Path-Coloring Algorithms for Plane Graphs

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Abstract

A path coloring of a graph G is a vertex coloring of G such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh [7], is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman [6] and Škrekovski [8], performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

1 Introduction

All graphs will be finite, simple, and undirected.

A *path coloring* of a graph G is a vertex coloring (not necessarily proper) of G such that each color class induces a disjoint union of paths. A graph G is *path k-colorable* if G admits a path coloring using k colors.

Broere & Mynhardt conjectured [1, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [7, Thm. 2] and by Goddard [5, Thm. 1].

Theorem 1.1 (Poh 1990, Goddard 1991)**.** *Let G be a planar graph. Then G is path 3-colorable.* □

It is easily shown that the "3" in Theorem 1.1 is best possible. In particular, Chartrand & Kronk [4] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

[*Check Chartrand-Kronk reference.*]

Hartman [6, Thm. 4.1] (see also Chappell & Hartman [3, Thm. 2.1]) proved a list-coloring generalization of Theorem 1.1. A graph G is *path k-choosable* if, whenever each vertex of G is assigned a list of k colors, there exists a path coloring of G in which each vertex receives a color from its list.

Theorem 1.2 (Hartman 1997)**.** *Let G be a planar graph. Then G is path 3-choosable.* □

Essentially the same technique was used by Škrekovski [8, Thm. 2.2b] to prove a result slightly weaker than Theorem 1.2.

We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane. A plane graph will generally be specified via a *rotation scheme*, which is a circular ordering, for each vertex v, of the edges incident with v. When the circular ordering of the edges around each vertex v is the order in which they appear—clockwise, say—around v in the plane, then the rotation scheme completely specifies the combinatorial embedding of the graph. Rotation scheme are particularly convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex v in clockwise order around v, and no additional data structures are required.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems.

Section 2 covers an algorithm based on Poh's proof of Theorem 1.1. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 3 covers an algorithm based Hartman's proof of Theorem 1.2, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [2].

2 Path Coloring: the Poh Algorithm

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3 Path List Coloring: the Hartman-Škrekovski Algorithm

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