, the graph reflects about the y-axis, but a change in sign does not affect its period. If $|k| > 1$, the graph is horizontally compressed, and if $|k| < 1$, it is horizontally stretched.

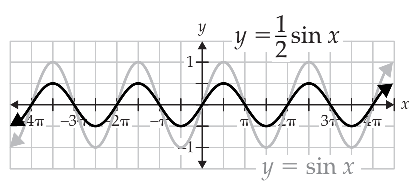
Vertical Scaling

We use the sine function as the parent function to discuss the concept of **vertical scaling**. This changes the amplitude of a function. We show $y = \sin x$ and $y = 3 \sin x$. $y = 3 \sin x$ is vertically taller than the parent function.



We show $y = \sin x$ and $y = \frac{1}{2} \sin x$.

$y = \frac{1}{2} \sin x$ is vertically shorter than the parent function. The constant multiplying the function either stretches it vertically, making the function's graph taller, or it squeezes it vertically, making it shorter.



Multiplying the function by a in $y = a \sin bx$ causes *vertical scaling*.

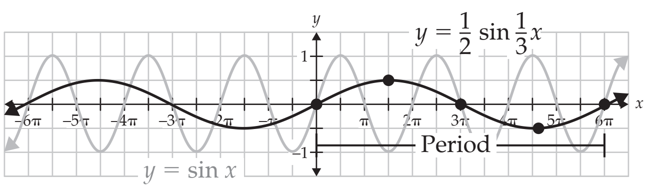
In general, the amplitude of $a \sin x$ or $a \cos x$ is $|a|$, the absolute value of a. We use the absolute value since the amplitude is a distance, and cannot be negative. When $|a|$ is greater than 1, it stretches the graph, making it taller than the parent graph. When $|a|$ is less than 1, it squeezes the graph, making it shorter than the parent graph. With $k(f(x))$, the graph scales vertically by the factor $|k|$. A negative value of k also reflects the graph through the x-axis.

MODEL PROBLEMS *continued*

2. Graph $y = \frac{1}{2}\sin\frac{1}{3}x$ compared to the parent function $y = \sin x$.

SOLUTION

Amplitude	$y = \frac{1}{2}\sin\frac{1}{3}x$ Amplitude $= \lvert a\rvert$ $a = \frac{1}{2}$	The coefficient of the sine is $\frac{1}{2}$, so $a = \frac{1}{2}$. Shrink the graph of $\sin x$ vertically by a factor of $\frac{1}{2}$, cutting its height in half.
Period	Period $= \frac{2\pi}{b}$ $b = \frac{1}{3}$ $\frac{2\pi}{b} = \frac{2\pi}{1/3} = 6\pi$	The value of b, the coefficient of the input x, is $\frac{1}{3}$. That means the period is stretched by a factor of 3, becoming 6π.



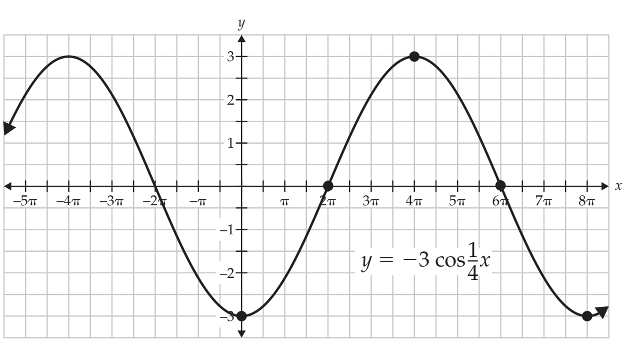
To stretch the graph horizontally, a period should start at the x-intercept $x = 0$ and finish at the x-intercept $x = 6\pi$. This is three times the length of the period of $\sin x$. Mark an x-intercept also at 3π, which is halfway. The maximum and minimum values occur halfway between the x-intercepts.

Plot these points. The y-values are $\frac{1}{2}$ and $-\frac{1}{2}$ at these points.

3. Graph $y = -3\cos\frac{1}{4}x$.

SOLUTION

Amplitude	$y = -3\cos\frac{1}{4}x$ Amplitude $= \lvert a\rvert$ $\lvert a\rvert = 3$	The amplitude of the graph is 3. This will help us plot points. Start with cos 0. It equals 1, and we multiply by -3 to calculate the y-coordinate. Since the amplitude is 3, we know this will be the lowest point on the graph.



x	y
0	-3
2π	0
4π	3
6π	0
8π	-3

With $x = 2\pi$, calculate $\cos\frac{\pi}{2}$, which is 0. Calculate for $x = 4\pi$. $\cos \pi$ equals -1. Multiply by -3 to get 3. This is the maximum height of the function since the amplitude is 3. Plot the points for $x = 6\pi$ and $x = 8\pi$. Connect the points with the curve.

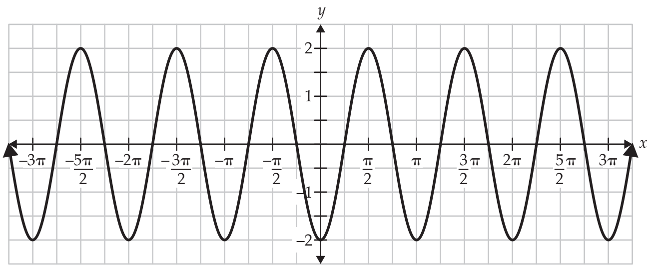
PRACTICE

1. What is the period of the function $y = -2 \sin 8x$?

A. $\frac{\pi}{8}$ C. 8

B. $\frac{\pi}{4}$ D. 16π

2. Which of the following functions best represents the graph below?



A. $y = -3 \cos 2x$

B. $y = -2 \cos \frac{1}{2} x$

C. $y = -2 \cos 2x$

D. $y = \frac{1}{2} \cos x$

3. $\sin \frac{\pi}{4} \approx 0.707$. What is $\sin \frac{9\pi}{4}$?

4. $\cos \frac{3\pi}{5} \approx -0.309$. What is $\cos \frac{13\pi}{5}$?

5. $\sin \frac{2\pi}{7} \approx 0.782$. What is $\sin \frac{16\pi}{7}$?

Exercises 6–8: Determine the amplitude of each function.

6. $24 \sin 3\pi x$

7. $6 \cos 9x$

8. $13 \sin 7\pi x$

Exercises 9–11: Determine the period of each function.

9. $5 \cos 3x$

10. $\frac{1}{2} \sin \pi x$

11. $2 \cos \frac{1}{5} x$

Exercises 12–20: Determine the amplitude, period, and maximum and minimum values for each function.

12. $f(x) = \cos 2x$

13. $f(x) = 2 \cos 3x$

14. $f(x) = 2 \cos 5x$

15. $f(x) = 4 \cos \pi x$

16. $f(x) = 3 \cos 3\pi x$

17. $f(x) = \frac{1}{4} \cos 4\pi x$

18. $f(x) = 2 \sin \pi x$

19. $f(x) = 6 \sin \frac{7}{3} x$

20. $f(x) = 7 \sin \frac{\pi}{3} x$

Exercises 21–24: State an equation of the form $y = a \sin bx$ that matches the description.

21. Maximum value is 2, minimum value is −2, and the period is 2π.

22. Maximum value is 4, minimum value is −4, and the period is π.

23. Maximum value is 0.5, minimum value is −0.5, and the period is 2.

24. Maximum value is 413, minimum value is −413, and the period is 0.1π.

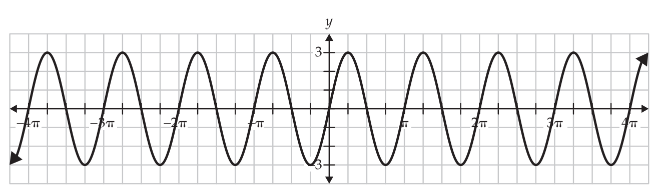
25. Graph $y = 2.5 \sin x$ from $x = -4\pi$ to 4π

26. Graph $y = 4 \cos \pi x$ from $x = -10$ to 10

27. Graph $y = 1.5 \cos 2x$ from $x = -\pi$ to π

28. Graph $y = 5 \sin 2\pi x$ from $x = -5$ to 5

29. What is the equation of the graph shown? State your answer using the sine function.



Translating Trigonometric Function Graphs

As with other functions, when a constant k is added to the parent function $f(x)$ to create the function $f(x) + k$, the graph is translated up by k. All the points are vertically shifted by k (and if k is negative, the graph shifts down).

With the graphs of the sine and cosine functions, we also discuss shifting the midline, since we draw points from there based on the graph's amplitude. The constant k, of course, shifts the midline by k units as well.

As with other graphs, when a constant is subtracted from the input of a function $f(x - h)$, the graph is translated to the right by h. This means that all points on the graph, including peaks and troughs, are translated by h units.

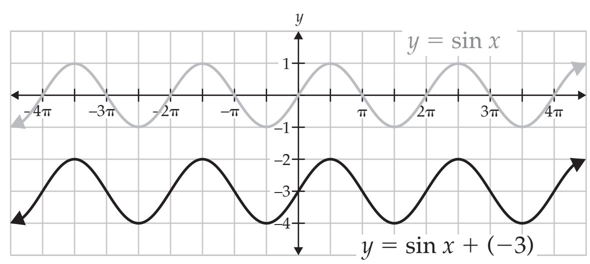
A constant that creates a horizontal translation of a sinusoidal function has a specific name, the *phase* of the function, and the resulting horizontal shift is called a **phase shift**. The graph is translated by the phase shift ***h***.

MODEL PROBLEMS

1. Graph $y = \sin x - 3$ compared to the parent function $y = \sin x$.

SOLUTION

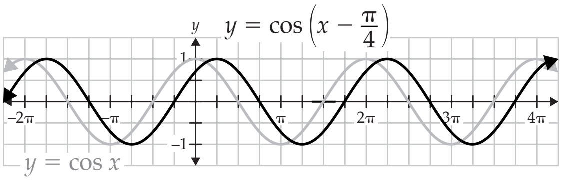
Determine sign of k	$y = \sin x + k$ $y = \sin x + (-3)$	Restate the equation so that the constant is added. This puts it in the form $f(x) + k$.
Translates graph down 3	Since the constant added to the function is negative, the graph shifts down 3 units.	



2. Graph $y = \cos\left(x - \frac{\pi}{4}\right)$ compared to the parent function $y = \cos x$.

SOLUTION

Translates graph horizontally by h	$y = \cos(x - h)$	The phase shift is the constant h subtracted from x. It translates the graph right if h is positive or left if h is negative.
Graph $\cos(x - h)$	$\cos\left(x - \frac{\pi}{4}\right)$	For the function $\cos\left(x - \frac{\pi}{4}\right)$, the phase shift h is $\frac{\pi}{4}$. The phase shift is positive, so it translates the graph to the right by the distance $\frac{\pi}{4}$.



Model Problems continue . . .

3. Graph a sine function with a phase shift of $-\frac{\pi}{2}$ compared to the parent function $y = \sin x$.

SOLUTION

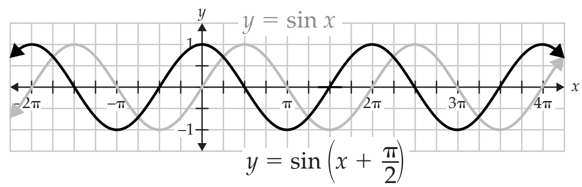
Identify phase shift, h

$$y = \sin(x - h)$$

$$y = \sin\left(x - \left(-\frac{\pi}{2}\right)\right) = \sin\left(x + \frac{\pi}{2}\right)$$

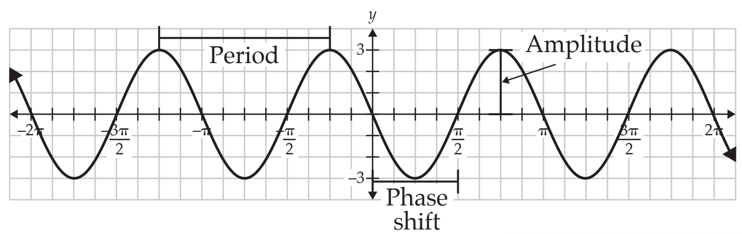
The phase shift is the constant subtracted from x, so in this case the phase shift is $-\frac{\pi}{2}$.

Graph



The phase shift is negative, so it translates the graph to the left by the distance $\frac{\pi}{2}$.

4. Write an equation for the graph using the sine function.



SOLUTION

Use sine as parent function

$$y = a \sin b(x - h)$$

The problem asks us to use sine as the parent function. We show the form of the function. We need to determine the values of a, b, and h.

This form of the trigonometric function uses variables for scaling (*a* and *b*) and translation (*h*).

Amplitude a

$$a = 3$$

The graph has not been vertically shifted. The amplitude of the graph is 3, the height of a peak. This gives us the value for a.

Use period to determine b

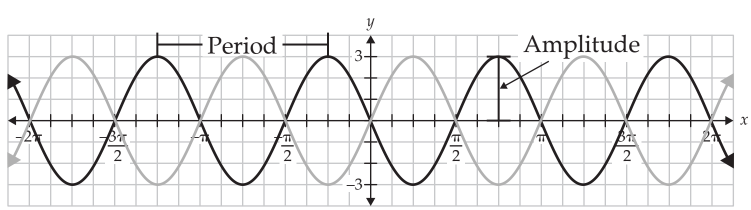
$$\text{Period} = \pi = \frac{2\pi}{b}$$

$$\frac{2\pi}{\pi} = 2$$

$$b = 2$$

The period of the graph is π. That is the distance between two adjacent peaks. The period is 2π divided by b. Write this as an equation and solve for b, which is 2.

Phase shift h



$$h = \frac{\pi}{2}$$

We show with a gray line the sine function with no phase shift. We need to shift the gray line graph. It will be shifted by the phase shift. The graph we are trying to match starts at 0 and decreases. The graph with no phase shift has that property at $\frac{\pi}{2}$. We need to shift the gray graph by $\frac{\pi}{2}$ to the right. This is the phase shift. This value equals h and it will be subtracted from x.

Substitute a, b, h

$$y = 3 \sin 2\left(x - \frac{\pi}{2}\right)$$

We now have the values for a, b, and h, which we substitute in the equation.

PRACTICE

1. Which of the following is true about the graph of $y = -12 + 19 \cos (x - 7)$ compared to the graph of $y = 19 \cos (x - 7)$?

A. It is shifted up by 7.
B. It is shifted down by 7.
C. It is shifted up by 12.
D. It is shifted down by 12.
E. It is shifted up by 19.
F. It is shifted down by 19.

2. Which of the following is true about the graph of $y = 4 - 15 \cos (x + 3)$ compared to the graph of $y = -15 \cos (x + 3)$?

A. It is shifted up by 3.
B. It is shifted down by 3.
C. It is shifted up by 4.
D. It is shifted down by 4.
E. It is shifted up by 15.
F. It is shifted down by 15.

3. Which of the following is true about the graph of $y = -25 - 8 \sin (x - 2)$ compared to the graph of $y = -8 \sin (x - 2)$?

A. It is shifted up by 2.
B. It is shifted down by 2.
C. It is shifted up by 8.
D. It is shifted down by 8.
E. It is shifted up by 25.
F. It is shifted down by 25.

4. Given the function $y = \sin x$, which of the following represents the translation of $\frac{\pi}{8}$ units to the right?

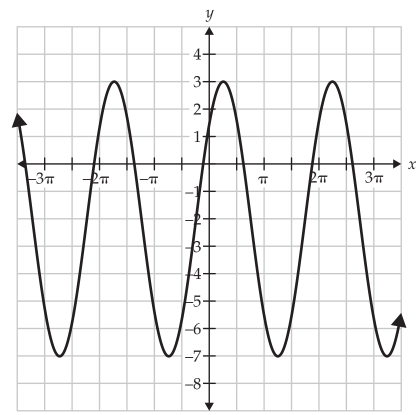
A. $\sin x - \frac{\pi}{8}$

B. $\sin x + \frac{\pi}{8}$

C. $\sin\left(x - \frac{\pi}{8}\right)$

D. $\sin\left(x + \frac{\pi}{8}\right)$

5. Given $f(x) = \sin x$, which of the following could represent the given graph?



A. $f(x) = 2f\left(x - \frac{\pi}{4}\right) - 5$

B. $f(x) = 2f\left(x + \frac{\pi}{2}\right) + 5$

C. $f(x) = 5f\left(x + \frac{\pi}{4}\right) - 2$

D. $f(x) = 5f\left(x - \frac{\pi}{2}\right) - 2$

6. Which of the following is equivalent to $\cos (2x - \pi)$?

A. $-\cos 2x$
B. $\cos 2x$
C. $2 \cos (x - \pi)$
D. $2 \cos\left(x - \frac{\pi}{2}\right)$

7. **MP 2, 7** Which function has a graph whose phase shift is twice the period?

A. $y = \cos 8\left(x - \frac{\pi}{2}\right)$

B. $y = \cos\left(x - \frac{\pi}{2}\right) + \frac{\pi}{4}$

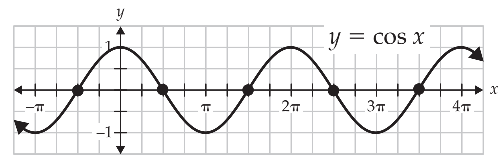
C. $y = \cos 4\left(x - \frac{\pi}{2}\right)$

D. $y = \frac{\pi}{4} \cos\left(x - \frac{\pi}{2}\right)$

Graph of the Tangent Function

To graph the tangent function, we could just plot points until we were sure we had a good idea of the shape of the curve. However, to reduce the number of points we have to plot, we consider some properties of the tangent.

Since the tangent is the sine divided by the cosine, we must consider values of x for which $\cos x$ is 0, since at these points the tangent is undefined. At these values, the graph of the tangent will have vertical asymptotes, lines which the graph approaches but never reaches. To identify these points, we graph the cosine function.



- Definition of tangent $\tan x = \dfrac{\sin x}{\cos x}$

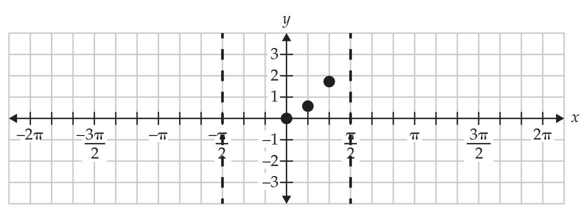
  Since the tangent is the sine divided by the cosine, it is undefined when the cosine has the value 0.

- Asymptotes for tangent $x = \ldots, -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots$

  $\cos x = 0$ at $\dfrac{\pi}{2}$, and then 0 again every π additional units left and right along the x-axis. Using the graph, we state the values of x that will result in asymptotes for the tangent.

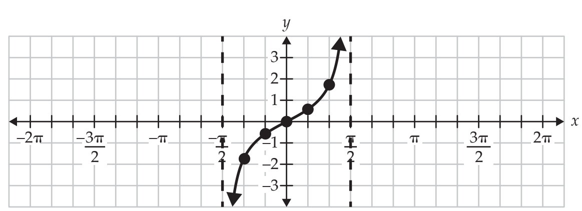
We also use the locations of the asymptotes to determine the period of the tangent function. Adjacent asymptotes are separated by π, so the period is π. We more formally derive the period next. Using our analysis, we need only five points to draw a graph.

- Draw asymptotes and plot points



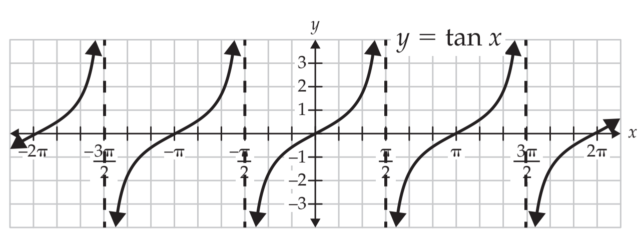
  We use the facts about the tangent function to graph it. We draw asymptotes based on our analysis above of their locations. We plot 3 points for the interval from 0 to the first asymptote. The function is undefined at $\dfrac{\pi}{2}$.

- Tangent is symmetric about origin



  We plot some points between $-\dfrac{\pi}{2}$ and 0, and we draw the curve that passes through these points. The graph is symmetric about the origin. This shows the function is odd.

- Period is π



  Finally, since the period of the tangent is π, we repeat the graph every π units.

LESSON 1.11

Volume

Volume is the amount of three-dimensional space an object occupies. As with surface area, there are formulas for the volumes of common figures.

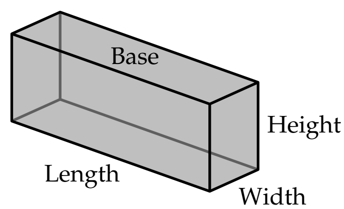
The units for volume are usually expressed as cubic units. This is because volume is a measure of capacity. We often use abbreviations such as in^3 or cm^3, but other units of volume include the gallon and liter.

Volume of Cubes and Rectangular Prisms

The general formula for the volume of any regular solid is $V = Bh$, where B is the area of the base and h is the height of the solid. Regular solids, such as rectangular prisms, cubes, cylinders, and more, have formulas specific to their properties, but they are all derived from the general formula given above. Below we see how the general formula is transformed into the specific formula for the volume of a rectangular prism.

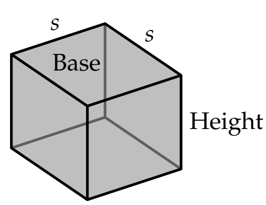
$V = Bh$
V = volume
B = area of base
h = height

$V = l \cdot w \cdot h$
V = volume
l = length
w = width
h = height



We can use the same general formula to determine the formula specific to the volume of a cube. Recall that the formula for the area of the base of a cube is s^2. As all the edges of a cube are the same length, multiplying s^2 by the height, s, yields s^3. Thus the volume of a cube is $V = s^3$.

$V = s^3$
s = edge length



MODEL PROBLEMS

1. A cube with edges of length s centimeters can hold N marbles. The density of marbles in this box can be determined using the expression

A. $(6s^2)N$

B. $\frac{6s^2}{N}$

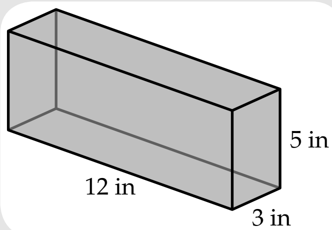
C. $\frac{s^3}{N}$

D. $\frac{N}{s^3}$

SOLUTION

The answer is D. Density is the amount divided by volume. The number of marbles is N, and the volume of a cube is s^3, so D reflects this division.

2. **MP 7** Determine the volume of the rectangular prism. Use two different formulas.



SOLUTION

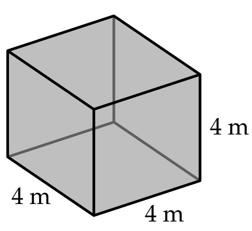
Formula 1

$V = l \cdot w \cdot h$	$V = 12 \cdot 3 \cdot 5$ $V = 180\text{ in}^3$	Substitute the values for the length, width, and height and multiply. The volume is 180 cubic inches.

Formula 2

$V = Bh$	$B = 12 \cdot 3 = 36$ $V = 36 \cdot 5$ $V = 180\text{ in}^3$	We could also calculate the area of its base first, and then multiply that area by the height. We get the same result.

3. What is the volume of the cube?



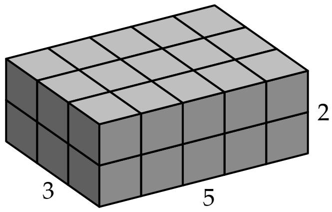
SOLUTION

Use formula	$V = s^3$ $V = 4^3$	To calculate the volume of this cube, substitute the length of a side, which is 4 m.
Multiply	$V = 64\text{ m}^3$	Complete the calculation. The volume of the cube is 64 cubic meters.

Model Problems continue . . .

MODEL PROBLEMS *continued*

4. What is the volume? Each cube has a volume of 8 cubic centimeters.



With a single cube, we multiply its length by its width by its height to calculate its volume. If we have many cubes, we can combine their volumes by multiplying.

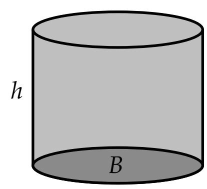
SOLUTION

Calculate the number of cubes	number of cubes $= 5 \cdot 3 \cdot 2 = 30$	To calculate the volume of this object, first calculate the number of cubes. There are 5 along the length of the object, 3 as we go back, and it is 2 high. Multiply these three values.
Multiply by volume of a single cube	$V = 30 \cdot 8 = 240 \text{ cm}^3$	To calculate the volume of the object, multiply the number of cubes, 30, by the volume of a cube, 8 cm^3.

Volume of Other Prisms and Cylinders

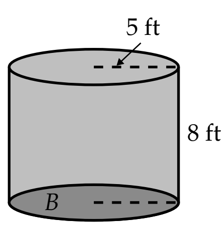
The volume of prisms and cylinders can be calculated as the product of the area of its base and its height.

$V = Bh$
$B =$ area of base
$h =$ height



MODEL PROBLEMS

1. Calculate the volume of the cylinder.



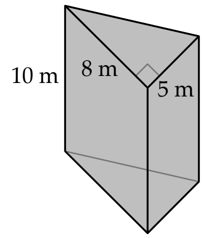
SOLUTION

Calculate the base area	$B = \pi r^2$ $B \approx 3.14 \cdot 5^2$	To calculate the volume of the cylinder, first calculate the area of its base, using the formula for the area of a circle. Substitute the radius shown in the diagram.
Multiply by height	$V = Bh$ $V \approx 3.14 \cdot 5^2 \cdot 8$ $V \approx 628 \text{ ft}^3$	The volume of a cylinder equals the product of its base area and height.

Model Problems continue . . .

MODEL PROBLEMS

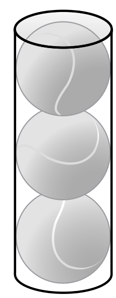
2. Calculate the volume of the triangular prism.



SOLUTION

Calculate the base area	B = area of triangle $B = \frac{1}{2} \cdot 5 \cdot 8 = 20\ \text{m}^2$	Use the formula for the area of a triangle to calculate the area of the base of the triangular prism. The area of a triangle equals one-half the triangle's base times its height.
Multiply by height	$V = Bh$ $V = 20 \cdot 10$ $V = 200\ \text{m}^3$	The volume of a prism equals the product of the area of its base and the prism's height. The base area is 20 m^2. Multiply that by the height, 10 m, and get 200 m^3.

3. Manufacturers package tennis balls in cylindrical cans. The cylinder is just wide enough to fit the ball and just large enough to hold 3 tennis balls stacked on top of each other. If each tennis ball has a diameter of 2.60 inches, what is the volume of air inside the cylindrical can?



SOLUTION

Determine the height of the cylinder	$h = 3 \cdot 2.6$ $h = 7.8$ inches	The height of the cylinder will be equal to the height of three tennis balls.
Find the volume of the cylinder	$V = \pi r^2 h$ $V = \pi(1.3)^2\,(7.8)$ $V \approx 41.4\ \text{in}^3$	Use the formula for the volume of a cylinder. Don't forget to calculate the radius of the tennis ball.
Find the volume of the tennis balls; there are 3 to a package	$V = 3\left(\frac{4}{3}\pi r^3\right)$ $V = 4\pi r^3$ $V = 4\pi(1.3)^3$ $V = 27.6\ \text{in}^3$	Multiply the formula for the volume of a sphere by 3 and simplify.
Subtract	$41.4 - 27.6 = 13.8\ \text{in}^3$	The volume of air in the cylinder is 13.8 in^3.

Volume of Pyramids and Cones

For the first volumes we worked with—cubes, regular prisms, and cylinders—we could multiply the base area and height. Finding the volume of a pyramid or cone is not quite so simple. However, we can use the same formula to describe the volume of a pyramid or of a cone.

$$V = \frac{1}{3}Bh$$

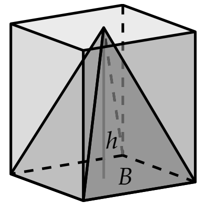
B = area of base
h = height

With the surface area formulas we discussed earlier, the pyramids and cones had to have regular polygons, like a square, as a base. With the volume formulas, we can drop that requirement.

We explain the $\frac{1}{3}$ in the volume formula:

A pyramid inside a prism with the same base area and height

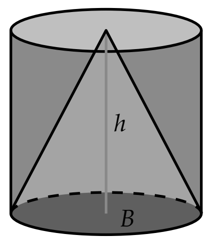
- Takes up $\frac{1}{3}$ the volume of the prism



If we were to fit a pyramid into a prism with the same base area and height as the pyramid, the pyramid would only take up one-third of the space inside the prism.

A cone inside a cylinder with the same base area and height

- Takes up $\frac{1}{3}$ the volume of the cylinder

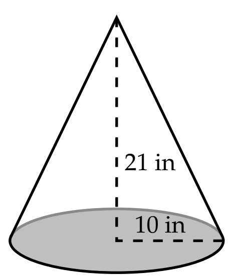


The same is true for a cone inside of a cylinder with the same base area and height.

Try it! One way to informally prove the relationship to yourself is to construct a pyramid and prism of the same base and height out of plastic or some other material that will hold water. Fill the pyramid with water and transfer the water to the prism. You should be able to do this exactly three times to completely fill the prism. The same works for a cone and a cylinder.

MODEL PROBLEMS

1. What is the cone's volume?



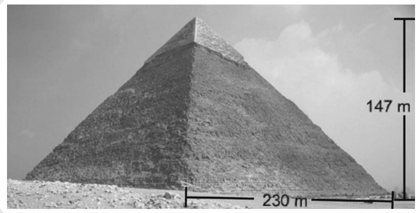
SOLUTION

Calculate the base area	$B = \pi r^2$ $B \approx 3.14 \cdot (10)^2 \approx 314 \text{ in}^2$	To calculate the volume of a cone, calculate the area of its base, using the formula for the area of a circle. Substitute the radius.
Multiply by height	$V = \frac{1}{3}Bh$ $V \approx \frac{1}{3}(314 \cdot 21) \approx 2198 \text{ in}^3$	The base area is 314 in^2, and the height is 21 in. Multiply and get 2198 in^3.

Model Problems continue . . .

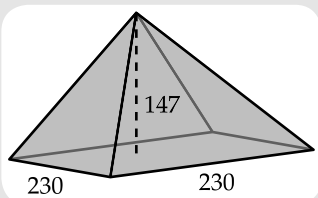
MODEL PROBLEMS *continued*

2. **MP 4** The Great Pyramid of Khufu is the last of the Seven Wonders of the Ancient World still standing. The pyramid is smaller now than when it was built due to erosion and other causes. It has a square base, and an estimate of its original dimensions appears in the diagram. What was its original volume?



SOLUTION

Draw diagram



Note that the height used for volume is not the slant height, but the vertical height from the base to the top.

Compute the base area	$B = s \cdot s$ $B = 230 \cdot 230$ $B = 52{,}900 \text{ m}^2$	Start by computing the area of the base. The problem says each side was about 230 meters, so substitute those values, and multiply.
Calculate the volume	$V = \frac{1}{3}Bh$ $V = \frac{1}{3}(52{,}900)(147)$ $V = 2{,}592{,}100 \text{ m}^2$	Apply the formula for the volume of a pyramid. Substitute the area of the base we just computed and the given vertical height. Notice that we do not need the slant height to calculate the pyramid's volume.

Volume of Spheres

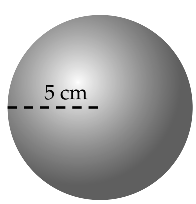
We write the equation for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

r = radius

MODEL PROBLEM

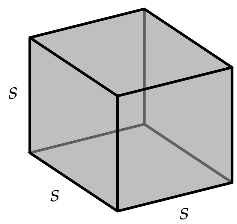
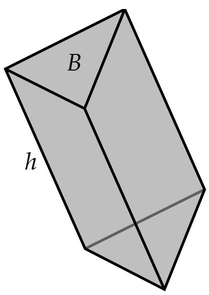
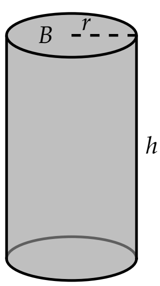
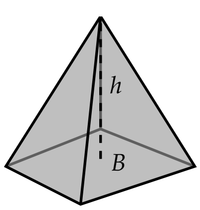
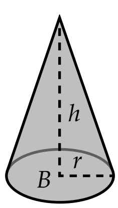
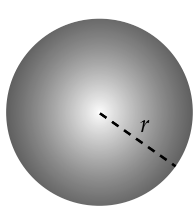
Find the volume of the sphere.



SOLUTION

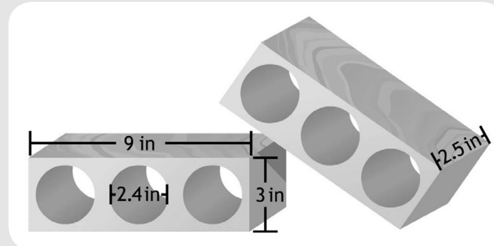
Formula	$V \approx \frac{4}{3}(3.14 \cdot 5^3)$	To calculate the volume of the sphere above, substitute the value of the radius.
Evaluate	$V \approx \frac{4}{3}(3.14 \cdot 125)$ $V \approx 523 \text{ cm}^3$	Then cube 5, and multiply. The volume of the sphere is about 523 cm^3.

Volume Summary

Volume	Formula	
Cube  	$V = s^3$	The volume of a cube equals the length of an edge, cubed.
Prism  	$V = Bh$	The volume of a prism can be calculated as the product of its base area and its height.
Cylinder  	$V = Bh$ $V = \pi r^2 h$	The volume of a cylinder also equals the product of its base area and height. The area of the base, since it is a circle, is πr^2.
Pyramid and cone   	$V = \frac{1}{3}Bh$	The volume of a pyramid and cone can be calculated with this formula. The height is the distance from the center of the base to the top, not the slant height.
Sphere  	$V = \frac{4}{3}\pi r^3$	This is the formula for the volume of a sphere.

MODEL PROBLEMS

1. **MP 1, 7** Calculate the mass of the two identical children's blocks to the nearest hundredth of a pound. The density of the wood is $\frac{0.02\text{ lbs}}{\text{in}^3}$.



SOLUTION

Calculate volume of a block, ignoring holes

$$V_{\text{one block}} = l \cdot w \cdot h$$
$$V_{\text{one block}} = 9 \cdot 2.5 \cdot 3$$
$$V_{\text{one block}} = 67.5\text{ in}^3$$

Start by calculating the volume of one block, ignoring the holes for now.

Calculate volume of holes

$$B = \pi r^2$$
$$V_{\text{holes}} = 3 \cdot Bh$$
$$V_{\text{holes}} \approx 3 \cdot 3.14 \cdot 1.2^2 \cdot 2.5$$
$$V_{\text{holes}} \approx 33.9\text{ in}^3$$

Now calculate the volume of the holes in one block. They are cylinders.

Subtract holes

$$V_{\text{one block}} - V_{\text{holes}} \approx 67.5 - 33.9$$
$$V_{\text{one block}} - V_{\text{holes}} \approx 33.6\text{ in}^3$$

Subtract the volume of the holes from the volume of the block if it were solid.

Two blocks

$$V_{\text{two blocks}} \approx 2 \cdot 33.6$$
$$V_{\text{two blocks}} \approx 67.2\text{ in}^3$$

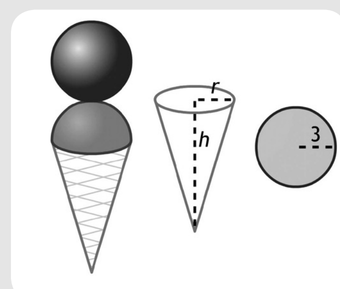
There are two blocks, so multiply by 2 the volume of one block with holes.

Weight

$$\text{mass} = \text{volume} \cdot \text{density}$$
$$\text{mass} \approx 67.2\text{ in}^3 \cdot 0.02\,\frac{\text{lbs}}{\text{in}^3}$$
$$\text{mass} \approx 1.34\text{ lbs}$$

Mass equals the product of volume and density. We were told the density. Multiply to calculate the mass.

2. **MP 2, 4** How much ice cream does the cone hold? Assume the cone itself is very thin, the bottom scoop is a perfect hemisphere, and the top scoop is a perfect sphere. The cone itself is also packed. The cone has a radius of 3 cm on top and a height of 10 cm. Estimate the volume of the cone to the nearest tenth of a cubic centimeter.



SOLUTION

Calculate the cone's base area

$$B = \pi r^2$$
$$B \approx 3.14(3)^2$$
$$B \approx 28.3\text{ cm}^2$$

Calculate the area of the base, using the area formula for a circle.

Calculate cone's volume

$$V = \frac{1}{3}Bh$$
$$V \approx \frac{1}{3}(28.3) \cdot 10$$
$$B = 94.3\text{ cm}^3$$

Use the formula for the volume of a cone.

Calculate volume of scoops

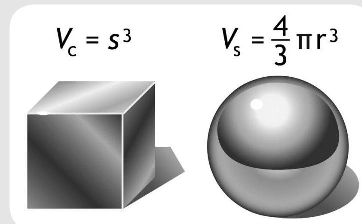
$$V = 1.5\left(\frac{4}{3}\pi r^3\right)$$
$$V \approx 1.5\left(\frac{4}{3} \cdot 3.14 \cdot (3)^3\right)$$
$$V \approx 169.6\text{ cm}^3$$

Calculate the volume of the scoops. We have one whole sphere and one half sphere, so we multiply the expression for the volume of a sphere by 1.5.

Model Problems continue . . .

Total	$V = \text{cone} + \text{scoops}$ $V \approx 94.3 + 169.6$ $V \approx 263.9 \text{ cm}^3$	The entire ice cream volume equals the sum of the ice cream in the cone and the ice cream in the scoops. Add the two volumes. The volume of ice cream is approximately 263.9 cubic centimeters. The volume 263.9 cubic centimeters equals about 16 cubic inches. That is a cube with sides of approximately 2.5 inches.

3. **MP 2, 3** An artist is asked to design an ornament that is 10 grams of copper covered by gold leaf. She considers a cube and a sphere as her two possible shapes. She wants to minimize the surface area so that she uses as little gold as possible. Which shape should she choose?



SOLUTION

Set volumes equal to each other

$$V_s = \frac{4}{3}\pi r^3$$
$$V_c = s^3$$
$$V_c = V_s$$
$$s^3 = \frac{4}{3}\pi r^3$$

We know that the volumes of the cube and sphere must be equal, so we use this fact to find the expression for the surface area of the cube in terms of *r*, the sphere's radius. The two volumes must be equal, since both shapes will contain 10 grams each.

Solve for *s*

$$s = \sqrt[3]{\frac{4}{3}\pi r^3}$$
$$s = r\sqrt[3]{\frac{4}{3}\pi}$$

Solve for *s*, the length of a side of the cube. Remove the *r* from the radical since the cube root of r^3 is r.

Surface area of cube

$$S_{\text{cube}} = 6s^2$$
$$S_{\text{cube}} = 6\left(r\sqrt[3]{\frac{4}{3}\pi}\right)^2$$
$$S_{\text{cube}} = 6r^2\left(\sqrt[3]{\frac{4}{3}\pi}\right)^2$$

Apply the formula for the surface area of a cube. Substitute the equation we just created for the length of a side.

Surface area of sphere

$$S_{\text{sphere}} = 4\pi r^2$$

This is the equation for the surface area of a sphere.

Ratio cube to sphere

$$\frac{S_{\text{cube}}}{S_{\text{sphere}}} = \frac{6r^2\left(\sqrt[3]{\frac{4}{3}\pi}\right)^2}{4\pi r^2}$$
$$\frac{S_{\text{cube}}}{S_{\text{sphere}}} = \frac{3\left(\sqrt[3]{\frac{4}{3}\pi}\right)^2}{2\pi}$$

State the ratio for the surface area of a cube to the surface area of a sphere.

Evaluate

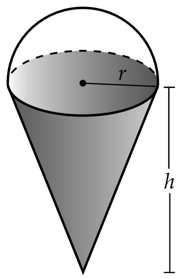
$$\frac{S_{\text{cube}}}{S_{\text{sphere}}} \approx 1.24$$

Evaluate. Take the cube root of $\frac{4}{3}\pi$, square it, and multiply it by 3. Divide that by 2π. The ratio is 1.24 to 1, so the cube has a greater surface area, about 24% greater. The sphere will be the less expensive shape, since it will require less gold to cover it.

Model Problems continue . . .

PRACTICE

1. What is the volume of the solid below?



A. $\pi r^3 h$

B. $\frac{2}{3}\pi r^3 + \frac{1}{3}hr^2$

C. $\frac{\pi r^2}{3}(2r + h)$

D. $\pi r^3 h + \frac{4}{3}hr^2$

2. The surface area of a cube is 1.5 mm^2. What is the volume of the cube?

A. 2.25 mm^3 C. 0.125 mm^3
B. 1.25 mm^3 D. 0.5625 mm^3

3. The formula $V = \frac{1}{3}\pi r^2 h$ can be used to find the volume of a

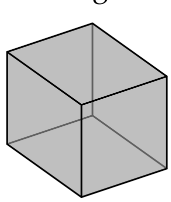
A. Cone. C. Pyramid.
B. Cylinder. D. Sphere.

4. The base of a pyramid is a trapezoid with bases of 3.56 feet and 6.12 feet, and a distance between the bases of 5.27 feet. The pyramid's height is 6 feet and 6 inches. What is the volume of the pyramid rounded to the nearest cubic foot? Hint: Start with the formula for the area of a trapezoid.

A. 26 cubic feet C. 168 cubic feet
B. 166 cubic feet D. 160 cubic feet

5. Find the volume of a cube with sides of length 3 centimeters. State your answer in cubic centimeters.

6. Calculate the volume of the object below in in cubic feet. The length of each side is 3 feet.



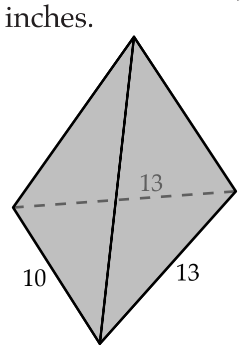
7. What is the volume of the object below, in cubic inches? The box is 8 inches high by 7 inches wide by 3 inches deep (or high). The figure is not drawn to scale.

8. The top face of a rectangular prism is 9 by 6 inches, and the height of the prism is 6 inches. There is a triangular hole in the top face that cuts all the way through the prism and through the bottom face. The triangle has a base of 2 inches and a height of 3 inches. What is the net volume of the prism, without the volume of the hole?

9. Sketch a figure whose volume could be found by subtracting the volume of a rectangular prism from the volume of a triangular prism.

10. Find the volume of a cone with a height of 5 centimeters. The radius of the base is 6 centimeters.

11. What is the volume of the object below? Its height is 7 inches.



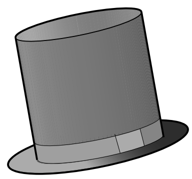
12. Find the volume of a cylinder with height 3 centimeters and radius 7 centimeters.

13. Find the volume of a cylinder with height 4 centimeters and radius 9 centimeters.

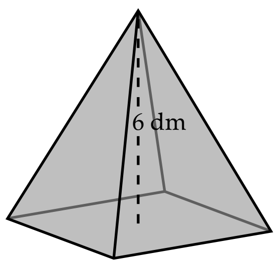
14. Find the volume of a prism with a triangular base. The base is a right triangle with sides of length 6 centimeters, 8 centimeters, and 10 centimeters, and the height of the prism is 7 centimeters.

Practice Problems continue . . .

15. Calculate the volume of the cylindrical portion of the hat below. State your answer to the nearest cubic centimeter. Its radius is 17 centimeters and its height is 30 centimeters.



16. What is the volume of the object shown in the figure, in cubic decimeters? Its base is 4 decimeters by 4 decimeters.



17. Paulina says it is not possible for a cone and a pyramid that have the same height to have the same volume, since their bases are different shapes. Do you agree with Paulina? Why or why not?

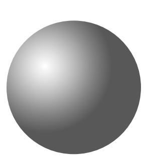
18. If you triple the radius of a cone without changing its height, what happens to its volume? Why?

19. Felix made a paperweight out of clay. The paperweight was in the shape of a square pyramid with a base of 3 inches on a side and a height of 2 inches. Sketch Felix's pyramid and find its volume.

20. You need to make the volume of a pyramid *k* times bigger without changing the dimensions of its base. What must you do to the height to cause the volume to increase?

21. Find the volume of a sphere with radius 8 centimeters.

22. What is the volume of the object below in the figure? Its radius is 4 millimeters.



23. What is the volume of the sculpture below? Assume the waffle cone is 4 feet tall, with a radius of 1.2 feet. Assume the ice cream is in the shape of a cone with a radius of 1.4 feet, and a height of 3 feet.

24. Sketch a building whose volume could be found by treating it as a rectangular prism topped by a rectangular pyramid.

25. Isabel is senior class president and is in charge of purchasing the class time capsule. The model she prefers is cylindrical with a hemisphere on each end. The entire capsule is 6 feet long, with the cylinder composing 5 feet of the length. The diameter of the capsule is 12 inches.

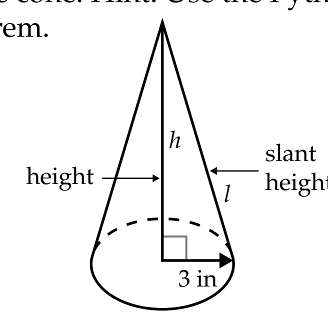
a What is the radius of the capsule and each of the hemispheres?

b Make a sketch of the capsule. Label all the dimensions.

c Calculate the capsule's volume.

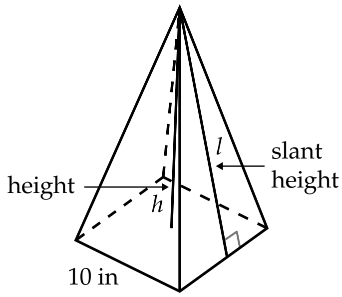
26. A box has a volume of 48 cubic units. Make a table listing integer combinations of length, width, and height that would produce this volume. Include at least five distinct combinations in the table.

27. The radius of the base of the cone below is 3 inches. The volume of the cone is 37.68 cubic inches. Find the slant height of the cone. Hint: Use the Pythagorean theorem.



Practice Problems continue . . .

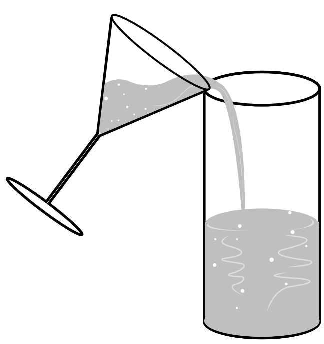
28. The base of the regular pyramid below is a square with a side length of 10 inches. The volume of the pyramid is 400 cubic inches. Find the slant height l of the pyramid.



29. The radius of the base of a cone is 6 meters. The lateral surface area of the cone is 188.4 square meters. What is the cone's volume?

30. Anthony ordered a truck of decorative bark to be delivered to his house. The truck dumped the bark on Anthony's driveway. The heap has the shape of a cone with a base radius of 15 feet and a slant height of 17 feet. How many cubic feet of bark did Anthony order? Disregard the space between individual bark pieces.

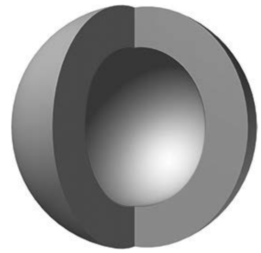
31. Caroline pours her soda from the cone-shaped glass to the cylinder-shaped one, as in the diagram below. The cone has a base radius of 2.1 inches, and a height of 3.2 inches. The base radius of the cylindrical glass is 1.8 inches. How many inches high will the soda be in the cylindrical glass?



32. How do you need to change the radius of a cone to increase its volume 49 times, if the height of the cone stays the same?

33. **MP 2, 3** If the radius of a sphere is doubled, how does the volume of the sphere change? Explain your reasoning.

34. A spherical capsule, like the one below, has an outer diameter of 18 meters and the thickness of its walls is 3 meters. What is the volume of the material used for the capsule?

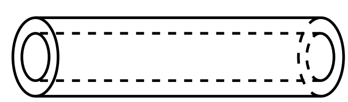


35. A tower has the shape of a cylinder with a height of 13 yards, with a hemisphere on top of the cylinder. The lateral surface area of the cylinder part is 530.66 square yards. What is the volume of the tower?

36. Select a complicated object. Make a sketch of the object, reducing it to four or more solids that have known formulas for volume.

37. Can you state a length of a cube where, when you compute its volume, the number for the volume is less than its surface area? Ignore the fact that the units (and quantities measured) differ.

38. **MP 5, 6** The density of a material is its mass per unit volume. For example, the density of gold is 19.32 grams per cubic centimeter. That means a gold cube with a side of 1 centimeter will have a mass of 19.32 grams. The density of lead is 11.4 grams per cubic centimeter. A lead pipe has an outer diameter of 7.9 cm, and an inner diameter of 3.3 cm. What is the mass, in grams, of a section of pipe that is 7 centimeters long? Use 3.14 for π and round your answer to the nearest whole gram.



39. The volume of a sphere is 179.5 mm^3. What is its radius to the nearest tenth of a micrometer?

Graphing Functions Answer Key

