

Unconditional Stability of the ADI scheme

MTSCS546

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Claim:

The 2D Douglas-Rachford scheme for the heat equation $u_t = \Delta u$ is unconditionally stable.

Proof:

Recall that the Douglas-Rachford scheme for the heat equation (using the notation of Morton) takes the form

$$(1 - \mu_x \delta_x^2) v^{n+1\star} = (1 + \mu_y \delta_y^2) v^n \quad (1a)$$

$$(1 - \mu_y \delta_y^2) v^{n+1} = v^{n+1\star} - \mu_y \delta_y^2 v^n \quad (1b)$$

Eliminating the intermediate solution $v^{n+1\star}$ leads to

$$(1 - \mu_x \delta_x^2)(1 - \mu_y \delta_y^2) v^{n+1} = (1 + \mu_x \mu_y \delta_x^2 \delta_y^2) v^n \quad (2)$$

Suppose the solution of the scheme is in the form

$$v_{j,\ell}^n = (\lambda)^n e^{i\xi j \Delta x} e^{i\eta \ell \Delta y}.$$

Substituting into the ADI scheme (2) leads to

$$\lambda \left(1 + 2\mu_x \sin^2 \frac{\theta_x}{2} \right) \left(1 + 2\mu_y \sin^2 \frac{\theta_y}{2} \right) = \left(1 - 4\mu_x \mu_y \sin^2 \frac{\theta_x}{2} \sin^2 \frac{\theta_y}{2} \right)$$

where

$$\theta_x = \xi \Delta x, \quad \theta_y = \eta \Delta y,$$

and we have used the trig identities

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}.$$

Solving for λ

$$\lambda = \frac{1 - 4\mu_x \mu_y \sin^2 \frac{\theta_x}{2} \sin^2 \frac{\theta_y}{2}}{(1 + 2\mu_x \sin^2 \frac{\theta_x}{2})(1 + 2\mu_y \sin^2 \frac{\theta_y}{2})} \quad (3)$$

$$= \frac{1 - 4\mu_x \mu_y \sin^2 \frac{\theta_x}{2} \sin^2 \frac{\theta_y}{2}}{1 + 4\mu_x \mu_y \sin^2 \frac{\theta_x}{2} \sin^2 \frac{\theta_y}{2} + 2\mu_x \sin^2 \frac{\theta_x}{2} + 2\mu_y \sin^2 \frac{\theta_y}{2}} \quad (4)$$

Since $0 \leq |\lambda| \leq 1$ whenever $\mu_x, \mu_y > 0$, the ADI scheme for the heat equation is unconditionally stable in 2D.