Path-Coloring Algorithms for Plane Graphs

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Abstract

A path coloring of a graph G is a vertex coloring of G such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh, is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman and Škrekovski, performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

1 Introduction

All graphs will be finite, simple, and undirected. See West [**?**] for graph theoretic terms.

A *path coloring* of a graph G is a vertex coloring (not necessarily proper) of G such that each color class induces a disjoint union of paths. A graph G is *path k-colorable* if G admits a path coloring using k colors.

Broere & Mynhardt conjectured [**?**, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [**?**, Thm. 2] and by Goddard [**?**, Thm. 1].

Theorem 1.1 (Poh 1990, Goddard 1991)**.** *If G is a planar graph, then G is path* 3*-colorable.* □

It is easily shown that the "3" in Theorem **??** is best possible. In particular, Chartrand & Kronk [**?**, Section 3] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

Hartman [**?**, Thm. 4.1] proved a list-coloring generalization of Theorem **??** (see also Chappell & Hartman [**?**, Thm. 2.1]). A graph G is *path k-choosable* if, whenever each vertex of G is assigned a list of k colors, there exists a path coloring of G in which each vertex receives a color from its list.

Theorem 1.2 (Hartman 1997)**.** *If G is a planar graph, then G is path* 3*-choosable.* □

Essentially the same technique was used by Škrekovski [**?**, Thm. 2.2b] to prove a result slightly weaker than Theorem **??**.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems. We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane.

In Section 2 we outline our graph representations and the basis for our computations of time complexity.

Section 3 covers an algorithm based on Poh's proof of Theorem **??**. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 4 covers an algorithm based Hartman's proof of Theorem **??**, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [**?**].

2 Graph Representatons and Time Complexity

We will represent a graph via *adjacency lists*: a list, for each vertex v, of the neighbors of v. A vertex can be represented by an integer $0 \ldots n-1$, where n is the order of the graph.

A plane graph will be specified via a *rotation scheme*: a circular ordering, for each vertex v, of the edges incident with v, in the order they appear around v in the plane embedding; this completely specifies the combinatorial embedding of the graph. Rotation schemes are convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex v in clockwise order around v; no additional data structures are required.

ZZZ Time Complexity ZZZ

ZZZ Augmented Adjacency Lists ZZZ

3 Path Coloring: the Poh Algorithm

ZZZ

4 Path List Coloring: the Hartman-Škrekovski Algorithm

ZZZ

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