Assignment 2

HMTH407/HFM307 (Partial Differential Equations)

September 2022

1. (10 marks) Solve the Cauchy problem for the heat equation

$$\begin{aligned} &u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0 \\ &u(x,0) = \phi(x), \end{aligned}$$

where

$$\phi(x) = \begin{cases} 5, & \text{if } |x| < 2, \\ 0, & \text{if } |x| > 2. \end{cases}$$

leaving your solution in terms of the error function.

2. (5 points) Use Fourier Transforms to solve the Cauchy problem for the heat equation

$$\begin{aligned} &u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0. \\ &u(x,0) = \phi(x). \end{aligned}$$

3. (10 points) Use Fourier Transforms to show that the solution of the Dirichlet problem for the Laplace equation in the upper-half plane

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0, \\ &u(x,0) = g(x). \end{aligned}$$

is given by

$$u(x,y) = \frac{y}{x} \int_{-\infty}^{\infty} \frac{f(\tau)}{(x-\tau)^2 + y^2} \, d\tau.$$

4. (10 points) Use the result from Question 4 to show that the solution of the Dirichlet problem for the Neumann problem for the Laplace equation

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad y > 0, \\ &u_y(x,0) = f(x). \end{aligned}$$

is given by

$$u(x,y) = \int_{-\infty}^{\infty} g(x-\xi) \ln(y^2 + \xi^2) \, d\xi + C.$$

Hint: Use the transformation $v = u_y$ and reduce the problem to a Dirichlet problem.

HFM only

5. (10 points) Logan 4.1: 1.

HMTH only

6. (10 points) Use Laplace Transforms to solve

$$\begin{aligned} &u_t = u_{xx}, \quad \text{on} \quad x > 0, \quad t > 0, \\ &u(0,t) = a, \quad u(x,0) = b. \end{aligned}$$

where a and b are constants.