Path-Coloring Algorithms for Plane Graphs

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Abstract

A path coloring of a graph G is a vertex coloring of G such that each color class induces a disjoint union of paths. We present two efficient algorithms to construct a path coloring of a plane graph.

The first algorithm, based on a proof of Poh, is given a plane graph; it produces a path coloring of the given graph using three colors.

The second algorithm, based on similar proofs by Hartman and Škrekovski, performs a list-coloring generalization of the above. The algorithm is given a plane graph and an assignment of lists of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available.

1 Introduction

All graphs will be finite, simple, and undirected. See West [10] for graph theoretic terms.

A *path coloring* of a graph G is a vertex coloring (not necessarily proper) of G such that each color class induces a disjoint union of paths. A graph G is *path k-colorable* if G admits a path coloring using k colors.

Broere & Mynhardt conjectured [2, Conj. 16] that every planar graph is path 3-colorable. This was proven independently by Poh [8, Thm. 2] and by Goddard [6, Thm. 1].

Theorem 1.1 (Poh 1990, Goddard 1991)**.** *If G is a planar graph, then G is path* 3*-colorable.* □

It is easily shown that the "3" in Theorem 1.1 is best possible. In particular, Chartrand & Kronk [5, Section 3] gave an example of a planar graph whose vertex set cannot be partitioned into two subsets, each inducing a forest.

Hartman [7, Thm. 4.1] proved a list-coloring generalization of Theorem 1.1 (see also Chappell & Hartman [4, Thm. 2.1]). A graph G is *path k-choosable* if, whenever each vertex of G is assigned a list of k colors, there exists a path coloring of G in which each vertex receives a color from its list.

Theorem 1.2 (Hartman 1997)**.** *If G is a planar graph, then G is path* 3*-choosable.* □

Essentially the same technique was used by Škrekovski [9, Thm. 2.2b] to prove a result slightly weaker than Theorem 1.2.

We discuss two efficient path-coloring algorithms based on proofs of the above theorems. We distinguish between a *planar* graph—one that can be drawn in the plane without crossing edges—and a *plane* graph—a graph with a given embedding in the plane.

In Section 2 we outline our graph representations and the basis for our computations of time complexity.

Section 3 covers an algorithm based on Poh's proof of Theorem 1.1. The algorithm is given a plane graph; it produces a path coloring of the given graph using three colors.

Section 4 covers an algorithm based Hartman's proof of Theorem 1.2, along with the proof of Škrekovski mentioned above. The algorithm is given a plane graph and an assignment of a list of three colors to each vertex; it produces a path coloring of the given graph in which each vertex receives a color from its list.

Implementations of both algorithms are available; see Bross [3].

2 Graph Representations and Time Complexity

We will represent a graph via *adjacency lists*: a list, for each vertex v, of the neighbors of v. A vertex can be represented by an integer $0 \dots n-1$, where n is the order of the graph.

A plane graph will be specified via a *rotation scheme*: a circular ordering, for each vertex v, of the edges incident with v, in the order they appear around v in the plane embedding; this completely specifies the combinatorial embedding of the graph. Rotation schemes are convenient when we represent a graph using adjacency lists; we simply order the adjacency list for each vertex v in clockwise order around v; no additional data structures are required.

We will assume an integer RAM model of computation. The input for each algorithm will be a triangulated plane graph with n vertices and m edges, represented via adjacency lists. The input size will be n, the number of vertices. Note that $\mathcal{O}(m) = \mathcal{O}(n)$, so it is equivalent to take the input size to be m, the number of edges. Moreover, arbitrary simple planar graphs may be plane embedded and triangulated in $\mathcal{O}(n)$ time, see [1].

In Section 4, given an edge uv, we will need an efficient operation to find v's entry in u's adjacency list from u's entry in v's list. An *augmented adjacency list* is an adjacency list such that for any edge uv, a reference to v's entry in u's list is stored in u's entry in v's list, and vice versa. Given an adjacency list representation of a graph, an augmented adjacency list representation may be constructed in $\mathcal{O}(m)$ time via the following procedure.

Algorithm 2.1.
Input: An adjacency list representation Adj of a graph G.
Output: An augmented adjacency list representation Adj' of G with the same rotation scheme as Adj.
Step 1: Construct an augmented adjacency list copy Adj' of Adj with the reference portion of each entry uninitialized.
Step 2: For each vertex v construct an array Wrk[v] storing vertex-reference pairs. For each v from 0 to $n-1$ iterate through Adj'[v]. For each neighbor u in Adj'[v] append the pair $(v, r_v(u))$ to Wrk[u] where $r_v(u)$ is a reference to u's entry in Adj'[v].
Step 3: For each v from $n-1$ to 0 iterate through Wrk[v]. Upon reaching each pair $(u, r_u(v))$ in Wrk[v] the last element of Wrk[u] will be $(v, r_v(u))$; for details on why this is, see the paragraphs below. Use $r_u(v)$ and $r_v(u)$ to look up and assign references for the edge uv in Adj'[u] and Adj'[v]. Remove $(v, r_v(u))$ from the back of Wrk[u].

After completing Step 2 in Implementation 2.1 the array Wrk[v] contains a pair $(u, r_u(v))$ for each neighbor of v, sorted in increasing order of by the neighbor u.

Let v be a vertex in G. Suppose that for each edge $uw \in E(G)$ such that $u < w$ and $v < w$, references have been initialized for uw in Adj'[u] and Adj'[w], and the pair $(w, r_w(u))$ has been removed from Wrk[u].

At a given iteration of Step 3 in Implementation 2.1 let v be the current vertex. For each edge $uw \in E(G)$ such that $u < w$ and $v < w$, prior iterations of Step 3 will have initialized the references for uw in Adj'[u] and Adj'[w], and also removed the pair $(w, r_w(u))$ from Wrk[u]. The current iteration will handle all edges $uv \in E(G)$ with $v > u$.

3 Path Coloring: the Poh Algorithm

We will first describe Poh's algorithm for path 3-coloring plane graphs.

Algorithm 3.1 (Poh 1990)**.**
Input: A weakly triangulated plane graph G with outer face a cycle $C = v_1, v_2, \ldots, v_k$ and a 2-coloring of C such that each color class induces a path, $P_1 = v_1, v_2, \ldots, v_l$ and $P_2 = v_{l+1}, v_{l+2}, \ldots, v_k$ respectively.
Output: An extension of the 2-coloring of C to a path 3-coloring of G such that no vertex in $G - C$ receives the same color as a neighbor of that vertex in C.

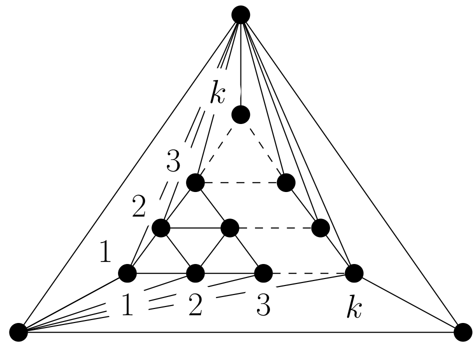

Figure 1: The collection of graphs $\{G_k\}_{k\in\mathbb{N}}$ on which Poh performs poorly.

Step 1: If $G = C$ then G is already path 3-colored and we are done. Otherwise there are two cases to consider.
Case 1.1: Suppose C is an induced subgraph of G. Let $u, w \in V(G) - V(C)$ such that the cycles u, v_1, v_k and w, v_l, v_{l+1} each exist and are faces of G; note that u and w are unique, but may not be distinct. Since C is induced and $G \neq C$, $G - C$ is connected. Let $P_3 = u_1, u_2, \ldots, u_r$ be a shortest u, w-path in $G - C$. Color each vertex of P_3 with the third color not used in the 2-coloring of C. Let $C_1 = v_1, v_2, \ldots, v_l, u_r, u_{r-1}, \ldots, u_1$ and $C_2 = u_1, u_2, \ldots, u_r, v_{l+1}, v_{l+2}, \ldots, v_k$.
Case 1.2: Suppose C is not an induced subgraph. Then there exists an edge $v_i v_j \in E(G) - E(C)$ such that $i \leq l < j$. Let $C_1 = v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_k$ and $C_2 = v_i, v_{i+1}, \ldots, v_j$.
Step 2: Apply Algorithm 3.1 separately to the maximal subgraph of G with outer face C_1 and to the maximal subgraph with outer face C_2.

Note that the graph G is finite and the recursive step applies the algorithm to two proper subgraphs of G. Therefore Algorithm 3.1 must terminate.

Let G be a triangulated plane graph. We may trivially path 2-color the outer triangle. Applying Poh's algorithm extends this coloring to a path 3-coloring of G.

A natural way to implement Poh's algorithm is to use a breadth-first search to to attempt to construct the shortest path of Case 1.1 and in the process locate a chord edge of Case 1.2 if no such path is possible.

Algorithm 3.2.
Input: A cycle $C = v_1, v_2, \ldots, v_k$ in a triangulated plane graph G, where G is represented by adjacency lists, and a 2-coloring of C such that each color class induces a path, respectively $P_1 = v_1, v_2, \ldots, v_l$ and $P_2 = v_l, v_{l+1}, \ldots, v_k$.
Output: An extension of the 2-coloring of C to a path 3-coloring of the maximal subgraph of G with outer face C.
Step 1: Iterate Adj$[v_1]$ to locate the vertex u immediately following v_k. Note that since G is triangulated, v_1, u, v_k is a face of G.
Case 1.1: If $u \in C$, then $u = v_{k-1}$, since G is triangulated, and C is not an induced cycle. We then apply Algorithm 3.2 to the cycle $C' = v_1, v_2, \ldots, v_{k-1}$.

Case 1.2: Perform a breadth-first of the maximal subgraph of G with outer face C, starting from the vertex u. Terminate the search upon locating a vertex w with adjacent neighbors $v_i \in P_1$ and $v_j \in P_2$ such that $i \neq 1$ or $j \neq k$. Backtrack along the breadth-first search to construct a minimal u,w-path $P_3 = u_1, u_2, \ldots, u_r$. Let $C_1 = v_1, v_2, \ldots, v_i, u_r, u_{r-1}, \ldots, u_1$ and $C_2 = u_1, u_2, \ldots, u_r, v_j, v_{j-1}, \ldots, v_k$. Apply Algorithm 3.2 separately to C_1 and C_2. If $i = l$ and $j = l+1$ then C was an induced cycle and we are done. Otherwise, also apply Algorithm 3.2 to $C_3 = v_i, v_{i+1}, \ldots, v_j$.

Unfortunately Algorithm 3.2 is not linear. To see this, consider the family of graphs $\{G_k\}_{k\in\mathbb{N}}$ depicted in Figure 1. Fix $k \in \mathbb{N}$ and note that $n = n(G_k) = k(k+1)/2 + 3$. Assume that the outer triangle is path 2-colored such that the top vertex is assigned a color distinct from the bottom two. At iteration i of Poh's algorithm the shortest path through the interior will be the path of length $l = k - i + 1$ directly along the base of the inner triangle. A breadth-first search of this inner triangle will hit all $l(l+1)/2$ vertices in order to find this path. Therefore the total number of operations performed will be

$$\Theta\left(\sum_{l=1}^{k} \frac{l(l+1)}{2}\right) = \Theta(n^{3/2}).$$

So Poh's algorithm with breadth-first search is $\Omega(n^{3/2})$.

However, the correctness of Poh's algorithm does not rely on locating the shortest uw-path, only on locating some induced uw-path. Below we will observe that modifying Poh's algorithm to locate induced paths will allow us produce a linear time implementation.

The general strategy will be to walk clockwise along the colored path $P_1 = v_1, v_2, \ldots, v_l$ in the outer cycle C, marking those vertices interior to C that have a neighbor in P_1. We will then construct an induced path $P_3 = u_1, u_2, \ldots, u_d$, consisting only of marked vertices, such that $C_1 = P_1 \cup P_3 \cup \{u_1v_1, u_dv_l\}$ is a cycle, and all marked vertices are contained in the maximal subgraph of G with outer face C_1.

When denoting vertices in a k-cycle it will be a useful convention to treat vertex indices mod $(k+1)$. From now on if we have a cycle $C = v_1, v_2, \ldots, v_k$, then for any $i \in \mathbb{Z}$ we will define $v_i = v_j$ where $j \in \{1, \ldots, k\}$ and $j \equiv i \mod (k+1)$.

4 Path List Coloring: the Hartman-Škrekovski Algorithm

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