Assignment 1

HMTH407/HFM307 (Partial Differential Equations)

September 2022

1. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous or linear homogeneous.

   (a)
   $$(\cos xy^2)u_x - y^2 u_{xxy} = \tan(x^2 + y^2)$$

   (b)
   $$\frac{u_x}{\sqrt{1+u_x^2}} + \frac{u_y}{\sqrt{1+u_y^2}} = 0.$$

2. Solve the equation
   $$(1+x^2)\, u_{xx} + u_{yy} = 0.$$
   Sketch some of the characteristic curves.

3. Solve the equation
   $$\sqrt{1-x^2}\, u_x + u_y = 0, \quad u(0,y) = y$$

4. Solve
   $$u_x + u_y + u = e^{x+y} \quad u(x,0) = 0.$$

5. Consider the one-dimensional wave equation
   $$u_{tt} - c^2 u_{xx} = 0, \quad c > 0$$

   (a) Show that the wave equation is hyperbolic.

   (b) By using the method of characteristics, deduce that the general formula for the wave equation is given by
   $$u(x,t) = F(x+ct) + G(x-ct),$$
   where F, G are arbitrary.

(c) The wave equation is supplemented with the initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

Deduce d'Alembert's formula:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds.$$

(d) Show that the d'Alembert formula above satisfies the wave equation. (You may assume that $f''(x)$ and $g'(x)$ exist and are continuous.

(e) Solve the wave equation

$$\begin{aligned} u_{tt} - 25u_{xx} &= 0, \\ u(x,0) &= e^{-x^2}, \\ u_t(x,0) &= 4 + x. \end{aligned}$$

6. Consider the equation $u_{xx} - 2u_{xy} + u_{yy} + 3u_x - u_y = 0$.

(a) What is its type (elliptic/parabolic/hyperbolic)?

(b) Reduce it to standard form.

(c) Find its general solution.