# A Path 3-List Coloring Algorithm for Plane Graphs

April 20, 2016

## Abstract

We present an algorithm to path 3-list-color plane graphs based on the work by Skrekovski [2] and Hartman [1].

## Introduction

All graphs discussed are assumed to be simple, undirected, and plane embeded. For plane embeddings we assume the edges around each vertex are arranged in clockwise order. For a graph $G$, let $V(G)$ denote its vertex set and $E(G)$ denote the edge set.

Using notation from [1], for $v \in V(G)$ we will denote the *neighborhood* of $v$ in $G$ as $N_G(x) = \{u \in V(G) \mid uv \in E(G)\}$. For $u, w \in N_G(v)$ we will use $[u, w]_v$ and $(u, w)_v$ to denote the ordered list of vertices between $u$ and $w$ in $N_G(v)$ in clockwise embedded order, inclusive and exclusive respectively. We will use $[u, w]'_v$ and $(u, w)'_v$ for the equivalent counterclockwise listings. If $C$ is the outer face of a weakly triangulated 2-connected graph then for $u, v \in C$ let $C[u, v]$ and $C(u, v)$ denote the set of vertices along the outer face in clockwise order between $u$ and $v$, inclusive and exclusive respectively.

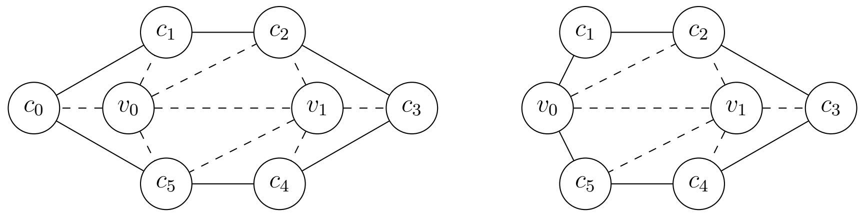
A *coloring* of a graph $G$ assigns a color to each vertex. If $L(v)$ assigns a list of $k$ colors to each vertex $v \in V(G)$, a *$k$-list-coloring* colors $G$ such that each $v \in V(G)$ is colored from $L(v)$. A *path $k$-list-coloring* is a $k$-list-coloring such that each color class induces a disjoint union of paths.

## Vertex Removal

In this section we present a lemma describing the effects of vertex removal on weakly triangulated plane graphs. This will be necessary to remove already colored paths from our graph. We consider removing a vertex from the outer face that has only two neighbors in the outer face (i.e. no chords).

**Lemma 1.** Let $G$ be a weakly triangulated 2-connected graph with $|V(G)| \geq 4$ and outer face $C = c_0c_1 \ldots c_n$ in clockwise embedded order. Then if $N_C(c_0) = \{c_1, c_n\}$, $G_0 = G - c_0$ is a weakly triangulated graph with $|V(G_0)| \geq 3$ and outer face $C_0 = c_1 \ldots c_n(c_1, c_n)'_{c_0}$.

*Proof.* First notice that $G_0$ is clearly weakly triangulated since $G$ was weakly triangulated and we removed a vertex from the outer face. Furthermore, $C_0$ is a cycle the path $(c_1, c_n)'_{c0}$ was disjoint from $C$. Therefore for each $u, v \in C$ there will be two vertex disjoint $uv$-paths. Since $G_0$ is weakly triangulated, each $v \in G_0 - C$ will have at least two vertex disjoint paths to vertices on the outer face. Thus, $G_0$ is 2-connected. $\square$




Removing $c_0$ with Lemma 1.

# Path 3-List Coloring Plane Graphs

In this section we present a correct algorithm for producing path 3-list coloring of a plane graph. The following theorem is equivalent to the results produced by Hartman in [1] and independantly by Skrekovski in [3]. The objective in restructuring the theorem is to empasize the mechanical operations that would take place in an algorithm implementation to produce such a coloring.

**Theorem 2.** Let $G$ be a weakly triangulated 2-connected graph with outer face $C = c_0c_1 \ldots c_n$ and let $x, y \in C$, not necessarily distinct. Suppose $L(v)$ assigns a list of colors to each $v \in V(G)$ such that

$$\begin{array}{ll} |L(v)| \geq 1 & \text{if } v = x \text{ or } v = y; \\ |L(v)| \geq 2 & \text{if } v \in C, v \neq x, v \neq y; \\ |L(v)| \geq 3 & \text{otherwise.} \end{array}$$

Let $P$ be a (potentially null) induced path in $G$ of vertices in $C[x, y]$. Assume there exists color $\alpha$ such that for all $v \in P$, $\alpha \in L(v)$ and $v$ has been colored $\alpha$. Also, for all $u \in C[x, y] - P$, $\alpha \notin L(u)$. If $x = y = p$, with $p$ the first vertex of $P$, then $p$ may be colored $\beta \neq \alpha$.

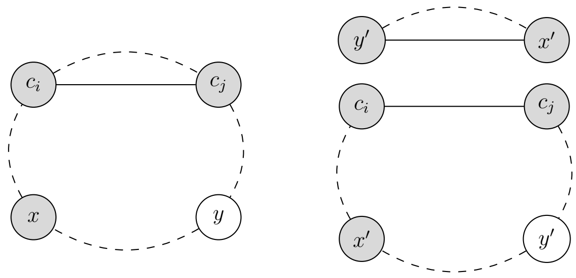
There exists a path list-coloring of $G$ from list assignment $L(v)$ such that $x$ and $y$ have at most one neighbor of the same color.

*Proof.* We proceed by induction on $|V(G)|$. If $|V(G)| \leq 3$ the statement is trivial. Let $|V(G)| \geq 4$ and suppose the statement holds for all graphs $G'$ with $|G'| < |G|$. Let $C = xc_1 \ldots c_n$ be the outer face of $G$.

Suppose path $P$ does is empty. We will construct an induced path $P$ starting with $x$ and satisfying the parameters of the Theorem statement. Let $\alpha$ be the first, and possibly only, color in $L(x)$. Initialize $P$ to be the singleton path $x$. Let $c_i \in C[x, y]$ be the current end of the path $P$. Let $c_j$ be the furthest vertex in $C(c_i, y]$ such that $\alpha \in L(c_j)$. If such a $c_j$ exists, append it to $P$ and consider this new end vertex. Otherwise we are done and we color each vertex in $P$ with $\alpha$. Note that $P$ is induced in $G$ by construction.

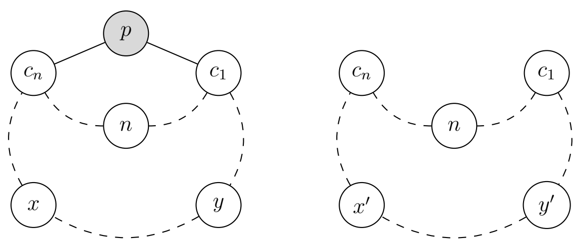
Therefore there exists a non-null path $P$. We will consider $G = G_0 \cup \ldots \cup G_m$ with each $G_i$ a weakly triangulated block. For each $G_i$ we will describe $x'$, $y'$, $P'$, and $L'(v)$ that satisfy the inductive hypothesis.

Case 1: Suppose $P$ takes a chord $c_ic_j$, $j > (i + 1)$, accross $C$. Then the inductive hypothesis holds for the subgraph $G_0$ bounded by $C[c_i, c_j]$ with $x = c_j$, $y = c_i$, and $P = c_jc_i$. Furthermore, the inductive hypothesis holds for the subgraph $G_1$ bounded by $C[c_j, c_i]$ with $P' = P$, $x' = x$, $y' = y$.



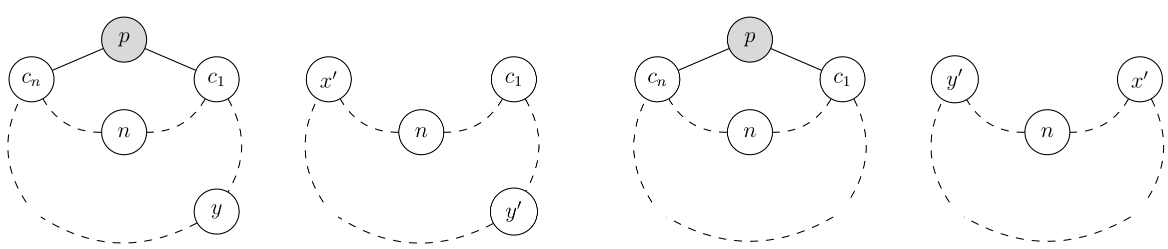

Case 1: $P$ takes a chord along $C$.

Case 2: Suppose $P$ does not take any chords of $C$ (i.e. $P$ is a continuous section of the outer face). We will remove $p$, the first vertex in the the path $P$, and show the inductive hypothesis holds for each of the remaining blocks of $G - p$. Let denote the vertices of the outer face in clockwise order as $C = pc_1 \dots c_n$.




Case 2.1: $p \neq x$, $p \neq y$.

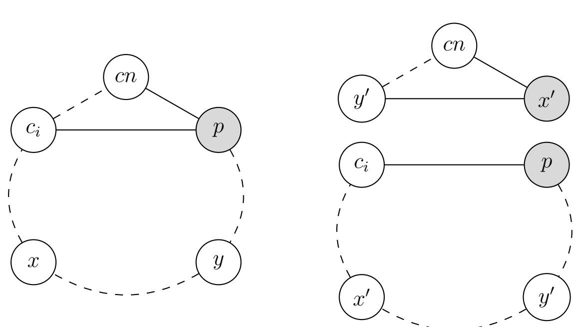
Case 2.1: Suppose $N_C(p) = \{c_1, c_n\}$. Let $\alpha$ be the color of $p$. For each vertex $n \in N_G(p) - P$ assign $L'(n) = L(n) \setminus \{\alpha\}$, and set $L'(v) = L(v)$ for all other vertices $v$. After removing $p$, there may be vertices in $P' = P - p$ and it must be maintained that $P'$ is between $x'$ and $y'$ clockwise on the outer face. If $x = y = p$ ($x$ and $y$ are removed), set $x' = c_1$ and $y' = c_n$. If $p = x$, $p \neq y$ ($x$ is removed) we set $x' = c_n$, otherwise $x' = x$. Similarly, if $y = p$ set $y' = c_1$, and otherwise $y' = y$. Since for any $v \in C[x, y]$, $\alpha \notin L(v)$, for all $n \in N_G(p)$ such that $n \in C[x, y]$, $|L'(n)| \geq 2$. Furthermore, for $n \in N_G(p)$, $n \notin C$, $|L'(v)| \geq 2$.




Case 2.1: $p = x$, $p \neq y$ (left), and Case 2.1: $p = x = y$ (right).

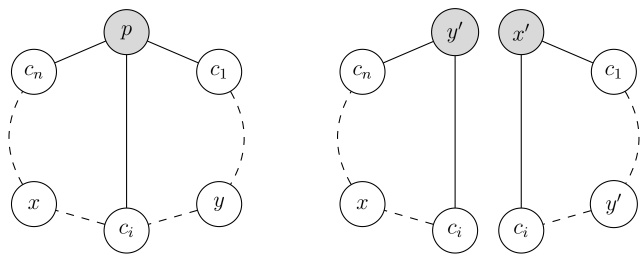
If $p \neq x$ and $p \neq y$, then clearly all $n \in N_G(p)$ fit these the above critera. Thus, using Lemma 1 the inductive hypothesis holds for $G_0 = G - p$, $x'$, $y'$, $P'$, and $L'(v)$. If $x = y = p$, we may have $|L'(c_n)| = 1$ and $|L'(c_1)| = 1$, but we have $x' = c_1$ and $y' = c_n$ so the hypothesis holds. In the case $x = p$ (or $y = p$), we may have $|L'(c_n)| = 1$ (or $|L'(c_1)| = 1$), but $x' = c_n$ (or $y' = c_1$) so the inductive hypothesis still holds.

Case 2.2: Suppose $|N_C(p)| > 2$. Let us select the largest $i \in \mathbb{Z}^+$ such that $c_i \in N_C(p)$ and $c_i \in C(c_1, c_n)$. Since $P$ is induced in $G$, $c_i \in C(p', c_n)$.



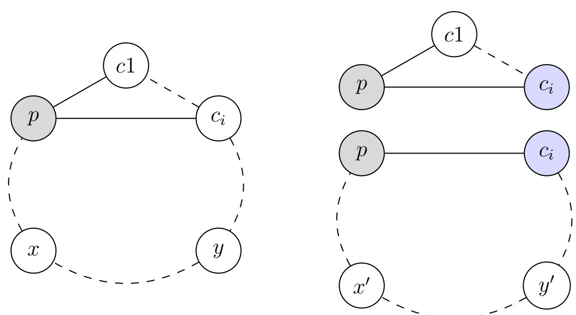

Case 2.2.1: $c_i \in C[x, c_n)$.

Case 2.2.1: Suppose $c_i \in C[x, c_n)$. Notice the inductive hypothesis holds for the to the subgraph $G_0$ bounded by $C[c_i, p]$ and the edge $pc_i$ with $P' = p$, $x' = p$ and $y' = c_i$. Furthermore, the inductive hypothesis holds for the subgraph $G_1$ bounded by $C[p, c_i]$ and the edge $pc_i$ with $P' = P$, $x' = x$, and $y' = y$. Therefore the coloring of $G = G_0 \cup G_1$ will be a path coloring.




Case 2.2.2: $c_i \in C(x, y]$.

Case 2.2.2: Suppose $c_i \in C[y, x)$. Notice the inductive hypothesis holds for the subgraph $G_0$ bounded by $C[c_i, p]$ and the edge $pc_i$ with $P' = p$, $x' = x$, and $y' = p$. Furthermore, the inductive hypothesis holds for the subgraph $G_1$ bounded by $C[p, c_i]$ and the edge $pc_i$, $P' = P$, $x' = p$, $y' = y$. Since the removal of $p$ in $G_0$ and $G_1$ will set $c_i$ to $x$ and $y$ respectively, $c_i$ will have at most one neighbor in each subgraph. Therefore the coloring of $G = G_0 \cup G_1$ will be a path coloring.




Case 2.2.3: $c_i \in C(y, p')$.

Case 2.2.3: Suppose $c_i \in C(p', y)$. This is similar to Case 2.2.1, but with the added dificulty that there might be remaining vertices of $P$ in the subgraph not containing $x$ and $y$. First notice that the inductive hypothesis holds for the subgraph $G_0$ bounded by $C[c_i, p]$ and the edge $pc_i$, $P' = p$, $x' = x$, $y' = y$. Notice that $c_i$ is now colored with some color $\beta$ by the coloring of $G_0$. Let $G_1$ be the subgraph bounded by $C[p, c_i]$ and the edge $pc_i$. Then the inductive hypothesis holds for for $G_1$ with $P' = c_iP$, $x' = c_i$, $y' = c_i$. Since $c_i$ will have no same color neighbors in $G_1$, the coloring of $G = G_0 \cup G_1$ will be a path coloring. $\square$

Since adding edges does not make a graph easier to color, we may add edges to weakly triangulate any graph. Then we may assign color lists of size 3 to each vertex, set $x$ and $y$ to be arbitrary vertices on the outer face, and apply Theorem 2.

# References

[1] Hartman, C., "Extremal problems in graph theory," Ph.D. thesis, Department of Mathematics, University of Illinois at Urbana-Champaign, 1997.

[2] Skrekovski, R., "List improper colourings of planar graphs," *Combinatorics, Probability and Computing*, vol. 8, pp. 293-299, 1999.