Path-Coloring Algorithms for Plane Graphs

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Abstract

A path coloring of a graph G is a vertex coloring of G such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh, is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman and Škrekovski, performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

1 Introduction

All graphs will be finite, simple, and undirected. See West [9] for graph theoretic terms.

A *path coloring* of a graph G is a vertex coloring (not necessarily proper) of G such that each color class induces a disjoint union of paths. A graph G is *path k-colorable* if G admits a path coloring using k colors.

Broere & Mynhardt conjectured [1, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [7, Thm. 2] and by Goddard [5, Thm. 1].

Theorem 1.1 (Poh 1990, Goddard 1991)**.** *If G is a planar graph, then G is path 3-colorable.* □

It is easily shown that the "3" in Theorem 1.1 is best possible. In particular, Chartrand & Kronk [4, Section 3] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

Hartman [6, Thm. 4.1] proved a list-coloring generalization of Theorem 1.1 (see also Chappell & Hartman [3, Thm. 2.1]). A graph G is *path k-choosable* if, whenever each vertex of G is assigned a list of k colors, there exists a path coloring of G in which each vertex receives a color from its list.

Theorem 1.2 (Hartman 1997)**.** *If G is a planar graph, then G is path 3-choosable.* □

Essentially the same technique was used by Škrekovski [8, Thm. 2.2b] to prove a result slightly weaker than Theorem 1.2.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems. We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane.

In Section 2 we outline our graph representations and the basis for our computations of time complexity.

Section 3 covers an algorithm based on Poh's proof of Theorem 1.1. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 4 covers an algorithm based Hartman's proof of Theorem 1.2, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [2].

2 Graph Representations and Time Complexity

We will represent a graph via *adjacency lists*: a list, for each vertex v, of the neighbors of v. A vertex can be represented by an integer $0 \ldots n-1$, where n is the order of the graph.

A plane graph will be specified via a *rotation scheme*: a circular ordering, for each vertex v, of the edges incident with v, in the order they appear around v in the plane embedding; this completely specifies the combinatorial embedding of the graph. Rotation schemes are convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex v in clockwise order around v; no additional data structures are required.

We will assume an integer RAM model of computation The input for each algorithm will be a connected planar graph with n vertices and m edges, represented via adjacency lists. The input size will be n, the number of vertices. Note that $\mathcal{O}(m) = \mathcal{O}(n)$.

In Section 4, given an edge uv, we will need an efficient operation to find v's entry in u's adjacency list from u's entry in v's list. An *augmented adjacency list* is an adjacency list such that for any edge uv, a reference to v's entry in u's list is stored in u's entry in v's list, and vice versa. Given an adjacency list representation of a graph, an augmented adjacency list representation may be constructed in $\mathcal{O}(m)$ time via the following algorithm.

Algorithm 2.1.
Input: An adjacency list representation Adj of a graph G

Step 1: Construct an augmented adjacency list Adj' copy of Adj with the reference portion of each entry uninitialized.

Step 2: For each vertex v construct an array Wrk[v] storing vertex-reference pairs. For each v from 0 to $n-1$ iterate through Adj'[v]. For each neighbor u in Adj'[v] let $r_{v,u}$ be a reference to u's entry in Adj'[v] and append the pair $(v, r_{v,u})$ to Wrk[u].

Step 3: For each v from $n-1$ to 0 iterate through Wrk[v]. At the pair $(u, r_{u,v})$ in Wrk[v] note that the last element of Wrk[u] is $(v, r_{v,u})$. Look up and assign references for the edge uv in Adj'. Remove $(v, r_{v,u})$ from the back of Wrk[u].

After completing Step 2 in Algorithm 2.1 the array Wrk[v] contains a pair $(u, r_{u,v})$ for each neighbor of v, sorted in increasing order of by the neighbor u. At a given vertex v in Step 3 in Algorithm 2.1, for each edge $uw \in E(G)$ such that $u < w$ and $v < w$, we shall have the initialized references for uw in Adj'[u] and Adj'[w] and removed the pair $(w, r_{w,u})$ from Wrk[u]. The next iteration of Step 3 will handle edges uv with $v > u$.

3 Path Coloring: the Poh Algorithm

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4 Path List Coloring: the Hartman-Škrekovski Algorithm

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