Solution for the Space Craft's Center of Mass

The purpose of this derivation is to find the center of mass of the space craft in its mechanical frame from the following known quantities:

- inertial properties of the space craft (i.e. mass and moment of inertia)
- locations of test masses and thrusters (in the mechanical frame of the space craft)
- accelerations of the test masses in response to the thrusters

For the purpose of this derivation, we use quantities and data for only one test mass and one thruster. Table 1 contains and describes all base variables used below. All positions are in the mechanical frame of the space craft.

Table 1: Variables used in derivation

Symbol	Description	Status
$m_{\rm sc}$	mass of space craft	known (measured)
$\mathcal{I}_{\rm sc}$	moment of inertia of space craft	known (measured)
$\mathbf{r}_{\rm tm}$	position of test mass	known (measured)
$\mathbf{r}_{\rm th}$	position of thruster	known (measured)
$\mathbf{r}_{\rm sc}$	position of space craft's center of mass	unknown
$\mathbf{a}_{\rm tm}$	linear acceleration of test mass (relative to space craft)	known (data)
$\mathbf{a}_{\rm sc}$	linear acceleration of space craft (absolute)	unknown
$\boldsymbol{\alpha}_{\rm tm}$	angular acceleration of test mass (relative to space craft)	known (data)
$\boldsymbol{\alpha}_{\rm sc}$	angular acceleration of space craft (absolute)	known ($=-\boldsymbol{\alpha}_{\rm tm}$)
$\mathbf{F}_{\rm th}$	force exerted by thruster on space craft	unknown
$\boldsymbol{\tau}_{\rm th}$	torque exerted by thruster on space craft	known ($=\mathcal{I}_{\rm sc}\boldsymbol{\alpha}_{\rm sc}$)

The torque exerted on the space craft by a thruster is given by

$$\boldsymbol{\tau}_{\rm th} = (\mathbf{r}_{\rm th} - \mathbf{r}_{\rm sc}) \times \mathbf{F}_{\rm th}. \tag{1}$$

Substituting the space craft's mass, moment of inertia, and accelerations yields

$$\mathcal{I}_{\rm sc}\boldsymbol{\alpha}_{\rm sc} = (\mathbf{r}_{\rm th} - \mathbf{r}_{\rm sc}) \times m_{\rm sc}\mathbf{a}_{\rm sc}. \tag{2}$$

Neglecting the Coriolis force and centripetal acceleration in the frame of the test mass, we can substitute measured accelerations of the test mass to get

$$-\mathcal{I}_{\rm sc}\boldsymbol{\alpha}_{\rm tm} = (\mathbf{r}_{\rm th} - \mathbf{r}_{\rm sc}) \times m_{\rm sc}\left(\mathbf{a}_{\rm tm} - \boldsymbol{\alpha}_{\rm tm} \times [\mathbf{r}_{\rm tm} - \mathbf{r}_{\rm sc}]\right). \tag{3}$$

Defining the vector

$$\mathbf{u} = \mathbf{a}_{\rm tm} - \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm tm}, \tag{4}$$

(3) can be written as

$$-m_{\rm sc}^{-1}\mathcal{I}_{\rm sc}\boldsymbol{\alpha}_{\rm tm} = (\mathbf{r}_{\rm th} - \mathbf{r}_{\rm sc}) \times \left(\mathbf{u} + \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}\right). \tag{5}$$

Separating known and unknown terms yields

$$\mathbf{r}_{\rm th} \times \mathbf{u} + m_{\rm sc}^{-1}\mathcal{I}_{\rm sc}\boldsymbol{\alpha}_{\rm tm} = \mathbf{r}_{\rm sc} \times \mathbf{u} + (\mathbf{r}_{\rm sc} - \mathbf{r}_{\rm th}) \times (\boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}) \tag{6}$$

Assuming that the solution space $\mathcal{R}_{\rm sc}$ of possible positions $\mathbf{r}_{\rm sc}$ satisfying (6) is a differentiable manifold, the equality in (6) does not change under a variation $\mathbf{r}_{\rm sc} \to \mathbf{r}_{\rm sc} + \delta\hat{\mathbf{p}}$ where δ is an infinitesimally small quantity and $\hat{\mathbf{p}}$ is a unit vector tangent to $\mathcal{R}_{\rm sc}$ at some valid $\mathbf{r}_{\rm sc}$. We can thus say that

$$\mathbf{r}_{\rm sc} \times \mathbf{u} + (\mathbf{r}_{\rm sc} - \mathbf{r}_{\rm th}) \times (\boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}) = (\mathbf{r}_{\rm sc} + \delta\hat{\mathbf{p}}) \times \mathbf{u} + (\mathbf{r}_{\rm sc} + \delta\hat{\mathbf{p}} - \mathbf{r}_{\rm th}) \times \left(\boldsymbol{\alpha}_{\rm tm} \times [\mathbf{r}_{\rm sc} + \delta\hat{\mathbf{p}}]\right). \tag{7}$$

Canceling out all terms not containing $\delta\hat{\mathbf{p}}$ on both sides of (7) yields:

$$0 = \delta\hat{\mathbf{p}} \times \mathbf{u} + (\mathbf{r}_{\rm sc} - \mathbf{r}_{\rm th}) \times (\boldsymbol{\alpha}_{\rm tm} \times \delta\hat{\mathbf{p}}) + \delta\hat{\mathbf{p}} \times (\boldsymbol{\alpha}_{\rm tm} \times [\mathbf{r}_{\rm sc} + \delta\hat{\mathbf{p}}]) \,. \tag{8}$$

Collecting terms by powers of δ,

$$0 = \delta\left(\hat{\mathbf{p}} \times \mathbf{u} + [\mathbf{r}_{\rm sc} - \mathbf{r}_{\rm th}] \times [\boldsymbol{\alpha}_{\rm tm} \times \hat{\mathbf{p}}] + \hat{\mathbf{p}} \times [\boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}]\right) + \delta^2\left(\hat{\mathbf{p}} \times [\boldsymbol{\alpha}_{\rm tm} \times \hat{\mathbf{p}}]\right). \tag{9}$$

As terms in different powers of δ must independently satisfy (9), the appearance of the δ^2 term therein implies

$$\hat{\mathbf{p}} = \hat{\boldsymbol{\alpha}}_{\rm tm}. \tag{10}$$

The terms in (9) which are in the first power of δ thus satisfy

$$0 = \hat{\boldsymbol{\alpha}}_{\rm tm} \times \mathbf{u} + (\mathbf{r}_{\rm sc} - \mathbf{r}_{\rm th}) \times (\boldsymbol{\alpha}_{\rm tm} \times \hat{\boldsymbol{\alpha}}_{\rm tm}) + \hat{\boldsymbol{\alpha}}_{\rm tm} \times (\boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}) = \hat{\boldsymbol{\alpha}}_{\rm tm} \times (\mathbf{u} + \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}) \,. \tag{11}$$

Removing the component in the cross product parallel to $\hat{\boldsymbol{\alpha}}_{\rm tm}$, we can say that

$$0 = \hat{\boldsymbol{\alpha}}_{\rm tm} \times (\mathbf{u} - [\mathbf{u} \cdot \hat{\boldsymbol{\alpha}}_{\rm tm}]\,\hat{\boldsymbol{\alpha}}_{\rm tm} + \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}) \equiv \hat{\boldsymbol{\alpha}}_{\rm tm} \times \left(\mathbf{u}^{\perp} + \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}\right), \tag{12}$$

where we have implicitly defined the vector

$$\mathbf{u}^{\perp} \equiv \mathbf{u} - (\mathbf{u} \cdot \hat{\boldsymbol{\alpha}}_{\rm tm})\,\hat{\boldsymbol{\alpha}}_{\rm tm} = \mathbf{a}_{\rm tm} - (\mathbf{a}_{\rm tm} \cdot \hat{\boldsymbol{\alpha}}_{\rm tm})\,\hat{\boldsymbol{\alpha}}_{\rm tm} - \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm tm}. \tag{13}$$

As all of $\mathbf{u}^{\perp} + \boldsymbol{\alpha}_{\rm tm} \times \mathbf{r}_{\rm sc}$ is perpendicular to $\hat{\boldsymbol{\alpha}}_{\rm tm}$, to satisfy (12) we must have

$$\mathbf{r}_{\rm sc} \times \boldsymbol{\alpha}_{\rm tm} = \mathbf{u}^{\perp}. \tag{14}$$

While this result does not allow us to solve for $\mathbf{r}_{\rm sc}$ explicitly, we can find the component

$$\mathbf{r}_{\rm sc}^{\perp} = \mathbf{r}_{\rm sc} - (\mathbf{r}_{\rm sc} \cdot \hat{\boldsymbol{\alpha}}_{\rm tm})\,\hat{\boldsymbol{\alpha}}_{\rm tm} \tag{15}$$

perpendicular to $\hat{\boldsymbol{\alpha}}_{\rm tm}$, which is given by:

$$\mathbf{r}_{\rm sc}^{\perp} = \frac{\left|\mathbf{u}^{\perp}\right|}{\left|\boldsymbol{\alpha}_{\rm tm}\right|}\hat{\boldsymbol{\alpha}}_{\rm tm} \times \hat{\mathbf{u}}^{\perp}. \tag{16}$$

We thus conclude that

$$\mathcal{R}_{\rm sc} = \mathbf{r}_{\rm sc}^{\perp} + \mathbb{R}\hat{\boldsymbol{\alpha}}_{\rm tm}. \tag{17}$$

Though one thruster does not provide enough information to determine $\mathbf{r}_{\rm sc}$, using data from multiple thrusters one can find the intersection of all $\mathcal{R}_{\rm sc}$ (or, in practice, the point of closest approach to all $\mathcal{R}_{\rm sc}$) to get the actual position $\mathbf{r}_{\rm sc}$.