A Path 3-List Coloring Algorithm for Plane Graphs

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Abstract

We present an algorithm to path 3-list-color plane graphs based on the work by Skrekovski [2] and Hartman [1]. Furthermore, we provide implementation details and an efficient implementation of the algorithm in the Boost Graph Library.

1 Introduction

All graphs discussed are assumed to be simple, undirected, and plane embeded. For plane embeddings we assume the edges around each vertex are arranged in clockwise order. For a graph G, let $V(G)$ denote its vertex set and $E(G)$ denote the edge set.

Using notation from [1], for $v \in V(G)$ we will denote the *neighborhood* of v in G as $N_G(x) = \{u \in V(G) \mid uv \in E(G)\}$. For $u, w \in N_G(v)$ we will use $[u, w]_v$ and $(u, w)_v$ to denote the ordered list of vertices between u and w in $N_G(v)$ in clockwise embedded order, inclusive and exclusive respectively. We will use $[u, w]'_v$ and $(u, w)'_v$ for the equivalent counterclockwise listings. Furthermore, if C is the outer face of a weakly triangulated 2-connected graph then for $u, v \in C$ let $C[u, v]$ denote the set of vertices along the outer face in clockwise order between u and v inclusive.

A *path* is a graph P with $V(P) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(P) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. It is common to refer to P as a v_1v_n-path. A graph G is connected if for every pair of vertices $u, v \in G$, there exists a uv-path subgraph of G. All graphs discussed in this project are connected. For a vertex $v \in G$, if removing v disconnects G, then v is known as a cutvertex. A graph is considered *biconnected* if it contains no cutvertices.

A *coloring* of a graph G assigns a color to each vertex. If $L(v)$ assigns a list of k colors to each vertex $v \in V(G)$, a *k-list-coloring* colors G such that each $v \in V(G)$ is colored from $L(v)$. A *path k-list-coloring* is a k-list-coloring such that each color class induces a disjoint union of paths.

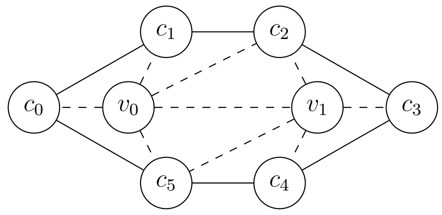
2 Vertex Removal

In this section we present two theorems describing the effects of vertex removal on weakly triangulated plane graphs. These will be necessary in later sections to remove already colored paths from our graph. First, we consider removing a vertex from the outer face that has only two neighbors in the outer face (i.e. no chords).

Theorem 1. Let G be a weakly triangulated biconnected plane graph with $|V(G)| \geq 4$ and outer face $C = c_0c_1 \ldots c_{n-1}$ in clockwise embedded order. Then $N_C(c_0) = \{c_1, c_{n-1}\}$ if and only if $G_0 = G - c_1$ is a weakly triangulated plane graph with $|V(G_0)| \geq 3$ and outer face $C_0 = c_1 \ldots c_{n-1}(c_1, c_{n-1})'_{c_0}$.

Proof. First notice that G_0 is clearly weakly triangulated since G was weakly triangulated and we removed a vertex from the outer face. Let $u, v \in G_0$. If $u, v \in C_0$ then there are clearly two edge disjoint uv-paths. If $u \in C_0$ and $v \notin C_0$, there are two edge disjoint paths v to C_0, thus two edge disjoint uv-paths. Finally, if

$u, v \notin C_0$ then we may find two edge disjoint paths from u to C_0 and two disjoint paths v to C_0 and join each pair using the outer cycle. Clearly in each case we have two edge disjoint uv-paths and G_0 is biconnected. □

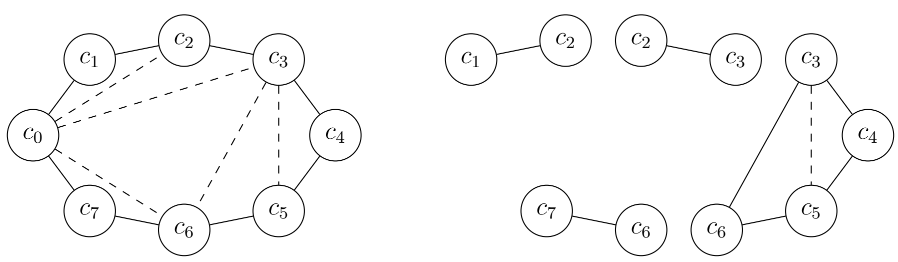


Removing c_0 with Theorem 1.

Using the previous statement we produce a stronger statement describing the result of removing any outer face vertex from a weakly triangulated graph. A *block* of a graph G is a maxmimal subgraph H of G such that H is biconnected or a complete graph on 1 or 2 vertices.

Theorem 2. Let G be a weakly triangulated biconnected plane graph with $|V(G)| \geq 3$ and outer face $C = c_0c_1 \ldots c_{n-1}$ in clockwise embedded order. Then $G - c_0$ may be divided up into one or more blocks $G_0, \ldots, G_k$ such that $G - c_0 = G_0 \cup G_1 \cup \ldots \cup G_k$, each G_i has $|V(G_i)| \geq 2$, and each G_i is weakly triangulated if $|V(G_i)| \geq 3$. Furthermore, each G_i has at most two cutvertices.

Proof. Notice if $|V(G)| = 3$, then $G - c_0$ is a complete graph on 2 vertices. Suppose $|V(G)| \geq 4$. If $N_C(c_0) = \{c_1, c_n\}$ then Theorem 1 applies. Otherwise, there exists $c_i \in N_C(c_0)$ distinct from c_1 and c_n. Let us then recurse on the subgraph bounded by $c_0c_1 \ldots c_i$ and the subgraph bounded by $c_ic_{i+1} \ldots c_nc_0$. Note that after removing c_0 each subgraph will only have the cutvertex c_i in common. □



Removing c_0 and disecting into blocks with Theorem 2.

Note that when applying Theorem 2, $G - c_0$ is biconnected (i.e. $G - c_0 = G_0$) if and only if Theorem 1 applies.

3 Path 3-List Coloring Plane Graphs

In this section we present a correct algorithm for producing path 3-list coloring of a plane graph. The following theorems are equivalent to results produced by Hartman in [1] and independantly by Skrekovski in [3]. The objective in restructuring the theorems in this way is to empasize the mechanical operations that would take place in an algorithm to produce such a coloring. In this way we prove correctness for our later algorithm, as well as structure the proof in such a way that the coloring procedure is immediately clear.

Theorem 3. Let G be a weakly triangulated plane graph with outer face $C = c_0c_1 \ldots c_{n-1}$ and let $x, y \in C$, not necessarily distinct. Suppose $L(v)$ assigns a list of colors to each $v \in V(G)$ such that

$$\begin{aligned} |L(v)| &\geq 1 \quad \text{if } v = x \text{ or } v = y; \\ |L(v)| &\geq 2 \quad \text{if } v \in C \setminus \{x, y\}; \\ |L(v)| &\geq 3 \quad \text{otherwise.} \end{aligned}$$

Optionally let P be a path of vertices in $C[x, y]$. If path P exists, assume there exists color α such that for all $v \in P$, $\alpha \in L(v)$ and for all $v \in C[x, y] \setminus P$, $\alpha \notin L(v)$.

Then we may path list color G from list assignment $L(v)$ such that x and y have at most one neighbor of the same color.

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 3$ the statement is trivial. Let $|V(G)| \geq 4$ and suppose the statement holds for all graphs G' with $|G'| < |G|$. Let $C = xc_1 \dots c_{n-1}$ be the outer face of G.

Suppose path P does not exist. Let α be the first, and possibly only, color in $L(x)$. We will construct a chordless path P of vertices along the outer face in $C[x, y]$, starting with x, such that we may color each vertex in P with the color α.

Initialize P to be the singleton path x. Let $c_i \in C[x, y]$ be the current end of the path P. Let c_j be the furthest vertex in $C[c_i, y]$ such that $\alpha \in L(c_j)$. If such a c_j exists, append it to P and recurse. Otherwise we are done.

Case 1: Suppose P takes a chord c_ic_j, $j > (i + 1)$, accross C. Then the inductive hypothesis holds for the graph bounded by $C[c_i, c_j]$ with $x = c_i$, $y = c_j$, and $P = c_ic_j$. Furthermore, if we denote the first and last vertices of P with p and p' respectively, the inductive holds for the graph bounded by P and $C(p', p)$ with our current x and y.

Case 2: Suppose P is chordless. We will proceed by removing p, the first vertex in the the path P, and showing the inductive hypothesis holds for each of the remaining blocks of $G \setminus \{p\}$. Let us denote $C = pc_1 \dots c_{n-1}$.
Case 2.1: Suppose $N_C(p) = \{c_1, c_{n-1}\}$. Let us color p with then by Lemma 1 clearly the inductive hypothesis holds for $G \setminus \{p\}$ with our current x and y and path $P \cap (G \setminus \{p\})$.

Case 2.2: Suppose $|N_C(p)| > 2$. Let us select largest i such that $c_i \in N_C(p)$, $c_i \neq c_1$, $c_i \neq c_{n-1}$. If $c_i \in C[y, p)$ then let us apply the inductive hypothesis to

□

Proof. Let $w \in N_C(z)$ be the closest vertex to y between z and y along the outer face, allowing $w = y$.

We will proceed by removing one vertex from P at a time. Let us set $L(v) = L(v) - \alpha$ for each $v \in N_{G-P}(x)$. Thus for $v \in N_{G-P}$, if $v \in C$ then $|L(v)| \geq 1$ and $|L(v)| \geq 2$ otherwise. Let w and z be the vertices immediately prior and immediately following x in C respectively. By applying Theorem 2 we may remove x.

We continue the above process, recursing on all blocks produced, until all path vertices are removed. Let $G - P = G_0, G_1, \dots, G_k$ be the blocks produced. We then proceed through each G_i from first split to last split.

Suppose $y \notin G_i$. If $G_i \cap P - x \neq \emptyset$ we color G_i with Theorem 3 using the cutvertices as x and y, and $G_i \cap P - x \neq \emptyset$ as the path. Otherwise we color G_i with Theorem 4 using cutvertices as x and y.

Suppose $y \in G_i$. By our construction of Path P we know either y is a cutvertex, or one cutvertex c has $|L(c)| \geq 2$. In this case we color G_i in the manner described above and find c has been colored some color, we will call ω. Then let us color . In this way we ensure that

Otherewise, by Theorem 2, let $G - p = G_0, G_1, \dots, G_k$ ordered such that G_0 is the block containing w and proceeding over until G_k is the block containing z. Clearly G_0 and G_k will have a single cutvertex and each G_i, $0 < i < k$, will have two. Furthermore, $c \in G_i$ is a cutvertex if and only if we had $c \in C \cap N_G(x)$ and therefore the cutvertices, and vertices w and z are exactly the vertices guaranteed $|L(v)| \geq 1$. All other vertices v on the outer face of each G_i will have $|L(v)| \geq 2$ and remaining interior vertices will still have $|L(v)| \geq 3$.

We shall proceed through each G_i from G_0 to G_k. Suppose $y \notin G_i$. If $P - x \cap G_i = \emptyset$ then we may apply Theorem 4 and color G_i. Otherwise apply Theorem 3 with graph G_i, path $P - x \cap G_i$, and

□

Theorem 4. Let G be a weakly triangulated plane graph with outer face C. Furthermore, let $x, y \in C$ and $L(v)$ assigns a list of colors to each $v \in V(G)$ such that

$$\begin{array}{ll} |L(v)| \geq 1 & \text{if } v = x \text{ or } v = y; \\ |L(v)| \geq 2 & \text{if } v \in C, v \neq x \text{ or } y; \\ |L(v)| \geq 3 & \text{otherwise.} \end{array}$$

Then G may be colored from the list assignment $L(v)$ such that each color class induces a disjoint union of paths and x and y each have at most one neighbor of the same color.

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| \leq 3$ the statement is trivial. Suppose $|V(G)| \geq 4$ and the statement holds for all graphs G', $|G'| < |G|$. Let $C = xc_1 \dots c_{n-1}$ be the outer face of G.

Let α be the first, and possibly only, color in $L(x)$. For $u, v \in C$, let $C[u, v]$ denote the set of vertices along the outer face in clockwise order between x and y inclusive. We will construct a chordless path P of vertices along the outer face in $C[x, y]$, starting with x, such that we may color each vertex in P with the color α.

Initialize P to be the singleton path x. Let $c_i \in C[x, y]$ be the current end of the path P. Let c_j be the furthest vertex in $C[c_i, y]$ such that $\alpha \in L(c_j)$. If such a c_j exists, append it to P and recurse. Otherwise we are done.

Suppose P takes a chord c_ic_j, $j > (i + 1)$, accross C. Then we may apply Theorem 3, with $x = c_i$ and $y = c_j$, to the weakly triangulated graph bounded by $C[c_i, c_j]$ and the edge c_ic_j. Then we may apply Theorem 3 to the weakly triangulated graph bounded by P and the remaining C, with our current x and y. □

References

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