Unconditional Stability of the ADI scheme

MTSCS546

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Claim:
The 2D Douglas-Rachford scheme for the heat equation $u_t = \Delta u$ is unconditionally stable.

Proof:
Recall that the Douglas-Rachford scheme for the heat equation (using the notation of Morton) takes the form

$$(1-\mu_x\delta_x^2)v^{n+1\star} = (1+\mu_y\delta_y^2)v^n \tag{1a}$$

$$(1-\mu_y\delta_y^2)v^{n+1} = v^{n+1\star} - \mu_y\delta_y^2 v^n \tag{1b}$$

Eliminating the intermediate solution $v^{n+1\star}$ leads to

$$(1-\mu_x\delta_x)(1-\mu_y\delta_y)v^{n+1} = (1+\mu_x\mu_y\delta_x^2\delta_y^2)v^n \tag{2}$$

Suppose the solution of the scheme is in the form

$$v_{j,\ell}^n = (\lambda)^n e^{i\xi j\Delta x} e^{i\eta\ell\Delta y}.$$

Substituting into the ADI scheme (2) leads to

$$\lambda\left(1+2\mu_x\sin^2\frac{\theta_x}{2}\right)\left(1+2\mu_x\sin^2\frac{\theta_y}{2}\right) = \left(1-4\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}\right)$$

where

$$\theta_x = \xi\Delta x, \quad \theta_y = \eta\Delta_y,$$

and we have used the trig identities

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad 1-\cos\theta = 2\sin^2\frac{\theta}{2}.$$

Solving for λ

$$\lambda = \frac{1-4\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}}{\left(1+2\mu_x\sin^2\frac{\theta_x}{2}\right)\left(1+2\mu_x\sin^2\frac{\theta_y}{2}\right)} \tag{3}$$

$$= \frac{1-4\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}}{1+4\mu_x\mu_y\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}+2\mu_x\sin^2\frac{\theta_x}{2}\sin^2\frac{\theta_y}{2}} \tag{4}$$

Since $0 \le |\lambda| \le 1$ whenever $\mu_x, \mu_y > 0$, the ADI scheme for the heat equation is unconditionally stable in 2D.