Classification

HMTH407/HFM307

September 2022

Find the general solution of

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$$

Solution:
In this case, $a = 4, 2b = 5$ and $c = 1$. So that

$$b^2 - ac = \left(\frac{5}{2}\right)^2 - (4)(1) = \frac{25}{4} - 4 > 0.$$

Hence the equation is hyperbolic. The characteristic equations are obtained from

$$\frac{dy}{dx} = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{4} = 1, \text{ or } \frac{1}{4}.$$

The characteristic equations are

$$\xi = y - x, \quad \eta = y - \frac{1}{4}x.$$

Using these curves, the original equation is transformed as follows:

$$u_x = w_\xi \xi_x + w_\eta \eta_x, \qquad u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{xx} = w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx}$$
$$u_{yy} = w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy}$$
$$u_{xy} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

In this case

$$\xi_x = -1, \;\; \xi_y = 1, \;\; \eta_x = -\frac{1}{4}, \;\; \eta_y = 1$$

and

$$\xi_{xx} = \xi_{yy} = \xi_{xy} = 0,$$
$$\eta_{xx} = \eta_{yy} = \eta_{xy} = 0.$$

Hence,

$$
\begin{aligned}
u_{xx} &= w_{\xi\xi} + \frac{1}{2}w_{\xi\eta} + \frac{1}{16}w_{\eta\eta} \\
u_{yy} &= w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \\
u_{xy} &= -w_{\xi\xi} - \frac{5}{4}w_{\xi\eta} - \frac{1}{4}w_{\eta\eta}.
\end{aligned}
$$

Combining, we have

$$
\begin{aligned}
4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y &= 4(w_{\xi\xi} + \frac{1}{2}w_{\xi\eta} + \frac{1}{16}w_{\eta\eta}) \\
&\quad +5(-w_{\xi\xi} - \frac{5}{4}w_{\xi\eta} - \frac{1}{4}w_{\eta\eta}) \\
&\quad +w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \\
&= -\frac{9}{4}w_{\xi\eta} - w_\xi - \frac{1}{4}w_\eta + w_\xi + w_\eta \\
&= -\frac{9}{4}w_{\xi\eta} + \frac{3}{4}w_\eta
\end{aligned}
$$

The original equation is transformed into

$$w_{\xi\eta} = \frac{1}{3}w_\eta - \frac{8}{9}.$$

Let $v = w_\eta$. Then

$$v_\xi = \frac{1}{3}v - \frac{8}{9}.$$

We solve this ODE using integrating factors. Multiplying the ODE by the integrating factor $e^{-\frac{1}{3}\xi}$, we get:

$$e^{-\frac{1}{3}\xi}v_\xi - \frac{1}{3}e^{-\frac{1}{3}\xi}v = -\frac{8}{9}e^{-\frac{1}{3}\xi}.$$

Using the product rule, we get

$$\frac{\partial}{\partial\xi}\left(e^{-\frac{1}{3}\xi}v\right) = -\frac{8}{9}e^{-\frac{1}{3}\xi}.$$

Integrating with respect to ξ,

$$e^{-\frac{1}{3}\xi}v = -\frac{8}{9}\int e^{-\frac{1}{3}\xi}\, d\xi = \frac{8}{3}e^{-\frac{1}{3}\xi} + g(\eta).$$

So that

$$v(\xi,\eta) = \frac{8}{3} + g(\eta)e^{\frac{1}{3}\xi}.$$

Since $v = w_\eta$, integrate v w.r.t η to get w

$$w(\xi,\eta) = \frac{8}{3}\eta + e^{\frac{1}{3}\xi}G(\eta) + F(\xi)$$

where

$$G(\eta) = \int g(\eta)\, d\eta$$

Substituting $\xi = y - x$ and $\eta = y - \frac{1}{4}x$, we obtain

$$u(x,t) = \frac{8}{3}\left(y - \frac{1}{4}x\right) + e^{\frac{1}{3}(y-x)} G\left(y - \frac{1}{4}x\right) + F(y-x)$$