# Path-Coloring Algorithms for Plane Graphs

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**Abstract**

A path coloring of a graph $G$ is a vertex coloring of $G$ such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh, is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman and Škrekovski, performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

## 1 Introduction

All graphs will be finite, simple, and undirected. See West [9] for graph theoretic terms.

A *path coloring* of a graph $G$ is a vertex coloring (not necessarily proper) of $G$ such that each color class induces a disjoint union of paths. A graph $G$ is *path $k$-colorable* if $G$ admits a path coloring using $k$ colors.

Broere & Mynhardt conjectured [1, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [7, Thm. 2] and by Goddard [5, Thm. 1].

**Theorem 1.1** (Poh 1990, Goddard 1991)**.** *If $G$ is a planar graph, then $G$ is path 3-colorable.* □

It is easily shown that the "3" in Theorem 1.1 is best possible. In particular, Chartrand & Kronk [4, Section 3] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

Hartman [6, Thm. 4.1] proved a list-coloring generalization of Theorem 1.1 (see also Chappell & Hartman [3, Thm. 2.1]). A graph $G$ is *path $k$-choosable* if, whenever each vertex of $G$ is assigned a list of $k$ colors, there exists a path coloring of $G$ in which each vertex receives a color from its list.

**Theorem 1.2** (Hartman 1997)**.** *If $G$ is a planar graph, then $G$ is path 3-choosable.* □

Essentially the same technique was used by Škrekovski [8, Thm. 2.2b] to prove a result slightly weaker than Theorem 1.2.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems. We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane.

In Section 2 we outline our graph representations and the basis for our computations of time complexity.

Section 3 covers an algorithm based on Poh's proof of Theorem 1.1. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 4 covers an algorithm based Hartman's proof of Theorem 1.2, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [2].

# 2 Graph Representatons and Time Complexity

We will represent a graph via *adjacency lists*: a list, for each vertex $v$, of the neighbors of $v$. A vertex can be represented by an integer $0 \dots n-1$, where $n$ is the order of the graph.

A plane graph will be specified via a *rotation scheme*: a circular ordering, for each vertex $v$, of the edges incident with $v$, in the order they appear around $v$ in the plane embedding; this completely specifies the combinatorial embedding of the graph. Rotation schemes are convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex $v$ in clockwise order around $v$; no additional data structures are required.

ZZZ Time Complexity ZZZ

ZZZ Augmented Adjacency Lists ZZZ

# 3 Path Coloring: the Poh Algorithm

ZZZ

# 4 Path List Coloring: the Hartman-Škrekovski Algorithm

ZZZ

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