

Path-Coloring Algorithms for Plane Graphs

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Abstract

A path coloring of a graph G is a vertex coloring of G such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh, is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman and Škrekovski, performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

1 Introduction

All graphs will be finite, simple, and undirected. See West [9] for graph theoretic terms.

A *path coloring* of a graph G is a vertex coloring (not necessarily proper) of G such that each color class induces a disjoint union of paths. A graph G is *path k -colorable* if G admits a path coloring using k colors.

Broere & Mynhardt conjectured [1, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [7, Thm. 2] and by Goddard [5, Thm. 1].

Theorem 1.1 (Poh 1990, Goddard 1991). *Let G be a planar graph. Then G is path 3-colorable.* \square

It is easily shown that the “3” in Theorem 1.1 is best possible. In particular, Chartrand & Kronk [4, Section 3] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

Hartman [6, Thm. 4.1] proved a list-coloring generalization of Theorem 1.1 (see also Chappell & Hartman [3, Thm. 2.1]). A graph G is *path k -choosable* if, whenever each vertex of G is assigned a list of k colors, there exists a path coloring of G in which each vertex receives a color from its list.

Theorem 1.2 (Hartman 1997). *Let G be a planar graph. Then G is path 3-choosable.* \square

Essentially the same technique was used by Škrekovski [8, Thm. 2.2b] to prove a result slightly weaker than Theorem 1.2.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems. We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane.

In Section 2 we outline our graph representations and the basis for our computations of time complexity.

Section 3 covers an algorithm based on Poh’s proof of Theorem 1.1. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 4 covers an algorithm based Hartman’s proof of Theorem 1.2, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [2].

2 Graph Representatons and Time Complexity

We will represent a graph via *adjacency lists*: a list, for each vertex v , of the neighbors of v . A vertex can be represented by an integer $0 \dots n - 1$, where n is the order of the graph.

A plane graph will be specified via a *rotation scheme*: a circular ordering, for each vertex v , of the edges incident with v , in the order they appear around v in the plane embedding; this completely specifies the combinatorial embedding of the graph. Rotation schemes are convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex v in clockwise order around v ; no additional data structures are required.

ZZZ Time Complexity ZZZ

ZZZ Augmented Adjacency Lists ZZZ

3 Path Coloring: the Poh Algorithm

ZZZ

4 Path List Coloring: the Hartman-Škrekovski Algorithm

ZZZ

References

- [1] I. Broere and C. M. Mynhardt, Generalized colorings of outerplanar and planar graphs, *Graph theory with applications to algorithms and computer science* (Kalamazoo, Mich., 1984), pp. 151–161, Wiley-Intersci. Publ., Wiley, New York, 1985.
- [2] A. Bross, *Implementing path coloring algorithms on planar graphs*, Masters Project, University of Alaska, 2017, available from http://github.com/permutationlock/path_coloring_bgl.
- [3] G. G. Chappell and C. Hartman, Path choosability of planar graphs, in preparation.
- [4] G. Chartrand and H. V. Kronk, The point-arboricity of planar graphs, *J. London. Math. Soc.* **44** (1969), 612–616.
- [5] W. Goddard, Acyclic colorings of planar graphs, *Discrete Math.* **91** (1991), no. 1, 91–94.
- [6] C. M. Hartman, *Extremal Problems in Graph Theory*, Ph.D. Thesis, University of Illinois, 1997.
- [7] K. S. Poh, On the linear vertex-arboricity of a planar graph, *J. Graph Theory* **14** (1990), no. 1, 73–75.
- [8] R. Škrekovski, List improper colourings of planar graphs, *Combin. Probab. Comput.* **8** (1999), no. 3, 293–299.
- [9] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2000.