A Path 3-List Coloring Algorithm for Plane Graphs

April 20, 2016

Abstract

We present an algorithm to path 3-list-color plane graphs based on the work by Skrekovski [2] and Hartman [1].

Introduction

All graphs discussed are assumed to be simple, undirected, and plane embeded. For plane embeddings we assume the edges around each vertex are arranged in clockwise order. For a graph G, let $V(G)$ denote its vertex set and $E(G)$ denote the edge set.

Using notation from [1], for $v \in V(G)$ we will denote the *neighborhood* of v in G as $N_G(x) = \{u \in V(G) \mid uv \in E(G)\}$. For $u, w \in N_G(v)$ we will use $[u, w]_v$ and $(u, w)_v$ to denote the ordered list of vertices between u and w in $N_G(v)$ in clockwise embedded order, inclusive and exclusive respectively. We will use $[u, w]'_v$ and $(u, w)'_v$ for the equivalent counterclockwise listings. If C is a cycle, for $u, v \in C$ let $C[u, v]$ and $C(u, v)$ denote the set of vertices between u and v in clockwise embedded order, inclusive and exclusive respectively.

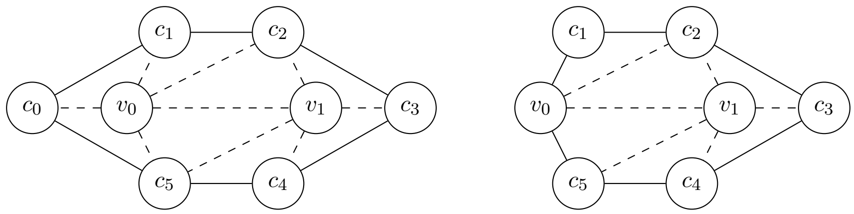
If $L(v)$ assigns a list of k colors to each vertex $v \in V(G)$, a *k-list-coloring* colors G such that each $v \in V(G)$ is colored from $L(v)$. A *path k-list-coloring* is a k-list-coloring such that each color class induces a disjoint union of paths.

Vertex Removal

In this section we present a lemma describing the effects of vertex removal on weakly triangulated plane graphs. This will be necessary to remove already colored paths from our graph. We consider removing a vertex from the outer face that has only two neighbors in the outer face (i.e. no chords).

Lemma 1. Let G be a weakly triangulated 2-connected graph with $|V(G)| \geq 4$ and outer face $C = c_0c_1 \dots c_n$ in clockwise embedded order. Then if $N_C(c_0) = \{c_1, c_n\}$, $G_0 = G - c_0$ is a weakly triangulated graph with $|V(G_0)| \geq 3$ and outer face $C_0 = c_1 \dots c_n(c_1, c_n)'_{c_0}$.

Proof. First notice that G_0 is clearly weakly triangulated since G was weakly triangulated and we removed a vertex from the outer face. Furthermore, C_0 is a cycle the path $(c_1, c_n)'_{c_0}$ was disjoint from C. Therefore for each $u, v \in C$ there will be two vertex disjoint uv-paths. Since G_0 is weakly triangulated, each $v \in G_0 - C$ will have at least two vertex disjoint paths to vertices on the outer face. Thus, G_0 is 2-connected. $\square$



Removing c_0 with Lemma 1.

Path 3-List Coloring Plane Graphs

In this section we present a correct algorithm for producing path 3-list coloring of a plane graph. The following theorem is equivalent to the results produced by Hartman in [1] and independantly by Skrekovski in [3]. The objective in restructuring the theorem is to emphasize the mechanical operations that would take place in an algorithm implementation to produce such a coloring.

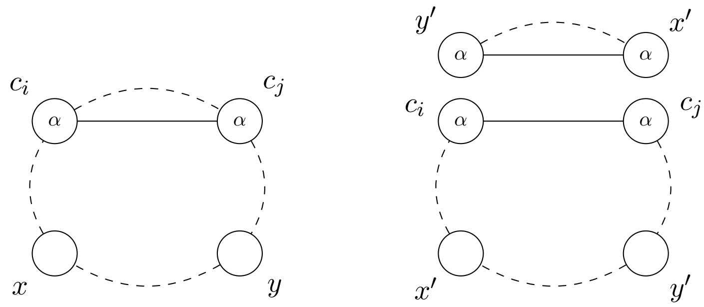
> **Theorem 2.** Let G be a weakly triangulated 2-connected graph with outer face $C = c_0c_1 \dots c_n$ and let $x, y \in C$, not necessarily distinct. Suppose $L(v)$ assigns a list of colors to each $v \in V(G)$ such that
>
> $$\begin{array}{ll} |L(v)| \geq 1 & \text{if } v = x \text{ or } v = y; \\ |L(v)| \geq 2 & \text{if } v \in C, v \neq x, v \neq y; \\ |L(v)| \geq 3 & \text{otherwise.} \end{array}$$
>
> Let P be a (potentially null) induced path in G of vertices in $C[x, y]$. Assume there exists color α such that for all $v \in P$, $\alpha \in L(v)$ and v has been colored α. Also, for all $u \in C[x, y] - P$, $\alpha \notin L(u)$. If $x = y = p$, with p the first vertex of P, then p may be colored $\beta \neq \alpha$.
>
> There exists a path list-coloring of G from list assignment $L(v)$ such that x and y have at most one neighbor of the same color and no neighbors of P recieve color α.

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 3$ the statement is trivial. Let $|V(G)| \geq 4$ and suppose the statement holds for all graphs G' with $|G'| < |G|$. Let $C = xc_1 \dots c_n$ be the outer face of G.

Suppose P is a null path. We will construct a new induced path P starting with x and satisfying the parameters of the Theorem statement. Let α be the first, and possibly only, color in $L(x)$. Initialize P to be the singleton path x. Let $c_i \in C[x, y]$ be the current end of the path P. Let c_j be the closest vertex to y in $N_C(c_i) \cap C(c_i, y]$ such that $\alpha \in L(c_j)$. If such a c_j exists, append it to P and consider this new end vertex. Otherwise we are done and we color each vertex in P with α.

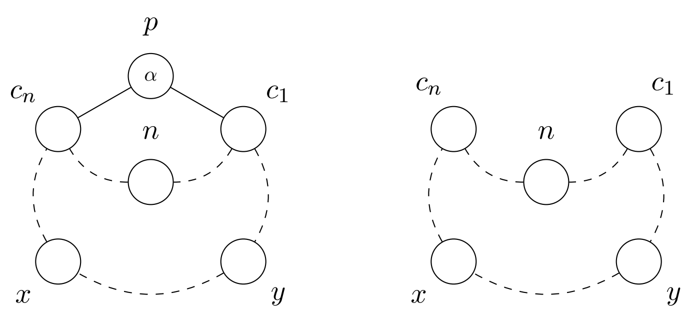
Suppose there exists a non-null induced path P. There are several cases to consider.

Case 1: Suppose P takes a chord $c_i c_j$, $j > (i+1)$, accross C. Then the inductive hypothesis holds for the subgraph G_0 bounded by $C[c_i, c_j]$ with $x = c_j$, $y = c_i$, and $P = c_j c_i$. Furthermore, the inductive hypothesis holds for the subgraph G_1 bounded by $C[c_j, c_i]$ with $P' = P$, $x' = x$, $y' = y$. Since G_0 and G_1 share only vertices in P, the coloring of $G = G_0 \cup G_1$ will be a path coloring.



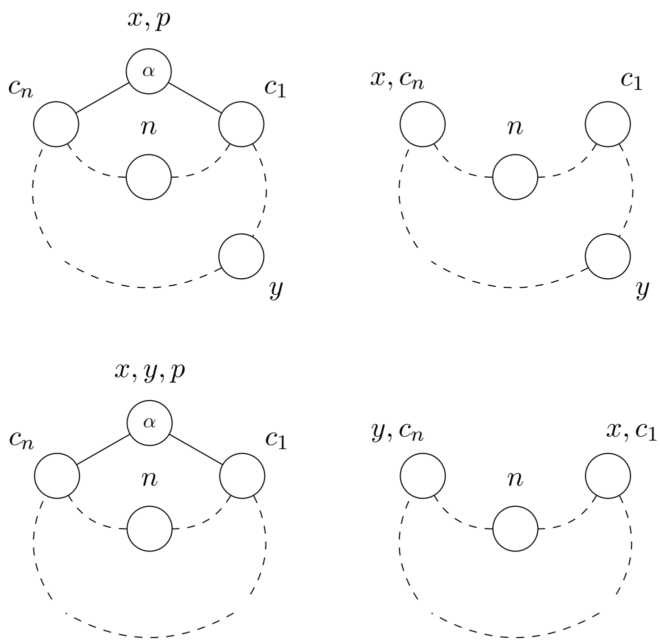
Case 1: P takes a chord along C.

Case 2: Suppose P does not take any chords of C (i.e. P is a continuous section of the outer face). We will remove p, the first vertex in the the path P, and show the inductive hypothesis holds for each of the remaining blocks of $G - p$. Let denote the vertices of the outer face in clockwise order as $C = pc_1 \ldots c_n$.



Case 2.1: $p \neq x$, $p \neq y$.

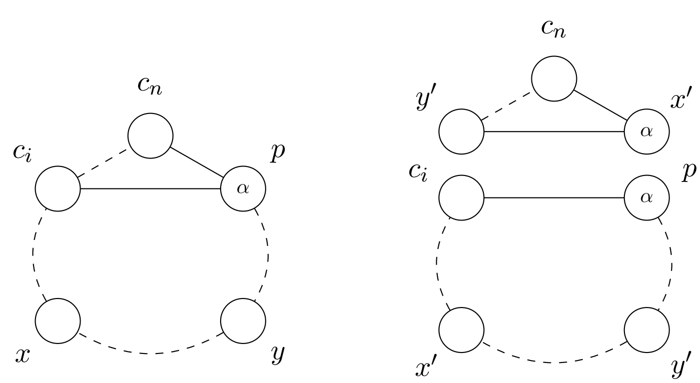
Case 2.1: Suppose $N_C(p) = \{c_1, c_n\}$. Let α be the color of p. For each vertex $n \in N_G(p) - P$ assign $L'(n) = L(n) \setminus \{\alpha\}$, and set $L'(v) = L(v)$ for all other vertices v. After removing p, there may be vertices in $P' = P - p$ and it must be maintained that P' is between x' and y' clockwise on the outer face. If $x = y = p$ (x and y are removed), set $x' = c_1$ and $y' = c_n$. If $p = x$, $p \neq y$ (x is removed) we set $x' = c_n$, otherwise $x' = x$. Similarly, if $y = p$ set $y' = c_1$, and otherwise $y' = y$. Since for any $v \in C[x, y]$, $\alpha \notin L(v)$, for all $n \in N_G(p)$ such that $n \in C[x, y]$, $|L'(n)| \geq 2$. Furthermore, for $n \in N_G(p)$, $n \notin C$, $|L'(v)| \geq 2$.



Case 2.1: $p = x$, $p \neq y$ (left), and Case 2.1: $p = x = y$ (right).

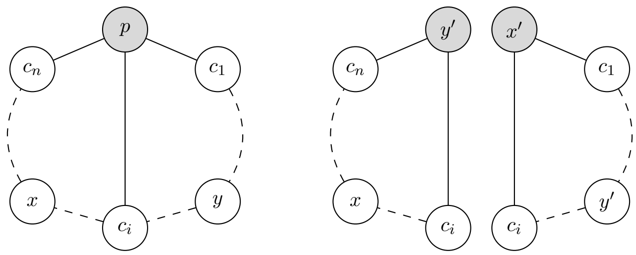
If $p \neq x$ and $p \neq y$, then clearly all $n \in N_G(p)$ fit these the above critera. Thus, using Lemma 1 the inductive hypothesis holds for $G_0 = G - p$, x', y', P', and $L'(v)$. If $x = y = p$, we may have $|L'(c_n)| = 1$ and $|L'(c_1)| = 1$, but we have $x' = c_1$ and $y' = c_n$ so the hypothesis holds. In the case $x = p$ (or $y = p$), we may have $|L'(c_n)| = 1$ (or $|L'(c_1)| = 1$), but $x' = c_n$ (or $y' = c_1$) so the inductive hypothesis still holds.

Case 2.2: Suppose $|N_C(p)| > 2$. Let us select the largest $i \in \mathbb{Z}^+$ such that $c_i \in N_C(p)$ and $c_i \in C(c_1, c_n)$. Since P is induced in G, $c_i \in C(p', c_n)$.



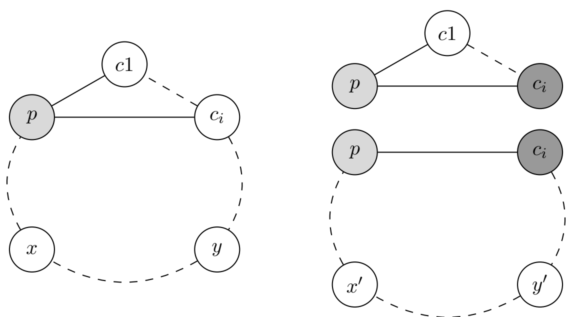
Case 2.2.1: $c_i \in C[x, c_n)$.

Case 2.2.1: Suppose $c_i \in C[x, c_n)$. Notice the inductive hypothesis holds for the to the subgraph G_0 bounded by $C[c_i, p]$ and the edge pc_i with $P' = p$, $x' = p$ and $y' = c_i$. Furthermore, the inductive hypothesis holds for the subgraph G_1 bounded by $C[p, c_i]$ and the edge pc_i with $P' = P$, $x' = x$, and $y' = y$. So the coloring of $G = G_0 \cup G_1$ is a path coloring.



Case 2.2.2: $c_i \in C(x, y]$.

Case 2.2.2: Suppose $c_i \in C[y, x)$. Notice the inductive hypothesis holds for the subgraph G_0 bounded by $C[c_i, p]$ and the edge pc_i with $P' = p$, $x' = x$, and $y' = p$. Furthermore, the inductive hypothesis holds for the subgraph G_1 bounded by $C[p, c_i]$ and the edge pc_i, $P' = P$, $x' = p$, $y' = y$. Since the removal of p in G_0 and G_1 will set c_i to x and y respectively, c_i will have at most one neighbor in each subgraph. Therefore the coloring of $G = G_0 \cup G_1$ is a path coloring.



Case 2.2.3: $c_i \in C(y, p')$.

Case 2.2.3: Suppose $c_i \in C(p', y)$. This is similar to Case 2.2.1, but with the added dificulty that there might be remaining vertices of P in the subgraph not containing x and y. First notice that the inductive hypothesis holds for the subgraph G_0 bounded by $C[c_i, p]$ and the edge pc_i, $P' = p$, $x' = x$, $y' = y$. Now c_i is colored with some color $\beta \neq \alpha$ by the path coloring of G_0. Let G_1 be the subgraph bounded by $C[p, c_i]$ and the edge pc_i. Then the inductive hypothesis holds for for G_1 with $P' = c_iP$, $x' = c_i$, $y' = c_i$. Since c_i will have no same color neighbors in G_1, the coloring of $G = G_0 \cup G_1$ is a path coloring. □

Since adding edges does not make a graph easier to color, we may add edges to weakly triangulate any graph. Then we may assign color lists of size 3 to each vertex, set x and y to be arbitrary vertices on the outer face, and apply Theorem 2.

References

[1] Hartman, C., "Extremal problems in graph theory," Ph.D. thesis, Department of Mathematics, University of Illinois at Urbana-Champaign, 1997.

[2] Skrekovski, R., "List improper colourings of planar graphs," *Combinatorics, Probability and Computing*, vol. 8, pp. 293-299, 1999.