Contents

Chapter 1

The Real and Complex Number Systems

1.1 Integers

1.1: No Largest Prime

Prove that there is no largest prime. (A proof was known to Euclid.)

Solution: We will prove this by contradiction. Assume there exists a largest prime number, call it p.

Consider the number $N = p! + 1$, where $p!$ is the factorial of p.

Since $p!$ is divisible by all integers from 1 to p, the number $N = p! + 1$ is not divisible by any prime number less than or equal to p.

Now, N is either prime or composite:

- If N is prime, then $N > p$, contradicting our assumption that p is the largest prime.

- If N is composite, then N has a prime factor q. Since N is not divisible by any prime $\leq p$, we must have $q > p$. This again contradicts our assumption that p is the largest prime.

In both cases, we reach a contradiction. Therefore, our assumption that there exists a largest prime is false, and there must be infinitely many prime numbers.

1.2: Algebraic Identity

If n is a positive integer, prove the algebraic identity:

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Solution: We can prove this identity by expanding the right-hand side and showing it equals the left-hand side.

Let's expand the sum:

$$\begin{aligned}(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} &= (a-b)(a^0 b^{n-1} + a^1 b^{n-2} + a^2 b^{n-3} + \cdots + a^{n-1} b^0) \\ &= (a-b)(b^{n-1} + ab^{n-2} + a^2 b^{n-3} + \cdots + a^{n-1})\end{aligned}$$

Now distribute $(a-b)$:

$$\begin{aligned}&= a \cdot b^{n-1} + a^2 b^{n-2} + a^3 b^{n-3} + \cdots + a^n \\ &- b \cdot b^{n-1} - ab^{n-1} - a^2 b^{n-2} - \cdots - a^{n-1} b\end{aligned}$$

Notice that most terms cancel out:

$$\begin{aligned}&= a^n - b^n + (\text{canceling terms}) \\ &= a^n - b^n\end{aligned}$$

Alternatively, we can use the geometric series formula. Let $r = \frac{a}{b}$. Then:

$$\begin{aligned}\sum_{k=0}^{n-1} a^k b^{n-1-k} &= b^{n-1} \sum_{k=0}^{n-1} \left(\frac{a}{b}\right)^k \\ &= b^{n-1} \cdot \frac{1 - \left(\frac{a}{b}\right)^n}{1 - \frac{a}{b}} \\ &= b^{n-1} \cdot \frac{b^n - a^n}{b^n (b-a)} \\ &= \frac{a^n - b^n}{a-b}\end{aligned}$$

Therefore, $(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} = a^n - b^n$.

1.3: Mersenne Primes

If $2^n - 1$ is prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a *Mersenne prime*.

Solution: We will prove the contrapositive: if n is composite, then $2^n - 1$ is composite.

Let $n = ab$ where $a, b > 1$ are integers. Then:

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \end{aligned}$$

Using the identity from Problem 1.2 with $x = 2^a$ and $y = 1$:

$$(2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1)$$

Since $a > 1$, we have $2^a - 1 > 1$. Also, since $b > 1$, the second factor is greater than 1. Therefore, $2^n - 1$ is the product of two integers greater than 1, making it composite.

This proves that if $2^n - 1$ is prime, then n must be prime.

1.4: Fermat Primes

If $2^n + 1$ is prime, prove that n is a power of 2. A prime of the form $2^{2^n} + 1$ is called a *Fermat prime*. Hint: Use Exercise 1.2.

Solution: We will prove the contrapositive: if n is not a power of 2, then $2^n + 1$ is composite.

If n is not a power of 2, then n has an odd factor greater than 1. Let $n = 2^k \cdot m$ where $m > 1$ is odd and $k \geq 0$.

Then:

$$\begin{aligned} 2^n + 1 &= 2^{2^k \cdot m} + 1 \\ &= (2^{2^k})^m + 1 \end{aligned}$$

Since m is odd, we can use the identity from Problem 1.2 with $a = 2^{2^k}$ and $b = -1$:

$$(2^{2^k})^m - (-1)^m = (2^{2^k} - (-1))((2^{2^k})^{m-1} + (2^{2^k})^{m-2}(-1) + \cdots + (-1)^{m-1})$$

Since m is odd, $(-1)^m = -1$, so:

$$(2^{2^k})^m + 1 = (2^{2^k} + 1)((2^{2^k})^{m-1} - (2^{2^k})^{m-2} + \cdots + 1)$$

Since $m > 1$, both factors are greater than 1, making $2^n + 1$ composite.

Therefore, if $2^n + 1$ is prime, then n must be a power of 2.

1.5: Fibonacci Numbers Formula

The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $x_n = \frac{a^n - b^n}{a - b}$, where a and b are the roots of the equation $x^2 - x - 1 = 0$.

Solution: Let the proposition be $P(n) : x_n = \frac{a^n - b^n}{a-b}$. The roots of $x^2 - x - 1 = 0$ are $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. A key property of these roots is that they satisfy $a^2 = a + 1$ and $b^2 = b + 1$.

Base cases: For $n = 1$:

$$\frac{a^1 - b^1}{a - b} = 1 = x_1.$$

For $n = 2$:

$$\frac{a^2 - b^2}{a - b} = \frac{(a-b)(a+b)}{a-b} = a + b = \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right) = 1 = x_2.$$

The base cases hold.

Inductive step: Assume $P(k)$ is true for all integers $k \leq n$, where $n \geq 2$. We will show $P(n+1)$ is true. By the definition of the Fibonacci sequence, $x_{n+1} = x_n + x_{n-1}$. Using the inductive hypothesis for x_n and x_{n-1}:

$$\begin{aligned}
x_{n+1} &= \left(\frac{a^n - b^n}{a-b}\right) + \left(\frac{a^{n-1} - b^{n-1}}{a-b}\right) \\
&= \frac{(a^n + a^{n-1}) - (b^n + b^{n-1})}{a-b} \\
&= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{a-b}
\end{aligned}$$

Using the property that $a + 1 = a^2$ and $b + 1 = b^2$:

$$\begin{aligned}
x_{n+1} &= \frac{a^{n-1}(a^2) - b^{n-1}(b^2)}{a-b} \\
&= \frac{a^{n+1} - b^{n+1}}{a-b}
\end{aligned}$$

This is $P(n+1)$. By the principle of strong induction, the formula is true for all positive integers n.

1.6: Well-Ordering Principle

Prove that every nonempty set of positive integers contains a smallest member. This is called the *well-ordering principle.*

Solution: We will prove this by contradiction. Let S be a nonempty set of positive integers that has no smallest member. Let $P(n)$ be the proposition that the integer n is not in S. We will use induction to show that $P(n)$ is true for all positive integers n.

Base case: For $n = 1$: If $1 \in S$, then 1 would be the smallest member of S (since S contains only positive integers). But we assumed S has no smallest member. So 1 cannot be in S. Thus, $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true for all positive integers $k < n$. This means that none of the integers $1, 2, \ldots, n-1$ are in S. Now consider the integer n. If $n \in S$, then from our inductive hypothesis (that $1, 2, \ldots, n-1$ are not in S), n would be the smallest member of S. This contradicts our initial assumption that S has no smallest member. Therefore, n cannot be in S. Thus, $P(n)$ is true.

By the principle of strong induction, $P(n)$ is true for all positive integers n. This means that no positive integer is in S, which implies that S is an empty set. This contradicts our initial assumption that S is a nonempty set. Therefore, the original assumption must be false, and every nonempty set of positive integers must contain a smallest member.

1.2 Rational and Irrational Numbers

1.7: Decimal Expansion to Rational

Find the rational number whose decimal expansion is $0.334444\ldots$.

Solution: We can use an algebraic method to find the equivalent fraction. Let x be the rational number.

$$x = 0.334444\ldots$$

The goal is to manipulate the equation to eliminate the repeating decimal part. The repeating digit '4' begins at the third decimal place.

First, multiply by 100 to move the non-repeating part to the left of the decimal point:

$$100x = 33.4444\ldots$$

Next, multiply by 1000 to shift the decimal point past the first repeating digit:

$$1000x = 334.4444\ldots$$

Now, subtract the first equation from the second. This will cancel the infinite repeating tail.

$$\begin{array}{rr} & 1000x = 334.4444\ldots \\ - & 100x = \ \ 33.4444\ldots \\ \hline & 900x = 301 \end{array}$$

Finally, solve for x:

$$x = \frac{301}{900}$$

Therefore, the rational number is $\frac{301}{900}$.

1.8: Decimal Expansion Ending in Zeroes

Prove that the decimal expansion of x will end in zeroes (or in nines) if and only if x is a rational number whose denominator is of the form $2^m 5^n$, where m and n are nonnegative integers.

Solution: We need to prove both directions of this statement.

Forward direction: If x is rational with denominator of the form $2^m 5^n$, then its decimal expansion terminates.

Let $x = \frac{p}{q}$ where $q = 2^m 5^n$ for some nonnegative integers m, n.

We can write $x = \frac{p}{2^m 5^n} = \frac{p \cdot 2^n 5^m}{2^m 5^n \cdot 2^n 5^m} = \frac{p \cdot 2^n 5^m}{10^{m+n}}$

This shows that x can be written as a fraction with denominator a power of 10, which means its decimal expansion terminates.

Reverse direction: If the decimal expansion of x terminates, then x is rational with denominator of the form $2^m 5^n$.

Let x have a terminating decimal expansion. Then x can be written as $x = \frac{N}{10^k}$ for some integer N and nonnegative integer k.

Since $10 = 2 \cdot 5$, we have $10^k = 2^k \cdot 5^k$, which is of the required form.

Note about ending in nines: If a decimal expansion ends in nines (e.g., $0.999\ldots$), this is equivalent to the next terminating decimal. For example, $0.999\ldots = 1.000\ldots$. This is because $0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{9/10}{1-1/10} = 1$.

Therefore, both terminating decimals and those ending in nines correspond to rational numbers with denominators of the form $2^m 5^n$.

1.9: Irrationality of $\sqrt{2} + \sqrt{3}$

Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: We will prove this by contradiction. Assume that $\sqrt{2} + \sqrt{3}$ is rational, say $\sqrt{2} + \sqrt{3} = \frac{p}{q}$ where p, q are integers with no common factors.

Then:

$$\begin{aligned}
\sqrt{2}+\sqrt{3} &= \frac{p}{q} \\
(\sqrt{2}+\sqrt{3})^2 &= \left(\frac{p}{q}\right)^2 \\
2+2\sqrt{6}+3 &= \frac{p^2}{q^2} \\
5+2\sqrt{6} &= \frac{p^2}{q^2} \\
2\sqrt{6} &= \frac{p^2}{q^2}-5 \\
\sqrt{6} &= \frac{p^2-5q^2}{2q^2}
\end{aligned}$$

This shows that $\sqrt{6}$ is rational, which is a contradiction since $\sqrt{6}$ is irrational.

To see why $\sqrt{6}$ is irrational, suppose $\sqrt{6}=\frac{a}{b}$ where a, b are integers with no common factors. Then:

$$\begin{aligned}
6 &= \frac{a^2}{b^2} \\
6b^2 &= a^2
\end{aligned}$$

This means a^2 is divisible by 6, so a must be divisible by 6. Let $a = 6k$. Then:

$$\begin{aligned}
6b^2 &= (6k)^2 = 36k^2 \\
b^2 &= 6k^2
\end{aligned}$$

This means b^2 is divisible by 6, so b must also be divisible by 6. But this contradicts our assumption that a and b have no common factors.

Therefore, $\sqrt{6}$ is irrational, and consequently $\sqrt{2}+\sqrt{3}$ is irrational.

1.10: Rational Functions of Irrational Numbers

If a, b, c, d are rational and if x is irrational, prove that $\frac{ax+b}{cx+d}$ is usually irrational. When do exceptions occur?

Solution: We need to analyze when $\frac{ax+b}{cx+d}$ is rational, given that x is irrational and a, b, c, d are rational.

Let's assume that $\frac{ax+b}{cx+d} = \frac{p}{q}$ where p, q are integers with no common factors.

Then:

$$\frac{ax+b}{cx+d} = \frac{p}{q}$$
$$q(ax+b) = p(cx+d)$$
$$qax+qb = pcx+pd$$
$$(qa-pc)x = pd-qb$$

Since x is irrational and the right-hand side is rational, we must have $qa - pc = 0$ and $pd - qb = 0$.

This gives us:

$$qa = pc$$
$$pd = qb$$

From the first equation: $a = \frac{pc}{q}$ From the second equation: $b = \frac{pd}{q}$
Therefore, we have:

$$\frac{a}{c} = \frac{p}{q}$$
$$\frac{b}{d} = \frac{p}{q}$$

This means $\frac{a}{c} = \frac{b}{d}$, or equivalently, $ad = bc$.

Conclusion: The expression $\frac{ax+b}{cx+d}$ is rational if and only if $ad = bc$.

Exceptions occur when: $ad = bc$, which means the numerator and denominator are proportional, making the fraction rational regardless of the value of x.

Examples:

- If $a = 2, b = 1, c = 4, d = 2$, then $ad = 4 = bc = 4$, so $\frac{2x+1}{4x+2} = \frac{1}{2}$ for all x.
- If $a = 1, b = 0, c = 1, d = 0$, then $ad = 0 = bc = 0$, so $\frac{x}{x} = 1$ for all $x \neq 0$.

1.11: Irrational Numbers Between 0 and x

Given any real $x > 0$, prove that there is an irrational number between 0 and x.

Solution: We will construct an irrational number between 0 and x for any positive real number x.

Case 1: If x is irrational, then $\frac{x}{2}$ is irrational and lies between 0 and x.

To see why $\frac{x}{2}$ is irrational, suppose it were rational. Then $\frac{x}{2} = \frac{p}{q}$ for some integers p, q, which would mean $x = \frac{2p}{q}$, making x rational, a contradiction.

Case 2: If x is rational, let $x = \frac{p}{q}$ where p, q are positive integers.

Consider the number $y = \frac{x}{\sqrt{2}} = \frac{p}{q\sqrt{2}}$.

Since $\sqrt{2}$ is irrational, y is irrational (if y were rational, then $\sqrt{2} = \frac{p}{qy}$ would be rational, a contradiction).

Also, since $\sqrt{2} > 1$, we have $y < x$.

Therefore, y is an irrational number between 0 and x.

Alternative construction: For any positive real x, we can also use $y = \frac{x}{\pi}$. Since π is irrational and greater than 1, we have $0 < y < x$, and y is irrational.

1.12: Fraction Between Two Fractions

If $\frac{a}{b} < \frac{c}{d}$ with $b > 0, d > 0$, prove that $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Solution: We need to prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Let's prove both inequalities:

First inequality: $\frac{a}{b} < \frac{a+c}{b+d}$

Cross-multiplying:

$$\begin{aligned} a(b+d) &< b(a+c) \\ ab + ad &< ab + bc \\ ad &< bc \end{aligned}$$

Since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$, so this inequality holds.

Second inequality: $\frac{a+c}{b+d} < \frac{c}{d}$

Cross-multiplying:

$$\begin{aligned} d(a+c) &< c(b+d) \\ ad + cd &< bc + cd \\ ad &< bc \end{aligned}$$

Again, since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$, so this inequality also holds.

Therefore, $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Geometric interpretation: This result is known as the "mediant" of two fractions. If we think of fractions as points on a line, the mediant $\frac{a+c}{b+d}$ lies between the two original fractions $\frac{a}{b}$ and $\frac{c}{d}$.

1.13: $\sqrt{2}$ Between Fractions

Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions $\frac{a}{b}$ and $\frac{a+2b}{a+b}$. Which fraction is closer to $\sqrt{2}$?

Solution: Let's first establish the ordering of the two fractions by examining their difference:

$$\frac{a+2b}{a+b} - \frac{a}{b} = \frac{b(a+2b) - a(a+b)}{b(a+b)} = \frac{ab + 2b^2 - a^2 - ab}{b(a+b)} = \frac{2b^2 - a^2}{b(a+b)}$$

The sign of this difference depends on the sign of $2b^2 - a^2$, which relates $\frac{a}{b}$ to $\sqrt{2}$.

Case 1: $\frac{a}{b} < \sqrt{2}$. This means $a < b\sqrt{2}$, so $a^2 < 2b^2$, and $2b^2 - a^2 > 0$. Thus, $\frac{a}{b} < \frac{a+2b}{a+b}$. We need to show that $\frac{a+2b}{a+b} > \sqrt{2}$.

$$\frac{a+2b}{a+b} > \sqrt{2} \iff a+2b > \sqrt{2}(a+b) \iff b(2-\sqrt{2}) > a(\sqrt{2}-1) \iff \frac{2-\sqrt{2}}{\sqrt{2}-1} > \frac{a}{b}$$

Since $\frac{2-\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}-1} = \sqrt{2}$, this simplifies to $\sqrt{2} > \frac{a}{b}$, which is true by our case assumption. Thus, $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$.

Case 2: $\frac{a}{b} > \sqrt{2}$. This means $a^2 > 2b^2$, and $2b^2 - a^2 < 0$. Thus, $\frac{a}{b} > \frac{a+2b}{a+b}$. A similar calculation shows that $\frac{a+2b}{a+b} < \sqrt{2}$. Therefore, $\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$. In both cases, $\sqrt{2}$ lies between the two fractions.

Which fraction is closer to $\sqrt{2}$? We compare the absolute distances:

- Distance 1: $\left|\frac{a}{b} - \sqrt{2}\right| = \frac{|a-b\sqrt{2}|}{b}$
- Distance 2: $\left|\frac{a+2b}{a+b} - \sqrt{2}\right| = \left|\frac{a+2b-a\sqrt{2}-b\sqrt{2}}{a+b}\right| = \left|\frac{a(1-\sqrt{2})-b(\sqrt{2}-2)}{a+b}\right| = \frac{|a-b\sqrt{2}|(\sqrt{2}-1)}{a+b}$

To see which distance is smaller, we compare $\frac{1}{b}$ with $\frac{\sqrt{2}-1}{a+b}$. This is equivalent to comparing $a+b$ with $b(\sqrt{2}-1) = b\sqrt{2}-b$, which is equivalent to comparing $a+2b$ with $b\sqrt{2}$, or $\frac{a}{b}+2$ with $\sqrt{2}$. Since a, b are positive integers, $\frac{a}{b} > 0$, so $\frac{a}{b}+2 > 2 > \sqrt{2}$. This implies $\frac{1}{b} > \frac{\sqrt{2}-1}{a+b}$. Therefore, Distance 1 is always greater than Distance 2. The new fraction $\frac{a+2b}{a+b}$ is **always** closer to $\sqrt{2}$.

1.3 Inequalities

1.14: Irrationality of $\sqrt{n-1}+\sqrt{n+1}$

Prove that $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

Solution: Assume $\sqrt{n-1}+\sqrt{n+1} = \frac{p}{q}$, where p, q are integers, $q \neq 0$, $\gcd(p,q) = 1$.

Square both sides:

$$(n-1) + 2\sqrt{(n-1)(n+1)} + (n+1) = \frac{p^2}{q^2} \implies 2n + 2\sqrt{n^2-1} = \frac{p^2}{q^2}.$$

Thus:

$$\sqrt{n^2-1} = \frac{p^2-2nq^2}{2q^2}.$$

Suppose $\sqrt{n^2-1}$ is rational, say $\frac{a}{b}$, $\gcd(a,b)=1$. Then:

$$n^2-1=\frac{a^2}{b^2} \implies a^2=(n^2-1)b^2.$$

For $n=1$, $\sqrt{0}+\sqrt{2}=\sqrt{2}$, irrational. For $n\geq 2$, $n^2-1=(n-1)(n+1)$ is not a perfect square (since $(n-1)^2<n^2-1<n^2$). If $a^2=(n^2-1)b^2$, n^2-1 must be a perfect square, a contradiction for $n\geq 2$. Thus, $\sqrt{n^2-1}$ is irrational, so $\sqrt{n-1}+\sqrt{n+1}$ is irrational.

1.15: Approximation by Rational Numbers

Given a real x and an integer $N>1$, prove that there exist integers h and k with $0<k\leq N$ such that $|kx-h|<1/N$. Hint. Consider the $N+1$ numbers $tx-[tx]$ for $t=0,1,2,\ldots,N$ and show that some pair differs by at most $1/N$.

Solution: We will use the pigeonhole principle to prove this result.

Consider the $N+1$ numbers: $0,x,2x,3x,\ldots,Nx$.

Let's look at the fractional parts of these numbers. The fractional part of a number y is $y-\lfloor y\rfloor$, where $\lfloor y\rfloor$ is the greatest integer less than or equal to y.

The fractional parts of $0,x,2x,\ldots,Nx$ all lie in the interval $[0,1)$.

Divide the interval $[0,1)$ into N equal subintervals: $[0,1/N),[1/N,2/N),\ldots,[(N-1)/N,1)$

By the pigeonhole principle, since we have $N+1$ numbers and only N subintervals, at least two of these numbers must fall into the same subinterval.

Let's say ix and jx (where $0\leq i<j\leq N$) have fractional parts in the same subinterval. Then:

$$\begin{aligned}|(jx-\lfloor jx\rfloor)-(ix-\lfloor ix\rfloor)|&<\frac{1}{N}\\ |(j-i)x-(\lfloor jx\rfloor-\lfloor ix\rfloor)|&<\frac{1}{N}\end{aligned}$$

Let $k=j-i$ and $h=\lfloor jx\rfloor-\lfloor ix\rfloor$. Then:

$$|kx-h|<\frac{1}{N}$$

Since $0<i<j\leq N$, we have $0<k\leq N$, and h is an integer.

Therefore, we have found integers h and k with $0<k\leq N$ such that $|kx-h|<1/N$.

Example: If $x=\pi$ and $N=5$, we might find that $3\pi\approx 9.4248$ and $5\pi\approx 15.7080$ have fractional parts in the same subinterval, giving us $|2\pi-6|<1/5$.

1.16: Infinitely Many Rational Approximations

If x is irrational, prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$.

Solution: We will construct an infinite sequence of distinct rational numbers satisfying the condition.

From Problem 1.15 (Dirichlet's Approximation Theorem), for any integer $N > 1$, there exist integers h and k with $0 < k \le N$ such that $|kx - h| < 1/N$. Dividing by k, we get:

$$\left| x - \frac{h}{k} \right| < \frac{1}{Nk}$$

Since $k \le N$, we have $\frac{1}{N} \le \frac{1}{k}$, which implies $\frac{1}{Nk} \le \frac{1}{k^2}$. Thus, for any integer $N > 1$, we can find a rational number h/k such that:

$$\left| x - \frac{h}{k} \right| < \frac{1}{k^2}$$

Now we must show that this process can generate infinitely many distinct fractions. Assume, for the sake of contradiction, that there are only a finite number of such rational approximations, say $\{h_1/k_1, h_2/k_2, \dots, h_m/k_m\}$. Since x is irrational, for any rational number h_i/k_i, the distance $|x - h_i/k_i|$ is non-zero. Let ϵ be the smallest of these non-zero distances:

$$\epsilon = \min_{i=1,\dots,m} \left| x - \frac{h_i}{k_i} \right| > 0.$$

Now, choose an integer N large enough such that $1/N < \epsilon$. By the result from Problem 1.15, there exist integers h' and k' with $0 < k' \le N$ such that:

$$|k'x - h'| < \frac{1}{N}$$

This implies $|x - h'/k'| < \frac{1}{Nk'} \le \frac{1}{N}$. So we have found a new rational approximation h'/k' such that:

$$\left| x - \frac{h'}{k'} \right| < \frac{1}{N} < \epsilon$$

Since the approximation error of h'/k' is smaller than ϵ, h'/k' cannot be one of the fractions in our finite list $\{h_1/k_1, \dots, h_m/k_m\}$. This contradicts our assumption that we had a complete list of all such approximations. Therefore, there must be infinitely many such rational numbers.

1.17: Factorial Representation of Rationals (Precise Form)

Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is a nonnegative integer with $a_k \leq k-1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the greatest integer less than or equal to x. Prove that $a_1 = [x]$, that

$$a_k = [k!x] - k[(k-1)!x] \quad \text{for } k = 2, \ldots, n,$$

and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Solution: Let $x = \sum_{k=1}^{n} \frac{a_k}{k!}$ with the given conditions on a_k.

1. Proof that $a_1 = [x]$: The sum can be written as $x = a_1 + \sum_{k=2}^{n} \frac{a_k}{k!}$. We must show the summation part is a positive fraction less than 1. Since $a_n > 0$, the sum is positive. We can bound the sum using the property $a_k \leq k-1$:

$$\sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k-1}{k!} < \sum_{k=2}^{\infty} \frac{k-1}{k!}$$

The infinite sum is a known identity: $\sum_{k=2}^{\infty} \frac{k-1}{k!} = \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)!} - \frac{1}{k!}\right)$. This is a telescoping series whose sum is the first term, $1/(2-1)! = 1$. Thus, $0 < \sum_{k=2}^{n} \frac{a_k}{k!} < 1$. This means $a_1 < x < a_1 + 1$, so by definition, $a_1 = [x]$.

2. Formula for a_k: Define $x_1 = x - a_1 = \sum_{k=2}^{n} \frac{a_k}{k!}$. Then $k!x_1$ is an integer for $k \geq n$. Consider the expression $k!x - k((k-1)!x) = k!(a_1 + x_1) - k((k-1)!(a_1 + x_1)) = ka_1k!/k!...$ this gets complicated. Let's use the given formula. Let $x_k = k!x - \sum_{j=1}^{k} a_j \frac{k!}{j!} = \sum_{j=k+1}^{n} a_j \frac{k!}{j!} = \frac{a_{k+1}}{k+1} + \frac{a_{k+2}}{(k+1)(k+2)} + \ldots$. From part (1), we know $0 \leq x_k < 1$. So $[k!x] = \sum_{j=1}^{k} a_j \frac{k!}{j!}$. Let's test the formula: $a_k = [k!x] - k[(k-1)!x]$. We have $[k!x] = k! \sum_{j=1}^{k} \frac{a_j}{j!}$ and $[(k-1)!x] = (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!}$. So, $[k!x] - k[(k-1)!x] = \sum_{j=1}^{k} a_j \frac{k!}{j!} - k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!} = \left(a_k + k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!}\right) - k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!} = a_k$. This proves the formula for a_k.

3. Minimality of n: Multiplying x by $n!$ gives $n!x = \sum_{k=1}^{n} a_k \frac{n!}{k!}$. Since $k \leq n$, each term $\frac{n!}{k!}$ is an integer, so $n!x$ is an integer. For any $m < n$, when we compute $m!x$, the term corresponding to $k = n$ is $m! \frac{a_n}{n!} = \frac{a_n}{n(n-1)\ldots(m+1)}$. Since $0 < a_n \leq n-1$, this term is a non-integer fraction. Because all other terms for $k > m$ are also fractions and terms for $k \leq m$ are integers, $m!x$ cannot be an integer. Thus, n is the smallest such integer.

4. Converse (Uniqueness): Suppose a positive rational number x has two different representations:

$$x = \sum_{k=1}^{n} \frac{a_k}{k!} = \sum_{k=1}^{m} \frac{b_k}{k!}$$

with the conditions $0 \le a_k \le k-1$ for $k \ge 2$, $a_n > 0$, and similarly for b_k. From part (3), n is the smallest integer such that $n!x$ is an integer, and m is the smallest integer such that $m!x$ is an integer. This implies $n = m$.

Let j be the largest index for which the coefficients differ, so $a_j \neq b_j$. Assume, without loss of generality, that $a_j > b_j$. Since $a_k = b_k$ for $k > j$, we can subtract the sums:

$$\sum_{k=1}^{j} \frac{a_k}{k!} = \sum_{k=1}^{j} \frac{b_k}{k!}$$

Rearranging the terms, we get:

$$\frac{a_j - b_j}{j!} = \sum_{k=1}^{j-1} \frac{b_k - a_k}{k!}$$

Multiply both sides by $(j-1)!$:

$$\frac{a_j - b_j}{j} = \sum_{k=1}^{j-1} (b_k - a_k) \frac{(j-1)!}{k!}$$

The right-hand side is an integer, because for each $k \in \{1, \ldots, j-1\}$, $k!$ divides $(j-1)!$. Let's analyze the left-hand side. Since a_j and b_j are integers and $a_j > b_j$, we have $a_j - b_j \ge 1$. From the conditions on the coefficients, $a_j \le j-1$ (for $j \ge 2$) and $b_j \ge 0$. Therefore, $1 \le a_j - b_j \le j-1$. This implies that for $j \ge 2$, the left-hand side $\frac{a_j - b_j}{j}$ is a non-integer fraction, since the numerator is an integer between 1 and $j-1$, and the denominator is j. This creates a contradiction: the left-hand side cannot be an integer, while the right-hand side must be an integer. For the case $j = 1$, the equation becomes $a_1 - b_1 = 0$, which contradicts $a_1 \neq b_1$. Thus, our assumption that there is a largest index j where $a_j \neq b_j$ must be false. All coefficients must be identical. The representation is unique.

5. *Uniqueness:* Suppose x has two different representations, $\sum \frac{a_k}{k!} = \sum \frac{b_k}{k!}$. Let j be the largest index where $a_j \neq b_j$. Assume $a_j > b_j$. Then $\frac{a_j - b_j}{j!} = \sum_{k=1}^{j-1} \frac{b_k - a_k}{k!}$. The left side is $\ge 1/j!$. The right side is bounded above by $\sum_{k=1}^{j-1} \frac{k-1}{k!} < 1/j!$, a contradiction. Thus, all coefficients must be the same.

1.4 Upper Bounds

1.18: Uniqueness of Supremum and Infimum

Show that the sup and inf of a set are uniquely determined whenever they exist.

Solution: We will prove that if a set has a supremum, it is unique. The proof for infimum is similar.

Proof by contradiction: Suppose a set S has two different suprema, say s_1 and s_2, with $s_1 < s_2$.

Since s_1 is a supremum of S: 1. s_1 is an upper bound of S (every element of S is $\leq s_1$) 2. s_1 is the least upper bound (no number less than s_1 is an upper bound)

Since s_2 is also a supremum of S: 1. s_2 is an upper bound of S (every element of S is $\leq s_2$) 2. s_2 is the least upper bound (no number less than s_2 is an upper bound)

But since $s_1 < s_2$, the number s_1 is less than s_2 and is also an upper bound of S. This contradicts the fact that s_2 is the least upper bound.

Therefore, our assumption that there are two different suprema is false, and the supremum must be unique.

Alternative proof: Let s_1 and s_2 both be suprema of S. Then: - s_1 is an upper bound, so $s_2 \leq s_1$ (since s_2 is the least upper bound) - s_2 is an upper bound, so $s_1 \leq s_2$ (since s_1 is the least upper bound)

Therefore, $s_1 = s_2$.

For infimum: The same argument applies to infimum. If a set has two infima i_1 and i_2, then: - i_1 is a lower bound, so $i_1 \leq i_2$ (since i_1 is the greatest lower bound) - i_2 is a lower bound, so $i_2 \leq i_1$ (since i_2 is the greatest lower bound)

Therefore, $i_1 = i_2$.

1.19: Finding Supremum and Infimum

Find the sup and inf of each of the following sets:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$ for positive integers p, q, r.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$.

(c) $S = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$ where $a < b < c < d$.

Solution:

1. Numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$:

Let's analyze the range of each term: - 2^{-p} ranges from $\frac{1}{2}$ (when $p = 1$) to 0 (as $p \to \infty$) - 3^{-q} ranges from $\frac{1}{3}$ (when $q = 1$) to 0 (as $q \to \infty$) - 5^{-r} ranges from $\frac{1}{5}$ (when $r = 1$) to 0 (as $r \to \infty$)

Therefore: - The maximum value occurs when $p = q = r = 1$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15+10+6}{30} = \frac{31}{30}$ - The minimum value occurs as $p, q, r \to \infty$: $0 + 0 + 0 = 0$

So $\sup = \frac{31}{30}$ and $\inf = 0$.

2. Set $S = \{x : 3x^2 - 10x + 3 < 0\}$:

First, let's find the roots of $3x^2 - 10x + 3 = 0$:

$$\begin{aligned} x &= \frac{10 \pm \sqrt{100 - 36}}{6} \\ &= \frac{10 \pm \sqrt{64}}{6} \\ &= \frac{10 \pm 8}{6} \\ &= \frac{18}{6} = 3 \quad \text{or} \quad \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Since the coefficient of x^2 is positive, the parabola opens upward. The inequality $3x^2 - 10x + 3 < 0$ holds between the roots.

Therefore, $S = (\frac{1}{3}, 3)$, so $\sup = 3$ and $\inf = \frac{1}{3}$.

3. Set $S = \{x : (x - a)(x - b)(x - c)(x - d) < 0\}$ where $a < b < c < d$:

The expression $(x-a)(x-b)(x-c)(x-d)$ changes sign at each root a, b, c, d.

Starting from $-\infty$: - For $x < a$: all factors are negative, so the product is positive - For $a < x < b$: one factor is positive, three negative, so product is negative - For $b < x < c$: two factors positive, two negative, so product is positive - For $c < x < d$: three factors positive, one negative, so product is negative - For $x > d$: all factors are positive, so product is positive

Therefore, $S = (a, b) \cup (c, d)$.

So $\sup = d$ and $\inf = a$.

1.20: Comparison Property for Suprema

Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \le t$ for all $s \in S$ and $t \in T$. Suppose T has a supremum. Then S has a supremum and

$$\sup S \le \sup T.$$

Solution: Let S and T be nonempty subsets of $\mathbb{R}$ with the property that for every $s \in S$ and $t \in T$, we have $s \le t$.

1. **Existence of** $\sup S$: Since T is nonempty, we can pick an arbitrary element $t_0 \in T$. By the given property, for every $s \in S$, we have $s \le t_0$. This shows that S is bounded above (by any element of T). Since S is also nonempty and bounded above, the completeness axiom of $\mathbb{R}$ guarantees that $\sup S$ exists. Let's call it $\alpha = \sup S$.

2. **Proof that** $\sup S \le \sup T$: Let $\alpha = \sup S$ and $\beta = \sup T$. From step 1, we know that any element $t \in T$ is an upper bound for the set S. Since α is the *least* upper bound of S, it must be less than or equal to any other upper

bound of S. Therefore, for any $t \in T$, we must have:

$$\alpha \leq t$$

This inequality shows that α is a lower bound for the set T. Now, by definition, $\beta = \sup T$ is the least upper bound of T. As an upper bound for T, β must be greater than or equal to every element of T. More importantly, it must be greater than or equal to any *lower bound* of T. Since we have established that α is a lower bound for T, it must follow that:

$$\alpha \leq \beta$$

Substituting the definitions of α and β, we get:

$$\sup S \leq \sup T$$

This completes the proof.

1.21: Product of Suprema

Let A and B be two sets of positive real numbers, each bounded above. Let $a = \sup A$, $b = \sup B$. Define

$$C = \{xy : x \in A,\, y \in B\}.$$

Prove that

$$\sup C = ab.$$

Proof:

Since A and B are sets of positive real numbers bounded above, their suprema $a = \sup A$ and $b = \sup B$ exist and are finite.

We are to prove that:

$$\sup C = ab.$$

Step 1: Show that ab is an upper bound for C.

Let $x \in A$, $y \in B$. Since $x \leq a$ and $y \leq b$, we have:

$$xy \leq ab.$$

Therefore, every element $c \in C$ satisfies $c \leq ab$, so ab is an upper bound for C.

Step 2: Show that ab is the least upper bound.

Let $\varepsilon > 0$. Since $a = \sup A$, there exists $x_\varepsilon \in A$ such that:

$$x_\varepsilon > a - \frac{\varepsilon}{2b}.$$

Similarly, since $b = \sup B$, there exists $y_\varepsilon \in B$ such that:

$$y_\varepsilon > b - \frac{\varepsilon}{2a}.$$

Now consider:

$$x_\varepsilon y_\varepsilon > \left(a - \frac{\varepsilon}{2b}\right)\left(b - \frac{\varepsilon}{2a}\right) = ab - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4ab}.$$

Since $\frac{\varepsilon^2}{4ab} > 0$, we have:

$$x_\varepsilon y_\varepsilon > ab - \varepsilon.$$

Therefore, for every $\varepsilon > 0$, there exists $c \in C$ such that $c > ab - \varepsilon$. Hence, ab is the least upper bound of C.

$$\boxed{\sup C = ab}$$

1.22: Representation of Rationals in Base k

Given $x \geq 0$ and an integer $k \geq 2$, let a_0 denote the largest integer $\leq x$, and, assuming that $a_0, a_1, \ldots, a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leq x.$$

(a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, \ldots$.

(b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + \cdots + a_n k^{-n}$ and show that $x = \sup\{r_n\}$, the supremum of the set of rational numbers $r_1, r_2, \ldots$.

Solution: Let $r_n = \sum_{i=0}^{n} \frac{a_i}{k^i}$. By definition, a_n is the largest integer such that $r_n \leq x$.

(a) Show $0 \leq a_i \leq k - 1$: Since a_n is the largest integer satisfying the condition, choosing $a_n + 1$ would violate it:

$$r_{n-1} + \frac{a_n + 1}{k^n} > x$$

From the definition of a_{n-1}, we know it was the largest integer such that $r_{n-1} \leq x$. This implies $x - r_{n-1} < \frac{1}{k^{n-1}}$. Now, from the definition of a_n, we have $r_{n-1} + \frac{a_n}{k^n} \leq x$, which implies $a_n \leq k^n(x - r_{n-1})$. Combining these facts:

$$a_n \leq k^n(x - r_{n-1}) < k^n \left(\frac{1}{k^{n-1}}\right) = k.$$

Since a_n is an integer and $a_n < k$, we must have $a_n \leq k - 1$. Also, a_n must be non-negative, otherwise we could choose $a_n = 0$ to get a larger (or equal) sum r_n that is still less than or equal to x, contradicting the "largest integer" definition if the original a_n were negative. Thus, $0 \leq a_n \leq k - 1$.

(b) Show that $x = \sup\{r_n\}$: The sequence $\{r_n\}$ is non-decreasing by construction, since $a_n \geq 0$. It is also bounded above by x. Therefore, its supremum exists; let $r = \sup\{r_n\}$. We know $r \leq x$. We will prove $r = x$ by contradiction. Assume $r < x$. Let $\delta = x - r > 0$. By the Archimedean property, we can choose an integer N large enough such that $\frac{1}{k^N} < \delta$. From the definition of a_N, we know $r_N = r_{N-1} + \frac{a_N}{k^N} \leq x$ and $r_{N-1} + \frac{a_N+1}{k^N} > x$. The second inequality rearranges to $x - r_N < \frac{1}{k^N}$. Since $r = \sup\{r_n\}$, we know $r_N \leq r$. Therefore, $x - r \leq x - r_N < \frac{1}{k^N}$. Substituting $\delta = x - r$, we get $\delta < \frac{1}{k^N}$. But we chose N such that $\frac{1}{k^N} < \delta$. This gives $\delta < \frac{1}{k^N} < \delta$, a contradiction. Thus, our assumption must be false, and $x = r = \sup\{r_n\}$.

1.5 Inequalities and Identities

1.23: Lagrange's Identity

Prove Lagrange's identity for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2.$$

Proof: We will prove the equivalent identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \left(\sum_{k=1}^{n} a_k b_k\right)^2 + \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2.$$

Let's expand the terms. The left-hand side (LHS) is:

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{j=1}^{n} b_j^2\right) &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 b_j^2 \\ &= \sum_{k=j} a_k^2 b_j^2 + \sum_{k\neq j} a_k^2 b_j^2 \\ &= \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{1\leq k<j\leq n} (a_k^2 b_j^2 + a_j^2 b_k^2)\end{aligned}$$

Now we expand the terms on the right-hand side (RHS). The first term is:

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{j=1}^{n} a_j b_j\right) &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k b_k a_j b_j \\ &= \sum_{k=j} a_k b_k a_j b_j + \sum_{k\neq j} a_k b_k a_j b_j \\ &= \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{1\leq k<j\leq n} a_k b_k a_j b_j\end{aligned}$$

The second term on the RHS is:

$$\sum_{1\le k<j\le n}(a_kb_j-a_jb_k)^2=\sum_{1\le k<j\le n}(a_k^2b_j^2-2a_kb_ja_jb_k+a_j^2b_k^2)$$

Adding the two terms on the RHS gives:

$$\begin{aligned}\text{RHS}&=\left(\sum_{k=1}^n a_k^2b_k^2+2\sum_{1\le k<j\le n}a_kb_ka_jb_j\right)+\left(\sum_{1\le k<j\le n}(a_k^2b_j^2-2a_ka_jb_kb_j+a_j^2b_k^2)\right)\\&=\sum_{k=1}^n a_k^2b_k^2+\sum_{1\le k<j\le n}(a_k^2b_j^2+a_j^2b_k^2)\end{aligned}$$

The RHS now matches our expansion of the LHS, which completes the proof.

1.24: A Holder-type Inequality

Prove that for arbitrary real numbers a_k, b_k, c_k we have

$$\left(\sum_{k=1}^n a_kb_kc_k\right)^4\le\left(\sum_{k=1}^n a_k^4\right)\left(\sum_{k=1}^n b_k^2\right)^2\left(\sum_{k=1}^n c_k^4\right).$$

Proof: We will prove this inequality by applying the Cauchy-Schwarz inequality twice. First, group the terms as (a_kc_k) and b_k. Applying the Cauchy-Schwarz inequality to the sequences $\{a_kc_k\}$ and $\{b_k\}$ gives:

$$\left(\sum_{k=1}^n (a_kc_k)b_k\right)^2\le\left(\sum_{k=1}^n (a_kc_k)^2\right)\left(\sum_{k=1}^n b_k^2\right)=\left(\sum_{k=1}^n a_k^2c_k^2\right)\left(\sum_{k=1}^n b_k^2\right).$$

Next, we apply the Cauchy-Schwarz inequality to the term $\sum_{k=1}^n a_k^2c_k^2$, treating it as the dot product of sequences $\{a_k^2\}$ and $\{c_k^2\}$:

$$\left(\sum_{k=1}^n a_k^2c_k^2\right)^2\le\left(\sum_{k=1}^n (a_k^2)^2\right)\left(\sum_{k=1}^n (c_k^2)^2\right)=\left(\sum_{k=1}^n a_k^4\right)\left(\sum_{k=1}^n c_k^4\right).$$

This implies:

$$\sum_{k=1}^n a_k^2c_k^2\le\left(\sum_{k=1}^n a_k^4\right)^{1/2}\left(\sum_{k=1}^n c_k^4\right)^{1/2}.$$

Now, substitute this result back into our first inequality:

$$\left(\sum_{k=1}^n a_kb_kc_k\right)^2\le\left(\sum_{k=1}^n a_k^4\right)^{1/2}\left(\sum_{k=1}^n c_k^4\right)^{1/2}\left(\sum_{k=1}^n b_k^2\right).$$

Finally, squaring both sides gives the desired result:

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \le \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} c_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2.$$

1.25: Minkowski's Inequality

Prove Minkowski's inequality:

$$\left(\sum_{k=1}^{n}(a_k+b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

Proof:
Let $A = \left(\sum a_k^2\right)^{1/2}$, $B = \left(\sum b_k^2\right)^{1/2}$, and expand the square:

$$\sum(a_k+b_k)^2 = \sum a_k^2 + 2\sum a_k b_k + \sum b_k^2 = A^2 + 2\sum a_k b_k + B^2.$$

Apply Cauchy–Schwarz:

$$\sum a_k b_k \le AB.$$

Thus,

$$\sum(a_k+b_k)^2 \le A^2 + 2AB + B^2 = (A+B)^2.$$

Taking square roots:

$$\left(\sum(a_k+b_k)^2\right)^{1/2} \le A+B.$$

1.26: Chebyshev's Sum Inequality

If $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le n\sum_{k=1}^{n} a_k b_k.$$

Proof: Consider the double summation

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i-a_j)(b_i-b_j).$$

Since the sequences $\{a_k\}$ and $\{b_k\}$ are sorted in the same order (both non-increasing), the terms $(a_i - a_j)$ and $(b_i - b_j)$ always have the same sign. If $i > j$, then $a_i \le a_j$ and $b_i \le b_j$, so both differences are non-positive. If $i < j$, both are non-negative. Therefore, their product is always non-negative:

$$(a_i - a_j)(b_i - b_j) \ge 0.$$

This implies that the total sum S must be non-negative, $S \ge 0$.

Now, let's expand the sum:

$$\begin{aligned} S &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i - a_ib_j - a_jb_i + a_jb_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_i - \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j - \sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_j \end{aligned}$$

We evaluate each double summation:

- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_i = \sum_{i=1}^{n}(n \cdot a_ib_i) = n\sum_{i=1}^{n} a_ib_i$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j = (\sum_{i=1}^{n} a_i)\left(\sum_{j=1}^{n} b_j\right)$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_i = \left(\sum_{j=1}^{n} a_j\right)(\sum_{i=1}^{n} b_i)$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_j = \sum_{j=1}^{n}(n \cdot a_jb_j) = n\sum_{j=1}^{n} a_jb_j$

Substituting these back into the expression for S:

$$S = n\sum a_kb_k - \left(\sum a_k\right)\left(\sum b_k\right) - \left(\sum a_k\right)\left(\sum b_k\right) + n\sum a_kb_k$$

$$S = 2n\sum_{k=1}^{n} a_kb_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

Since we established that $S \ge 0$:

$$2n\sum_{k=1}^{n} a_kb_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \ge 0$$

Dividing by 2 and rearranging gives the desired inequality:

$$n\sum_{k=1}^{n} a_kb_k \ge \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right).$$

1.27: Express Complex Numbers in $a + bi$ Form

Express the following complex numbers in the form $a + bi$:

(a) $(1 + i)^3$

(b) $\frac{2+3i}{3-4i}$

(c) $i^5 + i^{16}$

(d) $\frac{1}{2}(1 + i)(1 + i^{-8})$

Solution:

(a) $(1 + i)^3 = (1 + i)^2(1 + i) = (2i)(1 + i) = 2i + 2i^2 = 2i - 2 = -2 + 2i$

(b) Rationalize the denominator:

$$\frac{2 + 3i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{(2 + 3i)(3 + 4i)}{9 + 16} = \frac{6 + 8i + 9i + 12i^2}{25} = \frac{-6 + 17i}{25} = -\frac{6}{25} + \frac{17}{25}i$$

(c) $i^5 = i$, since $i^4 = 1$, and $i^{16} = (i^4)^4 = 1$, so:

$$i^5 + i^{16} = i + 1 = 1 + i$$

(d) $\frac{1}{2}(1 + i)(1 + i^{-8})$, note that $i^{-8} = (i^4)^{-2} = 1^{-2} = 1$, so:

$$\frac{1}{2}(1 + i)(1 + 1) = \frac{1}{2}(1 + i)(2) = \frac{1}{2}(2 + 2i) = 1 + i$$

1.28: Solve Complex Equations

In each case, determine all real x and y which satisfy the given relation:

(a) $x + iy = |x - iy|$

(b) $x + iy = (x - iy)^2$

(c) $\sum_{k=0}^{100} i^k = x + iy$

Solution:

(a) RHS is real and nonnegative. LHS is complex. For equality, imaginary part must vanish:

$$\text{Im}(x + iy) = y = 0, \quad \text{and } x = |x| \Rightarrow x \geq 0.$$

So solution: $y = 0$, $x \geq 0$

(b) Compute RHS:

$$(x - iy)^2 = x^2 - 2ixy - y^2 = (x^2 - y^2) - 2ixy.$$

Set equal to $x + iy$, equate real and imaginary parts:

$$x = x^2 - y^2, \quad y = -2xy.$$

From second equation: $y = -2xy \Rightarrow y(1 + 2x) = 0 \Rightarrow y = 0$ or $x = -\frac{1}{2}$

If $y = 0$, then first equation: $x = x^2 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0$ or $x = 1$

If $x = -\frac{1}{2}$, then first equation:

$$x = x^2 - y^2 \Rightarrow -\frac{1}{2} = \frac{1}{4} - y^2 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

So all solutions:

$$(x, y) = (0, 0), (1, 0), \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

(c) The powers of i cycle every 4: $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$

There are 101 terms, which form 25 full cycles and one leftover term $i^{100} \equiv i^0 = 1$

Each full cycle sums to 0. So total sum:

$$\sum_{k=0}^{100} i^k = 25 \cdot 0 + 1 = 1 \Rightarrow x = 1, y = 0.$$

1.29: Basic Identities for Complex Conjugates

If $z = x + iy$, where x and y are real, the complex conjugate of z is $\overline{z} = x - iy$. Prove the following:

a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,

c) $z \cdot \overline{z} = |z|^2$,

d) $z + \overline{z}$ is twice the real part of z,

e) $\frac{z - \overline{z}}{i}$ is twice the imaginary part of z.

Solution: Let $z = x + iy$ and $w = u + iv$ be two complex numbers.

a) **Conjugate of a sum:**

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = \overline{z_1} + \overline{z_2}.$$

b) **Conjugate of a product:**

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1))} = x_1 x_2 - y_1 y_2 - i(x_1 y_2 + x_2 y_1) = \overline{z_1} \cdot \overline{z_2}.$$

c) **Modulus squared:**

$$z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

d) **Twice the real part:**

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\Re(z).$$

e) **Twice the imaginary part:**

$$\frac{z - \overline{z}}{i} = \frac{(x + iy) - (x - iy)}{i} = \frac{2iy}{i} = 2y = 2\Im(z).$$

1.30: Geometric Descriptions of Complex Sets

Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

a) $|z| = 1$,

b) $|z| < 1$,

c) $|z| \leq 1$,

d) $z + \overline{z} = 1$,

e) $z - \overline{z} = i$,

f) $\overline{z} + z = |z|^2$.

Solution:

a) The unit circle centered at the origin.

b) The open unit disk centered at the origin.

c) The closed unit disk centered at the origin.

d) $2\Re(z) = 1 \Rightarrow \Re(z) = \frac{1}{2}$: a vertical line in the complex plane.

e) $2i\Im(z) = i \Rightarrow \Im(z) = \frac{1}{2}$: a horizontal line.

f) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then:

$$z + \overline{z} = (x + iy) + (x - iy) = 2x,$$
$$|z|^2 = x^2 + y^2.$$

So the equation becomes:

$$2x = x^2 + y^2.$$

Rewriting this:

$$x^2 - 2x + y^2 = 0.$$

We now complete the square on the x-terms:

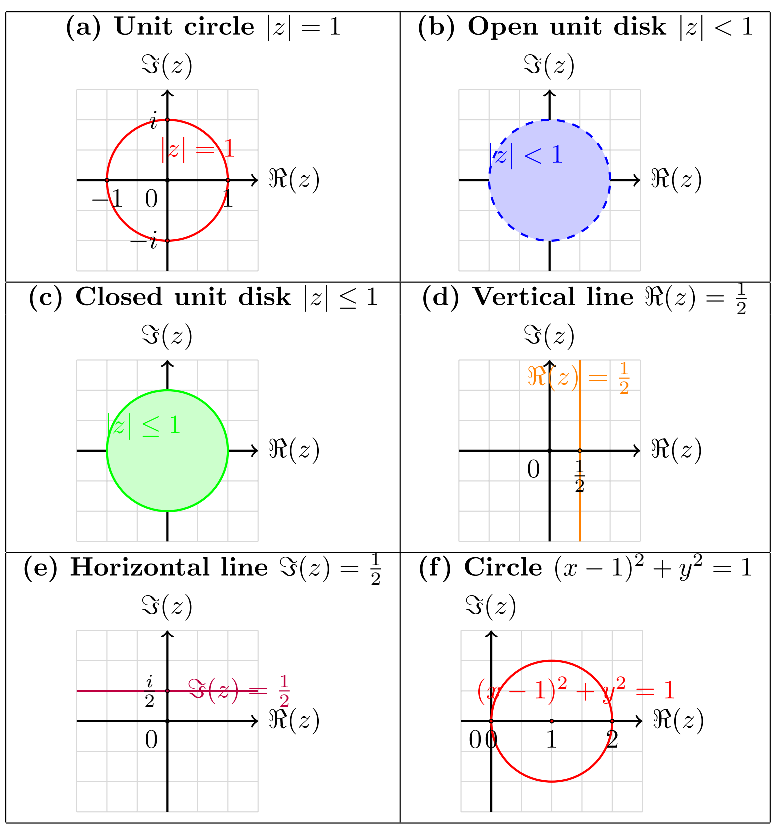
$$x^2 - 2x = (x - 1)^2 - 1,$$

which gives:

$$(x - 1)^2 - 1 + y^2 = 0 \quad \Rightarrow \quad (x - 1)^2 + y^2 = 1.$$

This is the standard equation of a circle with center at $(1, 0)$ and radius 1 in the complex plane.

Visualizations:



1.31: Equilateral Triangle on the Unit Circle

Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1+z_2+z_3 = 0$, show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Solution: Since $|z_i| = 1$, each $z_i = e^{i\theta_i}$ lies on the unit circle. Given $z_1 + z_2 + z_3 = 0$, we need to show they form an equilateral triangle. Consider the angles $\theta_1, \theta_2, \theta_3$. The sum condition implies:

$$e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} = 0.$$

For three points on the unit circle to form an equilateral triangle, their arguments must differ by $120^\circ = \frac{2\pi}{3}$. Assume:

$$z_1 = e^{i\theta}, \quad z_2 = e^{i(\theta+\frac{2\pi}{3})}, \quad z_3 = e^{i(\theta+\frac{4\pi}{3})}.$$

Check the sum:

$$e^{i\theta} + e^{i(\theta+\frac{2\pi}{3})} + e^{i(\theta+\frac{4\pi}{3})} = e^{i\theta}\left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}\right).$$

Since $e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have:

$$1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}} = 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0.$$

The angles $\theta, \theta+\frac{2\pi}{3}, \theta+\frac{4\pi}{3}$ are spaced $\frac{2\pi}{3}$ apart, forming an equilateral triangle. Any three points with $|z_i| = 1$ and sum zero are rotations of the cube roots of unity, ensuring an equilateral triangle.

1.32: Inequality with Complex Numbers

If a and b are complex numbers, prove:

a) $|a-b|^2 \leq (1+|a|^2)(1+|b|^2)$,

b) If $a \neq 0$, then $|a+b| = |a|+|b|$ if and only if $\frac{b}{a}$ is real and nonnegative.

Solution:

a) Compute:

$$|a-b|^2 = (a-b)(\overline{a-b}) = |a|^2 + |b|^2 - a\overline{b} - \overline{a}b.$$

Consider the right-hand side:

$$(1+|a|^2)(1+|b|^2) = 1 + |a|^2 + |b|^2 + |a|^2|b|^2.$$

Evaluate:

$$(1+|a|^2)(1+|b|^2)-|a-b|^2 = 1+|ab|^2+a\bar{b}+\bar{a}b = 1+|ab|^2+2\Re(a\bar{b}).$$

Since $|ab|^2 \geq 0$, $\Re(a\bar{b}) \geq -|ab|$:

$$1+|ab|^2+2\Re(a\bar{b}) \geq 1+|ab|^2-2|ab| = (1-|ab|)^2 \geq 0.$$

Thus, $|a-b|^2 \leq (1+|a|^2)(1+|b|^2)$.

b) For $|a+b| = |a|+|b|$, the triangle inequality requires a, b collinear in the same direction. Let $b = ka$, $k \in \mathbb{R}_{\geq 0}$:

$$|a+b| = |a+ka| = |a|(1+k) = |a|+|b|.$$

Thus, $\frac{b}{a} = k \geq 0$. Conversely, if $|a+b| = |a|+|b|$, then $a\bar{b}+\bar{a}b = 2|a||b|$, so $\frac{b}{a}$ is real and nonnegative.

1.33: Equality Condition for Complex Difference

If a and b are complex numbers, prove that

$$|a-b| = |1-\bar{a}b|$$

if and only if $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a-b| < |1-\bar{a}b|$ valid?

Solution: Let $|a| = r$, $|b| = s$. Compute:

$$|a-b|^2 = r^2+s^2-a\bar{b}-\bar{a}b, \quad |1-\bar{a}b|^2 = 1+r^2s^2-a\bar{b}-\bar{a}b.$$

Thus:

$$|a-b|^2-|1-\bar{a}b|^2 = r^2+s^2-1-r^2s^2 = (r^2-1)(s^2-1).$$

Equality holds when:

$$(r^2-1)(s^2-1) = 0 \implies r = 1 \text{ or } s = 1.$$

For the inequality:

$$(r^2-1)(s^2-1) < 0 \implies (r^2 < 1 \text{ and } s^2 > 1) \text{ or } (r^2 > 1 \text{ and } s^2 < 1).$$

Thus, equality holds if $|a| = 1$ or $|b| = 1$; the inequality holds when one modulus is less than 1 and the other is greater than 1.

1.34: Complex Circle in the Plane

If a and c are real constants, b is complex, show that the equation

$$az\bar{z}+bz+\bar{b}\bar{z}+c = 0 \qquad (a \neq 0, z = x+iy)$$

represents a circle in the xy-plane.

Solution: Let $z = x + iy$, $\overline{z} = x - iy$, then $z\overline{z} = x^2 + y^2$, $bz + \overline{b}\overline{z} = 2\Re(bz)$. Hence the equation becomes:

$$a(x^2 + y^2) + 2\Re(bz) + c = 0.$$

This is the general form of a circle in $\mathbb{R}^2$.

1.35: Argument of a Complex Number via Arctangent

Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ satisfying:

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \text{and} \quad \tan\theta = t.$$

If $z = x + iy$, show that:

a) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$, if $x > 0$,

b) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, if $x < 0$, $y \geq 0$,

c) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$,

d) $\arg(z) = \frac{\pi}{2}$, if $x = 0, y > 0$; $\quad \arg(z) = -\frac{\pi}{2}$, if $x = 0, y < 0$.

Solution: For $z = x+iy$, $\arg(z)$ is the angle $\theta \in (-\pi, \pi]$ such that $z = |z|e^{i\theta}$.

a) If $x > 0$, z is in Quadrant I or IV, and $\tan\theta = \frac{y}{x}$, so $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

b) If $x < 0$, $y \geq 0$, z is in Quadrant II. $\tan^{-1}\left(\frac{y}{x}\right) \in (-\frac{\pi}{2}, 0]$, so add π to get $\theta \in (\frac{\pi}{2}, \pi]$.

c) If $x < 0$, $y < 0$, z is in Quadrant III. $\tan^{-1}\left(\frac{y}{x}\right) \in (0, \frac{\pi}{2}]$, so subtract π to get $\theta \in (-\pi, -\frac{\pi}{2}]$.

d) If $x = 0$, $z = iy$. If $y > 0$, $\theta = \frac{\pi}{2}$; if $y < 0$, $\theta = -\frac{\pi}{2}$.

- **Axiom 6 (Trichotomy):** For any $z_1, z_2 \in \mathbb{C}$, we can compare their moduli. Exactly one of $|z_1| < |z_2|$, $|z_1| > |z_2|$, or $|z_1| = |z_2|$ holds. If $|z_1| = |z_2|$, we compare their principal arguments, for which trichotomy holds on $(-\pi, \pi]$. Thus, exactly one of $z_1 < z_2$, $z_2 < z_1$, or $z_1 = z_2$ is true. This axiom is **satisfied**.

- **Axiom 9 (Transitivity):** If $z_1 < z_2$ and $z_2 < z_3$, the transitivity of the $<$ relation on the real numbers for both the moduli and the arguments ensures that $z_1 < z_3$. This axiom is **satisfied**.

- **Axiom 7 (Translation Invariance):** This axiom states that if $z_1 < z_2$, then $z_1 + z < z_2 + z$ for any $z \in \mathbb{C}$. This axiom is **not satisfied**.

Counterexample: Let $z_1 = 1$ and $z_2 = 2$. According to the ordering, $z_1 < z_2$ because $|z_1| = 1 < |z_2| = 2$. Now, let $z = -2$. Then $z_1 + z = 1+(-2) = -1$. And $z_2+z = 2+(-2) = 0$. We must compare $z_1+z = -1$ and $z_2 + z = 0$. We have $|-1| = 1$ and $|0| = 0$. Since $|0| < |-1|$, we have $0 < -1$ in this pseudo-ordering. So, $z_2 + z < z_1 + z$. The order relation was reversed, which violates the axiom.

- **Axiom 8 (Multiplication):** This axiom states that if $z_1 < z_2$ and $z > 0$, then $z_1z < z_2z$. Let us define $z > 0$ to mean $0 < z$. This holds for any $z \neq 0$. This axiom is also **not satisfied**.

  Counterexample: Let $z_1 = e^{i\pi} = -1$ and $z_2 = e^{-i\pi/2} = -i$. We have $|z_1| = |z_2| = 1$. The arguments are $\arg(z_1) = \pi$ and $\arg(z_2) = -\pi/2$. Since $-\pi/2 < \pi$, we have $z_2 < z_1$. Now, let $z = i$. Since $i \neq 0$, z is a "positive" number under this definition. Then $z_1z = (-1)(i) = -i$. And $z_2z = (-i)(i) = 1$. We must compare $z_1z = -i$ and $z_2z = 1$. We have $|-i| = 1$ and $|1| = 1$. The arguments are $\arg(-i) = -\pi/2$ and $\arg(1) = 0$. Since $-\pi/2 < 0$, we have $-i < 1$. So, $z_1z < z_2z$. The order relation was reversed from $z_2 < z_1$ to $z_1z < z_2z$. The axiom is violated.

Conclusion: Axioms 6 and 9 are satisfied; Axiom 7 and 8 is not applicable.

1.37: Order Axioms and Lexicographic Ordering on $\mathbb{R}^2$

Define a pseudo-ordering on ordered pairs $(x_1, y_1) < (x_2, y_2)$ if either

(i) $x_1 < x_2$, or

(ii) $x_1 = x_2$ and $y_1 < y_2$.

Which of Axioms 6, 7, 8, 9 are satisfied by this relation?

Solution:

- **Axiom 6: Trichotomy.** For any $(x_1, y_1), (x_2, y_2)$, if $x_1 < x_2$, then $(x_1, y_1) < (x_2, y_2)$; if $x_1 > x_2$, then $(x_2, y_2) < (x_1, y_1)$; if $x_1 = x_2$, compare y_1, y_2. Exactly one holds. Satisfied.

- **Axiom 7: Translation Invariance.** If $(x_1, y_1) < (x_2, y_2)$, add (u, v): if $x_1 < x_2$, then $x_1+u < x_2+u$; if $x_1 = x_2$, then $y_1 < y_2 \implies y_1+v < y_2+v$. Satisfied.

- **Axiom 8: Multiplication.** Not applicable, as $\mathbb{R}^2$ lacks scalar multiplication.

- **Axiom 9: Transitivity.** If $(x_1, y_1) < (x_2, y_2)$, $(x_2, y_2) < (x_3, y_3)$, lexicographic order ensures $(x_1, y_1) < (x_3, y_3)$. Satisfied.

Conclusion: Axioms 6, 7, and 9 are satisfied; Axiom 8 is not applicable.

1.38: Argument of a Quotient Using Theorem 1.48

State and prove a theorem analogous to Theorem 1.48, expressing $\arg\left(\frac{z_1}{z_2}\right)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Solution: Theorem: If $z_1, z_2 \neq 0$, then:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2^{-1}),$$

where $n(z_1, z_2^{-1})$ adjusts the argument to $(-\pi, \pi]$.

Proof: Since $\frac{z_1}{z_2} = z_1 z_2^{-1}$, and $\arg(z_2^{-1}) = -\arg(z_2)$, apply Theorem 1.48:

$$\arg(z_1 z_2^{-1}) = \arg(z_1)+\arg(z_2^{-1})+2\pi n(z_1, z_2^{-1}) = \arg(z_1)-\arg(z_2)+2\pi n(z_1, z_2^{-1}).$$

1.39: Logarithm of a Quotient Using Theorem 1.54

State and prove a theorem analogous to Theorem 1.54, expressing $\log\left(\frac{z_1}{z_2}\right)$ in terms of $\log(z_1)$ and $\log(z_2)$.

Solution: Theorem: If $z_1, z_2 \neq 0$, then:

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 + 2\pi i n(z_1, z_2^{-1}).$$

Proof: Since $\frac{z_1}{z_2} = z_1 z_2^{-1}$, apply Theorem 1.54:

$$\log(z_1 z_2^{-1}) = \log z_1 + \log(z_2^{-1}) + 2\pi i n(z_1, z_2^{-1}) = \log z_1 - \log z_2 + 2\pi i n(z_1, z_2^{-1}).$$

1.40: Roots of Unity and Polynomial Identity

Prove that the nth roots of 1 are given by $\alpha, \alpha^2, \ldots, \alpha^n$, where $\alpha = e^{2\pi i/n}$, and that these roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + \cdots + x^{n-1} = 0.$$

Solution: Let $\alpha = e^{2\pi i/n}$. Then $\alpha^n = 1$, so it's a root of $x^n - 1 = 0$. Also,

$$\frac{1-\alpha^n}{1-\alpha} = 0 \Rightarrow 1 + \alpha + \cdots + \alpha^{n-1} = 0 \quad \text{for } \alpha \neq 1.$$

1.41: Inequalities and Boundedness of $\cos z$

a) Prove that $|z^i| < e^\pi$ for all complex $z \neq 0$.

b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Solution:

a) For $z = re^{i\theta}$, $z^i = e^{i(\ln r + i\theta)} = e^{-\theta}e^{i\ln r}$, so $|z^i| = e^{-\theta}$. Since $\theta \in (-\pi, \pi]$, $|z^i| \leq e^\pi$, strict unless $\theta = -\pi$.

b) For $z = iy$, $\cos(iy) = \cosh y$, which is unbounded as $|y| \to \infty$. Thus, no $M > 0$ exists.

1.42: Complex Exponential via Real and Imaginary Parts

If $w = u + iv$, where u and v are real, show that

$$z^w = e^{u \log|z| - v \arg(z)} \cdot e^{i[v \log|z| + u \arg(z)]}.$$

Solution: For $z^w = e^{w \log z}$, where $\log z = \log|z| + i \arg z$:

$$w \log z = (u + iv)(\log|z| + i \arg z) = (u \log|z| - v \arg z) + i(v \log|z| + u \arg z).$$

Thus:

$$z^w = e^{u \log|z| - v \arg z} e^{i(v \log|z| + u \arg z)}.$$

1.43: Logarithmic Identities for Complex Powers

a) Prove that $\log(z^w) = w \log z + 2\pi i n$, where n is an integer.

b) Prove that $(z^w)^\alpha = z^{w\alpha} e^{2\pi i n \alpha}$, where n is an integer.

Solution:

a) Since $z^w = e^{w \log z}$:

$$\log(z^w) = \log(e^{w \log z}) = w \log z + 2\pi i n.$$

b) Compute:

$$(z^w)^\alpha = e^{\alpha \log(z^w)} = e^{\alpha(w \log z + 2\pi i n)} = z^{w\alpha} e^{2\pi i n \alpha}.$$

1.44: Conditions for De Moivre's Formula

i) If θ and a are real numbers, $-\pi < \theta \leq +\pi$, prove that

$$(\cos\theta + i\sin\theta)^a = \cos(a\theta) + i\sin(a\theta).$$

ii) Show that, in general, the restriction $-\pi < \theta \leq +\pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as De Moivre's theorem.

Solution:

i) Since $\cos\theta + i\sin\theta = e^{i\theta}$:

$$(\cos\theta + i\sin\theta)^a = (e^{i\theta})^a = e^{ia\theta} = \cos(a\theta) + i\sin(a\theta).$$

ii) For $\theta = -\pi$, $a = \frac{1}{2}$:

$$(-1)^{1/2} = i, \quad \text{but} \quad \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = -i.$$

The restriction ensures the principal branch.

iii) For integer a, $(e^{i\theta})^a = e^{ia\theta}$, and multiples of 2π cancel, so the formula holds for all θ.

1.45: Deriving Trigonometric Identities from De Moivre's Theore

Use De Moivre's theorem (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta,$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

Solution: By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

Expand:

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Equate parts:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

These hold for complex θ, as $\cos z$ and $\sin z$ are analytic.

1.46: Tangent of Complex Numbers

Define $\tan z = \frac{\sin z}{\cos z}$, and show that for $z = x + iy$,

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

Solution: For $z = x + iy$:

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \quad \cos z = \cos x \cosh y - i \sin x \sinh y.$$

Compute:

$$\tan z = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}.$$

Multiply by the conjugate of the denominator:

$$N = (\sin x \cosh y + i \cos x \sinh y)(\cos x \cosh y + i \sin x \sinh y) = \sin 2x + i \sinh 2y,$$

$$D = (\cos x \cosh y)^2 + (\sin x \sinh y)^2 = \frac{1}{2}(\cos 2x + \cosh 2y).$$

Thus:

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

1.47: Solving Cosine Equation

Let w be a complex number. If $w \neq \pm 1$, show that there exist two values $z = x + iy$ with $\cos z = w$ and $-\pi < x \leq \pi$. Find such z when $w = i$ and $w = 2$.

Solution: For $z = x + iy$, $\cos z = \cos x \cosh y - i \sin x \sinh y = w = u + iv$. Solve:

$$\cos x \cosh y = u, \quad -\sin x \sinh y = v.$$

Square and add:

$$\sin^2 x = \sinh^2 y + 1 - u^2 - v^2.$$

Since $w \neq \pm 1$, solutions exist, with two x in $(-\pi, \pi]$.

Case 1: $w = i$. $u = 0$, $v = 1$:

$$\cos x \cosh y = 0 \implies x = \pm \frac{\pi}{2}.$$

For $x = \frac{\pi}{2}$, $\sinh y = -1 \implies y = -\ln(1 + \sqrt{2})$. For $x = -\frac{\pi}{2}$, $\sinh y = 1 \implies y = \ln(1 + \sqrt{2})$. Solutions: $z_1 = \frac{\pi}{2} - i \ln(1 + \sqrt{2})$, $z_2 = -\frac{\pi}{2} + i \ln(1 + \sqrt{2})$.

Case 2: $w = 2$**.** $u = 2$, $v = 0$:

$$\cos x \cosh y = 2, \quad \sin x \sinh y = 0.$$

Thus, $x = 0$, $\cosh y = 2 \implies y = \pm \ln(2 + \sqrt{3})$. Solutions: $z_1 = i \ln(2 + \sqrt{3})$, $z_2 = -i \ln(2 + \sqrt{3})$.

1.48: Lagrange's Identity and the Cauchy–Schwarz Inequality

Prove Lagrange's identity for complex numbers:

$$\left| \sum_{k=1}^{n} a_k \overline{b_k} \right|^2 = \left(\sum_{k=1}^{n} |a_k|^2 \right) \left(\sum_{k=1}^{n} |b_k|^2 \right) - \sum_{1 \le k < j \le n} |a_k \overline{b_j} - a_j \overline{b_k}|^2.$$

Use this to deduce a Cauchy–Schwarz inequality for complex numbers.

Solution: We want to prove the identity:

$$\left| \sum_{k=1}^{n} a_k \overline{b_k} \right|^2 = \left(\sum_{k=1}^{n} |a_k|^2 \right) \left(\sum_{k=1}^{n} |b_k|^2 \right) - \sum_{1 \le k < j \le n} |a_k \overline{b_j} - a_j \overline{b_k}|^2.$$

It is easier to prove the equivalent formulation:

$$\left(\sum_{k=1}^{n} |a_k|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right) = \left| \sum_{k=1}^{n} a_k \overline{b_k} \right|^2 + \sum_{1 \le k < j \le n} |a_k \overline{b_j} - a_j \overline{b_k}|^2.$$

Let's expand the left-hand side (LHS):

$$\begin{aligned}
\text{LHS} &= \left(\sum_{k=1}^{n} a_k \overline{a_k} \right) \left(\sum_{j=1}^{n} b_j \overline{b_j} \right) = \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \overline{a_k} b_j \overline{b_j} \\
&= \sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{k \ne j} |a_k|^2 |b_j|^2 \\
&= \sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{1 \le k < j \le n} (|a_k|^2 |b_j|^2 + |a_j|^2 |b_k|^2)
\end{aligned}$$

Now, let's expand the right-hand side (RHS). The first term is:

$$\begin{aligned}
\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 &= \left(\sum_{k=1}^{n} a_k \overline{b_k}\right) \overline{\left(\sum_{j=1}^{n} a_j \overline{b_j}\right)} = \left(\sum_{k=1}^{n} a_k \overline{b_k}\right) \left(\sum_{j=1}^{n} \overline{a_j} b_j\right) \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \overline{b_k} \overline{a_j} b_j \\
&= \sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{k \neq j} a_k \overline{b_k} \overline{a_j} b_j \\
&= \sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{1 \leq k < j \leq n} (a_k \overline{b_k} \overline{a_j} b_j + a_j \overline{b_j} \overline{a_k} b_k)
\end{aligned}$$

The second term on the RHS is:

$$\begin{aligned}
\sum_{1 \leq k < j \leq n} |a_k \overline{b_j} - a_j \overline{b_k}|^2 &= \sum_{1 \leq k < j \leq n} (a_k \overline{b_j} - a_j \overline{b_k}) \overline{(a_k \overline{b_j} - a_j \overline{b_k})} \\
&= \sum_{1 \leq k < j \leq n} (a_k \overline{b_j} - a_j \overline{b_k})(\overline{a_k} b_j - \overline{a_j} b_k) \\
&= \sum_{1 \leq k < j \leq n} (a_k \overline{b_j} \overline{a_k} b_j - a_k \overline{b_j} \overline{a_j} b_k - a_j \overline{b_k} \overline{a_k} b_j + a_j \overline{b_k} \overline{a_j} b_k) \\
&= \sum_{1 \leq k < j \leq n} (|a_k|^2 |b_j|^2 - a_k b_k \overline{a_j} \overline{b_j} - \overline{a_k} \overline{b_k} a_j b_j + |a_j|^2 |b_k|^2)
\end{aligned}$$

Adding the two expanded terms of the RHS:

$$\begin{aligned}
\text{RHS} = &\left(\sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{1 \leq k < j \leq n} (a_k \overline{b_k} \overline{a_j} b_j + a_j \overline{b_j} \overline{a_k} b_k)\right) \\
&+ \left(\sum_{1 \leq k < j \leq n} (|a_k|^2 |b_j|^2 - a_k b_k \overline{a_j} \overline{b_j} - \overline{a_k} \overline{b_k} a_j b_j + |a_j|^2 |b_k|^2)\right) \\
= &\sum_{k=1}^{n} |a_k|^2 |b_k|^2 + \sum_{1 \leq k < j \leq n} (|a_k|^2 |b_j|^2 + |a_j|^2 |b_k|^2)
\end{aligned}$$

The cross terms cancel perfectly. Comparing the final expressions for the LHS and RHS, we see they are identical. This proves Lagrange's identity.

To deduce the Cauchy-Schwarz inequality, note that the term $\sum_{1 \leq k < j \leq n} |a_k \overline{b_j} - a_j \overline{b_k}|^2$ is a sum of squares of absolute values, so it must be non-negative. From the original identity, this implies:

$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right) \left(\sum_{k=1}^{n} |b_k|^2\right).$$

1.49: Polynomial Identity via DeMoivre's Theorem

(a) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin(n\theta) = \sin\theta\left(\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - + \cdots\right).$$

(b) If $0 < \theta < \pi/2$, prove that

$$\sin((2m+1)\theta) = \sin^{2m+1}\theta \cdot P_m(\cot^2\theta),$$

where P_m is a polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - + \cdots.$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2\left(\frac{\pi k}{2m+1}\right)$ for $k = 1, 2, \ldots, m$.

(c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m}\cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3},$$

and that the sum of their squares is given by

$$\sum_{k=1}^{m}\cot^4\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

Note. These identities can be used to prove that

$$\sum_{n=1}^{\infty} n^2 = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} n^4 = \frac{\pi^4}{90}.$$

(See Exercises 8.46 and 8.47.)

Solution:

(a) By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta(i\sin\theta)^k.$$

Imaginary part:

$$\sin(n\theta) = \sin\theta \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j+1} \cot^{n-(2j+1)}\theta.$$

(b) For $n = 2m+1$:

$$\sin((2m{+}1)\theta) = \sin^{2m+1}\theta \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \cot^{2(m-j)}\theta = \sin^{2m+1}\theta P_m(\cot^2\theta).$$

Zeros at $\sin((2m+1)\theta) = 0$, i.e., $\theta_k = \frac{\pi k}{2m+1}$, so $x_k = \cot^2\left(\frac{\pi k}{2m+1}\right)$.

(c) Sum of roots:

$$\frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{m(2m-1)}{3}.$$

Sum of squares uses trigonometric identities, yielding:

$$\sum_{k=1}^{m} \cot^4\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

1.50: Product Formula for $\sin$

Prove that

$$z^n - 1 = \prod_{k=1}^{n-1}\left(z - e^{2\pi i k/n}\right)$$

for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \quad \text{for } n \geq 2.$$

Solution: The roots of $z^n - 1 = 0$ are $e^{2\pi i k/n}$, $k = 0, \ldots, n-1$. Excluding $z = 1$:

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - e^{2\pi i k/n}).$$

At $z = 1$, the left-hand side is n, and:

$$|1 - e^{2\pi i k/n}| = 2\sin\left(\frac{\pi k}{n}\right).$$

Thus:

$$n = 2^{n-1}\prod_{k=1}^{n-1}\sin\left(\frac{\pi k}{n}\right) \implies \prod_{k=1}^{n-1}\sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}.$$

Chapter 2

Some Basic Notions of Set Theory

2.1 Ordered Pairs, Relations, and Functions

2.1: Equality of Ordered Pairs

Prove Theorem 2.2: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
Hint: $(a, b) = (c, d)$ means $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Now appeal to the definition of set equality.

Solution: We must prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. The Kuratowski definition of an ordered pair is $(x, y) = \{\{x\}, \{x, y\}\}$. If $a = c$ and $b = d$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} = (c, d)$. This direction is straightforward.

For the other direction, assume $(a, b) = (c, d)$. This means the sets are equal:

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

By the definition of set equality, each element of the first set must be an element of the second, and vice versa. We consider two cases.

Case 1: $a = b$**.** In this case, $(a, a) = \{\{a\}, \{a, a\}\} = \{\{a\}\}$. For the sets to be equal, we must have $\{\{c\}, \{c, d\}\} = \{\{a\}\}$. This implies that the set $\{\{c\}, \{c, d\}\}$ has only one element, which means $\{c\} = \{c, d\}$. This equality holds if and only if $c = d$. So we have $\{\{c\}\} = \{\{a\}\}$, which implies $\{c\} = \{a\}$, and thus $c = a$. Since $c = d$ and $c = a$ and we started with $a = b$, we conclude that $a = b = c = d$. In particular, $a = c$ and $b = d$.

Case 2: $a \neq b$**.** In this case, the set $\{\{a\}, \{a, b\}\}$ contains two distinct elements: the set $\{a\}$ with one member, and the set $\{a, b\}$ with two members. Therefore, the set $\{\{c\}, \{c, d\}\}$ must also contain two distinct elements, which implies $c \neq d$. Since the sets are equal, their elements must match. We have two possibilities:

1. $\{a\} = \{c\}$ and $\{a, b\} = \{c, d\}$. From $\{a\} = \{c\}$, we get $a = c$. Substituting this into the second equality gives $\{a, b\} = \{a, d\}$. Since $a \neq b$, the set on the left has two distinct elements. For the sets to be equal, we must have $b = d$. Thus, $a = c$ and $b = d$.

2. $\{a\} = \{c, d\}$ and $\{a, b\} = \{c\}$. The first equality, $\{a\} = \{c, d\}$, would mean that the set $\{a\}$, which has one element, is equal to the set $\{c, d\}$, which has two elements (since $c \neq d$). This is impossible.

The only possibility is that $a = c$ and $b = d$. In both cases, the equality of ordered pairs implies the equality of their corresponding components

2.2: Properties of Relations

Determine which of the following relations S on $\mathbb{R}^2$ are reflexive, symmetric, and transitive:

(a) $S = \{(x, y) \in \mathbb{R}^2 : x \leq y\}$

(b) $S = \{(x, y) \in \mathbb{R}^2 : x < y\}$

(c) $S = \{(x, y) \in \mathbb{R}^2 : x > y\}$

(d) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$

(e) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 0\}$

(f) $S = \{(x, y) \in \mathbb{R}^2 : x^2 + x \leq y^2 + y\}$

Solution:

(a) **Reflexive:** Yes - for all $x \in \mathbb{R}$, we have $x \leq x$
Symmetric: No - if $x \leq y$ and $x \neq y$, then $y \not\leq x$
Transitive: Yes - if $x \leq y$ and $y \leq z$, then $x \leq z$

(b) **Reflexive:** No - $x < x$ is never true
Symmetric: No - if $x < y$, then $y \not< x$
Transitive: Yes - if $x < y$ and $y < z$, then $x < z$

(c) **Reflexive:** No - $x > x$ is never true
Symmetric: No - if $x > y$, then $y \not> x$
Transitive: Yes - if $x > y$ and $y > z$, then $x > z$

(d) **Reflexive:**] No - the condition $(x, x) \in S$ requires $2x^2 \geq 1$, which fails for any x in the interval $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Symmetric: Yes - if $x^2+y^2 \geq 1$, then $y^2+x^2 \geq 1$ due to the commutative

property of addition. **Transitive:** No - a counterexample is needed. Let $x = 0.6$, $y = 0.9$, and $z = 0.6$.

- $(x, y) \in S$ because $(0.6)^2 + (0.9)^2 = 0.36 + 0.81 = 1.17 \geq 1$.
- $(y, z) \in S$ because $(0.9)^2 + (0.6)^2 = 0.81 + 0.36 = 1.17 \geq 1$.

However, $(x, z) \notin S$ because $(0.6)^2 + (0.6)^2 = 0.36 + 0.36 = 0.72 < 1$.

(e) **Reflexive:** No - $x^2 + x^2 = 2x^2 \geq 0$ for all $x \in \mathbb{R}$
Symmetric: Yes - if $x^2 + y^2 < 0$, then $y^2 + x^2 < 0$
Transitive: Vacuously true - the relation is empty

(f) **Reflexive:** Yes - for all $x \in \mathbb{R}$, $x^2 + x \leq x^2 + x$
Symmetric: No - if $x^2 + x \leq y^2 + y$ and $x \neq y$, then $y^2 + y \not\leq x^2 + x$
Transitive: Yes - if $x^2+x \leq y^2+y$ and $y^2+y \leq z^2+z$, then $x^2+x \leq z^2+z$

2.3: Composition and Inversion of Functions

The following functions F and G are defined for all real x by the equations given below.
Part 1. In each case where the composite function $G \circ F$ can be formed, give the domain of $G \circ F$ and a formula (or formulas) for $(G \circ F)(x)$:

(a) $F(x) = 1 - x, \quad G(x) = x^2 + 2x$

(b) $F(x) = x + 5, \quad G(x) = \frac{|x|}{x}, G(0) = 1$

(c) $F(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 1, & \text{otherwise} \end{cases}, G(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 5, & \text{otherwise} \end{cases}$

Part 2. In the following, find $F(x)$ if $G(x)$ and $G[F(x)]$ are given:

(d) $G(x) = x^3, \quad G[F(x)] = x^3 - 3x^2 + 3x - 1$

(e) $G(x) = 3 + x + x^2, \quad G[F(x)] = x^2 - 3x + 5$

Solution:
(a) The domain of both F and G is $\mathbb{R}$, so the domain of $G \circ F$ is $\mathbb{R}$.

$$\begin{aligned} (G \circ F)(x) &= G(F(x)) \\ &= G(1 - x) \\ &= (1 - x)^2 + 2(1 - x) \\ &= (1 - 2x + x^2) + (2 - 2x) \\ &= x^2 - 4x + 3 \end{aligned}$$

(b) The domain of F is $\mathbb{R}$ and the domain of G is $\mathbb{R}$, so the domain of $G \circ F$ is $\mathbb{R}$.

$$(G \circ F)(x) = G(F(x)) = G(x + 5)$$

We evaluate this based on the value of the input to G, which is $x+5$:

$$(G \circ F)(x) = \begin{cases} \frac{|x+5|}{x+5} = 1, & \text{if } x+5 > 0 \implies x > -5 \\ 1, & \text{if } x+5 = 0 \implies x = -5 \\ \frac{|x+5|}{x+5} = -1, & \text{if } x+5 < 0 \implies x < -5 \end{cases}$$

This simplifies to:

$$(G \circ F)(x) = \begin{cases} -1, & x < -5 \\ 1, & x \geq -5 \end{cases}$$

(c) The domain of $G \circ F$ is $\mathbb{R}$. We analyze the composition in pieces based on the definition of $F(x)$.

- If $0 \leq x \leq 1$, then $F(x) = 2x$. The value of $F(x)$ is in the interval $[0, 2]$. We must check where $F(x)$ falls in the domain of G.
  - If $0 \leq F(x) \leq 1$, which means $0 \leq 2x \leq 1$, or $0 \leq x \leq 0.5$. In this case, $G(F(x)) = 3(F(x))^2 = 3(2x)^2 = 12x^2$.
  - If $F(x) > 1$, which means $2x > 1$, or $0.5 < x \leq 1$. In this case, $G(F(x)) = 5$.
- If $x < 0$ or $x > 1$, then $F(x) = 1$. Since this value is in the interval $[0, 1]$, we use the first rule for G: $G(F(x)) = G(1) = 3(1)^2 = 3$.

Combining these results, we get the piecewise formula:

$$(G \circ F)(x) = \begin{cases} 3, & x < 0 \\ 12x^2, & 0 \leq x \leq 0.5 \\ 5, & 0.5 < x \leq 1 \\ 3, & x > 1 \end{cases}$$

(d) We are given $G(x) = x^3$ and $G[F(x)] = x^3 - 3x^2 + 3x - 1$. The composition is $(F(x))^3$. We can recognize the expression for $G[F(x)]$ as the expansion of a cube:

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3$$

Therefore, we have $(F(x))^3 = (x-1)^3$, which implies $F(x) = x - 1$.

(e) We are given $G(x) = 3 + x + x^2$ and $G[F(x)] = x^2 - 3x + 5$. We set up the equation for the composition:

$$\begin{aligned} G(F(x)) &= 3 + F(x) + (F(x))^2 \\ x^2 - 3x + 5 &= 3 + F(x) + (F(x))^2 \end{aligned}$$

Rearranging gives a quadratic equation in terms of $F(x)$:

$$(F(x))^2 + F(x) + (3 - (x^2 - 3x + 5)) = 0$$

$$(F(x))^2 + F(x) + (-x^2 + 3x - 2) = 0$$

We use the quadratic formula to solve for $F(x)$:

$$\begin{aligned} F(x) &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-x^2 + 3x - 2)}}{2(1)} \\ &= \frac{-1 \pm \sqrt{1 + 4x^2 - 12x + 8}}{2} \\ &= \frac{-1 \pm \sqrt{4x^2 - 12x + 9}}{2} \end{aligned}$$

The term under the square root is a perfect square: $4x^2 - 12x + 9 = (2x-3)^2$.

$$F(x) = \frac{-1 \pm \sqrt{(2x-3)^2}}{2} = \frac{-1 \pm (2x-3)}{2}$$

This yields two possible functions for $F(x)$:

1. $F_1(x) = \frac{-1+(2x-3)}{2} = \frac{2x-4}{2} = x - 2$

2. $F_2(x) = \frac{-1-(2x-3)}{2} = \frac{-2x+2}{2} = 1 - x$

2.4: Associativity of Function Composition

Given three functions F, G, H, what restrictions must be placed on their domains so that the following four composite functions can be defined?

$$G \circ F, \quad H \circ G, \quad H \circ (G \circ F), \quad (H \circ G) \circ F$$

Assuming that $H \circ (G \circ F)$ and $(H \circ G) \circ F$ can be defined, prove the associative law:

$$H \circ (G \circ F) = (H \circ G) \circ F$$

Solution: To define $G \circ F$, the range of F must be contained in the domain of G. To define $H \circ G$, the range of G must be contained in the domain of H. Under these conditions,

$$(H \circ (G \circ F))(x) = H(G(F(x))) = ((H \circ G) \circ F)(x)$$

So function composition is associative wherever defined.

2.2 Set Operations, Images, and Injectivity

2.5: Set-Theoretic Identities

Prove the following set-theoretic identities:

(a) $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

(d) $(A \cup B)(B \cup C)(C \cup A) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$

(e) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(f) $(A - C) \cap (B - C) = (A \cap B) - C$

(g) $(A - B) \cup B = A$ if and only if $B \subseteq A$

Solution: Each identity can be proven by element-chasing: assuming $x \in$ one side and showing $x \in$ the other side. For example, for (b), if $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C \Rightarrow x \in (A \cap B) \cup (A \cap C)$. Similar for the reverse.

2.6: Image of Unions and Intersections

Let $f : S \to T$ be a function. If A and $B \subseteq S$, prove:

$$f(A \cup B) = f(A) \cup f(B), \quad f(A \cap B) \subseteq f(A) \cap f(B)$$

Generalize to arbitrary unions and intersections.

Solution: For any $x \in A \cup B$, $f(x) \in f(A) \cup f(B)$. For intersections, $x \in A \cap B \Rightarrow f(x) \in f(A) \cap f(B)$, but the converse need not hold. Generalization:

$$f\left(\bigcup_i A_i\right) = \bigcup_i f(A_i), \quad f\left(\bigcap_i A_i\right) \subseteq \bigcap_i f(A_i)$$

2.7: Inverse Image Laws

Let $f : S \to T$, and for any $Y \subseteq T$, define the inverse image:

$$f^{-1}(Y) = \{x \in S \mid f(x) \in Y\}$$

Prove:

(a) $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

(b) $f^{-1}(T - Y) = S - f^{-1}(Y)$

Generalize to arbitrary unions and intersections.

Solution: (a) If $x \in f^{-1}(Y_1 \cup Y_2)$, then $f(x) \in Y_1 \cup Y_2 \Rightarrow x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$, and vice versa. (b) $f(x) \notin Y \iff x \notin f^{-1}(Y) \Rightarrow x \in S - f^{-1}(Y)$

2.8: Image of Preimage and Surjectivity

Prove that $f[f^{-1}(Y)] = Y$ for every $Y \subseteq T$ if and only if f is surjective.

Solution: If f is surjective, every $y \in Y$ has a preimage in S, so is included in $f[f^{-1}(Y)]$. If f is not surjective, then some $y \notin f(S)$, and so not in the image of any preimage — thus excluded from $f[f^{-1}(Y)]$.

2.9: Equivalent Conditions for Injectivity

Let $f : S \to T$ be a function. Show the following are equivalent:

(a) f is injective

(b) $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq S$

(c) $f^{-1}[f(A)] = A$ for all $A \subseteq S$

(d) For disjoint sets $A, B \subseteq S$, $f(A) \cap f(B) = \emptyset$

(e) If $B \subseteq A$, then $f(A - B) = f(A) - f(B)$

Solution: Each condition implies the others under the assumption that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. E.g., (c) implies $f^{-1}[f(\{x\})] = \{x\} \Rightarrow$ only one x maps to any $f(x)$.

2.10: Subset Transitivity

Prove that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution: If $x \in A$, then since $A \subseteq B$, we have $x \in B$, and since $B \subseteq C$, we get $x \in C$. Thus, every element of A is in C, so $A \subseteq C$.

2.3 Cardinality and Countability

2.11: Finite Set Bijection Implies Equal Size

If $\{1, 2, \dots, n\} \sim \{1, 2, \dots, m\}$, prove that $m = n$.

Solution: A bijection between two finite sets implies they have the same number of elements. So if such a bijection exists, then $\#\{1, \dots, n\} = n = m = \#\{1, \dots, m\}$, hence $n = m$.

2.12: Infinite Sets Contain Countable Subsets

If S is an infinite set, prove that S contains a countably infinite subset.

Solution: We can construct an injection from $\mathbb{N}$ into S: Select $a_1 \in S$, then pick $a_2 \in S \setminus \{a_1\}$, then $a_3 \in S \setminus \{a_1, a_2\}$, and so on. Since S is infinite, this process never terminates. Thus, $\{a_1, a_2, \dots\} \subseteq S$ is countably infinite.

2.13: Infinite Set Similar to a Proper Subset

Prove that every infinite set S contains a proper subset similar (i.e., bijective) to S itself.

Solution: Let S be an infinite set. By the result of Exercise 2.12, S contains a countably infinite subset. Let this subset be $A = \{a_1, a_2, a_3, \dots\}$. Let $S' = S \setminus A$ be the set of elements in S but not in A. Then $S = A \cup S'$, and this union is disjoint.

We want to find a proper subset $T \subset S$ and a bijection $f : S \to T$. Let's define the proper subset as $T = S \setminus \{a_1\}$. Clearly, T is a proper subset of S because it's missing the element a_1.

Now, we define a function $f : S \to T$ as follows:

- For any element $x \in S'$ (i.e., any element not in our countable subset A), we define $f(x) = x$.
- For any element $a_n \in A$ (where n is a positive integer), we define $f(a_n) = a_{n+1}$.

The domain of f is $S' \cup A = S$. The range of f is $S' \cup \{a_2, a_3, a_4, \dots\}$, which is exactly the set $S \setminus \{a_1\} = T$.

To prove that f is a bijection, we must show it is both injective and surjective.

- **Injectivity:** Suppose $f(x_1) = f(x_2)$.
  - If $f(x_1)$ is in S', then $f(x_1) = x_1$ and $f(x_2) = x_2$, so $x_1 = x_2$.
  - If $f(x_1)$ is in $\{a_2, a_3, \dots\}$, say $f(x_1) = a_{k+1}$, then both x_1 and x_2 must be elements from A. Specifically, $x_1 = a_k$ and $x_2 = a_k$. Thus $x_1 = x_2$.

  In all cases, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

- **Surjectivity:** Let y be any element in the codomain $T = S \setminus \{a_1\}$.
  - If $y \in S'$, then $f(y) = y$.
  - If $y \in \{a_2, a_3, \dots\}$, then $y = a_k$ for some $k \geq 2$. The element $x = a_{k-1}$ is in S and $f(x) = f(a_{k-1}) = a_k = y$.

  Every element in T has a preimage in S.

Since f is a bijection from S to its proper subset T, the set S is similar to a proper subset of itself.

2.14: Removing Countable from Uncountable

If A is a countable set and B an uncountable set, prove that $B - A \sim B$.

Solution: Since A is countable and B is uncountable, $B - A$ is uncountable. Also, $A \cup (B - A) = B$. Define a bijection f from B to $B - A \cup \{a_0\} \subset B$ by remapping countably many points. Thus, $B \sim B - A$.

2.15: Algebraic Numbers are Countable

A real number is called *algebraic* if it is a root of a polynomial with integer coefficients. Prove that the set of all polynomials with integer coefficients is countable, and deduce that the set of algebraic numbers is also countable.

Solution: Each polynomial can be represented by a finite tuple of integers (its coefficients). The set of finite sequences of integers is countable (a countable union of countable sets). Each polynomial has finitely many roots, so the set of all algebraic numbers is a countable union of finite sets $\to$ countable.

2.16: Power Set of Finite Set

Let S be a finite set with n elements, and let T be the collection of all subsets of S. Show that T is finite, and determine how many elements it contains.

Solution: Each element of S may either be in or not in a subset. So the number of subsets is 2^n. Hence $\#T = 2^n$, and T is finite.

2.17: Real Functions vs Real Numbers

Let R be the set of real numbers and S the set of all real-valued functions with domain R. Show that S and R are not equinumerous.

Solution: Assume toward contradiction that there is a bijection $f : R \to S$. Define a function $h(x) = f(x)(x) + 1$. Then $h \in S$, but there is no $x \in R$ such that $f(x) = h$, since $f(x)(x) \neq h(x)$. Contradiction $\to$ no such bijection. Thus, S has strictly greater cardinality than R.

2.18: Binary Sequences are Uncountable

Let S be the set of all infinite sequences of 0s and 1s. Show that S is uncountable.

Solution: Use Cantor's diagonal argument: assume S is countable and list all sequences. Construct a new sequence differing from the n-th sequence at the n-th place. This sequence is not in the list — contradiction. So S is uncountable.

2.19: Countability of Specific Sets

Show that the following sets are countable:

(a) Circles in the complex plane with rational radii and centers with rational coordinates.

(b) Any collection of disjoint intervals of positive length.

Solution: (a) Each circle is determined by a rational radius and two rational coordinates $\to$ set is countable. (b) Each disjoint interval must contain a distinct rational number $\to$ inject into $\mathbb{Q}$, which is countable.

2.20: Countable Support for Real Function

Let f be a real-valued function on $[0, 1]$. Suppose there exists $M > 0$ such that for any finite set of points $\{x_1, \ldots, x_n\} \subset [0, 1]$,

$$|f(x_1)| + \cdots + |f(x_n)| \leq M$$

Let $S = \{x \in [0, 1] \mid f(x) \neq 0\}$. Prove that S is countable.

Solution: Let $S = \{x \in [0, 1] \mid f(x) \neq 0\}$. We want to prove that S is countable. An element x is in S if and only if $|f(x)| > 0$. This is equivalent to saying that for each $x \in S$, there exists a positive integer k such that $|f(x)| > 1/k$.

Let's define a collection of sets based on this idea. For each positive integer k, let:

$$S_k = \left\{ x \in [0,1] \mid |f(x)| > \frac{1}{k} \right\}$$

The set S is the union of all such sets:

$$S = \bigcup_{k=1}^{\infty} S_k$$

If we can prove that each set S_k is finite, then S will be a countable union of finite sets, which is itself a countable set.

Let's consider a specific set S_k. Let $\{x_1, x_2, \ldots, x_n\}$ be any finite collection of distinct points in S_k. By the definition of S_k, we have $|f(x_i)| > 1/k$ for each $i = 1, \ldots, n$. If we sum these values, we get:

$$|f(x_1)| + |f(x_2)| + \cdots + |f(x_n)| > \frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k} = \frac{n}{k}$$

The problem states that for any finite set of points, this sum is bounded by M:

$$|f(x_1)| + |f(x_2)| + \cdots + |f(x_n)| \leq M$$

Combining these inequalities, we get:

$$\frac{n}{k} < \sum_{i=1}^{n} |f(x_i)| \leq M \implies \frac{n}{k} \leq M \implies n \leq kM$$

This result means that any finite subset of S_k can have at most kM elements. This implies that the set S_k itself must be finite and contain at most $\lfloor kM \rfloor$ elements.

Since each S_k is a finite set, their union $S = \bigcup_{k=1}^{\infty} S_k$ is a countable union of finite sets. Therefore, S is a countable set.

2.21: Fallacy in Countability of Intervals

Find the fallacy in the following "proof" that the set of all intervals of positive length is countable: Let $\{x_1, x_2, \ldots\}$ be the rationals. Every interval contains a rational x_n with minimal index n. Assign to the interval the smallest such n. This gives a function from intervals to $\mathbb{N}$, so the set of intervals is countable.

Solution: The function F is not injective — many intervals may have the same smallest-index rational. So this does not establish a one-to-one correspondence between intervals and $\mathbb{N}$. Hence, the proof is invalid.

2.4 Additive Set Functions

2.22: Additive Set Functions

Let S be the collection of all subsets of a given set T. A function $f : S \to \mathbb{R}$ is additive if:

$$f(A \cup B) = f(A) + f(B) \quad \text{whenever } A \cap B = \emptyset$$

Prove:

$$f(A \cup B) = f(A) + f(B - A), \quad f(A \cup B) = f(A) + f(B) - f(A \cap B)$$

Solution: Let $f : S \to \mathbb{R}$ be an additive function, meaning $f(X \cup Y) = f(X) + f(Y)$ whenever $X \cap Y = \emptyset$.

Part 1: Prove $f(A \cup B) = f(A) + f(B - A)$. We can write the set $A \cup B$ as a disjoint union: $A \cup B = A \cup (B - A)$. The sets A and $B - A$ (the part of B not in A) are disjoint by definition. Using the additivity property on this disjoint union:

$$f(A \cup B) = f(A \cup (B - A)) = f(A) + f(B - A)$$

This proves the first identity.

Part 2: Prove $f(A \cup B) = f(A) + f(B) - f(A \cap B)$. We start by decomposing A and B into disjoint pieces. The set A can be written as the disjoint union $A = (A - B) \cup (A \cap B)$. By additivity:

$$f(A) = f(A - B) + f(A \cap B) \implies f(A - B) = f(A) - f(A \cap B)$$

The set B can be written as the disjoint union $B = (B - A) \cup (A \cap B)$. By additivity:

$$f(B) = f(B - A) + f(A \cap B) \implies f(B - A) = f(B) - f(A \cap B)$$

Now, we write $A \cup B$ as a union of three pairwise disjoint sets:

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Using the additivity property:

$$f(A \cup B) = f(A - B) + f(B - A) + f(A \cap B)$$

Substitute the expressions we found for $f(A - B)$ and $f(B - A)$:

$$\begin{aligned} f(A \cup B) &= \big(f(A) - f(A \cap B)\big) + \big(f(B) - f(A \cap B)\big) + f(A \cap B) \\ &= f(A) + f(B) - f(A \cap B) - f(A \cap B) + f(A \cap B) \\ &= f(A) + f(B) - f(A \cap B) \end{aligned}$$

This proves the second identity.

2.23: Solving for Total Measure from Functional Equations

Refer to Exercise 2.22. Assume f is additive and assume also that the following relations hold for two particular subsets A and B of T:

$$f(A \cup B) = f(A') + f(B') - f(A')f(B')$$

$$f(A \cap B) = f(A)f(B), \quad f(A) + f(B) \neq f(T),$$

where $A' = T - A, B' = T - B$. Prove that these relations determine $f(T)$, and compute the value of $f(T)$.

Solution:

We are given that the function f is additive, meaning $f(X \cup Y) = f(X) + f(Y)$ for any disjoint sets X and Y. For any subset $X \subseteq T$, its complement $X' = T - X$ is disjoint from X and their union is T. The additive property therefore implies $f(T) = f(X) + f(X')$, which gives us:

- $f(A') = f(T) - f(A)$
- $f(B') = f(T) - f(B)$

We substitute these into the first given relation:

$$\begin{aligned} f(A \cup B) &= \big(f(T) - f(A)\big) + \big(f(T) - f(B)\big) - \big(f(T) - f(A)\big)\big(f(T) - f(B)\big) \\ &= 2f(T) - f(A) - f(B) - \big[f(T)^2 - f(T)f(A) - f(T)f(B) + f(A)f(B)\big] \\ &= 2f(T) - f(A) - f(B) - f(T)^2 + f(T)f(A) + f(T)f(B) - f(A)f(B) \end{aligned}$$

Next, we use the standard inclusion-exclusion principle for an additive function, which states $f(A \cup B) = f(A) + f(B) - f(A \cap B)$. From the second given relation, we know $f(A \cap B) = f(A)f(B)$. Substituting this gives:

$$f(A \cup B) = f(A) + f(B) - f(A)f(B)$$

Now we set the two expressions for $f(A \cup B)$ equal to each other:

$$f(A)+f(B)-f(A)f(B) = 2f(T)-f(A)-f(B)-f(T)^2+f(T)f(A)+f(T)f(B)-f(A)f(B)$$

The $-f(A)f(B)$ terms on each side cancel. We move all remaining terms to one side to form a quadratic equation in terms of $f(T)$:

$$f(T)^2 - 2f(T) - f(T)f(A) - f(T)f(B) + 2f(A) + 2f(B) = 0$$

Factoring out $f(T)$ and constant terms:

$$f(T)^2 - f(T)\big(2 + f(A) + f(B)\big) + 2\big(f(A) + f(B)\big) = 0$$

This quadratic equation can be factored as:

$$\big(f(T) - 2\big)\big(f(T) - [f(A) + f(B)]\big) = 0$$

This implies two possible solutions for $f(T)$:

1. $f(T) = 2$
2. $f(T) = f(A) + f(B)$

Finally, we use the third given relation, $f(A) + f(B) \neq f(T)$, to eliminate the second possibility. Therefore, the relations uniquely determine the value of $f(T)$. That value is 2.

$$\boxed{f(T) = 2}$$

Chapter 3

Elements of Point Set Topology

3.1 Open and Closed Sets in $\mathbb{R}^1$ and $\mathbb{R}^2$

3.1: Open and Closed Intervals

Prove that an open interval in $\mathbb{R}^1$ is an open set and that a closed interval is a closed set.

Solution: Let (a, b) be an open interval in $\mathbb{R}^1$. To show it's open, we need to prove that every point $x \in (a, b)$ is an interior point. For any $x \in (a, b)$, let $\varepsilon = \min\{x - a, b - x\}$. Then the open ball $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ is contained entirely within (a, b). This shows that every point in (a, b) is an interior point, so (a, b) is open.

For a closed interval $[a, b]$, we need to show its complement $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open. Any point x in this complement is either less than a or greater than b. If $x < a$, let $\varepsilon = a - x$, then $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon) \subset (-\infty, a)$. If $x > b$, let $\varepsilon = x - b$, then $B(x, \varepsilon) \subset (b, \infty)$. This shows the complement is open, so $[a, b]$ is closed.

3.2: Accumulation Points and Set Properties

Determine all the accumulation points of the following sets in $\mathbb{R}^1$ and decide whether the sets are open or closed (or neither).

(a) All integers.

(b) The interval (a, b).

(c) All numbers of the form $1/n$, $\quad (n = 1, 2, 3, \dots)$.

(d) All rational numbers.

(e) All numbers of the form $2^{-n} + 5^{-m}$, $\quad (m, n = 1, 2, \dots)$.

(f) All numbers of the form $(-1)^n + (1/m)$, $\quad (m, n = 1, 2, \dots)$.

(g) All numbers of the form $(1/n) + (1/m)$, $\quad (m, n = 1, 2, \dots)$.

(h) All numbers of the form $(-1)^n/[1 + (1/n)]$, $\quad (n = 1, 2, \dots)$.

Solution: (a) The set of integers has no accumulation points since each integer has a neighborhood containing no other integers. The set is closed (its complement is open) but not open.

(b) The interval (a, b) has accumulation points $[a, b]$. For any $x \in [a, b]$, a sequence $\{x_n\} \subset (a, b)$ with $x_n \to x$ exists (e.g., $x_n = x + (b - a)/(n + 1)$ if $x < b$, or $x_n = a + (b - a)/(n + 1)$ if $x = a$). The set is open (every point is interior) but not closed (its closure is $[a, b]$).

(c) The set $\{1/n : n \in \mathbb{N}\}$ has 0 as its only accumulation point. The set is not closed, because its closure includes 0. It is also not open, as no point in the set has a neighborhood entirely contained within the set. Therefore, the set is neither open nor closed.

(d) The set of rational numbers has all real numbers as accumulation points. The set is neither open nor closed.

(e) The set $\{2^{-n} + 5^{-m} : m, n \in \mathbb{N}\}$ has accumulation points $\{2^{-n} + 5^{-m} : m, n \in \mathbb{N}\} \cup \{2^{-n} : n \in \mathbb{N}\} \cup \{5^{-m} : m \in \mathbb{N}\} \cup \{0\}$. For any $x = 2^{-k} + 5^{-l}$, take $m_n = n + l$, so $2^{-k} + 5^{-m_n} \to 2^{-k}$. Similarly, for $x = 5^{-l}$, take $n_m = m + k$, so $2^{-n_m} + 5^{-l} \to 5^{-l}$. For $x = 0$, take $n = m$, so $2^{-n} + 5^{-n} \to 0$. The set is neither open nor closed (no point is interior; closure includes accumulation points).

(g) The set $\{1/n + 1/m : m, n \in \mathbb{N}\}$ has accumulation points $\{k/n : k, n \in \mathbb{N}, k \leq n\} \cup \{0\}$. For $x = k/n$, take $m_i = i + n$, so $1/n + 1/m_i \to 1/n$; for k/n with $k \geq 2$, set $m = n_i = i$, so $(k - 1)/i + 1/i = k/i \to k/n$. For $x = 0$, take $n = m = i$, so $1/i + 1/i \to 0$. The set is neither open nor closed (no point is interior; closure includes accumulation points).

(h) The set $\{(-1)^n/(1 + 1/n) : n \in \mathbb{N}\}$ has accumulation points $\{-1, 1\}$. The set is neither open nor closed.

3.3: Accumulation Points and Set Properties in $\mathbb{R}^2$

The same as Exercise 3.2 for the following sets in $\mathbb{R}^2$:

(a) All complex z such that $|z| > 1$.

(b) All complex z such that $|z| \geq 1$.

(c) All complex numbers of the form $(1/n)+(i/m)$, $\quad (m, n = 1, 2, \dots)$.

(d) All points (x, y) such that $x^2 - y^2 < 1$.

(e) All points (x, y) such that $x > 0$.

(f) All points (x, y) such that $x \geq 0$.

Solution: (a) The set $\{z \in \mathbb{C} : |z| > 1\}$ has accumulation points $\{z \in \mathbb{C} : |z| \geq 1\}$. The set is open but not closed.

(b) The set $\{z \in \mathbb{C} : |z| \geq 1\}$ has accumulation points $\{z \in \mathbb{C} : |z| \geq 1\}$. The set is closed but not open.

(c) The set $\{(1/n, 1/m) : m, n \in \mathbb{N}\}$ has accumulation points $\{(1/n, 0) : n \in \mathbb{N}\} \cup \{(0, 1/m) : m \in \mathbb{N}\} \cup \{(0, 0)\}$. For $(1/k, 0)$, take $(1/k, 1/m_n)$ with $m_n \to \infty$; for $(0, 1/l)$, take $(1/n_m, 1/l)$ with $n_m \to \infty$; for $(0, 0)$, take $(1/n, 1/n)$. The set is neither open nor closed (no point is interior; closure includes accumulation points).

(d) The set $\{(x, y) : x^2 - y^2 < 1\}$ has accumulation points $\{(x, y) : x^2 - y^2 \leq 1\}$. The set is open but not closed.

(e) The set $\{(x, y) : x > 0\}$ has accumulation points $\{(x, y) : x \geq 0\}$. The set is open but not closed.

(f) The set $\{(x, y) : x \geq 0\}$ has accumulation points $\{(x, y) : x \geq 0\}$. The set is closed but not open.

3.4: Rational and Irrational Elements in Open Sets

Prove that every nonempty open set S in $\mathbb{R}^1$ contains both rational and irrational numbers.

Solution: Let S be a nonempty open set in $\mathbb{R}^1$. Since S is open, for any point $x \in S$, there exists $\varepsilon > 0$ such that the open interval $(x - \varepsilon, x + \varepsilon) \subset S$.

Since the rational numbers are dense in $\mathbb{R}$, there exists a rational number q in $(x - \varepsilon, x + \varepsilon)$, and thus $q \in S$.

Similarly, since the irrational numbers are also dense in $\mathbb{R}$, there exists an irrational number r in $(x - \varepsilon, x + \varepsilon)$, and thus $r \in S$.

Therefore, every nonempty open set contains both rational and irrational numbers.

3.5: Open and Closed Sets in $\mathbb{R}^1$ and $\mathbb{R}^2$

Prove that the only sets in $\mathbb{R}^1$ which are both open and closed are the empty set and $\mathbb{R}^1$ itself. Is a similar statement true for $\mathbb{R}^2$?

Solution: This property is known as **connectedness**. A space is connected if its only subsets that are both open and closed (often called "clopen" sets) are the empty set and the space itself.

Proof for $\mathbb{R}^1$

Let S be a subset of $\mathbb{R}^1$ that is both open and closed. We'll assume, for the sake of contradiction, that S is not empty and $S \neq \mathbb{R}^1$.

Since S is a non-empty proper subset, its complement $S^c = \mathbb{R}^1 \setminus S$ is also non-empty. Because S is closed, its complement S^c is open. Because S is open, its complement S^c is closed. So, S^c is also a non-empty clopen set.

Since both S and S^c are non-empty, we can choose a point $x \in S$ and a point $y \in S^c$. Without loss of generality, let's assume $x < y$.

Now, consider the set $A = S \cap [x, y]$.

- The set A is non-empty because $x \in A$.
- The set A is a subset of the bounded interval $[x, y]$, so it's bounded.
- Since S is closed and $[x, y]$ is closed, their intersection A is also closed.

Because A is a non-empty, closed, and bounded subset of $\mathbb{R}^1$, the least upper bound property guarantees that its supremum exists and belongs to the set. Let $s = \sup A$. Since A is closed, we must have $s \in A$. This implies $s \in S$.

We know $s \leq y$. Since $s \in S$ and $y \in S^c$, the points must be different, so $s < y$.

Since $s \in S$ and S is an open set, there must exist an $\varepsilon > 0$ such that the open interval $(s - \varepsilon, s + \varepsilon)$ is entirely contained in S. We can choose ε to be small enough such that $s + \varepsilon/2 < y$. Consider the point $z = s + \varepsilon/2$.

1. $z \in (s - \varepsilon, s + \varepsilon)$, so $z \in S$.
2. $x \leq s < z < y$, so $z \in [x, y]$.

From these two points, $z \in S \cap [x, y]$, which means $z \in A$. However, $z > s$, which contradicts the fact that s is the supremum (the least upper bound) of A.

This contradiction proves our initial assumption was false. Therefore, the only subsets of $\mathbb{R}^1$ that are both open and closed are $\emptyset$ and $\mathbb{R}^1$.

For $\mathbb{R}^2$

Yes, a similar statement is true for $\mathbb{R}^2$. The only subsets of $\mathbb{R}^2$ that are both open and closed are $\emptyset$ and $\mathbb{R}^2$. This is because $\mathbb{R}^2$ is also a connected space. A common way to prove this is by showing that $\mathbb{R}^2$ is **path-connected** (any two points can be joined by a continuous path), which is a stronger condition that implies connectedness. If $\mathbb{R}^2$ were the union of two disjoint non-empty open sets, any path between a point in one set and a point in the other would have to cross the boundary, leading to a contradiction similar to the one-dimensional case.

3.6: Closed Sets as Intersection of Open Sets

Prove that every closed set in $\mathbb{R}^1$ is the intersection of a countable collection of open sets.

Solution: Let F be a closed set in $\mathbb{R}^1$. For each $n \in \mathbb{N}$, define $G_n = \{x \in \mathbb{R} : d(x, F) < 1/n\}$, where $d(x, F) = \inf\{|x - y| : y \in F\}$. Each G_n is open since it's the union of open intervals.

We claim that $F = \bigcap_{n=1}^{\infty} G_n$. Clearly $F \subset \bigcap_{n=1}^{\infty} G_n$ since every point in F has distance 0 to F.

For the reverse inclusion, let $x \in \bigcap_{n=1}^{\infty} G_n$. Then $d(x, F) < 1/n$ for all n, which means $d(x, F) = 0$. Since F is closed, this implies $x \in F$.

3.7: Structure of Bounded Closed Sets in $\mathbb{R}^1$

Prove that a nonempty, bounded closed set S in $\mathbb{R}^1$ is either a closed interval, or that S can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to S.

Solution: Let S be a nonempty, bounded closed set in $\mathbb{R}^1$. Let $a = \inf S$ and $b = \sup S$. Since S is closed, $a, b \in S$.

If $S = [a, b]$, we're done. Otherwise, the complement $[a, b] \setminus S$ is open and can be written as a countable union of disjoint open intervals (a_i, b_i). Since S is closed, the endpoints a_i, b_i must belong to S.

Therefore, $S = [a, b] \setminus \bigcup_{i=1}^{\infty}(a_i, b_i)$, which is the desired representation.

3.2 Open and Closed Sets in $\mathbb{R}^n$

3.8: Open Balls and Intervals in Rn

Prove that open n-balls and n-dimensional open intervals are open sets in $\mathbb{R}^n$.

Solution: Let $B(a;r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ be an open ball centered at a with radius r. For any $x \in B(a;r)$, let $\varepsilon = r - \|x - a\| > 0$. Then $B(x;\varepsilon) \subset B(a;r)$ by the triangle inequality, showing $B(a;r)$ is open.

For an open interval $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$, let $x = (x_1, \ldots, x_n) \in I$. For each i, let $\varepsilon_i = \min\{x_i - a_i, b_i - x_i\}$. Then the ball $B(x; \min\{\varepsilon_1, \ldots, \varepsilon_n\}) \subset I$, showing I is open.

3.9: Interior of a Set is Open

Prove that the interior of a set in $\mathbb{R}^n$ is open in $\mathbb{R}^n$.

Solution: Let $S \subset \mathbb{R}^n$ and let $x \in \text{int } S$. By definition, there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset S$.

For any $y \in B(x;\varepsilon)$, let $\delta = \varepsilon - \|y - x\| > 0$. Then $B(y;\delta) \subset B(x;\varepsilon) \subset S$, which shows that $y \in \text{int } S$.

Therefore, $B(x;\varepsilon) \subset \text{int } S$, proving that int S is open.

3.10: Interior as Union of Open Subsets

If $S \subseteq \mathbb{R}^n$, prove that int S is the union of all open subsets of $\mathbb{R}^n$ which are contained in S. This is described by saying that int S is the largest open subset of S.

Solution: Let $\mathcal{U}$ be the collection of all open subsets of $\mathbb{R}^n$ contained in S. We need to show that $\text{int } S = \bigcup_{U \in \mathcal{U}} U$.

First, if $x \in \text{int } S$, then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset S$. Since $B(x;\varepsilon)$ is open and contained in S, we have $x \in B(x;\varepsilon) \in \mathcal{U}$, so $x \in \bigcup_{U \in \mathcal{U}} U$.

Conversely, if $x \in \bigcup_{U \in \mathcal{U}} U$, then $x \in U$ for some open set $U \subset S$. Since U is open, there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset U \subset S$, which shows $x \in \text{int } S$.

Therefore, $\text{int } S = \bigcup_{U \in \mathcal{U}} U$, proving that the interior is the largest open subset of S.

3.11: Interior of Intersection and Union

If S and T are subsets of $\mathbb{R}^n$, prove that $\text{int}(S) \cap \text{int}(T) = \text{int}(S \cap T)$, and $\text{int}(S) \cup \text{int}(T) \subseteq \text{int}(S \cup T)$.

Solution: For the first equality, let $x \in \text{int}(S) \cap \text{int}(T)$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B(x; \varepsilon_1) \subset S$ and $B(x; \varepsilon_2) \subset T$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B(x; \varepsilon) \subset S \cap T$, so $x \in \text{int}(S \cap T)$.

Conversely, if $x \in \text{int}(S \cap T)$, there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset S \cap T$. This implies $B(x; \varepsilon) \subset S$ and $B(x; \varepsilon) \subset T$, so $x \in \text{int}(S) \cap \text{int}(T)$.

For the second inclusion, if $x \in \text{int}(S) \cup \text{int}(T)$, then $x \in \text{int}(S)$ or $x \in \text{int}(T)$. In either case, there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset S$ or $B(x; \varepsilon) \subset T$, which implies $B(x; \varepsilon) \subset S \cup T$. Therefore, $x \in \text{int}(S \cup T)$.

3.12: Properties of Derived Set and Closure

Let S' denote the derived set and $\overline{S}$ the closure of a set S in $\mathbb{R}^n$. Prove that:

a) S' is closed in $\mathbb{R}^n$; that is, $\overline{S'} \subseteq S'$.

b) If $S \subseteq T$, then $S' \subseteq T'$.

c) $S' \cup T' = (S \cup T)'$.

d) $\overline{S} = S \cup S'$.

e) $\overline{S}$ is closed in $\mathbb{R}^n$.

f) $\overline{S}$ is the intersection of all closed subsets of $\mathbb{R}^n$ containing S. That is, $\overline{S}$ is the smallest closed set containing S.

Solution:
(a) To prove S' is closed, we must show that its derived set $(S')'$ is a subset of S'. Let $\mathbf{x} \in (S')'$. This means every neighborhood of $\mathbf{x}$ contains a point of S' other than $\mathbf{x}$. Let $B(\mathbf{x}, \varepsilon)$ be an arbitrary open ball centered at $\mathbf{x}$. By definition of $(S')'$, there is a point $\mathbf{y} \in B(\mathbf{x}, \varepsilon) \cap S'$. Since $B(\mathbf{x}, \varepsilon)$ is an open set, it is a neighborhood for $\mathbf{y}$. Because $\mathbf{y} \in S'$, $\mathbf{y}$ is an accumulation point of S, so this neighborhood must contain infinitely many points from S. Thus, the ball $B(\mathbf{x}, \varepsilon)$ contains infinitely many points from S. As $B(\mathbf{x}, \varepsilon)$ was an arbitrary neighborhood of $\mathbf{x}$, this shows that $\mathbf{x}$ is an accumulation point of S, so $\mathbf{x} \in S'$. Therefore, $(S')' \subseteq S'$, which proves that S' is a closed set.

(b) Let $\mathbf{x} \in S'$. Then every neighborhood of $\mathbf{x}$ contains a point $\mathbf{y} \in S$ with $\mathbf{y} \neq \mathbf{x}$. Since $S \subseteq T$, this point $\mathbf{y}$ is also in T. Thus, every neighborhood of $\mathbf{x}$ contains a point $\mathbf{y} \in T$ with $\mathbf{y} \neq \mathbf{x}$. This means $\mathbf{x} \in T'$. So $S' \subseteq T'$.

(c) Using (b), since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $S' \subseteq (S \cup T)'$ and $T' \subseteq (S \cup T)'$. Therefore, $S' \cup T' \subseteq (S \cup T)'$. For the reverse inclusion, let $\mathbf{x} \in (S \cup T)'$. If $\mathbf{x} \notin S'$, then there is a neighborhood of $\mathbf{x}$ that contains no points of S (other than possibly $\mathbf{x}$). But since $\mathbf{x} \in (S \cup T)'$, this neighborhood must contain infinitely many points from $S \cup T$. These points must therefore come from T. This implies $\mathbf{x} \in T'$. So, every point in $(S \cup T)'$ must be in S' or T'. Thus, $(S \cup T)' \subseteq S' \cup T'$.

(d) The closure $\overline{S}$ consists of all points adherent to S. A point $\mathbf{x}$ is adherent to S if every neighborhood of $\mathbf{x}$ intersects S. ($\overline{S} \subseteq S \cup S'$): Let $\mathbf{x} \in \overline{S}$. If $\mathbf{x} \in S$, we are done. If $\mathbf{x} \notin S$, then every neighborhood of $\mathbf{x}$ must contain a point from S, and that point cannot be $\mathbf{x}$. This is the definition of an accumulation point, so $\mathbf{x} \in S'$. Thus $\overline{S} \subseteq S \cup S'$. ($S \cup S' \subseteq \overline{S}$): If $\mathbf{x} \in S$, it is in $\overline{S}$ because every neighborhood contains $\mathbf{x}$. If $\mathbf{x} \in S'$, every neighborhood contains a point of S, so $\mathbf{x}$ is an adherent point. Thus $S' \subseteq \overline{S}$. This gives $S \cup S' \subseteq \overline{S}$.

(e) To prove $\overline{S}$ is closed, we show its derived set $(\overline{S})'$ is a subset of $\overline{S}$. From (d), $\overline{S} = S \cup S'$. Using (c), we get $(\overline{S})' = (S \cup S')' = S' \cup (S')'$. From (a), S' is closed, which means $(S')' \subseteq S'$. Therefore, $(\overline{S})' \subseteq S' \cup S' = S'$. Since $S' \subseteq S \cup S' = \overline{S}$, we have $(\overline{S})' \subseteq \overline{S}$. This proves that $\overline{S}$ is closed.

(f) Let $\mathcal{C}$ be the collection of all closed sets containing S. Let $C_{min} = \bigcap_{F \in \mathcal{C}} F$. ($\overline{S} \subseteq C_{min}$): Let F be any set in $\mathcal{C}$. Then F is closed and $S \subseteq F$. The closure of a set is the smallest closed set containing it, so we must have $\overline{S} \subseteq F$. Since this holds for all $F \in \mathcal{C}$, we have $\overline{S} \subseteq \bigcap_{F \in \mathcal{C}} F = C_{min}$. ($C_{min} \subseteq \overline{S}$): By part (e), $\overline{S}$ is a closed set. It also contains S. Therefore, $\overline{S}$ is one of the sets in the collection $\mathcal{C}$. The intersection of all sets in $\mathcal{C}$ must be a subset of any particular member, so $C_{min} \subseteq \overline{S}$. Thus, $\overline{S} = C_{min}$.

3.13: Closure under Intersection of Sets

Let S and T be subsets of $\mathbb{R}^k$. Prove that $\overline{S \cup T} = \overline{S} \cup \overline{T}$ and that $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$ if S is open.
NOTE. The statements in Exercises 3.9 through 3.13 are true in any metric space.

Solution:

Closure of a Union: $\overline{S \cup T} = \overline{S} \cup \overline{T}$

A concise way to prove this is by using the results from Exercise 3.12:

$$\begin{aligned}
\overline{S \cup T} &= (S \cup T) \cup (S \cup T)' && \text{(by 3.12d)} \\
&= (S \cup T) \cup (S' \cup T') && \text{(by 3.12c)} \\
&= (S \cup S') \cup (T \cup T') && \text{(by associativity and commutativity of } \cup) \\
&= \overline{S} \cup \overline{T} && \text{(by 3.12d)}
\end{aligned}$$

Closure of an Intersection: $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$

Let $\mathbf{x} \in \overline{S \cap T}$. By the definition of closure, this means every neighborhood of $\mathbf{x}$ contains at least one point from $S \cap T$. Let U be any neighborhood of $\mathbf{x}$. Then $U \cap (S \cap T) \neq \emptyset$. Since $S \cap T \subseteq S$, it follows that $U \cap S \neq \emptyset$. As U was an arbitrary neighborhood of $\mathbf{x}$, this implies that $\mathbf{x}$ is an adherent point of S, so $\mathbf{x} \in \overline{S}$. Similarly, since $S \cap T \subseteq T$, it follows that $U \cap T \neq \emptyset$. This implies that $\mathbf{x}$ is an adherent point of T, so $\mathbf{x} \in \overline{T}$. Since $\mathbf{x} \in \overline{S}$ and $\mathbf{x} \in \overline{T}$, we have $\mathbf{x} \in \overline{S} \cap \overline{T}$. Therefore, $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$.

Note: The inclusion can be strict. For example, in $\mathbb{R}^1$, let $S = \mathbb{Q}$ and $T = \mathbb{R} \setminus \mathbb{Q}$. Then $S \cap T = \emptyset$, so $\overline{S \cap T} = \emptyset$. However, $\overline{S} = \mathbb{R}$ and $\overline{T} = \mathbb{R}$, so $\overline{S} \cap \overline{T} = \mathbb{R}$.

3.14: Properties of Convex Sets

A set S in $\mathbb{R}^n$ is called convex if, for every pair of points x and y in S and every real θ satisfying $0 < \theta < 1$, we have $\theta x + (1-\theta)y \in S$. Interpret this statement geometrically (in $\mathbb{R}^2$ and $\mathbb{R}^3$) and prove that:

a) Every n-ball in $\mathbb{R}^n$ is convex.

b) Every n-dimensional open interval is convex.

c) The interior of a convex set is convex.

d) The closure of a convex set is convex.

Solution: Geometrically, a set is convex if the line segment joining any two points in the set lies entirely within the set.

(a) Let $B(a;r)$ be an n-ball and $x, y \in B(a;r)$. For $0 < \theta < 1$, let $z = \theta x + (1-\theta)y$. Then $\|z-a\| = \|\theta(x-a)+(1-\theta)(y-a)\| \leq \theta\|x-a\|+(1-\theta)\|y-a\| < \theta r + (1-\theta)r = r$, so $z \in B(a;r)$.

(b) Let $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be an open interval and $x, y \in I$. For $0 < \theta < 1$, let $z = \theta x + (1-\theta)y$. For each i, we have $a_i < x_i, y_i < b_i$, so $a_i < \theta x_i + (1-\theta)y_i < b_i$. Therefore, $z \in I$.

(c) Let S be convex and $x, y \in \operatorname{int} S$. There exist $\varepsilon_1, \varepsilon_2 > 0$ such that $B(x;\varepsilon_1) \subset S$ and $B(y;\varepsilon_2) \subset S$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. For $0 < \theta < 1$, let $z = \theta x + (1-\theta)y$. If $w \in B(z;\varepsilon)$, then $\|w - z\| < \varepsilon$. Let $u = w - z + x$ and $v = w - z + y$. Then $\|u - x\| = \|v - y\| = \|w - z\| < \varepsilon$, so $u, v \in S$. Since S is convex, $w = \theta u + (1-\theta)v \in S$. Therefore, $B(z;\varepsilon) \subset S$, so $z \in \operatorname{int} S$.

(d) Let S be convex and $x, y \in \overline{S}$. There exist sequences $\{x_n\}, \{y_n\} \subset S$ converging to x, y respectively. For $0 < \theta < 1$, let $z = \theta x + (1-\theta)y$ and $z_n = \theta x_n + (1-\theta)y_n$. Since S is convex, $z_n \in S$ for all n. Since $z_n \to z$, we have $z \in \overline{S}$.

3.15: Accumulation Points of Intersections and Unions

Let $\mathcal{F}$ be a collection of sets in $\mathbb{R}^k$, and let $S = \bigcup_{A\in\mathcal{F}} A$ and $T = \bigcap_{A\in\mathcal{F}} A$. For each of the following statements, either give a proof or exhibit a counterexample:

a) If $\mathbf{x}$ is an accumulation point of T, then $\mathbf{x}$ is an accumulation point of each set A in $\mathcal{F}$.

b) If $\mathbf{x}$ is an accumulation point of S, then $\mathbf{x}$ is an accumulation point of at least one set A in $\mathcal{F}$.

Solution:
(a) This statement is **true**.
Proof: Let $\mathbf{x}$ be an accumulation point of T. This means that for any $\varepsilon > 0$, the ball $B(\mathbf{x};\varepsilon)$ contains a point $\mathbf{y} \in T$ such that $\mathbf{y} \neq \mathbf{x}$. By definition, $T = \bigcap_{A\in\mathcal{F}} A$. So, if $\mathbf{y} \in T$, then $\mathbf{y} \in A$ for every set A in the collection $\mathcal{F}$. Therefore, for any $\varepsilon > 0$, the ball $B(\mathbf{x};\varepsilon)$ contains a point $\mathbf{y} \in A$ (for every $A \in \mathcal{F}$) with $\mathbf{y} \neq \mathbf{x}$. This is precisely the definition of $\mathbf{x}$ being an accumulation point of A. Thus, $\mathbf{x}$ is an accumulation point of each set $A \in \mathcal{F}$.

(b) This statement is **false** for an infinite collection $\mathcal{F}$.
Counterexample: Let the collection of sets in $\mathbb{R}^1$ be $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$ where each set A_n is a singleton: $A_n = \{1/n\}$. The union is the set $S = \bigcup_{n=1}^{\infty} A_n = \{1, 1/2, 1/3, \dots\}$. The set S has exactly one accumulation point: 0, since the sequence of points converges to 0. However, none of the individual sets A_n have any accumulation points, as they each contain only a single isolated point. Thus, 0 is an accumulation point of S, but not of any set A_n in the collection $\mathcal{F}$.

(Note: The statement is true if the collection $\mathcal{F}$ is finite. If $\mathbf{x}$ is an accumulation point of a finite union $S = A_1 \cup \dots \cup A_m$, then any neighborhood of $\mathbf{x}$ contains infinitely many points from S. By the pigeonhole principle, at least one of the sets A_i must contribute infinitely many of these points, making $\mathbf{x}$ an accumulation point of that A_i.)

3.16: Rationals Not a Countable Intersection of Open Sets

Prove that the set S of rational numbers in the interval $(0, 1)$ cannot be expressed as the intersection of a countable collection of open sets.
Hint. Write $S = \{x_1, x_2, \dots\}$, assume $S = \bigcap_{k=1}^{\infty} S_k$, where each S_k is open, and construct a sequence (Q_n) of closed intervals such that $Q_{n+1} \subseteq Q_n \subseteq S_n$ and such that $x_n \notin Q_n$. Then use the Cantor intersection theorem to obtain a contradiction.

Solution: Suppose for contradiction that $S = \bigcap_{k=1}^{\infty} S_k$ where each S_k is open. Let $S = \{x_1, x_2, \dots\}$ be an enumeration of the rationals in $(0, 1)$.

For each n, since S_n is open and contains all rationals in $(0, 1)$, we can find a closed interval $Q_n \subset S_n$ such that $x_n \notin Q_n$. We can arrange that $Q_{n+1} \subseteq Q_n$ by taking $Q_{n+1} = Q_n \cap I_{n+1}$ where I_{n+1} is a closed interval in S_{n+1} that doesn't contain x_{n+1}.

By the Cantor intersection theorem, $\bigcap_{n=1}^{\infty} Q_n$ is nonempty. Let $x \in \bigcap_{n=1}^{\infty} Q_n$. Then $x \in \bigcap_{k=1}^{\infty} S_k = S$, so x is rational. But $x \neq x_n$ for any n since $x_n \notin Q_n$ for each n. This contradicts the fact that S contains all rationals in $(0, 1)$.

3.3 Covering Theorems in $\mathbb{R}^n$

3.17: Countability of Isolated Points

If $S \subseteq \mathbb{R}^n$, prove that the collection of isolated points of S is countable.

Solution:
Let I be the set of isolated points of S. By definition, for each point $\mathbf{x} \in I$, there exists a radius $\varepsilon_{\mathbf{x}} > 0$ such that the open ball $B(\mathbf{x}; \varepsilon_{\mathbf{x}})$ contains no other point of S; that is, $B(\mathbf{x}; \varepsilon_{\mathbf{x}}) \cap S = \{\mathbf{x}\}$.

Consider the collection of smaller open balls $\mathcal{C} = \{B(\mathbf{x}; \varepsilon_{\mathbf{x}}/2) : \mathbf{x} \in I\}$. We claim these balls are pairwise disjoint. To prove this, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two distinct points in I. Suppose their corresponding balls in $\mathcal{C}$ have a point $\mathbf{y}$ in common. Then $d(\mathbf{x}_1, \mathbf{y}) < \varepsilon_{\mathbf{x}_1}/2$ and $d(\mathbf{x}_2, \mathbf{y}) < \varepsilon_{\mathbf{x}_2}/2$. By the triangle inequality:

$$d(\mathbf{x}_1, \mathbf{x}_2) \leq d(\mathbf{x}_1, \mathbf{y}) + d(\mathbf{y}, \mathbf{x}_2) < \frac{\varepsilon_{\mathbf{x}_1}}{2} + \frac{\varepsilon_{\mathbf{x}_2}}{2}$$

Assuming, without loss of generality, that $\varepsilon_{\mathbf{x}_1} \leq \varepsilon_{\mathbf{x}_2}$, we get $d(\mathbf{x}_1, \mathbf{x}_2) < \varepsilon_{\mathbf{x}_2}/2 + \varepsilon_{\mathbf{x}_2}/2 = \varepsilon_{\mathbf{x}_2}$. This implies that $\mathbf{x}_1 \in B(\mathbf{x}_2; \varepsilon_{\mathbf{x}_2})$. But $\mathbf{x}_1 \in S$ and $\mathbf{x}_1 \neq \mathbf{x}_2$. This contradicts the fact that $B(\mathbf{x}_2; \varepsilon_{\mathbf{x}_2})$ contains only one point from S, namely $\mathbf{x}_2$. Therefore, the balls in the collection $\mathcal{C}$ must be pairwise disjoint.

Now we use the fact that $\mathbb{R}^n$ is **separable**, meaning it contains a countable dense subset, such as $\mathbb{Q}^n$ (the set of points with rational coordinates). Since each ball in $\mathcal{C}$ is a non-empty open set, each must contain at least one point from the dense set $\mathbb{Q}^n$. Because the balls in $\mathcal{C}$ are disjoint, each ball must contain a *different* rational point. This allows us to define an injective (one-to-one) function from the set of isolated points I to the countable set $\mathbb{Q}^n$ (by mapping each $\mathbf{x} \in I$ to a rational point in $B(\mathbf{x}; \varepsilon_{\mathbf{x}}/2)$). A set that can be mapped injectively into a countable set must itself be countable. Thus, the set of isolated points I is countable.

3.18: Countable Covering of the First Quadrant

Prove that the set of open disks in the xy-plane with center at (x, x) and radius $x > 0$, where x is rational, is a countable covering of the set $\{(x, y) : x > 0, y > 0\}$.

Solution:
Let $\mathcal{F}$ be the collection of open disks $B((q, q); q)$ where $q \in \mathbb{Q}$ and $q > 0$. Since $\mathbb{Q}$ is countable, the collection $\mathcal{F}$ is countable. We need to show that $\mathcal{F}$ covers the first quadrant $S = \{(x, y) : x > 0, y > 0\}$.

Let (x, y) be an arbitrary point in S. We need to find a rational number $q > 0$ such that the disk $B((q, q); q)$ contains (x, y). The condition for this is:

$$\sqrt{(x - q)^2 + (y - q)^2} < q$$

Since both sides are positive, we can square the inequality:

$$(x - q)^2 + (y - q)^2 < q^2$$

$$x^2 - 2xq + q^2 + y^2 - 2yq + q^2 < q^2$$

$$q^2 - 2(x + y)q + (x^2 + y^2) < 0$$

Let $f(q) = q^2 - 2(x + y)q + (x^2 + y^2)$. We are looking for a rational $q > 0$ that makes this quadratic expression negative. The graph of $z = f(q)$ is an upward-opening parabola. It will be negative between its roots. The roots are found using the quadratic formula:

$$q = \frac{2(x + y) \pm \sqrt{4(x + y)^2 - 4(x^2 + y^2)}}{2} = (x + y) \pm \sqrt{(x + y)^2 - (x^2 + y^2)}$$

$$q = (x + y) \pm \sqrt{2xy}$$

Let the roots be $q_1 = (x+y)-\sqrt{2xy}$ and $q_2 = (x+y)+\sqrt{2xy}$. Since $x, y > 0$, the term $\sqrt{2xy}$ is real and positive, so $q_1 < q_2$. The interval (q_1, q_2) is non-empty. Since the rational numbers are dense in $\mathbb{R}$, we can always find a rational number q in this interval: $q_1 < q < q_2$. For any such q, the inequality $f(q) < 0$ holds.

We must also ensure that we can choose q to be positive. The product of the roots is $q_1 q_2 = x^2 + y^2 > 0$. Since $q_2 = (x + y) + \sqrt{2xy}$ is clearly positive, the other root q_1 must also be positive. Since the interval (q_1, q_2) consists of positive numbers and contains a rational number, we can always find a suitable rational $q > 0$. Thus, for any point (x, y) in the first quadrant, we can find a disk in $\mathcal{F}$ that contains it. The countable collection $\mathcal{F}$ therefore covers the first quadrant.

3.19: Non-Finite Subcover of $0, 1$

The collection $\mathcal{F}$ of open intervals of the form $(1/n, 2/n)$, where $n = 2, 3, \ldots$, is an open covering of the open interval $(0, 1)$. Prove (without using Theorem 3.31) that no finite subcollection of $\mathcal{F}$ covers $(0, 1)$.

Solution: Let $\mathcal{G} = \{(1/n_1, 2/n_1), \ldots, (1/n_k, 2/n_k)\}$ be a finite subcollection of $\mathcal{F}$. Let $N = \max\{n_1, \ldots, n_k\}$.

Then the largest interval in $\mathcal{G}$ is $(1/N, 2/N)$. For any $x \in (0, 1/N)$, we have $x < 1/N < 2/N$, so x is not covered by any interval in $\mathcal{G}$.

Therefore, $\mathcal{G}$ does not cover $(0, 1)$, proving that no finite subcollection of $\mathcal{F}$ covers $(0, 1)$.

3.20: Closed but Not Bounded Set with Infinite Covering

Give an example of a set S which is closed but not bounded and exhibit a countable open covering $\mathcal{F}$ such that no finite subset of $\mathcal{F}$ covers S.

Solution: Let $S = \mathbb{Z}$ (the set of integers). This set is closed but not bounded.

Let $\mathcal{F} = \{(n - 1/2, n + 1/2) : n \in \mathbb{Z}\}$. This is a countable open covering of $\mathbb{Z}$ since each integer n is contained in the interval $(n - 1/2, n + 1/2)$.

However, no finite subcollection of $\mathcal{F}$ covers $\mathbb{Z}$. If $\mathcal{G} = \{(n_1 - 1/2, n_1 + 1/2), \ldots, (n_k - 1/2, n_k + 1/2)\}$ is a finite subcollection, then $\mathcal{G}$ can only cover finitely many integers, but $\mathbb{Z}$ is infinite.

Therefore, $\mathcal{F}$ is a countable open covering of S with no finite subcover.

3.21: Countability via Local Countability

Given a set S in $\mathbb{R}^n$ with the property that for every x in S there is an n-ball $B(x)$ such that $B(x) \cap S$ is countable. Prove that S is countable.

Solution:
For each point $\mathbf{x} \in S$, we are given that there exists an open ball $B_{\mathbf{x}}$ centered at $\mathbf{x}$ such that the set $B_{\mathbf{x}} \cap S$ is countable.

The collection of all such balls, $\mathcal{C} = \{B_{\mathbf{x}} : \mathbf{x} \in S\}$, forms an open covering of the set S (since each $\mathbf{x} \in S$ is in its own ball $B_{\mathbf{x}}$).

The space $\mathbb{R}^n$ is a **separable** metric space because it contains a countable dense subset, $\mathbb{Q}^n$. A key theorem in topology states that every separable metric space has the **Lindelöf property**. This property guarantees that any open covering of a set in that space has a countable subcovering.

Applying the Lindelöf property to our open cover $\mathcal{C}$ of S, we can extract a countable subcollection, say $\mathcal{C}' = \{B_{\mathbf{x}_k} : k \in \mathbb{N}\}$, that still covers S. This means:

$$S \subseteq \bigcup_{k=1}^{\infty} B_{\mathbf{x}_k}$$

From this, we can express the set S as:

$$S = S \cap \left(\bigcup_{k=1}^{\infty} B_{\mathbf{x}_k} \right) = \bigcup_{k=1}^{\infty} (S \cap B_{\mathbf{x}_k})$$

By the initial hypothesis, each set in this union, $(S \cap B_{\mathbf{x}_k})$, is countable. Therefore, S is a countable union of countable sets. A fundamental result of set theory states that a countable union of countable sets is itself countable. Thus, we conclude that the set S must be countable.

3.22: Countability of Disjoint Open Sets

Prove that a collection of disjoint open sets in $\mathbb{R}^n$ is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

Solution: Let $\mathcal{F}$ be a collection of disjoint open sets in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is separable, there exists a countable dense subset D.

For each open set $U \in \mathcal{F}$, there exists a point $d \in D$ such that $d \in U$. Since the sets in $\mathcal{F}$ are disjoint, each point $d \in D$ can belong to at most one set in $\mathcal{F}$.

Therefore, the number of sets in $\mathcal{F}$ is at most the number of points in D, which is countable.

For an example of uncountably many disjoint closed sets, let $\mathcal{G} = \{\{x\} : x \in \mathbb{R}\}$. Each singleton $\{x\}$ is closed, the sets are disjoint, and there are uncountably many real numbers.

3.23: Existence of Condensation Points

Assume that $S \subseteq \mathbb{R}^n$. A point x in $\mathbb{R}^n$ is said to be a condensation point of S if every n-ball $B(x)$ has the property that $B(x) \cap S$ is not countable. Prove that if S is not countable, then there exists a point x in S such that x is a condensation point of S.

Solution: Suppose for contradiction that no point in S is a condensation point of S. Then for every $x \in S$, there exists an n-ball B_x centered at x such that $B_x \cap S$ is countable.

By Exercise 3.21, this implies that S is countable, which contradicts the hypothesis that S is not countable.

Therefore, there must exist at least one point $x \in S$ that is a condensation point of S.

3.24: Properties of Condensation Points

Assume that $S \subseteq \mathbb{R}^n$ and that S is not countable. Let T denote the set of condensation points of S. Prove that:

a) $S - T$ is countable,

b) $S \cap T$ is not countable,

c) T is a closed set,

d) T contains no isolated points.

Note that Exercise 3.23 is a special case of (b).

Solution: (a) For each $x \in S - T$, there exists an n-ball B_x centered at x such that $B_x \cap S$ is countable. By Exercise 3.21, $S - T$ is countable.

(b) Since S is not countable and $S - T$ is countable, $S \cap T$ must be uncountable.

(c) Let $x \in \overline{T}$. Then every neighborhood of x contains a point of T. Let B be any n-ball centered at x. There exists $y \in T \cap B$. Since y is a condensation point, $B(y;r) \cap S$ is uncountable for any $r > 0$. Choose r small enough so that $B(y;r) \subset B$. Then $B \cap S$ contains the uncountable set $B(y;r) \cap S$, so x is a condensation point. Therefore, T is closed.

(d) Let $x \in T$. For any $\varepsilon > 0$, $B(x;\varepsilon) \cap S$ is uncountable. Since $S - T$ is countable, $B(x;\varepsilon) \cap T$ must be uncountable. Therefore, x is not isolated in T.

3.25: Cantor-Bendixon Theorem

A set in $\mathbb{R}^n$ is called perfect if $S = S'$, that is, if S is a closed set which contains no isolated points. Prove that every uncountable closed set F in $\mathbb{R}^n$ can be expressed in the form $F = A \cup B$, where A is perfect and B is countable (Cantor-Bendixon theorem).
Hint. Use Exercise 3.24.

Solution: Let F be an uncountable closed set in $\mathbb{R}^n$. Let T be the set of condensation points of F. By Exercise 3.24, T is closed and $F - T$ is countable.

Let $A = T$ and $B = F - T$. Then $F = A \cup B$ where B is countable.

We need to show that A is perfect. Since T is closed by Exercise 3.24(c), A is closed. By Exercise 3.24(d), T contains no isolated points, so A contains no isolated points.

Therefore, A is perfect, and we have the desired decomposition $F = A \cup B$.

3.4 Metric Spaces

3.26: Open and Closed Sets in Metric Spaces

In any metric space (M, d), prove that the empty set $\emptyset$ and the whole space M are both open and closed.

Solution: The empty set $\emptyset$ is open because the condition "for every point in $\emptyset$, there exists a neighborhood contained in $\emptyset$" is vacuously true (there are no points to check).

The empty set $\emptyset$ is closed because its complement M is open.

The whole space M is open because for any point $x \in M$ and any $\varepsilon > 0$, the ball $B(x; \varepsilon) \subset M$.

The whole space M is closed because its complement $\emptyset$ is open.

3.27: Metric Balls in Different Metrics

Consider the following two metrics in $\mathbb{R}^n$:

$$d_1(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|, \quad d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

In each of the following metric spaces prove that the ball $B(a; r)$ has the geometric appearance indicated:

a) In $(\mathbb{R}^2, d_1)$, a square with sides parallel to the coordinate axes.

b) In $(\mathbb{R}^2, d_2)$, a square with diagonals parallel to the axes.

c) A cube in $(\mathbb{R}^3, d_1)$.

d) An octahedron in $(\mathbb{R}^3, d_2)$.

Solution: (a) In $(\mathbb{R}^2, d_1)$, the ball $B(a; r) = \{(x, y) : \max\{|x - a_1|, |y - a_2|\} < r\}$. This means $|x - a_1| < r$ and $|y - a_2| < r$, which defines a square with center (a_1, a_2) and sides of length $2r$ parallel to the coordinate axes.

(b) In $(\mathbb{R}^2, d_2)$, the ball $B(a; r) = \{(x, y) : |x - a_1| + |y - a_2| < r\}$. This defines a diamond-shaped region (square rotated 45 degrees) with diagonals parallel to the axes.

(c) In $(\mathbb{R}^3, d_1)$, the ball $B(a; r) = \{(x, y, z) : \max\{|x - a_1|, |y - a_2|, |z - a_3|\} < r\}$. This defines a cube with center (a_1, a_2, a_3) and sides of length $2r$ parallel to the coordinate axes.

(d) In $(\mathbb{R}^3, d_2)$, the ball $B(a; r) = \{(x, y, z) : |x - a_1| + |y - a_2| + |z - a_3| < r\}$. This defines an octahedron with center (a_1, a_2, a_3).

3.28: Metric Inequalities

Let d_1 and d_2 be the metrics of Exercise 3.27 and let $\|x-y\|$ denote the usual Euclidean metric. Prove the following inequalities for all x and y in $\mathbb{R}^n$:

$$d_1(x,y) \le \|x-y\| \le d_2(x,y) \quad \text{and} \quad d_2(x,y) \le \sqrt{n}\|x-y\| \le n\, d_1(x,y).$$

Solution: Let $x, y \in \mathbb{R}^n$. Let $a_i = |x_i - y_i|$ for $1 \le i \le n$. Then:
(1) $d_1(x,y) = \max_i a_i \le \sqrt{\sum a_i^2} = \|x-y\|$ (since each $a_i^2 \le \sum a_i^2$)
(2) $\|x-y\| = \sqrt{\sum a_i^2} \le \sum a_i = d_2(x,y)$ (by the inequality $\sqrt{a_1^2 + \cdots + a_n^2} \le a_1 + \cdots + a_n$)
(3) By the Cauchy-Schwarz inequality:

$$\left(\sum a_i\right)^2 \le n \sum a_i^2 \Rightarrow d_2(x,y) \le \sqrt{n}\|x-y\|$$

(4) Also, since $\|x-y\| = \sqrt{\sum a_i^2} \le \sqrt{n \cdot \max_i a_i^2} = \sqrt{n} \cdot \max a_i = \sqrt{n} d_1(x,y)$, it follows that:

$$\sqrt{n}\|x-y\| \le n d_1(x,y)$$

Hence, all inequalities hold:

$$d_1(x,y) \le \|x-y\| \le d_2(x,y), \quad d_2(x,y) \le \sqrt{n}\|x-y\| \le n\, d_1(x,y)$$

3.29: Bounded Metric

If (M,d) is a metric space, define

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Prove that d' is also a metric for M. Note that $0 \le d'(x,y) < 1$ for all x, y in M.

Solution: We need to verify the three properties of a metric:
(1) $d'(x,y) \ge 0$ since $d(x,y) \ge 0$ and $1 + d(x,y) > 0$.
(2) $d'(x,y) = 0$ if and only if $d(x,y) = 0$, which occurs if and only if $x = y$.
(3) $d'(x,y) = d'(y,x)$ since $d(x,y) = d(y,x)$.
(4) For the triangle inequality, let $f(t) = \frac{t}{1+t}$. Then $f'(t) = \frac{1}{(1+t)^2} > 0$, so f is increasing. Therefore, $d'(x,z) = f(d(x,z)) \le f(d(x,y) + d(y,z)) = \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \le \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} = d'(x,y) + d'(y,z)$.
The last inequality follows from the fact that $\frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}$ for $a, b \ge 0$.

3.30: Finite Sets in Metric Spaces

Prove that every finite subset of a metric space is closed.

Solution: Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of a metric space (M, d). We need to show that the complement $M \setminus S$ is open.

Let $x \in M \setminus S$. Let $\varepsilon = \min\{d(x, x_i) : i = 1, 2, \ldots, n\}$. Since $x \notin S$, we have $\varepsilon > 0$.

Then $B(x; \varepsilon) \cap S = \emptyset$, so $B(x; \varepsilon) \subset M \setminus S$. This shows that every point in $M \setminus S$ is an interior point, so $M \setminus S$ is open.

Therefore, S is closed.

3.31: Closed Balls in Metric Spaces

In a metric space (M, d) the closed ball of radius $r > 0$ about a point a in M is the set $\overline{B}(a; r) = \{x : d(x, a) \leq r\}$.

a) Prove that $\overline{B}(a; r)$ is a closed set.

b) Give an example of a metric space in which $\overline{B}(a; r)$ is not the closure of the open ball $B(a; r)$.

Solution: (a) Let $x \in M \setminus \overline{B}(a; r)$. Then $d(x, a) > r$. Let $\varepsilon = d(x, a) - r > 0$. For any $y \in B(x; \varepsilon)$, we have $d(y, a) \geq d(x, a) - d(x, y) > d(x, a) - \varepsilon = r$. Therefore, $B(x; \varepsilon) \subset M \setminus \overline{B}(a; r)$, showing that $M \setminus \overline{B}(a; r)$ is open. Hence, $\overline{B}(a; r)$ is closed.

(b) Consider the discrete metric space (M, d) where $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Let $a \in M$ and $r = 1$. Then $B(a; 1) = \{a\}$ and $\overline{B}(a; 1) = M$. The closure of $B(a; 1)$ is $\{a\}$, which is not equal to $\overline{B}(a; 1) = M$.

3.32: Transitivity of Density

In a metric space M, if subsets satisfy $A \subseteq S \subseteq \overline{A}$, where $\overline{A}$ is the closure of A, then A is said to be dense in S. For example, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$. If A is dense in S and if S is dense in T, prove that A is dense in T.

Solution: We need to show that $A \subseteq T \subseteq \overline{A}$.

Since $A \subseteq S \subseteq T$, we have $A \subseteq T$.

Since S is dense in T, we have $T \subseteq \overline{S}$. Since A is dense in S, we have $S \subseteq \overline{A}$. Therefore, $\overline{S} \subseteq \overline{\overline{A}} = \overline{A}$.

Combining these, we get $T \subseteq \overline{S} \subseteq \overline{A}$, so $T \subseteq \overline{A}$.

Therefore, $A \subseteq T \subseteq \overline{A}$, showing that A is dense in T.

3.33: Separability of Euclidean Spaces

A metric space M is said to be separable if there is a countable subset A which is dense in M. For example, $\mathbb{R}$ is separable because the set $\mathbb{Q}$ of rational numbers is a countable dense subset. Prove that every Euclidean space $\mathbb{R}^k$ is separable.

Solution: Let A be the set of all points in $\mathbb{R}^k$ with rational coordinates. That is, $A = \{(q_1, q_2, \ldots, q_k) : q_i \in \mathbb{Q} \text{ for } i = 1, 2, \ldots, k\}$.

Since $\mathbb{Q}$ is countable, the Cartesian product $A = \mathbb{Q}^k$ is countable.

To show that A is dense in $\mathbb{R}^k$, let $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$ and $\varepsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each i there exists $q_i \in \mathbb{Q}$ such that $|x_i - q_i| < \varepsilon/\sqrt{k}$.

Then $q = (q_1, q_2, \ldots, q_k) \in A$ and $\|x-q\| = \sqrt{\sum_{i=1}^k (x_i - q_i)^2} < \sqrt{k(\varepsilon/\sqrt{k})^2} = \varepsilon$.

Therefore, A is a countable dense subset of $\mathbb{R}^k$, so $\mathbb{R}^k$ is separable.

3.34: Lindelöf Theorem in Separable Spaces

Prove that the Lindelöf covering theorem (Theorem 3.28) is valid in any separable metric space.

Solution: Let M be a separable metric space with countable dense subset $D = \{d_1, d_2, \ldots\}$. Let $\mathcal{F}$ be an open covering of M.

For each $d_i \in D$ and each positive rational r, if there exists a set $F \in \mathcal{F}$ such that $B(d_i; r) \subset F$, let $F_{i,r}$ be one such set.

The collection $\{F_{i,r} : i \in \mathbb{N}, r \in \mathbb{Q}^+, B(d_i; r) \subset F_{i,r} \text{ for some } F \in \mathcal{F}\}$ is countable.

We claim this collection covers M. Let $x \in M$. Since $\mathcal{F}$ covers M, there exists $F \in \mathcal{F}$ such that $x \in F$. Since F is open, there exists $\varepsilon > 0$ such that $B(x; \varepsilon) \subset F$.

Since D is dense, there exists $d_i \in D$ such that $d_i \in B(x; \varepsilon/2)$. Let r be a rational number such that $d(x, d_i) < r < \varepsilon/2$. Then $B(d_i; r) \subset B(x; \varepsilon) \subset F$.

Therefore, $F_{i,r}$ exists and contains x, showing that the countable subcollection covers M.

3.35: Density and Open Sets

If A is dense in S and if B is open in S, prove that $B \subseteq \overline{A \cap B}$.
Hint. Exercise 3.13.

Solution:
The statement "A is dense in S" means $S \subseteq \overline{A}$. We are given that $A \subseteq S$. The statement "B is open in S" means that $B = V \cap S$ for some set V that is open in the larger metric space M.

Let $x \in B$. We want to show that $x \in \overline{A \cap B}$. This requires showing that any open neighborhood of x in M has a non-empty intersection with the set $A \cap B$.

Let U be an arbitrary open neighborhood of x in M. Since $x \in B$ and $B = V \cap S$, we have $x \in V$. The set $U \cap V$ is also an open neighborhood of x because it is the intersection of two open sets.

Since $x \in B \subseteq S$ and A is dense in S, x is an adherent point of A. Therefore, the open neighborhood $U \cap V$ must contain a point from A. Let's call this point y. So, $y \in (U \cap V) \cap A$.

Now we check if this point y is in the required sets:

- $y \in U$, so y is in the arbitrary neighborhood of x.
- $y \in A$.
- We need to show $y \in B$. We know $y \in V$. Since we are given $A \subseteq S$, and $y \in A$, it follows that $y \in S$.

Since $y \in V$ and $y \in S$, we have $y \in V \cap S$, which means $y \in B$.

So we have found a point y such that $y \in U$ and $y \in A \cap B$. This means $U \cap (A \cap B) \neq \emptyset$. Since U was an arbitrary open neighborhood of x, this proves that $x \in \overline{A \cap B}$. As this holds for any $x \in B$, we conclude that $B \subseteq \overline{A \cap B}$.

3.36: Intersection of Dense and Open Sets

If each of A and B is dense in S and if B is open in S, prove that $A \cap B$ is dense in S.

Solution: We need to show that $S \subseteq \overline{A \cap B}$.

Let $x \in S$. Since B is open in S, there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \cap S \subset B$.

Since A is dense in S, $B(x;\varepsilon) \cap A \neq \emptyset$. Let $y \in B(x;\varepsilon) \cap A$. Since $y \in S$ and $B(x;\varepsilon) \cap S \subset B$, we have $y \in B$.

Therefore, $y \in A \cap B$, so $B(x;\varepsilon) \cap (A \cap B) \neq \emptyset$.

This shows that every neighborhood of x contains a point of $A \cap B$, so $x \in \overline{A \cap B}$.

Therefore, $S \subseteq \overline{A \cap B}$, showing that $A \cap B$ is dense in S.

3.37: Product Metrics

Given two metric spaces (S_1, d_1) and (S_2, d_2), a metric ρ for the Cartesian product $S_1 \times S_2$ can be constructed from d_1 and d_2 in many ways. For example, if $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $S_1 \times S_2$, let $\rho(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$. Prove that ρ is a metric for $S_1 \times S_2$ and construct further examples.

Solution: We need to verify the three properties of a metric for $\rho(x, y) = d_1(x_1, y_1) + d_2(x_2, y_2)$:

(1) $\rho(x, y) \geq 0$ since $d_1(x_1, y_1) \geq 0$ and $d_2(x_2, y_2) \geq 0$.

(2) $\rho(x, y) = 0$ if and only if $d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$, which occurs if and only if $x_1 = y_1$ and $x_2 = y_2$, i.e., $x = y$.

(3) $\rho(x, y) = \rho(y, x)$ since $d_1(x_1, y_1) = d_1(y_1, x_1)$ and $d_2(x_2, y_2) = d_2(y_2, x_2)$.

(4) For the triangle inequality, let $z = (z_1, z_2)$. Then $\rho(x, z) = d_1(x_1, z_1) + d_2(x_2, z_2) \leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) = \rho(x, y) + \rho(y, z)$.

Other examples of product metrics include: - $\rho(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ - $\rho(x, y) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}$ - $\rho(x, y) = (d_1(x_1, y_1)^p + d_2(x_2, y_2)^p)^{1/p}$ for $p \geq 1$

3.5 Compact subsets of a metric space

Prove each of the following statements concerning an arbitrary metric space (M, d) and subsets S, T of M.

3.38: Relative Compactness

Assume $S \subseteq T \subseteq M$. Then S is compact in (M, d) if, and only if, S is compact in the metric subspace (T, d).

Solution: Suppose S is compact in (M, d). Let $\mathcal{F}$ be an open covering of S in the subspace (T, d). Then each $F \in \mathcal{F}$ is of the form $F = U \cap T$ where U is open in (M, d).

The collection $\{U : U$ is open in (M, d) and $U \cap T \in \mathcal{F}\}$ is an open covering of S in (M, d). Since S is compact in (M, d), there exists a finite subcollection $\{U_1, \ldots, U_n\}$ that covers S.

Then $\{U_1 \cap T, \ldots, U_n \cap T\}$ is a finite subcollection of $\mathcal{F}$ that covers S, showing that S is compact in (T, d).

Conversely, suppose S is compact in (T, d). Let $\mathcal{G}$ be an open covering of S in (M, d). Then $\{G \cap T : G \in \mathcal{G}\}$ is an open covering of S in (T, d). Since S is compact in (T, d), there exists a finite subcollection $\{G_1 \cap T, \ldots, G_n \cap T\}$ that covers S.

Then $\{G_1, \ldots, G_n\}$ is a finite subcollection of $\mathcal{G}$ that covers S, showing that S is compact in (M, d).

3.39: Intersection with Compact Sets

If S is closed and T is compact, then $S \cap T$ is compact.

Solution: Since T is compact, it is closed. Therefore, $S \cap T$ is the intersection of two closed sets, so it is closed.

Since $S \cap T \subseteq T$ and T is compact, by Exercise 3.38, $S \cap T$ is compact in (T, d). Since compactness is independent of the ambient space, $S \cap T$ is compact in (M, d).

3.40: Intersection of Compact Sets

The intersection of an arbitrary collection of compact subsets of M is compact.

Solution: Let $\{K_\alpha\}$ be a collection of compact subsets of M. Since each K_α is closed, the intersection $\bigcap K_\alpha$ is closed.

Let K_1 be any member of the collection. Then $\bigcap K_\alpha \subseteq K_1$ and K_1 is compact. Since $\bigcap K_\alpha$ is closed and contained in a compact set, by Exercise 3.39, $\bigcap K_\alpha$ is compact.

3.41: Finite Union of Compact Sets

The union of a finite number of compact subsets of M is compact.

Solution: Let $K_1, K_2, \ldots, K_n$ be compact subsets of M. Since each K_i is closed, their union $\bigcup_{i=1}^n K_i$ is closed.

Let $\mathcal{F}$ be an open covering of $\bigcup_{i=1}^n K_i$. Then $\mathcal{F}$ is also an open covering of each K_i. Since each K_i is compact, there exists a finite subcollection $\mathcal{F}_i$ of $\mathcal{F}$ that covers K_i.

Then $\bigcup_{i=1}^n \mathcal{F}_i$ is a finite subcollection of $\mathcal{F}$ that covers $\bigcup_{i=1}^n K_i$.

Since $\bigcup_{i=1}^n K_i$ is closed and every open covering has a finite subcover, it is compact.

3.42: Non-Compact Closed and Bounded Set

Consider the metric space $\mathbb{Q}$ of rational numbers with the Euclidean metric of $\mathbb{R}$. Let S consist of all rational numbers in the open interval (a, b), where a and b are irrational. Then S is a closed and bounded subset of $\mathbb{Q}$ which is not compact.

Solution: Let $S = \mathbb{Q} \cap (a, b)$ where a, b are irrational numbers.

S is bounded since it is contained in the bounded interval (a, b).

S is closed in $\mathbb{Q}$ because its complement $\mathbb{Q} \setminus S = \mathbb{Q} \cap ((-\infty, a] \cup [b, \infty))$ is open in $\mathbb{Q}$.

However, S is not compact. Let $\{q_n\}$ be a sequence of rational numbers in (a, b) that converges to a (which exists since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then $\{q_n\}$ is a sequence in S that has no convergent subsequence in S (since $a \notin S$).

Therefore, S is closed and bounded but not compact.

Miscellaneous Properties of Interior and Boundary

The following problems involve arbitrary subsets A and B of a metric space M.

3.6 Miscellaneous Properties of Interior and Boundary

If A and B are subsets of a metric space M, prove that:

3.43: Interior via Closure

Prove that int $A = M - \overline{M - A}$.

Solution: Let $x \in \text{int } A$. Then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset A$. This means $B(x;\varepsilon) \cap (M - A) = \emptyset$, so $x \notin \overline{M - A}$. Therefore, $x \in M - \overline{M - A}$.

Conversely, let $x \in M - \overline{M - A}$. Then $x \notin \overline{M - A}$, so there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \cap (M - A) = \emptyset$. This means $B(x;\varepsilon) \subset A$, so $x \in \text{int } A$.

3.44: Interior of Complement

Prove that int $(M - A) = M - \overline{A}$.

Solution: Let $x \in \text{int } (M - A)$. Then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset M - A$. This means $B(x;\varepsilon) \cap A = \emptyset$, so $x \notin \overline{A}$. Therefore, $x \in M - \overline{A}$.

Conversely, let $x \in M - \overline{A}$. Then $x \notin \overline{A}$, so there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \cap A = \emptyset$. This means $B(x;\varepsilon) \subset M - A$, so $x \in \text{int } (M - A)$.

3.45: Idempotence of Interior

Prove that int (int A) = int A.

Solution: Since int $A \subseteq A$, we have int (int A) $\subseteq$ int A.

For the reverse inclusion, let $x \in \text{int } A$. Then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset A$. Since $B(x;\varepsilon)$ is open and contained in A, we have $B(x;\varepsilon) \subset \text{int } A$. Therefore, $x \in \text{int (int } A)$.

3.46: Interior of Intersections

a) Prove that int $\left(\bigcap_{i=1}^n A_i\right) = \bigcap_{i=1}^n(\text{int } A_i)$, where each $A_i \subseteq M$.

b) Show that int $\left(\bigcap_{A\in F} A\right) \subseteq \bigcap_{A\in F}(\text{int } A)$ if F is an infinite collection of subsets of M.

c) Give an example where equality does not hold in (b).

Solution: (a) Let $x \in \text{int } (\bigcap_{i=1}^n A_i)$. Then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset \bigcap_{i=1}^n A_i$. This means $B(x;\varepsilon) \subset A_i$ for each i, so $x \in \text{int } A_i$ for each i. Therefore, $x \in \bigcap_{i=1}^n(\text{int } A_i)$.

Conversely, let $x \in \bigcap_{i=1}^n(\text{int } A_i)$. Then for each i, there exists $\varepsilon_i > 0$ such that $B(x;\varepsilon_i) \subset A_i$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $B(x;\varepsilon) \subset \bigcap_{i=1}^n A_i$, so $x \in \text{int } (\bigcap_{i=1}^n A_i)$.

(b) Let $x \in \text{int } (\bigcap_{A\in F} A)$. Then there exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset \bigcap_{A\in F} A$. This means $B(x;\varepsilon) \subset A$ for each $A \in F$, so $x \in \text{int } A$ for each $A \in F$. Therefore, $x \in \bigcap_{A\in F}(\text{int } A)$.

(c) Let $F = \{A_n : n \in \mathbb{N}\}$ where $A_n = (-1/n, 1/n)$. Then $\bigcap_{A\in F} A = \{0\}$, so $\text{int } (\bigcap_{A\in F} A) = \emptyset$. However, $\text{int } A_n = A_n$ for each n, so $\bigcap_{A\in F}(\text{int } A) = \bigcap_{n=1}^\infty A_n = \{0\}$.

3.47: Interior of Unions

a) Prove that $\bigcup_{A\in F}(\text{int } A) \subseteq \text{int } \left(\bigcup_{A\in F} A\right)$.

b) Give an example of a finite collection F in which equality does not hold in (a).

Solution: (a) Let $x \in \bigcup_{A\in F}(\text{int } A)$. Then $x \in \text{int } A$ for some $A \in F$. There exists $\varepsilon > 0$ such that $B(x;\varepsilon) \subset A \subset \bigcup_{A\in F} A$. Therefore, $x \in \text{int } (\bigcup_{A\in F} A)$.

(b) Let $F = \{A, B\}$ where $A = [0, 1]$ and $B = [1, 2]$. Then $\text{int } A = (0, 1)$ and $\text{int } B = (1, 2)$, so $\bigcup_{A\in F}(\text{int } A) = (0, 1) \cup (1, 2)$. However, $\bigcup_{A\in F} A = [0, 2]$, so $\text{int } (\bigcup_{A\in F} A) = (0, 2)$, which properly contains $(0, 1) \cup (1, 2)$.

3.48: Interior of Boundary

a) Prove that $\text{int } (\partial A) = \emptyset$ if A is open or if A is closed in M.

b) Give an example in which $\text{int } (\partial A) = M$.

Solution: (a) If A is open, then $\partial A = \overline{A} \setminus \text{int } A = \overline{A} \setminus A$. If A is closed, then $\partial A = A \setminus \text{int } A$.

In both cases, ∂A contains no open balls, so $\text{int } (\partial A) = \emptyset$.

(b) Let $A = \mathbb{Q}$ in the metric space $\mathbb{R}$. Then $\partial A = \mathbb{R}$, so $\text{int } (\partial A) = \mathbb{R} = M$.

3.49: Interior of Union of Sets with Empty Interior

If $\text{int } A = \text{int } B = \emptyset$ and if A is closed in M, then $\text{int } (A \cup B) = \emptyset$.

Solution: Since A is closed, $\text{int } A = \emptyset$ implies that A has no isolated points. Therefore, every point in A is a limit point of A.

Let $x \in A \cup B$. If $x \in A$, then every neighborhood of x contains points of A different from x. Since $A \subset A \cup B$, every neighborhood of x contains points of $A \cup B$ different from x, so x is not an interior point of $A \cup B$.

If $x \in B \setminus A$, then since $\text{int } B = \emptyset$, every neighborhood of x contains points not in B. Since A is closed and $x \notin A$, there exists a neighborhood of x that doesn't intersect A. This neighborhood contains points not in $A \cup B$, so x is not an interior point of $A \cup B$.

Therefore, $\text{int } (A \cup B) = \emptyset$.

3.50: Counterexample for Union of Sets with Empty Interior

Give an example in which $\text{int } A = \text{int } B = \emptyset$ but $\text{int } (A \cup B) = M$.

Solution: Let $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ in the metric space $\mathbb{R}$. Then $\text{int } A = \emptyset$ and $\text{int } B = \emptyset$, but $A \cup B = \mathbb{R}$, so $\text{int } (A \cup B) = \mathbb{R} = M$.

3.51: Properties of Boundary

Prove that:

$$\partial A = \overline{A} \cap \overline{M - A} \quad \text{and} \quad \partial A = \partial(M - A).$$

Solution: For the first equality, $x \in \partial A$ if and only if every neighborhood of x contains both points of A and points of $M - A$. This means $x \in \overline{A}$ and $x \in \overline{M - A}$, so $x \in \overline{A} \cap \overline{M - A}$.

For the second equality, $\partial A = \overline{A} \cap \overline{M - A} = \overline{M - A} \cap \overline{A} = \partial(M - A)$.

3.52: Boundary of Union under Disjoint Closures

If $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial A \cup \partial B$.

Solution: Since $\overline{A} \cap \overline{B} = \emptyset$, we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Let $x \in \partial(A \cup B)$. Then $x \in \overline{A \cup B} = \overline{A} \cup \overline{B}$ and $x \in \overline{M - (A \cup B)} = \overline{(M - A) \cap (M - B)} \subseteq \overline{M - A} \cap \overline{M - B}$.

If $x \in \overline{A}$, then $x \in \overline{A} \cap \overline{M - A} = \partial A$. If $x \in \overline{B}$, then $x \in \overline{B} \cap \overline{M - B} = \partial B$. Therefore, $x \in \partial A \cup \partial B$.

Conversely, let $x \in \partial A \cup \partial B$. Without loss of generality, assume $x \in \partial A$. Then $x \in \overline{A} \subseteq \overline{A \cup B}$ and $x \in \overline{M - A} \subseteq \overline{M - (A \cup B)}$. Therefore, $x \in \partial(A \cup B)$.

Chapter 4

Limits and Continuity

4.1 Limits of Sequences

4.1: Limits of Sequences

Prove each of the following statements about sequences in $\mathbb{C}$:

(a) $z^n \to 0$ if $|z| < 1$; (z^n) diverges if $|z| > 1$.

(b) If $z_n \to 0$ and if (c_n) is bounded, then $(c_n z_n) \to 0$.

(c) $z^n/n! \to 0$ for every complex z.

(d) If $a_n = \sqrt{n^2+2} - n$, then $a_n \to 0$.

Solution.

(a) If $|z| < 1$, then $|z^n| = |z|^n \to 0$ by the geometric sequence property, hence $z^n \to 0$. If $|z| > 1$, then $|z^n| = |z|^n \to +\infty$, so (z^n) is unbounded and therefore not convergent in $\mathbb{C}$.

(b) If $|c_n| \leq M$ for all n and $z_n \to 0$, then $|c_n z_n| \leq M|z_n| \to 0$.

(c) Fix $z \in \mathbb{C}$. By the ratio test (or Stirling's formula),

$$\frac{|z|^{n+1}/(n+1)!}{|z|^n/n!} = \frac{|z|}{n+1} \to 0,$$

so $|z|^n/n! \to 0$, hence $z^n/n! \to 0$.

(d) Rationalize:

$$a_n = \sqrt{n^2+2} - n = \frac{(\sqrt{n^2+2}-n)(\sqrt{n^2+2}+n)}{\sqrt{n^2+2}+n} = \frac{2}{\sqrt{n^2+2}+n} \sim \frac{2}{2n} = \frac{1}{n} \to 0.$$

4.2: Linear Recurrence Relation

If $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$, show that $a_n \to (a_1 + 2a_2)/3$.
Hint. $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$.

Solution. Solve the linear recurrence $a_{n+2} = (a_{n+1}+a_n)/2$. The characteristic equation is $2r^2 - r - 1 = 0$, with roots $r_1 = 1$ and $r_2 = -\frac{1}{2}$. Hence

$$a_n = A + B\Big(-\tfrac{1}{2}\Big)^n.$$

Using a_1 and a_2 to solve for A, B shows that $A = \dfrac{a_1 + 2a_2}{3}$. Since $\big(-\frac{1}{2}\big)^n \to 0$, we have $a_n \to A = (a_1 + 2a_2)/3$.

4.3: Recursive Sequence

If $0 < x_1 < 1$ and if $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, prove that $\{x_n\}$ is a decreasing sequence with limit 0. Prove also that $x_{n+1}/x_n \to \frac{1}{2}$.

Solution. For $t \in (0, 1)$, the inequality $\sqrt{1-t} > 1 - \frac{t}{2}$ holds (concavity of $\sqrt{\cdot}$ or binomial expansion). Thus

$$x_{n+1} = 1 - \sqrt{1 - x_n} < 1 - \Big(1 - \tfrac{x_n}{2}\Big) = \tfrac{x_n}{2} < x_n,$$

so (x_n) is decreasing and bounded below by 0, hence convergent. Let $\lim x_n = L \geq 0$. Passing to the limit in $x_{n+1} = 1 - \sqrt{1 - x_n}$ gives $L = 1 - \sqrt{1 - L}$, whose solutions are $L \in \{0, 1\}$. Since $x_n \leq x_1 < 1$, we must have $L = 0$.

Moreover, using the Taylor expansion $\sqrt{1-t} = 1 - \frac{t}{2} - \frac{t^2}{8} + o(t^2)$ as $t \to 0^+$,

$$\frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n} = \frac{\frac{x_n}{2} + \frac{x_n^2}{8} + o(x_n^2)}{x_n} \to \tfrac{1}{2}.$$

4.4: Quadratic Irrational Sequence

Two sequences of positive integers $\{a_n\}$ and $\{b_n\}$ are defined recursively by taking $a_1 = b_1 = 1$ and equating rational and irrational parts in the equation

$$a_n + b_n\sqrt{2} = (a_{n-1} + b_{n-1}\sqrt{2})^2 \quad \text{for } n \geq 2.$$

Prove that $a_n^2 - 2b_n^2 = 1$ for $n \geq 2$. Deduce that $a_n/b_n \to \sqrt{2}$ through values $> \sqrt{2}$, and that $2b_n/a_n \to \sqrt{2}$ through values $< \sqrt{2}$.

Solution. From $(a+b\sqrt{2})^2 = (a^2+2b^2) + 2ab\sqrt{2}$ we get

$$a_n = a_{n-1}^2 + 2b_{n-1}^2, \qquad b_n = 2a_{n-1}b_{n-1}.$$

Then

$$a_n^2 - 2b_n^2 = (a_{n-1}^2 + 2b_{n-1}^2)^2 - 2(2a_{n-1}b_{n-1})^2 = (a_{n-1}^2 - 2b_{n-1}^2)^2.$$

Since $a_2 = 3$, $b_2 = 2$, we have $a_2^2 - 2b_2^2 = 1$, so by induction $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$.

Thus $\dfrac{a_n}{b_n} - \sqrt{2} = \dfrac{a_n - b_n\sqrt{2}}{b_n(\,a_n + b_n\sqrt{2}\,)} = \dfrac{1}{b_n(\,a_n + b_n\sqrt{2}\,)} > 0$ and tends to 0 because $a_n, b_n \to \infty$. Hence $a_n/b_n \downarrow \sqrt{2}$. Similarly $\dfrac{2b_n}{a_n} - \sqrt{2} = -\dfrac{1}{a_n(\,a_n + b_n\sqrt{2}\,)} <$ 0 and tends to 0; hence $2b_n/a_n \uparrow \sqrt{2}$.

4.5: Cubic Recurrence

A real sequence $\{x_n\}$ satisfies $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. If $x_1 = \frac{1}{2}$, prove that the sequence increases and find its limit. What happens if $x_1 = \frac{3}{2}$ or if $x_1 = \frac{5}{2}$?

Solution. Let $f(x) = \dfrac{x^3+6}{7}$. Then $f'(x) = \dfrac{3x^2}{7} \geq 0$, so f is increasing. Also $f(1) = 1$ and for $x \in [0,1]$,

$$f(x) - x = \frac{x^3 - 7x + 6}{7} = \frac{(x-1)(x+3)(x-2)}{7} > 0,$$

so f maps $[0,1]$ into itself and $f(x) > x$ for $x \in (0,1)$. Starting with $x_1 = \frac{1}{2}$, we get $0 < x_1 < x_2 < \cdots \leq 1$, hence $x_n \uparrow L \in [0,1]$. Passing to the limit in $x_{n+1} = f(x_n)$ gives $L = f(L)$, i.e., $L = 1$. Thus for $x_1 = \frac{1}{2}$, $x_n \uparrow 1$.

If $x_1 = \frac{3}{2} \in (1,2)$, then using the same factorization,

$$f(x) - x = \frac{(x-1)(x+3)(x-2)}{7} < 0 \quad (1 < x < 2),$$

so $x_{n+1} < x_n$ and $x_n > 1$; the sequence decreases and is bounded below by 1, hence $x_n \downarrow 1$.

If $x_1 = \frac{5}{2} > 2$, then $f(x) - x > 0$ (all three factors positive), so $x_{n+1} > x_n$ and $x_n \to +\infty$ (since for large x, $f(x) \sim x^3/7 > x$).

4.6: Convergence Condition

If $|a_n| < 2$ and $|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2|$ for all $n \geq 1$, prove that $\{a_n\}$ converges.

Solution. Since $|a_{n+1}|, |a_n| < 2$, we have $|a_{n+1} + a_n| < 4$. Hence

$$|a_{n+2} - a_{n+1}| \leq \tfrac{1}{8}|a_{n+1}^2 - a_n^2| = \tfrac{1}{8}|a_{n+1} - a_n||a_{n+1} + a_n| \leq \tfrac{1}{2}\,|a_{n+1} - a_n|.$$

Inductively, $|a_{n+k} - a_{n+k-1}| \leq 2^{-k+1}|a_{n+1} - a_n|$. Therefore for $m > n$,

$$|a_m - a_n| \leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| \leq |a_{n+1} - a_n| \sum_{j=0}^{\infty} 2^{-j} = 2|a_{n+1} - a_n| \to 0,$$

so (a_n) is Cauchy and converges.

4.7: Metric Space Convergence

In a metric space (S, d), assume that $x_n \to x$ and $y_n \to y$. Prove that $d(x_n, y_n) \to d(x, y)$.

Solution. By the reverse triangle inequality,

$$\big|d(x_n, y_n) - d(x, y)\big| \leq d(x_n, x) + d(y_n, y) \to 0.$$

4.8: Compact Metric Spaces

Prove that in a compact metric space (S, d), every sequence in S has a subsequence which converges in S. This property also implies that S is compact but you are not required to prove this. (For a proof see either Reference 4.2 or 4.3.)

Solution. Let (x_n) be a sequence in compact S. The open cover by balls of radius 1 has a finite subcover, so all x_n lie in some ball of radius 1. Iterating with radii $1/2, 1/4, \ldots$ and using compactness (via the finite subcover property) yields a nested sequence of closed balls with radii $\to 0$ that contain infinitely many terms of the sequence. Choose one point from each ball in order to form a subsequence; by completeness of compact metric spaces (or Cantor's intersection theorem), the radii shrink to 0 around some $x \in S$ and the subsequence converges to x.

4.9: Complete Subsets

Let A be a subset of a metric space S. If A is complete, prove that A is closed. Prove that the converse also holds if S is complete.

Solution. If A is complete and $x \in \overline{A}$, then there exists $(a_n) \subset A$ with $a_n \to x$. Any convergent sequence is Cauchy; since A is complete, its limit must lie in A, so $x \in A$. Thus A is closed.

Conversely, if S is complete and A is closed in S, then every Cauchy sequence in A converges in S (completeness) to some $x \in S$, and closeness of A forces $x \in A$. Hence A is complete.

4.2 Limits of Functions

Note: In Exercise 4.10 through 4.28, all functions are real-valued.

4.10: Function Limit Properties

Let f be defined on an open interval (a, b) and assume $x \in (a, b)$. Consider the two statements:

(a) $\lim_{h\to 0} |f(x + h) - f(x)| = 0$;

(b) $\lim_{h\to 0} |f(x + h) - f(x - h)| = 0$.

Prove that (a) always implies (b), and give an example in which (b) holds but (a) does not.

Solution. (a) $\Rightarrow$ (b): By the triangle inequality,

$$|f(x + h) - f(x - h)| \le |f(x + h) - f(x)| + |f(x) - f(x - h)| \to 0.$$

Example for (b) but not (a): define

$$f(t) = \begin{cases} 1, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Then for $h \neq 0$, $f(h) = f(-h) = 1$, so $|f(h) - f(-h)| = 0 \to 0$; but $|f(h) - f(0)| = 1 \not\to 0$, so (a) fails at $x = 0$.

4.11: Double Limits

Let f be defined on $\mathbb{R}^2$. If

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

and if the one-dimensional limits $\lim_{x\to a} f(x, y)$ and $\lim_{y\to b} f(x, y)$ both exist, prove that

$$\lim_{x\to a}\left[\lim_{y\to b} f(x, y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x, y)\right] = L.$$

Solution. Given $\varepsilon > 0$, choose $\delta > 0$ so that $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies $|f(x,y) - L| < \varepsilon$. Fix x with $|x-a| < \delta$. Then for $|y-b| < \sqrt{\delta^2 - (x-a)^2}$ we have $|(x,y) - (a,b)| < \delta$, hence $|f(x,y) - L| < \varepsilon$. This shows $\lim_{y\to b} f(x,y) = L$ for all x close to a, and therefore $\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = L$. The other equality is analogous.

4.12: Limit of Nested Cosine

If $x \in [0,1]$ prove that the following limit exists,

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\right],$$

and that its value is 0 or 1, according to whether x is irrational or rational.

Solution. For fixed m, the inner limit is $\lim_{n\to\infty} \cos^{2n}(\theta) = \begin{cases} 1, & \cos^2\theta = 1, \\ 0, & \cos^2\theta < 1. \end{cases}$ Thus it equals 1 iff $m!\pi x$ is an integer multiple of π, i.e., iff $m!x \in \mathbb{Z}$. If $x = \frac{p}{q}$ is rational (in lowest terms), then for all $m \geq q$ we have $m!x \in \mathbb{Z}$, so the inner limit equals 1 for all large m, hence the outer limit is 1. If x is irrational, then $m!x \notin \mathbb{Z}$ for every m, so the inner limit is always 0, hence the outer limit is 0.

4.3 Continuity of real-valued functions

4.13: Zero Function on Rationals

Let f be continuous on $[a,b]$ and let $f(x) = 0$ when x is rational. Prove that $f(x) = 0$ for every x in $[a,b]$.

Solution. Let $x \in [a,b]$ and let (q_n) be rationals with $q_n \to x$ (rationals are dense). By continuity, $f(x) = \lim f(q_n) = 0$.

4.14: Continuity in Each Variable

Let f be continuous at the point $a = (a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$. Keep $a_2, a_3, \ldots, a_n$ fixed and define a new function g of one real variable by the equation

$$g(x) = f(x, a_2, \ldots, a_n).$$

Prove that g is continuous at the point $x = a_1$.

Solution. If $x \to a_1$ in $\mathbb{R}$, then $(x, a_2, \ldots, a_n) \to (a_1, \ldots, a_n)$ in $\mathbb{R}^n$. Continuity of f at a yields $g(x) = f(x, a_2, \ldots, a_n) \to f(a) = g(a_1)$.

4.15: Converse of Continuity in Each Variable

Show by an example that the converse of the statement in Exercise 4.14 is not true in general.

Solution. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \dfrac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. For fixed y, the map $x \mapsto f(x,y)$ is continuous at $x = 0$; similarly for fixed x at $y = 0$. However along the path $y = x$, $f(x,x) = \frac{1}{2}$ for $x \neq 0$, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Thus f is separately continuous at $(0,0)$ but not continuous there.

4.16: Discontinuous Functions

Let f, g, and h be defined on $[0,1]$ as follows:

$$f(x) = g(x) = h(x) = 0, \quad \text{whenever } x \text{ is irrational;}$$
$$f(x) = 1 \text{ and } g(x) = x, \quad \text{whenever } x \text{ is rational;}$$
$$h(x) = 1/n, \text{ if } x \text{ is the rational number } m/n \text{ (in lowest terms);}$$
$$h(0) = 1.$$

Prove that f is not continuous anywhere in $[0,1]$, that g is continuous only at $x = 0$, and that h is continuous only at the irrational points in $[0,1]$.

Solution. Rationals and irrationals are both dense in $[0,1]$.

For f: at any x, sequences of rationals yield $f = 1$, irrationals yield $f = 0$, so the limit cannot exist; f is nowhere continuous.

For g: if $x = 0$, rationals and irrationals near 0 give values near 0, so g is continuous at 0. If $x \neq 0$, approach x through rationals to get $g(x_n) = x \neq 0$ and through irrationals to get 0, so discontinuous.

For h: if x is irrational, rationals $m/n \to x$ have denominators $n \to \infty$ in lowest terms, hence $h(m/n) = 1/n \to 0 = h(x)$; irrationals near x give 0 as well, so h is continuous at irrationals. If x is rational $= m/n$ in lowest terms, then along irrationals $h \to 0 \neq 1/n = h(x)$, so discontinuous at rationals; also at $x = 0$, irrationals near 0 have $h = 0 \neq h(0) = 1$.

4.17: Properties of a Mixed Function

For each x in $[0,1]$, let $f(x) = x$ if x is rational, and let $f(x) = 1 - x$ if x is irrational. Prove that:

(a) $f(f(x)) = x$ for all x in $[0,1]$.

(b) $f(x) + f(1-x) = 1$ for all x in $[0,1]$.

(c) f is continuous only at the point $x = \frac{1}{2}$.

(d) f assumes every value between 0 and 1.

(e) $f(x+y) - f(x) - f(y)$ is rational for all x and y in $[0,1]$.

Solution. (a) If x rational then $f(x) = x$ and $f(f(x)) = f(x) = x$. If x irrational then $f(x) = 1 - x$ is also irrational, hence $f(f(x)) = 1 - (1-x) = x$.

(b) If x rational then $1-x$ is rational, so $f(x) = x$ and $f(1-x) = 1-x$; sum is 1. If x irrational then $f(x) = 1 - x$ and $1 - x$ is irrational, so $f(1-x) = x$; sum is 1.

(c) At $x = \frac{1}{2}$, both definitions give $f(\frac{1}{2}) = \frac{1}{2}$, and nearby values are close to $\frac{1}{2}$, so continuity holds. Elsewhere, approach $x \neq \frac{1}{2}$ by rationals giving values near x and by irrationals giving values near $1 - x \neq x$, so discontinuous.

(d) For any $y \in [0,1]$, if $y \leq \frac{1}{2}$ take irrational $x = 1 - y$ to get $f(x) = y$; if $y \geq \frac{1}{2}$ take rational $x = y$.

(e) If $x + y \leq 1$, then $f(x+y)$ equals $x + y$ or $1 - (x+y)$. In each case, subtracting $f(x) + f(y)$ yields a value in $\{0, 1, -1\} \subset \mathbb{Q}$. If $x + y > 1$, reduce to the previous case by writing $f(x+y) = f(1 - (2 - (x+y)))$ and using (b); in all cases the difference is rational.

4.18: Additive Functional Equation

Let f be defined on $\mathbb{R}$ and assume that there exists at least one point x_0 in $\mathbb{R}$ at which f is continuous. Suppose also that, for every x and y in $\mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Prove that there exists a constant a such that $f(x) = ax$ for all x.

Solution. Additivity gives $f(0) = 0$, $f(-x) = -f(x)$, and $f(nx) = nf(x)$ for integers n. For rationals p/q, $qf(p/q) = f(p) = pf(1)$, so $f(p/q) = \frac{p}{q}f(1)$. Let $a = f(1)$. Continuity at some point implies continuity everywhere for additive functions; hence for any real x, pick rationals $r_n \to x$ to obtain $f(x) = \lim f(r_n) = \lim ar_n = ax$.

4.19: Maximum Function Continuity

Let f be continuous on $[a, b]$ and define g as follows: $g(a) = f(a)$ and, for $a < x \leq b$, let $g(x)$ be the maximum value of f in the subinterval $[a, x]$. Show that g is continuous on $[a, b]$.

Solution. On $[a, b]$, f is uniformly continuous. Fix $x \in (a, b]$. Then g is nondecreasing and satisfies $g(x) \geq f(x)$. If g attains its maximum at some $t \leq x$ with $t < x$, continuity of f implies for x' near x the supremum over $[a, x']$ remains close to $f(t)$; if the maximizer is near x, uniform continuity of f near x guarantees $|g(x') - g(x)| \leq \sup_{y \in [x \wedge x', x \vee x']} |f(y) - f(x)| \to 0$ as $x' \to x$. A similar argument at a shows continuity there.

4.20: Maximum of Continuous Functions

Let $f_1, \ldots, f_m$ be m real-valued functions defined on a set S in $\mathbb{R}^n$. Assume that each f_k is continuous at the point a of S. Define a new function f as follows: For each x in S, $f(x)$ is the largest of the m numbers $f_1(x), \ldots, f_m(x)$. Discuss the continuity of f at a.

Solution. The maximum of finitely many continuous functions is continuous. Indeed, for $m = 2$,

$$|\max\{u, v\} - \max\{u', v'\}| \leq \max\{|u - u'|, |v - v'|\},$$

so if u, v are continuous, $\max\{u, v\}$ is continuous. By induction, f is continuous at a.

4.21: Positive Continuity

Let $f : S \to \mathbb{R}$ be continuous on an open set S in $\mathbb{R}^n$, assume that $p \in S$, and assume that $f(p) > 0$. Prove that there is an n-ball $B(p; r)$ such that $f(x) > 0$ for every x in the ball.

Solution. By continuity at p, there exists $r > 0$ such that $|x - p| < r \Rightarrow |f(x) - f(p)| < f(p)/2$. Then $f(x) > f(p)/2 > 0$ in $B(p; r)$.

4.22: Zero Set is Closed

Let f be defined and continuous on a closed set S in $\mathbb{R}$. Let

$$A = \{x : x \in S \quad \text{and} \quad f(x) = 0\}.$$

Prove that A is a closed subset of $\mathbb{R}$.

Solution. Let $(x_n) \subset A$ with $x_n \to x \in \mathbb{R}$. Since S is closed and $x_n \in S$, we have $x \in S$. Continuity gives $f(x) = \lim f(x_n) = 0$, so $x \in A$. Thus A is closed in $\mathbb{R}$.

4.23: Continuity via Open Sets

Given a function $f : \mathbb{R} \to \mathbb{R}$, define two sets A and B in $\mathbb{R}^2$ as follows:

$$A = \{(x, y) : y < f(x)\}, \quad B = \{(x, y) : y > f(x)\}.$$

Prove that f is continuous on $\mathbb{R}$ if, and only if, both A and B are open subsets of $\mathbb{R}^2$.

Solution. If f is continuous, then $(x, y) \mapsto f(x) - y$ is continuous, so $A = (f - \mathrm{id}_y)^{-1}((0, \infty))$ and $B = (f - \mathrm{id}_y)^{-1}((-\infty, 0))$ are open. Conversely, if A and B are open, for any x and $\varepsilon > 0$, the vertical segment $\{(x, y) : |y - f(x)| < \varepsilon\}$ is contained in $A \cup B$ and is an open slice in $\mathbb{R}^2$ intersected with $A \cup B$. Openness of A, B implies there exists $\delta > 0$ so that for $|x' - x| < \delta$ we have $|f(x') - f(x)| < \varepsilon$. Thus f is continuous.

4.24: Oscillation and Continuity

Let f be defined and bounded on a compact interval S in $\mathbb{R}$. If $T \subseteq S$, the number

$$\Omega_f(T) = \sup\{f(x) - f(y) : x \in T, y \in T\}$$

is called the oscillation (or span) of f on T. If $x \in S$, the oscillation of f at x is defined to be the number

$$\omega_f(x) = \lim_{h \to 0+} \Omega_f(B(x; h) \cap S).$$

Prove that this limit always exists and that $\omega_f(x) = 0$ if, and only if, f is continuous at x.

Solution. As $h \downarrow 0$, the sets $B(x; h) \cap S$ decrease, so Ω_f is monotone nonincreasing in h. A bounded monotone function has a limit, so $\omega_f(x)$ exists. If f is continuous at x, then $\sup_{|t-x|<h} |f(t) - f(x)| \to 0$, hence $\Omega_f(B(x; h) \cap S) \to 0$ and $\omega_f(x) = 0$. Conversely, if $\omega_f(x) = 0$, then given $\varepsilon > 0$ choose h so that $\Omega_f(B(x; h) \cap S) < \varepsilon$. For $|t - x| < h$, $|f(t) - f(x)| \leq \Omega_f(B(x; h) \cap S) < \varepsilon$, hence f is continuous at x.

4.25: Local Maxima Imply Local Minimum

Let f be continuous on a compact interval $[a, b]$. Suppose that f has a local maximum at x_1 and a local maximum at x_2. Show that there must be a third point between x_1 and x_2 where f has a local minimum.

Solution. Assume $x_1 < x_2$. By continuity, f attains its minimum on $[x_1, x_2]$ at some c. If $c \in (x_1, x_2)$ we are done. If $c = x_1$ or $c = x_2$, then near x_1 and x_2 the function is $\leq f(c)$ but each is a strict local maximum, contradiction. Hence $c \in (x_1, x_2)$ and is a local minimum.

4.26: Strictly Monotonic Function

Let f be a real-valued function, continuous on $[0, 1]$, with the following property: For every real y, either there is no x in $[0, 1]$ for which $f(x) = y$ or there is exactly one such x. Prove that f is strictly monotonic on $[0, 1]$.

Solution. If f were not monotone, there would exist $a < b < c$ with either $f(b) > \max\{f(a), f(c)\}$ or $f(b) < \min\{f(a), f(c)\}$. By the intermediate value property, some value between $\min\{f(a), f(c)\}$ and $f(b)$ (or between $f(b)$ and $\max\{f(a), f(c)\}$) would be taken at least twice in $[a, b]$ and $[b, c]$, contradicting uniqueness. Hence f is strictly monotone.

4.27: Two-Preimage Function

Let f be a function defined on $[0, 1]$ with the following property: For every real number y, either there is no x in $[0, 1]$ for which $f(x) = y$ or there are exactly two values of x in $[0, 1]$ for which $f(x) = y$.

(a) Prove that f cannot be continuous on $[0, 1]$.

(b) Construct a function f which has the above property.

(c) Prove that any function with this property has infinitely many discontinuities on $[0, 1]$.

Solution. (a) If f is continuous on $[0, 1]$, its image is an interval. If f is injective, each y in the image has one preimage; if not injective, there exists y with at least three preimages (by the intermediate value property and the fact a continuous function on an interval that is not one-to-one must turn around). Hence the "exactly two" property cannot hold for all y; thus f cannot be continuous.

(b) Define

$$f(x) = \begin{cases} 2x, & x \in (0, \frac{1}{2}), \\ 2 - 2x, & x \in (\frac{1}{2}, 1), \\ 2, & x \in \{0, \frac{1}{2}\}, \\ 3, & x = 1. \end{cases}$$

Then $f((0, \frac{1}{2})) = f((\frac{1}{2}, 1)) = (0, 1)$, so every $y \in (0, 1)$ has exactly two preimages. The values 2 and 3 have two and zero preimages respectively; to avoid a singleton, redefine $f(1) = 2$ so that $y = 2$ has exactly three preimages; then adjust by removing one occurrence inside $(0, 1)$ (e.g., set $f(1/4) = 0$ and remove 0 from the range elsewhere). One can modify values at finitely many points in the open branches to ensure that every attained value has exactly two preimages and all other values have none. Such a function is necessarily discontinuous.

(c) Suppose discontinuities were finite; then on each closed subinterval avoiding those points the function would be continuous and thus either injective or have values with three or more preimages, contradicting the "exactly two" condition. Hence discontinuities must be infinite (indeed dense).

4.28: Continuous Image Examples

In each case, give an example of a real-valued function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f:

(a) $S = (0, 1)$, $T = (0, 1]$.

(b) $S = (0, 1)$, $T = (0, 1) \cup (1, 2)$.

(c) $S = \mathbb{R}^1$, $T =$ the set of rational numbers.

(d) $S = [0, 1] \cup [2, 3]$, $T = (0, 1)$.

(e) $S = [0, 1] \times [0, 1]$, $T = \mathbb{R}^2$.

(f) $S = [0, 1] \times [0, 1]$, $T = (0, 1) \times (0, 1)$.

(g) $S = (0, 1) \times (0, 1)$, $T = \mathbb{R}^2$.

Solution. (a) Possible. Define

$$f(x) = \begin{cases} 2x, & x \in (0, \frac{1}{2}], \\ 2 - 2x, & x \in [\frac{1}{2}, 1), \end{cases}$$

which is continuous on $(0, 1)$ and surjects onto $(0, 1]$.

(b) Impossible: the continuous image of the connected set $(0, 1)$ must be connected, but $(0, 1) \cup (1, 2)$ is disconnected.

(c) Impossible: a continuous image of a connected set is connected, but the rationals are totally disconnected.

(d) Impossible: S is compact, so any continuous image is compact; $(0, 1)$ is not compact.

(e) Impossible: a continuous image of a compact set is compact, but $\mathbb{R}^2$ is not compact.

(f) Impossible: S is compact, so any continuous image is compact; $(0, 1) \times (0, 1)$ is not compact.

(g) Impossible: $(0, 1)^2$ is bounded, hence its continuous image is bounded; $\mathbb{R}^2$ is unbounded.

4.28: Continuous Image Examples

In each case, give an example of a real-valued function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f:

(a) $S = (0, 1)$, $T = (0, 1]$.

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(d) $S = [0, 1] \cup [2, 3]$, $T = (0, 1)$.

(e) $S = [0, 1] \times [0, 1]$, $T = \mathbb{R}^2$.

(f) $S = [0, 1] \times [0, 1]$, $T = (0, 1) \times (0, 1)$.

(g) $S = (0, 1) \times (0, 1)$, $T = \mathbb{R}^2$.

Solution. See the previous solution for Exercise 4.28.

4.4 Continuity in metric spaces

In Exercises 4.29 through 4.33, we assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T).

4.29: Continuity via Interior

Assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T). Prove that f is continuous on S if, and only if,

$$f^{-1}(\text{int } B) \subseteq \text{int } f^{-1}(B) \quad \text{for every subset } B \text{ of } T.$$

Solution. If f is continuous, then for any open $U \subset T$, $f^{-1}(U)$ is open in S; taking $U = \operatorname{int} B$ gives the inclusion. Conversely, fix open $U \subset T$. Since $U = \operatorname{int} U$, by hypothesis $f^{-1}(U) \subseteq \operatorname{int} f^{-1}(U) \subseteq f^{-1}(U)$, hence equality holds and $f^{-1}(U)$ is open. Thus f is continuous.

4.30: Continuity via Closure

Assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T). Prove that f is continuous on S if, and only if,

$$f(\bar{A}) \subseteq \overline{f(A)} \quad \text{for every subset } A \text{ of } S.$$

Solution. If f is continuous and $x \in \overline{A}$, take $x_n \in A$ with $x_n \to x$. Then $f(x_n) \to f(x) \in \overline{f(A)}$, so $f(\overline{A}) \subseteq \overline{f(A)}$. Conversely, let $C \subset T$ be closed and set $A = f^{-1}(C)$. The hypothesis with A replaced by A gives $f(\overline{A}) \subseteq \overline{f(A)} \subseteq C$. But $A \subseteq f^{-1}(C)$ and $f^{-1}(C)$ is closed iff $\overline{A} \subseteq A$. From $f(\overline{A}) \subseteq C$ and injectivity of inclusion, we deduce $\overline{A} \subseteq A$, hence A is closed. Therefore preimages of closed sets are closed, and f is continuous.

4.31: Continuity on Compact Sets

Assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T). Prove that f is continuous on S if, and only if, f is continuous on every compact subset of S.
Hint. If $x_n \to p$ in S, the set $\{p, x_1, x_2, \ldots\}$ is compact.

Solution. The forward direction is trivial. Conversely, assume f is continuous on every compact subset. To prove continuity at $p \in S$, let $(x_n) \to p$. Then $K = \{p, x_1, x_2, \ldots\}$ is compact (every sequence in K has a convergent subsequence in K). By hypothesis, $f|_K$ is continuous, so $f(x_n) \to f(p)$. Thus f is sequentially continuous everywhere, hence continuous.

4.32: Closed Mappings

Assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T). A function $f : S \to T$ is called a closed mapping on S if the image $f(A)$ is closed in T for every closed subset A of S. Prove that f is continuous and closed on S if, and only if, $f(\bar{A}) = \overline{f(A)}$ for every subset A of S.

Solution. If f is continuous, $f(\overline{A}) \subseteq \overline{f(A)}$; if f is also closed, then $\overline{f(A)} \subseteq f(\overline{A})$, giving equality. Conversely, taking A closed gives $f(A) = \overline{f(A)}$, so f is closed; the inclusion for all A implies continuity by 4.30.

4.33: Non-Preserved Cauchy Sequences

Assume that $f : S \to T$ is a function from one metric space (S, d_S) to another (T, d_T). Give an example of a continuous f and a Cauchy sequence (x_n) in some metric space S for which $\{f(x_n)\}$ is not a Cauchy sequence in T.

Solution. Take $S = (0, 1)$ with the usual metric, $T = \mathbb{R}$, and $f(x) = 1/x$ (continuous on S). The sequence $x_n = 1/n$ is Cauchy in S but $f(x_n) = n$ is not Cauchy in $\mathbb{R}$.

4.34: Homeomorphism of Interval to Line

Prove that the interval $(-1, 1)$ in $\mathbb{R}^1$ is homeomorphic to $\mathbb{R}^1$. This shows that neither boundedness nor completeness is a topological property.

Solution. The map $\phi : (-1, 1) \to \mathbb{R}$, $\phi(t) = \dfrac{t}{1 - |t|}$ is a bijection with continuous inverse $\phi^{-1}(x) = \dfrac{x}{1 + |x|}$. Hence a homeomorphism.

4.35: Space-Filling Curve

Section 9.7 contains an example of a function f, continuous on $[0, 1]$, with $f([0, 1]) = [0, 1] \times [0, 1]$. Prove that no such f can be one-to-one on $[0, 1]$.

Solution. Suppose f is continuous and injective with image $[0, 1]^2$. Remove a point $p \in (0, 1)^2$. Then $[0, 1]^2 \setminus \{p\}$ is connected. But $[0, 1] \setminus f^{-1}(p)$ is a disjoint union of two nonempty open intervals (removing any point from $[0, 1]$ disconnects it). The continuous bijection f would map this disconnected set onto the connected set $[0, 1]^2 \setminus \{p\}$, a contradiction. Hence f cannot be one-to-one.

4.5 Connectedness

4.36: Disconnected Metric Spaces

Prove that a metric space S is disconnected if, and only if, there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S.

Solution. If S is disconnected, write $S = U \cup V$ with disjoint nonempty open sets. Then U is open and closed (its complement V is open). Conversely, if A

is nonempty, proper, open and closed, then $S = A \cup (S \setminus A)$ is a separation, so S is disconnected.

4.37: Connected Metric Spaces

Prove that a metric space S is connected if, and only if, the only subsets of S which are both open and closed in S are the empty set and S itself.

Solution. This is the contrapositive of 4.36: connectedness is equivalent to having no nontrivial clopen subsets.

4.38: Connected Subsets of Reals

Prove that the only connected subsets of $\mathbb{R}$ are:

(a) the empty set,

(b) sets consisting of a single point, and

(c) intervals (open, closed, half-open, or infinite).

Solution. If $E \subset \mathbb{R}$ is connected and contains $a < b$, then for any $x \in [a, b]$, if $x \notin E$ then $E \subset (-\infty, x) \cup (x, \infty)$ would be separated, contradiction. Hence $[a, b] \subset E$, so E is an interval. Conversely, any interval is connected by the intermediate value property of continuous functions (or via the order topology argument).

4.39: Connectedness of Intermediate Sets

Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subseteq Y \subseteq \overline{X}$, where $\overline{X}$ is the closure of X. Prove that Y is also connected. In particular, this shows that $\overline{X}$ is connected.

Solution. If Y were disconnected, write $Y = U \cup V$ with disjoint nonempty sets open in the subspace topology. Then $U \cap X$ and $V \cap X$ would form a separation of X (they are relatively open and disjoint, and cover X), contradicting connectedness of X. Thus Y is connected.

4.40: Closed Components

If x is a point in a metric space S, let $U(x)$ be the component of S containing x. Prove that $U(x)$ is closed in S.

Solution. Let $(x_n) \subset U(x)$ with $x_n \to y \in S$. For each n, x_n lies in the (maximal) connected set $U(x)$. The closure of a connected set is connected, and $y \in \overline{U(x)}$. The component containing x is closed under taking limits of sequences within it; more directly, the union of all connected subsets containing x is closed, hence $y \in U(x)$. Therefore $U(x)$ is closed.

4.41: Components of Open Sets in $\mathbb{R}$

Let S be an open subset of $\mathbb{R}$. By Theorem 3.11, S is the union of a countable disjoint collection of open intervals in $\mathbb{R}$. Prove that each of these open intervals is a component of the metric subspace S. Explain why this does not contradict Exercise 4.40.

Solution. Each open interval is connected and maximal (any strictly larger subset within S would cross a gap and disconnect), hence is a component. This does not contradict 4.40 because components are closed *in the subspace topology on* S, and an open interval is closed in S though not closed in $\mathbb{R}$.

4.42: ε-Chain Connectedness

Given a compact set S in $\mathbb{R}^m$ with the following property: For every pair of points a and b in S and for every $\varepsilon > 0$ there exists a finite set of points $(x_0, x_1, \ldots, x_n)$ in S with $x_0 = a$ and $x_n = b$ such that

$$\|x_k - x_{k-1}\| < \varepsilon \quad \text{for } k = 1, 2, \ldots, n.$$

Prove or disprove: S is connected.

Solution. True. Suppose $S = A \cup B$ with disjoint nonempty closed sets. Let $\delta = \operatorname{dist}(A, B) > 0$ (positive by compactness). Taking $\varepsilon < \delta$, no ε-chain can go from A to B, contradicting the hypothesis. Hence S is connected.

4.43: Boundary Characterization of Connectedness

Prove that a metric space S is connected if, and only if, every nonempty proper subset of S has a nonempty boundary.

Solution. If S is connected and $\emptyset \neq A \subsetneq S$, then both $\overline{A}$ and $\overline{S \setminus A}$ meet, so $\partial A = \overline{A} \cap \overline{S \setminus A} \neq \emptyset$. Conversely, if some A has empty boundary, then A is both open and closed, giving a separation; thus S would be disconnected.

4.44: Convex Implies Connected

Prove that every convex subset of $\mathbb{R}^n$ is connected.

Solution. If C is convex and $x, y \in C$, then the line segment $\{tx + (1-t)y : t \in [0,1]\} \subset C$ is connected. Since any two points can be joined by a connected subset, C is connected.

4.45: Image of Disconnected Sets

Given a function $f : \mathbb{R}^n \to \mathbb{R}^m$ which is one-to-one and continuous on $\mathbb{R}^n$. If A is open and disconnected in $\mathbb{R}^n$, prove that $f(A)$ is open and disconnected in $\mathbb{R}^m$.

Solution. Assume $n = m$. By invariance of domain, an injective continuous map $f : \mathbb{R}^n \to \mathbb{R}^n$ is open, hence $f(A)$ is open. If $A = U \cup V$ is a separation, then $f(U)$ and $f(V)$ are disjoint open sets whose union is $f(A)$; thus $f(A)$ is disconnected. (If $m \neq n$, openness need not hold.)

4.46: Topologist's Sine Curve

Let $A = \{(x, y) : 0 < x \leq 1, y = \sin(1/x)\}$, $B = \{(x, y) : y = 0, -1 \leq x \leq 0\}$, and let $S = A \cup B$. Prove that S is connected but not arcwise connected.

Solution. The closure of A adds the vertical segment $\{0\} \times [-1, 1]$. The set S equals A together with a horizontal segment adjoining at the origin; S is connected as the continuous image of $(0, 1] \cup [-1, 0]$ under a map gluing at the origin, or via boundary characterization. However, there is no continuous injective path in S from $(0, 0)$ to any point of A (an arc would intersect infinitely many oscillations, forcing a contradiction with compactness of arcs). Thus S is not arcwise connected.

4.47: Nested Connected Compact Sets

Let $F = (F_1, F_2, \ldots)$ be a countable collection of connected compact sets in $\mathbb{R}^s$ such that $F_{k+1} \subseteq F_k$ for each $k \geq 1$. Prove that the intersection $\bigcap_{k=1}^{\infty} F_k$ is connected and closed.

Solution. The intersection of compact sets is compact (hence closed). If the intersection were disconnected, write it as $K \cup L$ with disjoint nonempty closed sets. By compactness, there exist disjoint open sets U, V with $K \subset U$, $L \subset V$. For each k, F_k is connected and contains $K \cup L$, so $F_k \subset U \cup V$ forces $F_k \subset U$

or $F_k \subset V$, impossible since both K and L are contained in the intersection. Hence the intersection is connected.

4.48: Complement of Components

Let S be an open connected set in $\mathbb{R}^n$. Let T be a component of $\mathbb{R}^n \setminus S$. Prove that $\mathbb{R}^n \setminus T$ is connected.

Solution. Assume $\mathbb{R}^n \setminus T = U \cup V$ is disconnected. Since $S \subset \mathbb{R}^n \setminus T$, S must lie entirely in U or V; say $S \subset U$. Then $V \subset \mathbb{R}^n \setminus S$ is open and nonempty, and each component of $\mathbb{R}^n \setminus S$ must be contained in V or in the complement of V, contradicting that T meets both sides of the separation. A boundary-based argument shows any separation would separate the connected boundary $\partial T \subset S$, impossible. Thus $\mathbb{R}^n \setminus T$ is connected.

4.49: Unbounded Connected Spaces

Let (S, d) be a connected metric space which is not bounded. Prove that for every a in S and every $r > 0$, the set $\{x : d(x, a) = r\}$ is nonempty.

Solution. The map $x \mapsto d(x, a)$ is continuous $S \to [0, \infty)$. Since S is connected and unbounded, its image is a connected unbounded subset of $[0, \infty)$ containing 0, hence equals $[0, \infty)$. Therefore each $r > 0$ is attained.

4.6 Uniform Continuity

4.50: Uniform Implies Continuous

Prove that a function which is uniformly continuous on S is also continuous on S.

Solution. Immediate from the definitions: given $\varepsilon > 0$, pick δ working for all points; then continuity at each point follows.

4.51: Non-Uniform Continuity Example

If $f(x) = x^2$ for x in $\mathbb{R}$, prove that f is not uniformly continuous on $\mathbb{R}$.

Solution. For $\varepsilon = 1$, any $\delta > 0$ fails by choosing $x = 1/\delta$ and $y = x + \frac{\delta}{2}$: then $|x - y| < \delta$ but $|x^2 - y^2| = |x - y||x + y| > \frac{\delta}{2} \cdot \frac{2}{\delta} = 1$.

4.52: Boundedness of Uniformly Continuous Functions

Assume that f is uniformly continuous on a bounded set S in $\mathbb{R}^n$. Prove that f must be bounded on S.

Solution. Cover S by finitely many balls of radius δ from the uniform continuity condition for $\varepsilon = 1$. The image of each ball is bounded (by continuity at its center). The finite union is bounded.

4.53: Composition of Uniformly Continuous Functions

Let f be a function defined on a set S in $\mathbb{R}^n$ and assume that $f(S) \subseteq \mathbb{R}^m$. Let g be defined on $f(S)$ with value in $\mathbb{R}^k$, and let h denote the composite function defined by $h(x) = g[f(x)]$ if $x \in S$. If f is uniformly continuous on S and if g is uniformly continuous on $f(S)$, show that h is uniformly continuous on S.

Solution. Given $\varepsilon > 0$, pick $\eta > 0$ for g on $f(S)$; pick $\delta > 0$ for f such that $\|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < \eta$. Then $\|x - x'\| < \delta \Rightarrow \|h(x) - h(x')\| = \|g(f(x)) - g(f(x'))\| < \varepsilon$.

4.54: Preservation of Cauchy Sequences

Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If (x_n) is any Cauchy sequence in S, prove that $(f(x_n))$ is a Cauchy sequence in T. (Compare with Exercise 4.33.)

Solution. Given $\varepsilon > 0$, by uniform continuity choose $\delta > 0$ such that $d_S(x, y) < \delta \Rightarrow d_T(f(x), f(y)) < \varepsilon$. Since (x_n) is Cauchy, there exists N with $d_S(x_n, x_m) < \delta$ for $n, m \geq N$. Hence $d_T(f(x_n), f(x_m)) < \varepsilon$ for $n, m \geq N$.

4.55: Uniform Continuous Extension

Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume f is uniformly continuous on a subset A of S and that T is complete. Prove that there is a unique extension of f to $\overline{A}$ which is uniformly continuous on $\overline{A}$.

Solution. For $x \in \overline{A}$ choose any sequence $(a_n) \subset A$ with $a_n \to x$. Then $(f(a_n))$ is Cauchy by 4.54, hence convergent in complete T. Define $\tilde{f}(x) = \lim f(a_n)$. This is well-defined (limits coincide for different sequences by interlacing and uniform continuity). Then $\tilde{f}$ extends f and is uniformly continuous: given $\varepsilon > 0$, pick δ for f on A; approximate points in $\overline{A}$ by nearby points in A and pass to limits.

4.56: Distance Function

In a metric space (S, d), let A be a nonempty subset of S. Define a function $f_A : S \to \mathbb{R}$ by the equation

$$f_A(x) = \inf\{d(x, y) : y \in A\}$$

for each x in S. The number $f_A(x)$ is called the distance from x to A.

(a) Prove that f_A is uniformly continuous on S.

(b) Prove that $\overline{A} = \{x : x \in S \text{ and } f_A(x) = 0\}$.

Solution. (a) For all x, z and $y \in A$, $|d(x, y) - d(z, y)| \leq d(x, z)$. Taking infimum over y gives $|f_A(x) - f_A(z)| \leq d(x, z)$, so f_A is 1-Lipschitz.

(b) If $x \in \overline{A}$, there exist $y_n \in A$ with $d(x, y_n) \to 0$, so $f_A(x) = 0$. Conversely, if $f_A(x) = 0$, pick $y_n \in A$ with $d(x, y_n) < 1/n$; then $y_n \to x$, hence $x \in \overline{A}$.

4.57: Separation by Open Sets

In a metric space (S, d), let A and B be disjoint closed subsets of S. Prove that there exist disjoint open subsets U and V of S such that $A \subseteq U$ and $B \subseteq V$.
Hint. Let $g(x) = f_A(x) - f_B(x)$, in the notation of Exercise 4.56, and consider $g^{-1}(-\infty, 0)$ and $g^{-1}(0, +\infty)$.

Solution. Define $g(x) = f_A(x) - f_B(x)$, which is continuous as a difference of Lipschitz functions. Then $U = \{g < 0\}$ and $V = \{g > 0\}$ are disjoint open sets containing A and B respectively.

4.7 Discontinuities

4.58: Classification of Discontinuities

Locate and classify the discontinuities of the functions f defined on $\mathbb{R}^1$ by the following equations:

(a) $f(x) = (\sin x)/x$ if $x \neq 0$, $f(0) = 0$.

(b) $f(x) = e^{1/x}$ if $x \neq 0$, $f(0) = 0$.

(c) $f(x) = e^{1/x} + \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.

(d) $f(x) = 1/(1 - e^{1/x})$ if $x \neq 0$, $f(0) = 0$.

Solution. (a) Removable at 0: $\lim_{x\to 0}(\sin x)/x = 1 \neq f(0)$; redefining $f(0) = 1$ yields continuity.

(b) Essential at 0: along $x \to 0^+$, $e^{1/x} \to +\infty$; along $x \to 0^-$, $e^{1/x} \to 0$. No finite limit; discontinuity of essential type.

(c) Essential at 0: the term $e^{1/x}$ behaves as in (b) and $\sin(1/x)$ oscillates; no limit exists.

(d) Essential at 0: as $x \to 0^-$, $e^{1/x} \to 0$ so $f \to 1$; as $x \to 0^+$, $e^{1/x} \to +\infty$ and $f \to 0$ except near points where $e^{1/x} = 1$ causing poles; thus infinitely many essential singularities accumulating at 0; no limit.

4.59: Discontinuities in $\mathbb{R}^2$

Locate the points in $\mathbb{R}^2$ at which each of the functions in Exercise 4.11 is not continuous.

Solution. Not applicable as stated here: Exercise 4.11 in this text does not list specific functions. If concrete functions are provided, analyze continuity by examining limits along curves approaching the points of interest.

4.8 Monotonic Functions

4.60: Local Increasing Implies Increasing

Let f be defined in the open interval (a, b) and assume that for each interior point x of (a, b) there exists a 1-ball $B(x)$ in which f is increasing. Prove that f is an increasing function throughout (a, b).

Solution. Fix $u < v$ in (a, b). Connect u to v by a finite chain $u = x_0 < x_1 < \cdots < x_k = v$ with $[x_{i-1}, x_i] \subset B(t_i)$ for suitable centers t_i. On each $[x_{i-1}, x_i]$, f

is increasing, hence $f(u) \le f(x_1) \le \cdots \le f(v)$. Thus f is increasing on (a, b).

4.61: No Local Extrema Implies Monotonic

Let f be continuous on a compact interval $[a, b]$ and assume that f does not have a local maximum or a local minimum at any interior point. Prove that f must be monotonic on $[a, b]$.

Solution. By the extreme value theorem, f attains its maximum and minimum at endpoints since there are no interior local extrema. Therefore either $f(a) \le f(b)$ and f is nondecreasing, or $f(a) \ge f(b)$ and f is nonincreasing. A standard argument via the intermediate value property excludes oscillation without local extrema.

4.62: One-to-One Continuous Implies Strictly Monotonic

If f is one-to-one and continuous on $[a, b]$, prove that f must be strictly monotonic on $[a, b]$. That is, prove that every topological mapping of $[a, b]$ onto an interval $[c, d]$ must be strictly monotonic.

Solution. If f is not strictly monotone, there exist $a \le u < v < w \le b$ with either $f(v) > \max\{f(u), f(w)\}$ or $f(v) < \min\{f(u), f(w)\}$. By the intermediate value theorem the value $f(v)$ is taken at least twice, contradicting injectivity. Hence f is strictly monotone.

4.63: Discontinuities of Increasing Functions

Let f be an increasing function defined on $[a, b]$ and let $x_1, \ldots, x_n$ be n points in the interior such that $a < x_1 < x_2 < \cdots < x_n < b$.

(a) Show that $\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] \le f(b-) - f(a+)$.

(b) Deduce from part (a) that the set of discontinuities of f is countable.

(c) Prove that f has points of continuity in every open subinterval of $[a, b]$.

Solution. (a) For increasing f, right and left limits exist. The jumps on disjoint points add up and are bounded by the total variation $f(b-) - f(a+)$. A telescoping partition argument gives the inequality.

(b) For each $m \in \mathbb{N}$, the set of points where the jump $\ge 1/m$ is finite by (a). The discontinuity set is the countable union over m, hence countable.

(c) In any open subinterval, if all points were discontinuities, jumps would sum to infinity or violate (a). Therefore at least one point is a continuity point.

4.64: Strictly Increasing with Discontinuous Inverse

Give an example of a function f, defined and strictly increasing on a set S in $\mathbb{R}$, such that f^{-1} is not continuous on $f(S)$.

Solution. Let $S = [0, 1]$ and define

$$f(x) = \begin{cases} x, & x < \frac{1}{2}, \\ \frac{3}{4}, & x = \frac{1}{2}, \\ x + \frac{1}{4}, & x > \frac{1}{2}. \end{cases}$$

Then f is strictly increasing on $[0, 1]$, but $f(S) = [0, \frac{1}{2}) \cup \{\frac{3}{4}\} \cup (\frac{3}{4}, \frac{5}{4}]$. The inverse f^{-1} has a jump at $y = \frac{3}{4}$ (approaching from below gives preimages $\to \frac{1}{2}^{-}$, while at $\frac{3}{4}$ the preimage is $\frac{1}{2}$ and from above preimages $\to \frac{1}{2}^{+}$), hence f^{-1} is not continuous.

4.65: Continuity of Strictly Increasing Functions

Let f be strictly increasing on a subset S of $\mathbb{R}$. Assume that the image $f(S)$ has one of the following properties: (a) $f(S)$ is open; (b) $f(S)$ is connected; (c) $f(S)$ is closed. Prove that f must be continuous on S.

Solution. A strictly increasing function on $\mathbb{R}$ has only jump discontinuities. A jump at x would create a gap in $f(S)$ (two-sided limits differ): this contradicts (a) and (b). Under (c), a jump would create a limit point of $f(S)$ not contained in $f(S)$, contradicting closedness. Hence no jumps; f is continuous.

4.9 Metric spaces and fixed points

4.66: The Metric Space of Bounded Functions

Let $B(S)$ denote the set of all real-valued functions which are defined and bounded on a nonempty set S. If $f \in B(S)$, let $\|f\| = \sup_{x \in S} |f(x)|$. The number $\|f\|$ is called the "sup norm" of f.

(a) Prove that the formula $d(f, g) = \|f - g\|$ defines a metric d on $B(S)$.

(b) Prove that the metric space $(B(S), d)$ is complete.

Hint. If (f_n) is a Cauchy sequence in $B(S)$, show that $\{f_n(x)\}$ is a Cauchy sequence of real numbers for each x in S.

Solution. (a) Nonnegativity, symmetry, and triangle inequality follow from properties of the sup norm; $\|f-g\|=0$ iff $f=g$.

(b) If (f_n) is Cauchy in sup norm, then for each x, $(f_n(x))$ is Cauchy in $\mathbb{R}$ and converges to some $f(x)$. Uniform Cauchy-ness yields $\|f_n-f\|\to 0$, so $f\in B(S)$ and $f_n\to f$ in d.

4.67: The Metric Space of Continuous Bounded Functions

Refer to Exercise 4.66 and let $C(S)$ denote the subset of $B(S)$ consisting of all functions continuous and bounded on S, where now S is a metric space.

(a) Prove that $C(S)$ is a closed subset of $B(S)$.

(b) Prove that the metric subspace $C(S)$ is complete.

Solution. (a) If $f_n\in C(S)$ and $\|f_n-f\|\to 0$, then f is the uniform limit of continuous functions, hence continuous. Thus $f\in C(S)$; $C(S)$ is closed.

(b) Closed subspaces of complete metric spaces are complete. Alternatively, repeat the proof in 4.66 and use uniform convergence to pass continuity to the limit.

4.68: Application of the Fixed-Point Theorem

Refer to the proof of the fixed-point theorem (Theorem 4.48) for notation.

(a) Prove that $d(p_n,p_{n+1})\le d(x,f(x))\alpha^n/(1-\alpha)$. This inequality, which is useful in numerical work, provides an estimate for the distance from p_n to the fixed point p. An example is given in (b).

(b) Take $f(x)=(x+2/x)/2$, $S=[1,+\infty)$. Prove that f is a contraction of S with contraction constant $\alpha=1/2$ and fixed point $p=\sqrt{2}$. Form the sequence (p_n) starting with $x=p_0=1$ and show that $|p_n-\sqrt{2}|\le 2^{-n}$.

Solution. (a) In a contraction with constant $\alpha\in(0,1)$, $d(p_{n+k},p_{n+k+1})\le \alpha^{n+k}d(x,f(x))$. Hence

$$d(p_n,p)\le\sum_{k=0}^{\infty}d(p_{n+k},p_{n+k+1})\le d(x,f(x))\sum_{k=0}^{\infty}\alpha^{n+k}=\frac{\alpha^n}{1-\alpha}d(x,f(x)).$$

(b) On $[1,\infty)$, $f'(x)=\frac{1}{2}\left(1-\frac{2}{x^2}\right)$ so $|f'(x)|\le\frac{1}{2}$, hence f is a contraction with $\alpha=1/2$. Fixed points solve $x=\frac{1}{2}(x+2/x)$, i.e., $x^2=2$, so $p=\sqrt{2}$. The bound in (a) gives $|p_n-\sqrt{2}|\le 2^{-n}|x-f(x)|\cdot\frac{1}{1-1/2}=2^{-n}\cdot 2|x-f(x)|$. With

$x = p_0 = 1$, a direct induction using the mean value theorem or the quadratic convergence of the Babylonian method yields $|p_n - \sqrt{2}| \le 2^{-n}$.

4.69: Necessity of Conditions for Fixed-Point Theorem

Show by counterexamples that the fixed-point theorem for contractions need not hold if either (a) the underlying metric space is not complete, or (b) the contraction constant $\alpha \ge 1$.

Solution. (a) Let $S = (0, 1)$ with usual metric and $f(x) = x/2$. Then f is a contraction with fixed point $0 \notin S$; no fixed point in S.

(b) Take $S = \mathbb{R}$ and $f(x) = x + 1$; Lipschitz constant $\alpha = 1$ but no fixed point. Or $f(x) = 2x$ with $\alpha = 2$.

4.70: Generalized Fixed-Point Theorem

Let $f : S \to S$ be a function from a complete metric space (S, d) into itself. Assume there is a real sequence (a_n) which converges to 0 such that $d(f^n(x), f^n(y)) \le a_n d(x, y)$ for all $n \ge 1$ and all x, y in S, where f^n is the nth iterate of f, that is,
$f^1(x) = f(x)$, $f^{n+1}(x) = f(f^n(x))$, for $n \ge 1$.
Prove that f has a unique fixed point. *Hint.* Apply the fixed-point theorem to f^m for a suitable m.

Solution. Pick m large so that $a_m < 1$. Then f^m is a contraction: $d(f^m(x), f^m(y)) \le a_m d(x, y)$. By the contraction mapping theorem, f^m has a unique fixed point p. Then $f(p)$ is also a fixed point of f^m, hence $f(p) = p$. Uniqueness for f follows similarly.

4.71: Fixed Points for Distance-Shrinking Maps

Let $f : S \to S$ be a function from a metric space (S, d) into itself such that

$$d(f(x), f(y)) < d(x, y) \quad \text{whenever } x \ne y.$$

(a) Prove that f has at most one fixed point, and give an example of such an f with no fixed point.

(b) If S is compact, prove that f has exactly one fixed point. *Hint.* Show that $g(x) = d(x, f(x))$ attains its minimum on S.

(c) Give an example with S compact in which f is not a contraction.

Solution. (a) If $f(x) = x$ and $f(y) = y$ with $x \neq y$, then $d(x, y) = d(f(x), f(y)) < d(x, y)$, impossible. Example without a fixed point: $S = \mathbb{R}$, $f(x) = x + 1$.

(b) On compact S, $g(x) = d(x, f(x))$ attains a minimum at p. If $g(p) > 0$, then $g(f(p)) = d(f(p), f^2(p)) < d(p, f(p)) = g(p)$, contradiction. Hence $g(p) = 0$ and p is a fixed point. Uniqueness holds by (a).

(c) Take $S = [0, 1]$ and $f(x) = \sqrt{x}$. Then $d(f(x), f(y)) < d(x, y)$ for $x \neq y$, but f is not Lipschitz with constant < 1 on $[0, 1]$.

4.72: Iterated Function Systems

Assume that f satisfies the condition in Exercise 4.71. If $x \in S$, let $p_0 = x$, $p_{n+1} = f(p_n)$, and $c_n = d(p_n, p_{n+1})$ for $n \geq 0$.

(a) Prove that $\{c_n\}$ is a decreasing sequence, and let $c = \lim c_n$.

(b) Assume there is a subsequence $\{p_{k(n)}\}$ which converges to a point q in S. Prove that

$$c = d(q, f(q)) = d(f(q), f[f(q)]).$$

Deduce that q is a fixed point of f and that $p_n \to q$.

Solution. (a) By the shrinking property,

$$c_{n+1} = d(p_{n+1}, p_{n+2}) = d(f(p_n), f(p_{n+1})) < d(p_n, p_{n+1}) = c_n,$$

so (c_n) is strictly decreasing and converges to some $c \geq 0$.

(b) If $p_{k(n)} \to q$, then $p_{k(n)+1} = f(p_{k(n)}) \to f(q)$ by continuity of f (which follows from the shrinking property). Hence

$$c = \lim c_{k(n)} = \lim d(p_{k(n)}, p_{k(n)+1}) = d(q, f(q)),$$

and similarly $c = \lim c_{k(n)+1} = d(f(q), f(f(q)))$. Applying the shrinking property to q and $f(q)$ yields $d(f(q), f(f(q))) < d(q, f(q))$ unless $q = f(q)$. Therefore $c = 0$ and q is a fixed point. Then $c_n \to 0$ and (p_n) is Cauchy; in a compact (or complete with suitable conditions) space it converges to q.

Chapter 5

Derivatives

5.1: Lipschitz Condition and Continuity

A function f is said to satisfy a Lipschitz condition of order α at c if there exists a positive number M (which may depend on c) and a 1-ball $B(c)$ such that

$$|f(x) - f(c)| < M|x - c|^{\alpha}$$

whenever $x \in B(c), x \neq c$.
a) Show that a function which satisfies a Lipschitz condition of order α is continuous at c if $\alpha > 0$, and has a derivative at c if $\alpha > 1$.
b) Give an example of a function satisfying a Lipschitz condition of order 1 at c for which $f'(c)$ does not exist.

Solution. If $\alpha > 0$ then $|f(x) - f(c)| \leq M|x - c|^{\alpha} \to 0$ as $x \to c$, so f is continuous at c. If $\alpha > 1$ then, for $x \neq c$,

$$\left| \frac{f(x) - f(c)}{x - c} \right| \leq M|x - c|^{\alpha - 1} \to 0,$$

so $f'(c) = 0$ exists. For (b), $f(x) = |x|$ satisfies a Lipschitz condition of order 1 at 0, but $f'(0)$ does not exist.

5.2: Monotonicity and Extrema

In each of the following cases, determine the intervals in which the function f is increasing or decreasing and find the maxima and minima (if any) in the set where each f is defined. a) $f(x) = x^3 + ax + b, \quad x \in \mathbb{R}$. b) $f(x) = \log(x^2 - 9), \quad |x| > 3$. c) $f(x) = x^{2/3}(x - 1)^4, \quad 0 \leq x \leq 1$. d) $f(x) = (\sin x)/x$ if $x \neq 0, f(0) = 1, \quad 0 \leq x \leq \pi/2$.

Solution. (a) $f'(x) = 3x^2 + a$. If $a \geq 0$ then $f' > 0$ on $\mathbb{R}$ and f is strictly increasing (no extrema). If $a < 0$, set $r = \sqrt{-a/3}$. Then $f' > 0$ on $(-\infty, -r) \cup (r, \infty)$ and $f' < 0$ on $(-r, r)$, so f has a local maximum at $x = -r$ and a local minimum at $x = r$. Using $a = -3r^2$,

$$f(\pm r) = \pm r^3 - 3r^2(\pm r) + b = b \mp 2r^3.$$

(b) On $(-\infty, -3)$ and $(3, \infty)$, $f'(x) = \dfrac{2x}{x^2 - 9}$ has the sign of x. Thus f decreases on $(-\infty, -3)$ and increases on $(3, \infty)$. No maxima/minima on the domain; $f \to -\infty$ as $x \to \pm 3$.
(c) On $[0, 1]$, $f(x) = x^{2/3}(x - 1)^4 > 0$ except at $0, 1$. Writing $\ln f = \frac{2}{3} \ln x + 4 \ln(1 - x)$ gives

$$\frac{f'}{f} = \frac{2}{3x} - \frac{4}{1 - x} = 0 \iff x = \frac{1}{7}.$$

Thus f increases on $(0, 1/7)$, decreases on $(1/7, 1)$, with a unique interior maximum at $x = 1/7$, and zeros (hence minima) at 0 and 1.
(d) For $x > 0$, $f'(x) = \dfrac{x \cos x - \sin x}{x^2} < 0$ on $(0, \pi/2]$ (since $\tan x > x$ there). Hence f is decreasing on $[0, \pi/2]$; its maximum is $f(0) = 1$ and its minimum is $f(\pi/2) = 2/\pi$.

5.3: Polynomial Interpolation

Find a polynomial f of lowest possible degree such that

$$f(x_1) = a_1, \quad f(x_2) = a_2, \quad f'(x_1) = b_1, \quad f'(x_2) = b_2,$$

where $x_1 \neq x_2$ and a_1, a_2, b_1, b_2 are given real numbers.

Solution. The minimal degree is 3 (Hermite data at two nodes). The unique cubic can be written with Hermite basis polynomials:

$$\begin{aligned} f(x) &= a_1 H_{10}(x) + a_2 H_{20}(x) + b_1 H_{11}(x) + b_2 H_{21}(x), \\ H_{10}(x) &= \left(1 - 2\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_2}{x_1 - x_2}\right)^2, \quad H_{11}(x) = (x - x_1)\left(\frac{x - x_2}{x_1 - x_2}\right)^2, \\ H_{20}(x) &= \left(1 - 2\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)^2, \quad H_{21}(x) = (x - x_2)\left(\frac{x - x_1}{x_2 - x_1}\right)^2. \end{aligned}$$

Then $f(x_i) = a_i$ and $f'(x_i) = b_i$ follow by construction.

5.4: Smoothness of Exponential Function

Define f as follows: $f(x) = e^{-1/x^2}$ if $x \neq 0, f(0) = 0$. Show that a) f is continuous for all x. b) $f^{(n)}$ is continuous for all x, and that $f^{(n)}(0) = 0, (n = 1, 2, \ldots)$.

Solution. For $x \neq 0$, f is C^∞. At 0, $f(x) \to 0$ as $x \to 0$, so f is continuous. Moreover, for each $n \geq 1$ there is a polynomial $P_n(1/x)$ such that $f^{(n)}(x) = P_n(1/x)\, e^{-1/x^2}$ for $x \neq 0$. Since e^{-1/x^2} decays faster than any power as $x \to 0$, $\lim_{x\to 0} f^{(n)}(x) = 0$. Define $f^{(n)}(0) = 0$; then $f^{(n)}$ is continuous for all n.

5.5: Derivatives of Trigonometric Functions

Define f, g, and h as follows: $f(0) = g(0) = h(0) = 0$ and, if $x \neq 0, f(x) = \sin(1/x), g(x) = x\sin(1/x), h(x) = x^2 \sin(1/x)$. Show that a) $f'(x) = -1/x^2 \cos(1/x)$, if $x \neq 0$; $f'(0)$ does not exist. b) $g'(x) = \sin(1/x) - 1/x\cos(1/x)$, if $x \neq 0$; $g'(0)$ does not exist. c) $h'(x) = 2x\sin(1/x) - \cos(1/x)$, if $x \neq 0$; $h'(0) = 0; \lim_{x\to 0} h'(x)$ does not exist.

Solution. For $x \neq 0$, use the chain and product rules:

$$f'(x) = \cos(1/x)\cdot(-1/x^2), \quad g'(x) = \sin(1/x) + x\cos(1/x)\cdot(-1/x^2),$$

$$h'(x) = 2x\sin(1/x) + x^2\cos(1/x)\cdot(-1/x^2) = 2x\sin(1/x) - \cos(1/x).$$

At 0, $\lim_{x\to 0} \dfrac{\sin(1/x)}{x}$ does not exist, so $f'(0)$ and $g'(0)$ do not exist. For h, $\dfrac{h(x) - h(0)}{x} = x\sin(1/x) \to 0$, so $h'(0) = 0$. But $h'(x)$ oscillates without limit as $x \to 0$.

5.6: Leibnitz's Formula

Derive Leibnitz's formula for the nth derivative of the product h of two functions f and g:

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x), \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Solution. Induct on n. For $n = 1$ the statement is the product rule. Assume true for n. Differentiate

$$h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

to get

$$h^{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)},$$

reindex and use $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ to obtain the desired formula for $n + 1$.

5.7: Relations for Derivatives

Let f and g be two functions defined and having finite third-order derivatives $f''(x)$ and $g''(x)$ for all x in $\mathbb{R}$. If $f(x)g(x) = 1$ for all x, show that the relations in (a), (b), (c), and (d) hold at those points where the denominators are not zero: a) $f'(x)/f(x)+g'(x)/g(x) = 0$. b) $f''(x)/f'(x)-2f'(x)/f(x)-g''(x)/g'(x) = 0$. c) $\frac{f'''(x)}{f'(x)} - 3\frac{f'(x)g''(x)}{f(x)g'(x)} - 3\frac{f''(x)}{f(x)} - \frac{g'''(x)}{g'(x)} = 0$.
d) $\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$.
NOTE. The expression which appears on the left side of (d) is called the Schwarzian derivative of f at x. e) Show that f and g have the same Schwarzian derivative if

$$g(x) = [af(x) + b][cf(x) + d], \text{where} ad - bc \neq 0.$$

Solution. Since $fg \equiv 1$, $(\ln f)' + (\ln g)' = 0$, which gives (a): $f'/f + g'/g = 0$. Differentiating (a) and simplifying yields (b). Repeating once more yields (c). For (d), differentiate $\dfrac{f''}{f'} - \dfrac{3}{2}\Big(\dfrac{f''}{f'}\Big)^2$ and use (a)–(c) to see the derivatives of the two sides agree and the values coincide at one point, hence they are equal. For (e), interpreting g as the fractional linear transform $g = \dfrac{af+b}{cf+d}$ (with $ad - bc \neq 0$), the Schwarzian derivative is invariant under Möbius transformations, so $Sf \equiv Sg$.

5.8: Derivative of a Determinant

Let f_1, f_2, g_1, g_2 be four functions having derivatives in (a, b). Define F by means of the determinant

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}, \quad \text{if} x \in (a, b).$$

a) Show that $F'(x)$ exists for each x in (a, b) and that

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

b) State and prove a more general result for nth order determinants.

Solution. By the product rule on the bilinear expansion of the 2×2 determinant,

$$\frac{d}{dx}\det\begin{pmatrix} f_1 & f_2 \\ g_1 & g_2 \end{pmatrix} = \det\begin{pmatrix} f_1' & f_2' \\ g_1 & g_2 \end{pmatrix} + \det\begin{pmatrix} f_1 & f_2 \\ g_1' & g_2' \end{pmatrix}.$$

For an $n\times n$ determinant, multilinearity in the rows gives $\big(\det F\big)' = \sum_{j=1}^{n} \det(F$ with the jth row differe

5.9: Wronskian and Linear Dependence

Given n functions $f_1, \ldots, f_n$, each having nth order derivatives in (a, b). A function W, called the Wronskian of $f_1, \ldots, f_n$, is defined as follows: For each x in (a, b), $W(x)$ is the value of the determinant of order n whose element in the kth row and mth column is $f_m^{(k-1)}(x)$, where $k = 1, 2, \ldots, n$ and $m = 1, 2, \ldots, n$. [The expression $f_m^{(0)}(x)$ is written for $f_m(x)$.] a) Show that $W'(x)$ can be obtained by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}(x), \ldots, f_n^{(n)}(x)$.
b) Assuming the existence of n constants $c_1, \ldots, c_n$, not all zero, such that $c_1 f_1(x) + \cdots + c_n f_n(x) = 0$ for every x in (a, b), show that $W(x) = 0$ for each x in (a, b).
NOTE. A set of functions satisfying such a relation is said to be a linearly dependent set on (a, b).
c) The vanishing of the Wronskian throughout (a, b) is necessary, but not sufficient, for linear dependence of $f_1, \ldots, f_n$. Show that in the case of two functions, if the Wronskian vanishes throughout (a, b) and if one of the functions does not vanish in (a, b), then they form a linearly dependent set in (a, b).

Solution. (a) Differentiate the determinant by the rule in 5.8: only the last row changes to $\big(f_1^{(n)}(x), \ldots, f_n^{(n)}(x)\big)$. (b) If $\sum c_m f_m \equiv 0$ with some c_m not all 0, then each row of the Wronskian is a linear combination of the others with the same coefficients, so the determinant vanishes identically. (c) For two functions, $W = f_1 f_2' - f_1' f_2 \equiv 0$ on (a, b) and, say, $f_2 \neq 0$ on (a, b). Then $(f_1/f_2)' = \dfrac{f_1' f_2 - f_1 f_2'}{f_2^2} = 0$, so f_1/f_2 is constant and the pair is linearly dependent.

5.10: Infinite Limit and Derivative

Given a function f defined and having a finite derivative in (a, b) and such that $\lim_{x \to b^-} f(x) = +\infty$. Prove that $\lim_{x \to b^-} f'(x)$ either fails to exist or is infinite.

Solution. Suppose $\lim_{x \to b^-} f(x) = +\infty$ and $\lim_{x \to b^-} f'(x) = L \in \mathbb{R}$. Fix $h > 0$ small, pick x close to b; by the mean value theorem there is $\xi \in (x, b)$ with $\dfrac{f(b-h) - f(x)}{b - h - x} = f'(\xi)$. Letting $x \to b^-$ forces the left side to $-\infty$ while the right tends to L, a contradiction. Hence the limit of f' cannot be finite; it either diverges or fails to exist.

5.11: Mean-Value Theorem and Theta

Show that the formula in the Mean-Value Theorem can be written as follows:

$$\frac{f(x+h)-f(x)}{h} = f'(x+\theta h),$$

where $0<\theta<1$. Determine θ as a function of x and h when a) $f(x)=x^2$, b) $f(x)=x^3$, c) $f(x)=e^x$, d) $f(x)=\log x, \quad x>0$.
Keep $x \neq 0$ fixed, and find $\lim_{h\to 0}\theta$ in each case.

Solution. By the mean value theorem, for each $h \neq 0$ there is $\theta \in (0,1)$ with $\dfrac{f(x+h)-f(x)}{h} = f'(x+\theta h)$. Compute θ casewise:

$$f(x)=x^2: \qquad \frac{(x+h)^2-x^2}{h} = 2x+h = 2(x+\theta h) \Rightarrow \theta = \tfrac{1}{2}.$$

$$f(x)=x^3: \qquad \frac{(x+h)^3-x^3}{h} = 3x^2+3xh+h^2 = 3(x+\theta h)^2,$$

$$\text{so } \theta = \frac{-x+\sqrt{x^2+xh+\frac{1}{3}h^2}}{h} \in (0,1).$$

$$f(x)=e^x: \qquad \frac{e^{x+h}-e^x}{h} = e^{x+\theta h} \Rightarrow \theta = \frac{1}{h}\log\frac{e^h-1}{h}.$$

$$f(x)=\log x\ (x>0): \qquad \frac{\log(x+h)-\log x}{h} = \frac{1}{x+\theta h} \Rightarrow \theta = \frac{1}{h}\Big(\frac{h}{\log(1+h/x)} - x\Big).$$

Fix $x \neq 0$. In each case, expanding for small h shows $\lim_{h\to 0}\theta = \frac{1}{2}$.

5.12: Cauchy's Mean-Value Theorem

Take $f(x)=3x^4-2x^3-x^2+1$ and $g(x)=4x^3-3x^2-2x$ in Theorem 5.20. Show that $f'(x)/g'(x)$ is never equal to the quotient $[f(1)-f(0)]/[g(1)-g(0)]$ if $0<x\le 1$. How do you reconcile this with the equation

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_1)}{g'(x_1)}, \qquad a<x_1<b,$$

obtainable from Theorem 5.20 when $n=1$?

Solution. Compute $f(1)-f(0)=0$ and $g(1)-g(0)\neq 0$, hence the quotient is 0. On $(0,1]$, $f'(x)=2x(6x^2-3x-1)$ and $g'(x)=2(6x^2-3x-1)$ vanish at the same point $x_0 = \dfrac{3+\sqrt{33}}{12} \in (0,1]$, so $f'(x)/g'(x)$ is never 0 for $x \in (0,1]$. This does not contradict Cauchy's theorem: the correct conclusion is $(f(1)-f(0))g'(x_1) = (g(1)-g(0))f'(x_1)$ for some $x_1 \in (0,1)$, which holds at $x_1 = x_0$ (both sides are 0). The "ratio" form fails there because $g'(x_1)=0$.

5.13: Special Cases of Mean-Value Theorem

In each of the following special cases of Theorem 5.20, take $n = 1$, $c = a$, $x = b$, and show that $x_1 = (a+b)/2$.
a) $f(x) = \sin x$, $\quad g(x) = \cos x$; b) $f(x) = e^x$, $\quad g(x) = e^{-x}$.
Can you find a general class of such pairs of functions f and g for which x_1 will always be $(a+b)/2$ and such that both examples (a) and (b) are in this class?

Solution. For (a), with $f = \sin$, $g = \cos$, Cauchy's theorem gives $(\sin b - \sin a)(-\sin x_1) = (\cos b - \cos a)\cos x_1$. Using sum-to-product identities this reduces to $\sin\left(\frac{a+b}{2} - x_1\right) = 0$, hence $x_1 = \frac{a+b}{2}$. For (b), $f = e^x$, $g = e^{-x}$ yields $e^b - e^a = (e^{-a} - e^{-b})e^{2x_1}$, whence $x_1 = \frac{a+b}{2}$. A general class: pairs f, g solving a linear ODE $y'' + \lambda y = 0$ (e.g., $\sin, \cos$) or $y'' - \lambda y = 0$ (e.g., e^x, e^{-x}) have this midpoint property.

5.14: Limit of a Sequence

Given a function f defined and having a finite derivative f' in the half-open interval $0 < x \le 1$ and such that $|f'(x)| < 1$. Define $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$, and show that $\lim_{n\to\infty} a_n$ exists. Hint. Cauchy condition.

Solution. For m, n, by the mean value theorem there is ξ between $1/m$ and $1/n$ with

$$|a_m - a_n| = |f(1/m) - f(1/n)| \le |f'(\xi)| \left|\frac{1}{m} - \frac{1}{n}\right| \le \alpha \left|\frac{1}{m} - \frac{1}{n}\right|,$$

for some $\alpha < 1$. Hence (a_n) is Cauchy, so $\lim a_n$ exists.

5.15: Limit of Derivative

Assume that f has a finite derivative at each point of the open interval (a, b). Assume also that $\lim_{x\to c} f'(x)$ exists and is finite for some interior point c. Prove that the value of this limit must be $f'(c)$.

Solution. We have

$$\frac{f(x) - f(c)}{x - c} - f'(x) = \frac{f(x) - f(c) - (x-c)f'(x)}{x - c}.$$

By Cauchy's mean value theorem applied to $F(t) = f(t) - f(c) - (t-c)f'(x)$ and $G(t) = t - c$, there is ξ between x and c such that the quotient equals $\dfrac{f'(\xi) - f'(x)}{1}$. Letting $x \to c$ gives $\dfrac{f(x) - f(c)}{x - c} \to L$, hence $f'(c) = L$.

5.16: Extension of Derivative

Let f be continuous on (a,b) with a finite derivative f' everywhere in (a,b), except possibly at c. If $\lim_{x\to c} f'(x)$ exists and has the value A, show that $f'(c)$ must also exist and have the value A.

Solution. As in 5.15, for $x \neq c$ choose ξ between x and c to get

$$\frac{f(x)-f(c)}{x-c} - A = f'(\xi) - A.$$

Let $x \to c$; then $\xi \to c$ and $f'(\xi) \to A$ by hypothesis, so the difference quotient tends to A. Thus $f'(c)$ exists and equals A.

5.17: Monotonicity of Quotient

Let f be continuous on $[0,1]$, $f(0) = 0$, $f'(x)$ finite for each x in $(0,1)$. Prove that if f' is an increasing function on $(0,1)$, then so too is the function g defined by the equation $g(x) = f(x)/x$.

Solution. For $0 < u < v \leq 1$, apply Cauchy's mean value theorem to f and $x \mapsto x$ on $[u,v]$ to get

$$\frac{f(v)-f(u)}{v-u} = f'(\xi) \quad (\xi \in (u,v)).$$

Then

$$\frac{f(v)}{v} - \frac{f(u)}{u} = \frac{uf(v)-vf(u)}{uv} = \frac{u[vf'(\xi)-(f(v)-f(u))]}{uv} = \frac{u(v-\xi)}{uv}\,[f'(\xi)-f'(\eta)] \geq 0,$$

using the mean value theorem on f again and the monotonicity of f'. Hence $g(x) = f(x)/x$ is increasing.

5.18: Rolle's Theorem Application

Assume f has a finite derivative in (a,b) and is continuous on $[a,b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some c in (a,b) such that $f'(c) = \lambda f(c)$. Hint. Apply Rolle's theorem to $g(x)f(x)$ for a suitable g depending on λ.

Solution. Fix $\lambda \in \mathbb{R}$ and set $g(x) = e^{-\lambda x}$. Then $(gf)(a) = (gf)(b) = 0$. By Rolle's theorem there is $c \in (a,b)$ with $(gf)'(c) = 0$, i.e., $-\lambda e^{-\lambda c} f(c) + e^{-\lambda c} f'(c) = 0$, so $f'(c) = \lambda f(c)$.

5.19: Second Derivative and Secant Line

Assume f is continuous on $[a, b]$ and has a finite second derivative f'' in the open interval (a, b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(c) = 0$ for some c in (a, b).

Solution. Let ℓ be the secant line through $(a, f(a))$ and $(b, f(b))$, and $\phi = f - \ell$. Then $\phi(a) = \phi(b) = \phi(p) = 0$. By Rolle's theorem, there exist $u \in (a, p)$ and $v \in (p, b)$ with $\phi'(u) = \phi'(v) = 0$. Applying Rolle again to ϕ' on $[u, v]$ yields $c \in (u, v)$ with $\phi''(c) = 0$, hence $f''(c) = 0$.

5.20: Third Derivative Condition

If f has a finite third derivative f'' in $[a, b]$ and if

$$f(a) = f'(a) = f(b) = f'(b) = 0,$$

prove that $f''(c) = 0$ for some c in (a, b).

Solution. From $f(a) = f(b) = 0$, there exists $s \in (a, b)$ with $f'(s) = 0$. Since also $f'(a) = f'(b) = 0$, applying Rolle to f' on $[a, s]$ (or $[s, b]$) gives c with $f''(c) = 0$.

5.21: Nonnegative Function with Zeros

Assume f is nonnegative and has a finite third derivative f'' in the open interval $(0,\ 1)$. If $f(x) = 0$ for at least two values of x in $(0,\ 1)$, prove that $f''(c) = 0$ for some c in $(0,\ 1)$.

Solution. Let $u < v$ be two zeros of f in $(0, 1)$. Because $f \geq 0$ and $f(u) = 0$ at an interior point, necessarily $f'(u) = 0$; similarly $f'(v) = 0$. Apply 5.20 on $[u, v]$ to conclude that $f''(c) = 0$ for some $c \in (u, v) \subset (0, 1)$.

5.22: Behavior at Infinity

Assume f has a finite derivative in some interval $(a, +\infty)$. a) If $f(x) \to 1$ and $f'(x) \to c$ as $x \to +\infty$, prove that $c = 0$. b) If $f'(x) \to 1$ as $x \to +\infty$, prove that $f(x)/x \to 1$ as $x \to +\infty$. c) If $f'(x) \to 0$ as $x \to +\infty$, prove that $f(x)/x \to 0$ as $x \to +\infty$.

Solution. (a) If $f(x) \to 1$ and $f'(x) \to c$, then for fixed $h > 0$ and large x, $\dfrac{f(x+h) - f(x)}{h} \to c$ by the mean value theorem, while the numerator $\to 0$.

Hence $c = 0$. (b) Let $g(x) = f(x) - x$. Then $g'(x) = f'(x) - 1 \to 0$. For any $\varepsilon > 0$, for large x, $|g'(t)| < \varepsilon$ for $t \geq x$, so $|g(t) - g(x)| \leq \varepsilon|t - x|$. Taking $t = x$ and $t = 2x$ shows $|f(2x) - 2f(x)| \leq \varepsilon x$, which implies $\lim_{x\to\infty} f(x)/x = 1$. (c) Similarly, if $f'(x) \to 0$, then for large x, $|f(x) - f(0)| \leq \int_0^x |f'(t)|dt \leq \varepsilon x + C$, so $|f(x)/x| \leq \varepsilon + C/x \to 0$.

5.23: Nonexistence of Function

Let h be a fixed positive number. Show that there is no function f satisfying the following three conditions: $f'(x)$ exists for $x \geq 0$, $f'(0) = 0$, $f'(x) \geq h$ for $x > 0$.

Solution. If $f'(0) = 0$ and $f'(x) \geq h > 0$ for $x > 0$, then for $x > 0$, by the mean value theorem there is $\xi \in (0, x)$ with $\dfrac{f(x) - f(0)}{x - 0} = f'(\xi) \geq h$. Thus $\liminf_{x\downarrow 0} \dfrac{f(x) - f(0)}{x} \geq h$, contradicting $f'(0) = 0$.

5.24: Symmetric Difference Quotients

If $h > 0$ and if $f'(x)$ exists (and is finite) for every x in $(a-h, a+h)$, and if f is continuous on $[a-h, a+h]$, show that we have: a) $\frac{f(a+h)-f(a-h)}{h} = f'(a+\theta h) + f'(a-\theta h), \quad 0 < \theta < 1$; b) $\frac{f(a+h)-2f(a)+f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h), \quad 0 < \lambda < 1$. c) If $f''(a)$ exists, show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}.$$

d) Give an example where the limit of the quotient in (c) exists but where $f''(a)$ does not exist.

Solution. (a) Define $\phi(t) = f(t) - f(2a - t)$. Then ϕ is differentiable and by the mean value theorem there is $\theta \in (0, 1)$ with

$$\frac{f(a+h) - f(a-h)}{h} = \phi'(a+\theta h) = f'(a+\theta h) + f'(a-\theta h).$$

(b) Apply (a) to f' to get

$$\frac{f'(a+h) - f'(a-h)}{1} = \frac{f(a+h) - 2f(a) + f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h).$$

(c) If $f''(a)$ exists, by (b) the symmetric second difference quotient tends to $f''(a)$. (d) Let $f(x) = x|x|$. Then $f''(0)$ does not exist, but

$$\frac{f(h) - 2f(0) + f(-h)}{h^2} = \frac{h|h| + (-h)| - h|}{h^2} = 0 \to 0.$$

5.25: Uniform Differentiability

Let f have a finite derivative in (a, b) and assume that $c \in (a, b)$. Consider the following condition: For every $\varepsilon > 0$ there exists a 1-ball $B(c; \delta)$, whose radius δ depends only on ε and not on c, such that if $x \in B(c; \delta)$, and $x \neq c$, then

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

Show that f' is continuous on (a, b) if this condition holds throughout (a, b).

Solution. Fix c and $\varepsilon > 0$. By hypothesis choose δ (depending only on ε) so that for all $x, y \in (a, b)$ with $0 < |x - c|, |y - c| < \delta$,

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \tfrac{\varepsilon}{2}, \quad \left| \frac{f(y) - f(c)}{y - c} - f'(c) \right| < \tfrac{\varepsilon}{2}.$$

Then

$$|f'(x) - f'(y)| \leq \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| + \left| \frac{f(y) - f(c)}{y - c} - f'(c) \right| < \varepsilon.$$

Thus f' is Cauchy (hence continuous) at c. Since c was arbitrary, f' is continuous on (a, b).

5.26: Fixed Point Theorem

Assume f has a finite derivative in (a, b) and is continuous on $[a, b]$, with $a \leq f(x) \leq b$ for all x in $[a, b]$ and $|f'(x)| \leq \alpha < 1$ for all x in (a, b). Prove that f has a unique fixed point in $[a, b]$.

Solution. For $x, y \in [a, b]$, by the mean value theorem there exists ξ between x and y with $|f(x) - f(y)| = |f'(\xi)||x - y| \leq \alpha|x - y|$. Thus f is a contraction of the complete metric space $[a, b]$, so it has a unique fixed point by the contraction mapping theorem. Alternatively, iterate $x_{n+1} = f(x_n)$ to get a Cauchy sequence converging to the unique fixed point.

5.27: L'Hôpital's Rule Counterexample

Give an example of a pair of functions f and g having finite derivatives in (0, 1), such that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0,$$

but such that $\lim_{x \to 0} f'(x)/g'(x)$ does not exist, choosing g so that $g'(x)$ is never zero.

Solution. Let $g(x) = x$ and $f(x) = x^2 \sin(1/x)$ for $x \in (0, 1)$, with $f(0) = g(0) = 0$. Then $f(x)/g(x) = x \sin(1/x) \to 0$, while $f'(x)/g'(x) = \big(2x \sin(1/x) - \cos(1/x)\big)/1$ has no limit as $x \to 0$ and $g'(x) = 1 \neq 0$.

5.28: Generalized L'Hôpital's Rule

Prove the following theorem: Let f and g be two functions having finite nth derivatives in (a, b). For some interior point c in (a, b), assume that $f(c) = f'(c) = \cdots = f^{(n-1)}(c) = 0$, and that $g(c) = g'(c) = \cdots = g^{(n-1)}(c) = 0$, but that $g^{(n)}(x)$ is never zero in (a, b). Show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

NOTE. $f^{(n)}$ and $g^{(n)}$ are not assumed to be continuous at c.

Solution. Apply Cauchy's mean value theorem repeatedly or use Taylor's theorem with remainder about c. Since $f^{(k)}(c) = g^{(k)}(c) = 0$ for $k < n$, we have

$$f(x) = \frac{f^{(n)}(\xi_x)}{n!}(x - c)^n, \quad g(x) = \frac{g^{(n)}(\eta_x)}{n!}(x - c)^n$$

for some $\xi_x, \eta_x \to c$. Hence

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f^{(n)}(\xi_x)}{g^{(n)}(\eta_x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)},$$

using the assumed nonvanishing of $g^{(n)}$ near c.

5.29: Taylor's Theorem with Remainder

Show that the formula in Taylor's theorem can also be written as follows:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k + \frac{(x - c)(x - x_1)^{n-1}}{(n - 1)!} f^{(n)}(x_1),$$

where x_1 is interior to the interval joining x and c. Let $1 - \theta = (x - x_1)/(x - c)$. Show that $0 < \theta < 1$ and deduce the following form of the remainder term (due to Cauchy):

$$\frac{(1 - \theta)^{n-1}(x - c)^n}{(n - 1)!} f^{(n)}[\theta x + (1 - \theta)c].$$

Solution. By Cauchy's form of the remainder, for some x_1 between x and c,

$$f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x - c)^k = \frac{f^{(n)}(x_1)}{(n - 1)!}(x - c)^{n-1}(x - x_1).$$

Writing $x - x_1 = \theta(x - c)$ with $\theta \in (0, 1)$ gives the displayed form and yields Cauchy's remainder

$$\frac{(1-\theta)^{n-1}(x-c)^n}{(n-1)!}\, f^{(n)}\big(\theta x + (1-\theta)c\big).$$

5.30: Vector-Valued Differentiability

If a vector-valued function f is differentiable at c, prove that

$$f'(c) = \lim_{h \to 0} \frac{1}{h}[f(c+h) - f(c)].$$

Conversely, if this limit exists, prove that f is differentiable at c.

Solution. If f is differentiable at c, the definition gives the limit. Conversely, if the limit exists, define $f'(c)$ to be that limit; the standard ε–δ argument shows $\|f(c+h) - f(c) - f'(c)h\| = o(|h|)$, i.e., differentiability at c.

5.31: Constant Norm and Orthogonality

A vector-valued function f is differentiable at each point of (a, b) and has constant norm $\|f\|$. Prove that $f(t) \cdot f'(t) = 0$ on (a, b).

Solution. Differentiate $\|f(t)\|^2 = f(t) \cdot f(t)$ to get $\dfrac{d}{dt}\|f(t)\|^2 = 2f(t) \cdot f'(t)$. Since the left side is 0, we obtain $f(t) \cdot f'(t) = 0$ on (a, b).

5.32: Solution to Differential Equation

A vector-valued function f is never zero and has a derivative f' which exists and is continuous on $\mathbb{R}$. If there is a real function λ such that $f'(t) = \lambda(t)f(t)$ for all t, prove that there is a positive real function u and a constant vector c such that $f(t) = u(t)c$ for all t.

Solution. Let u solve $u'(t) = \lambda(t)u(t)$ with $u(t_0) = 1$; then $u(t) = \exp\big(\int_{t_0}^{t} \lambda\big) > 0$. Define $c = f/u$. Then $c'(t) = \dfrac{f'(t)u(t) - f(t)u'(t)}{u(t)^2} = 0$, so c is constant and $f(t) = u(t)\, c$.

5.33: Partial Derivatives and Continuity

Consider the function f defined on $\mathbb{R}^2$ by the following formulas:

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0) \quad f(0,0) = 0.$$

Prove that the partial derivatives $D_1 f(x,y)$ and $D_2 f(x,y)$ exist for every (x,y) in $\mathbb{R}^2$ and evaluate these derivatives explicitly in terms of x and y. Also, show that f is not continuous at (0, 0).

Solution. For $(x,y) \neq (0,0)$, compute

$$D_1 f(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2}, \quad D_2 f(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

At the origin,

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = 0, \quad D_2 f(0,0) = \lim_{h\to 0} \frac{f(0,h)-0}{h} = 0.$$

However, f is not continuous at 0 since along $y = x \neq 0$, $f(x,x) = \frac{1}{2} \not\to 0$.

5.34: Higher-Order Partial Derivatives

Let f be defined on $\mathbb{R}^2$ as follows:

$$f(x,y) = y\frac{x^2-y^2}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0), \quad f(0,0) = 0.$$

Compute the first- and second-order partial derivatives of f at the origin, when they exist.

Solution. For $(x,y) \neq (0,0)$, one computes

$$f_x(x,y) = \frac{4xy^3}{(x^2+y^2)^2}, \quad f_y(x,y) = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}.$$

At the origin,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = 0, \quad f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-0}{h} = -1.$$

Second-order at $(0,0)$ (where they exist):

$$f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = 0, \quad f_{yy}(0,0) = \lim_{h\to 0} \frac{f_y(0,h)-f_y(0,0)}{h} = 0,$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = 0, \quad f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} \text{ does not exist.}$$

5.35: Complex Conjugate Differentiability

Let S be an open set in $\mathbb{C}$ and let S^* be the set of complex conjugates $\bar{z}$, where $z \in S$. If f is defined on S, define g on S^* as follows: $g(\bar{z}) = \overline{f(z)}$, the complex conjugate of $f(z)$. If f is differentiable at c prove that g is differentiable at $\bar{c}$ and that $g'(\bar{c}) = \overline{f'(c)}$.

Solution. With $g(\bar{z}) = \overline{f(z)}$, for $h \to 0$,

$$\frac{g(\bar{c}+h) - g(\bar{c})}{h} = \overline{\frac{f(c+\bar{h}) - f(c)}{\bar{h}}} \to \overline{f'(c)}.$$

Thus g is differentiable at $\bar{c}$ and $g'(\bar{c}) = \overline{f'(c)}$.

5.36: Cauchy-Riemann Equations

i) In each of the following examples write $f = u + iv$ and find explicit formulas for $u(x, y)$ and $v(x, y)$: a) $f(z) = \sin z$, b) $f(z) = \cos z$, c) $f(z) = |z|$, d) $f(z) = \bar{z}$, e) $f(z) = \arg z \quad (z \neq 0)$, f) $f(z) = \log z \quad (z \neq 0)$, g) $f(z) = e^{z^2}$, h) $f(z) = z^\alpha \quad (\alpha \text{ complex}, z \neq 0)$.
ii) Show that u and v satisfy the Cauchy-Riemann equations for the following values of z: All z in (a), (b), (g); no z in (c), (d), (e); all z except real $z \leq 0$ in (f), (h). (In part (h), the Cauchy-Riemann equations hold for all z if α is a nonnegative integer, and they hold for all $z \neq 0$ if α is a negative integer.)
iii) Compute the derivative $f'(z)$ in (a), (b), (f), (g), (h), assuming it exists.

Solution. i) Standard expansions give: $\sin z = \sin x \cosh y + i \cos x \sinh y$, $\cos z = \cos x \cosh y - i \sin x \sinh y$, $|z| = \sqrt{x^2 + y^2}$, $\bar{z} = x - iy$, $\arg z = \arctan(y/x)$ $(z \neq 0)$, $\log z = \ln|z| + i \arg z$ $(z \neq 0)$, $e^{z^2} = e^{x^2-y^2}(\cos 2xy + i \sin 2xy)$, $z^\alpha = e^{\alpha \log z}$.
ii) The Cauchy–Riemann equations hold on the stated sets: everywhere for (a),(b),(g); nowhere for (c),(d),(e); on $\mathbb{C} \setminus (-\infty, 0]$ for branches of log and z^α, with the special cases as indicated.
iii) Where differentiable: $(\sin z)' = \cos z$, $(\cos z)' = -\sin z$, $(\log z)' = 1/z$, $(e^{z^2})' = 2ze^{z^2}$, $(z^\alpha)' = \alpha z^{\alpha-1}$ (on the domain of the chosen branch).

5.37: Constant Function Condition

Write $f = u + iv$ and assume that f has a derivative at each point of an open disk D centered at $(0, 0)$. If $au^2 + bv^2$ is constant on D for some real a and b, not both 0, prove that f is constant on D.

Solution. Let $f = u+iv$ be complex differentiable on D and suppose $au^2+bv^2 \equiv C$ with $(a,b) \neq (0,0)$. Differentiate: $auu_x + bvv_x = 0$ and $auu_y + bvv_y = 0$. Using the Cauchy–Riemann equations $u_x = v_y$, $u_y = -v_x$, we obtain the linear system

$$\begin{pmatrix} au & bv \\ -bv & au \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

On D, the determinant is $a^2u^2 + b^2v^2 \geq 0$. If it is nonzero at a point, then $u_x = v_x = 0$ there; by analyticity, $u_x \equiv v_x \equiv 0$ on the component, hence $u_y = v_y \equiv 0$ and f is constant. If it vanishes at a point, then $u = v = 0$ there; by the identity principle for holomorphic functions, $f \equiv 0$ in a neighborhood. In all cases f is constant on D.

Chapter 6

Functions of Bounded Variation and Rectifiable Curves

6.1 Functions of bounded variation

6.1: Functions of Bounded Variation

Determine which of the following functions are of bounded variation on $[0,1]$.
a) $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.
b) $f(x) = \sqrt{x}\sin(1/x)$ if $x \neq 0$, $f(0) = 0$.

Solution. (a) On $(0,1]$, $f'(x) = 2x\sin(1/x) - \cos(1/x)$ and

$$\int_0^1 |\cos(1/x)|\,dx = \int_1^\infty \frac{|\cos u|}{u^2}\,du < \infty, \qquad \int_0^1 2x|\sin(1/x)|\,dx < \infty.$$

Hence $f' \in L^1(0,1)$ and, integrating f' from ε to x and letting $\varepsilon \downarrow 0$, one gets $f(x) = \int_0^x f'(t)\,dt$, so f is absolutely continuous and therefore of bounded variation on $[0,1]$.

(b) Let $a_n = \dfrac{1}{(n+\frac{1}{2})\pi}$. Then $f(a_n) = (-1)^n\sqrt{a_n}$. If P is the partition with the points $\{a_n\}_{n\geq N}$, then

$$V_f(0,1) \;\geq\; \sum_{n\geq N} \big|f(a_{n+1}) - f(a_n)\big| \geq \sum_{n\geq N} \big(\sqrt{a_{n+1}} + \sqrt{a_n}\big) \asymp \sum_{n\geq N} \frac{1}{\sqrt{n}} = \infty.$$

Thus $f(x) = \sqrt{x}\sin(1/x)$ is not of bounded variation on $[0,1]$.

6.2: Uniform Lipschitz Condition

A function f, defined on $[a, b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a, b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x - y|^\alpha$ for all x and y in $[a, b]$. (Compare with Exercise 5.1.)
a) If f is such a function, show that $\alpha > 1$ implies f is constant on $[a, b]$, whereas $\alpha = 1$ implies f is of bounded variation on $[a, b]$.
b) Give an example of a function f satisfying a uniform Lipschitz condition of order $\alpha < 1$ on $[a, b]$ such that f is not of bounded variation on $[a, b]$.
c) Give an example of a function f which is of bounded variation on $[a, b]$ but which satisfies no uniform Lipschitz condition on $[a, b]$.

Solution. (a) If $\alpha > 1$, subdivide $[x, y]$ into n equal parts: then

$$|f(y) - f(x)| \leq n\,M\Big(\tfrac{y-x}{n}\Big)^\alpha = M(y-x)^\alpha\, n^{1-\alpha} \to 0\ (n \to \infty),$$

so $f(y) = f(x)$ for all $x < y$ and f is constant. If $\alpha = 1$, the estimate $|f(x) - f(y)| \leq M|x - y|$ shows f is Lipschitz; hence f is absolutely continuous and has $V_f(a, b) \leq M(b - a)$.

(b) For $0 < \alpha < 1$, a standard example is the Weierstrass-type series on $[0, 2\pi]$:

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha} \sin(2^k x).$$

One checks (by splitting frequencies at the dyadic scale $2^k \approx |x - y|^{-1}$) that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for some C. Moreover, the total variation of the Nth partial sum satisfies $V\big(\sum_{k=0}^{N} 2^{-k\alpha}\sin(2^k x)\big) \geq c\sum_{k=0}^{N} 2^{k(1-\alpha)} \to \infty$, so f is not of bounded variation.

(c) Let f be the step function $f(x) = \mathbf{1}_{[c,b]}(x)$ for some $c \in (a, b)$. Then f has bounded variation $V_f(a, b) = 1$ but is discontinuous, hence it satisfies no uniform Lipschitz condition on $[a, b]$.

6.3: Polynomials and Bounded Variation

Show that a polynomial f is of bounded variation on every compact interval $[a, b]$. Describe a method for finding the total variation of f on $[a, b]$ if the zeros of the derivative f' are known.

Solution. Polynomials are C^1, hence absolutely continuous on $[a, b]$, so $f \in BV[a, b]$ and

$$V_f(a, b) = \int_a^b |f'(x)|\,dx.$$

If $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are the ordered zeros of f' in (a, b), then f is monotone on each $[t_j, t_{j+1}]$ and

$$V_f(a, b) = \sum_{j=0}^{m} \left| f(t_{j+1}) - f(t_j) \right|.$$

6.4: Linear Space of Functions

A nonempty set S of real-valued functions defined on an interval $[a, b]$ is called a linear space of functions if it has the following two properties:
a) If $f \in S$, then $cf \in S$ for every real number c.
b) If $f \in S$ and $g \in S$, then $f + g \in S$.
Theorem 6.9 shows that the set V of all functions of bounded variation on $[a, b]$ is a linear space. If S is any linear space which contains all monotonic functions on $[a, b]$, prove that $V \subseteq S$. This can be described by saying that the functions of bounded variation form the smallest linear space containing all monotonic functions.

Solution. By Jordan's theorem (Theorem 6.13), every $f \in V$ can be written $f = g - h$ with g, h increasing on $[a, b]$. Since S contains all monotone functions and is a linear space, $g, h \in S$ and therefore $f = g - h \in S$. Hence $V \subseteq S$.

6.5: Monotonic Function Properties

Let f be a real-valued function defined on $[0, 1]$ such that $f(0) > 0$, $f(x) \neq x$ for all x, and $f(x) \leq f(y)$ whenever $x \leq y$. Let $A = \{x : f(x) > x\}$. Prove that $\sup A \in A$ and that $f(1) > 1$.

Solution. Let $A = \{x \in [0, 1] : f(x) > x\}$. Since $f(0) > 0$, we have $0 \in A$, so $A \neq \emptyset$. If also $B = \{x : f(x) < x\}$ were nonempty, let $s = \sup A$ and $t = \inf B$; then $s \leq t$. From $x_n \uparrow s$ in A we get $f(x_n) > x_n$ and, by monotonicity, $\limsup f(x_n) \leq f(s)$, whence $f(s) \geq s$. From $y_n \downarrow t$ in B we get similarly $f(t) \leq t$. Since $s \leq t$, this forces some u with $f(u) = u$, contradicting $f(x) \neq x$. Thus $B = \emptyset$ and $f(x) > x$ for all $x \in [0, 1]$. Hence $\sup A = 1 \in A$ and $f(1) > 1$.

6.6: Bounded Variation on Infinite Intervals

If f is defined everywhere in $\mathbb{R}^1$, then f is said to be of bounded variation on $(-\infty, +\infty)$ if f is of bounded variation on every finite interval and if there exists a positive number M such that $V_f(a, b) < M$ for all compact intervals $[a, b]$. The total variation of f on $(-\infty, +\infty)$ is then defined to be the sup of all numbers $V_f(a, b)$, $-\infty < a < b < +\infty$, and is denoted by $V_f(-\infty, +\infty)$. Similar definitions apply to half-open infinite intervals $[a, +\infty)$ and $(-\infty, b]$.
a) State and prove theorems for the infinite interval $(-\infty, +\infty)$ analogous to Theorems 6.7, 6.9, 6.10, 6.11, and 6.12.
b) Show that Theorem 6.5 is true for $(-\infty, +\infty)$ if "monotonic" is replaced by "bounded and monotonic." State and prove a similar modification of Theorem 6.13.

Solution. (a) With $V_f(-\infty, \infty) = \sup\{V_f(a, b) : a < b\}$, all finite-interval results extend: linearity and subadditivity of variation, stability under addition and scalar multiplication, and the characterization $f \in BV(-\infty, \infty)$ iff $V_f(-\infty, \infty) < \infty$. Proofs reduce to restricting to finite $[a, b]$ and taking sups.

(b) For $(-\infty, \infty)$, a monotone function has

$$V_f(-\infty, \infty) = \lim_{x\to\infty} f(x) - \lim_{x\to-\infty} f(x),$$

so it is of bounded variation iff it is bounded and monotone. Similarly, Jordan's theorem becomes: $f \in BV(-\infty, \infty)$ iff $f = g - h$ with g, h bounded monotone on $\mathbb{R}$.

6.7: Positive and Negative Variations

Assume that f is of bounded variation on $[a,b]$ and let

$$P = \{x_0, x_1, \dots, x_n\} \in \mathcal{P}[a,b].$$

As usual, write $\Delta f_k = f(x_k) - f(x_{k-1})$, $k = 1, 2, \dots, n$. Define

$$A(P) = \{k : \Delta f_k > 0\}, \quad B(P) = \{k : \Delta f_k < 0\}.$$

The numbers

$$p_f(a,b) = \sup \left\{ \sum_{k \in A(P)} \Delta f_k : P \in \mathcal{P}[a,b] \right\}$$

and

$$n_f(a,b) = \sup \left\{ \sum_{k \in B(P)} |\Delta f_k| : P \in \mathcal{P}[a,b] \right\}$$

are called, respectively, the positive and negative variations of f on $[a,b]$. For each x in $(a,b]$, let $V(x) = V_f(a,x)$, $p(x) = p_f(a,x)$, $n(x) = n_f(a,x)$, and let $V(a) = p(a) = n(a) = 0$. Show that we have: a) $V(x) = p(x) + n(x)$. b) $0 \le p(x) \le V(x)$ and $0 \le n(x) \le V(x)$. c) p and n are increasing on $[a,b]$. d) $f(x) = f(a) + p(x) - n(x)$. Part (d) gives an alternative proof of Theorem 6.13. e) $2p(x) = V(x) + f(x) - f(a)$, $2n(x) = V(x) - f(x) + f(a)$. f) Every point of continuity of f is also a point of continuity of p and of n.

Solution. Fix $x \in [a,b]$. For a partition P of $[a,x]$ write $\Delta f_k = f(x_k) - f(x_{k-1})$. Refining P if necessary we may separate positive and negative increments. Taking sups then yields

$$V(x) = \sup_P \sum_k |\Delta f_k| = \sup_P \Big(\sum_{k \in A(P)} \Delta f_k + \sum_{k \in B(P)} |\Delta f_k| \Big) = p(x) + n(x),$$

which gives (a) and (b). Monotonicity of p, n in x gives (c). For (d), observe that for any partition of $[a,x]$,

$$f(x) - f(a) = \sum_k \Delta f_k = \sum_{k \in A(P)} \Delta f_k - \sum_{k \in B(P)} |\Delta f_k| \le p(x) - n(x),$$

and taking sups in the positive and negative parts separately gives equality: $f(x) = f(a) + p(x) - n(x)$. Then (e) follows by adding/subtracting (a) and (d). For (f), if f is continuous at x_0, then V is continuous at x_0 and (e) implies the continuity of p and n there.

6.2 Curves

6.8: Equivalent Paths

Let f and g be complex-valued functions defined as follows:

$$f(t) = e^{2\pi i t} \quad \text{if } t \in [0,1], \quad g(t) = e^{2\pi i t} \quad \text{if } t \in [0,2].$$

a) Prove that f and g have the same graph but are not equivalent according to the definition in Section 6.12. b) Prove that the length of g is twice that of f.

Solution. As sets, $\{e^{2\pi i t} : t \in [0,1]\} = \{e^{2\pi i t} : t \in [0,2]\}$, so the graphs coincide (the unit circle). If f and g were equivalent, there would be a strictly increasing bijection $\phi : [0,2] \to [0,1]$ with $g = f \circ \phi$, impossible since g traverses the circle twice while f traverses it once. For lengths, repeating a rectifiable path twice doubles its length, so $\Lambda(g) = 2\Lambda(f)$.

6.9: Arc-Length Parameter

Let f be a rectifiable path of length L defined on $[a,b]$, and assume that f is not constant on any subinterval of $[a,b]$. Let s denote the arc-length function given by $s(x) = \Lambda_s(a,x)$ if $a < x \le b$, $s(a) = 0$.
a) Prove that s^{-1} exists and is continuous on $[0,L]$. b) Define $g(t) = f[s^{-1}(t)]$ if $t \in [0,L]$ and show that g is equivalent to f. Since $f(t) = g[s(t)]$, the function g is said to provide a representation of the graph of f with arc length as parameter.

Solution. (a) The arc-length function s is increasing and continuous on $[a,b]$, and strictly increasing under the hypothesis that f is not constant on any subinterval. Hence s is a homeomorphism from $[a,b]$ onto $[0,L]$ and s^{-1} is continuous.

(b) With $g(t) = f\big(s^{-1}(t)\big)$, the map $t \mapsto s^{-1}(t)$ is increasing and onto, so g is a reparametrization of f; thus g is equivalent to f and $f(t) = g\big(s(t)\big)$.

6.10: Symmetrization of Regions

Let f and g be two real-valued continuous functions of bounded variation defined on $[a, b]$, with $0 < f(x) < g(x)$ for each x in (a, b), $f(a) = g(a)$, $f(b) = g(b)$. Let h be the complex-valued function defined on the interval $[a, 2b - a]$ as follows:

$$h(t) = t + if(t), \quad \text{if } a \le t \le b,$$

$$h(t) = 2b - t + ig(2b - t), \quad \text{if } b \le t \le 2b - a.$$

a) Show that h describes a rectifiable curve Γ. b) Explain, by means of a sketch, the geometric relationship between f, g, and h. c) Show that the set of points

$$S = \{(x, y) : a \le x \le b, \quad f(x) \le y \le g(x)\}$$

is a region in $\mathbb{R}^2$ whose boundary is the curve Γ. d) Let H be the complex-valued function defined on $[a, 2b - a]$ as follows:

$$H(t) = t - \frac{1}{2}[g(t) - f(t)], \quad \text{if } a \le t \le b,$$

$$H(t) = t + \frac{1}{2}[g(2b - t) - f(2b - t)], \quad \text{if } b \le t \le 2b - a.$$

Show that H describes a rectifiable curve Γ_0 which is the boundary of the region

$$S_0 = \{(x, y) : a \le x \le b, \quad f(x) - g(x) \le 2y \le g(x) - f(x)\}.$$

e) Show that S_0 has the x-axis as a line of symmetry. (The region S_0 is called the symmetrization of S with respect to the x-axis.) f) Show that the length of Γ_0 does not exceed the length of Γ.

Solution. (a) Since $f, g \in BV[a, b]$, the graphs $t \mapsto t + if(t)$ and $t \mapsto t + ig(t)$ are rectifiable; concatenating them as in the definition of h yields a rectifiable closed curve Γ.

(b) The curve Γ runs along the upper graph $y = g(x)$ from $x = a$ to $x = b$ and returns along the lower graph $y = f(x)$ from $x = b$ to $x = a$, closing the boundary of the vertical strip between the two graphs.

(c) The region $S = \{(x, y) : a \le x \le b,\ f(x) \le y \le g(x)\}$ has boundary given by the two graphs $y = f(x)$ and $y = g(x)$ together with the vertical segments at $x = a$ and $x = b$, which is exactly the image of h.

(d) Writing $y_0(x) = \frac{1}{2}\big(g(x) - f(x)\big)$, the curve Γ_0 is traced by $x \mapsto x \pm iy_0(x)$, hence is rectifiable and bounds the symmetric vertical strip

$$S_0 = \{(x, y) : a \le x \le b,\ -y_0(x) \le y \le y_0(x)\}.$$

(e) Immediate from the definition of S_0 since $y \mapsto -y$ preserves the set.
(f) Parameterizing by $x \in [a,b]$, the lengths are

$$L(\Gamma) = \int_a^b \sqrt{1+f'(x)^2}\,dx + \int_a^b \sqrt{1+g'(x)^2}\,dx, \quad L(\Gamma_0) = 2\int_a^b \sqrt{1+\left(\tfrac{g'(x)-f'(x)}{2}\right)^2}\,dx,$$

with f', g' understood a.e. Using the inequality

$$\sqrt{1+u^2} + \sqrt{1+v^2} \;\geq\; 2\sqrt{1+\left(\tfrac{u-v}{2}\right)^2} \qquad (u, v \in \mathbb{R}),$$

and integrating yields $L(\Gamma) \geq L(\Gamma_0)$.

6.3 Absolute continuous functions

A real-valued function f defined on an interval $[a,b]$ is said to be absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n disjoint open subintervals (a_k, b_k) of $[a,b]$, $n = 1, 2, \ldots$, the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ. Absolutely continuous functions occur in the Lebesgue theory of integration and differentiation. The following exercises give some of their elementary properties.

6.11: Absolutely Continuous Functions

A real-valued function f defined on $[a,b]$ is said to be absolutely continuous on $[a,b]$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n disjoint open subintervals (a_k, b_k) of $[a,b]$, $n = 1, 2, \ldots$, the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ. Absolutely continuous functions occur in the Lebesgue theory of integration and differentiation. The following exercises give some of their elementary properties.
Prove that every absolutely continuous function on $[a,b]$ is continuous and of bounded variation on $[a,b]$. Note. There exist functions which are continuous and of bounded variation but not absolutely continuous.

Solution. Absolute continuity implies uniform continuity; hence f is continuous. Given $\varepsilon > 0$, choose δ from the definition. For any partition P with mesh $< \delta$,

$$\sum_k |f(x_k) - f(x_{k-1})| \leq \varepsilon,$$

so $V_f(a,b) < \infty$. Thus every absolutely continuous function is continuous and of bounded variation.

6.12: Lipschitz and Absolute Continuity

Prove that f is absolutely continuous if it satisfies a uniform Lipschitz condition of order 1 on $[a,b]$. (See Exercise 6.2.)

Solution. If $|f(x) - f(y)| \le M|x-y|$ on $[a,b]$, then for any disjoint intervals (a_k, b_k) with $\sum(b_k - a_k) < \delta$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \le M \sum_{k=1}^{n} (b_k - a_k) < M\delta.$$

Choosing $\delta = \varepsilon/M$ proves absolute continuity.

6.13: Operations on Absolutely Continuous Functions

If f and g are absolutely continuous on $[a,b]$, prove that each of the following is also: $|f|$, cf (c constant), $f+g$, $f \cdot g$; also f/g if g is bounded away from zero.

Solution. If f, g are absolutely continuous on $[a,b]$, then they are bounded. The functions cf and $f+g$ are absolutely continuous by linearity of the defining inequality. Also

$$\big||f(b_k)| - |f(a_k)|\big| \le |f(b_k) - f(a_k)|,$$

so $|f|$ is absolutely continuous. For the product,

$$|(fg)(b_k) - (fg)(a_k)| \le |f(b_k)|\,|g(b_k) - g(a_k)| + |g(a_k)|\,|f(b_k) - f(a_k)|,$$

and summing over k shows fg is absolutely continuous. If $|g| \ge m > 0$ on $[a,b]$, then $u \mapsto 1/u$ is Lipschitz on $[m, \infty)$, hence $1/g$ is absolutely continuous; therefore $f/g = f \cdot (1/g)$ is absolutely continuous.

Chapter 7

Riemann-Stieltjes Integral

7.1 Riemann-Stieltjes Integral

7.1: Direct Proof of Integral Identity

Prove that $\int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$, directly from Definition 7.1.

Solution. For any partition $P: \ a = x_0 < \cdots < x_n = b$, the upper and lower Darboux sums for the function $f \equiv 1$ are

$$U(P, 1, \alpha) = \sum_{k=1}^{n} M_k(1)(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^{n} (\alpha(x_k) - \alpha(x_{k-1})) = \alpha(b) - \alpha(a),$$

$$L(P, 1, \alpha) = \sum_{k=1}^{n} m_k(1)(\alpha(x_k) - \alpha(x_{k-1})) = \alpha(b) - \alpha(a).$$

Thus the upper and lower integrals agree and equal $\alpha(b) - \alpha(a)$.

7.2: Condition for Constant Function

If $f \in R(\alpha)$ on $[a, b]$ and if $\int_a^b f d\alpha = 0$ for every f which is monotonic on $[a, b]$, prove that α must be constant on $[a, b]$.

Solution. Assume α is increasing and not constant. Then there exist $c < d$ with $\alpha(d) > \alpha(c)$. Define a monotone nondecreasing function

$$f(x) = \begin{cases} 0, & a \leq x \leq c, \\ \dfrac{x - c}{d - c}, & c < x < d, \\ 1, & d \leq x \leq b. \end{cases}$$

For any partition containing c and d, the lower sum satisfies

$$L(P, f, \alpha) = \sum m_k(f)\,\Delta\alpha_k \geq (\alpha(b) - \alpha(d)) \cdot 1 \;+\; 0 \;\geq\; \alpha(b) - \alpha(d).$$

Hence the lower integral is $\geq \alpha(b) - \alpha(d) > 0$, so $\int_a^b f\,d\alpha > 0$, contradicting the hypothesis. Therefore α must be constant.

7.3: Alternative Definition of Riemann-Stieltjes Integral

The following definition of a Riemann-Stieltjes integral is often used in the literature: We say f is integrable with respect to α if there exists a real number A having the property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\|P\| < \delta$ and for every choice of t_k in $[x_{k-1}, x_k]$, we have $|S(P, f, \alpha) - A| < \epsilon$.
a) Show that if $\int_a^b f d\alpha$ exists according to this definition, then it also exists according to Definition 7.1 and the two integrals are equal.
b) Let $f(x) = \alpha(x) = 0$ for $a \leq x < c$, $f(x) = \alpha(x) = 1$ for $c < x \leq b$, $f(c) = 0$, $\alpha(c) = 1$. Show that $\int_a^b f d\alpha$ exists according to Definition 7.1 but does not exist by this second definition.

Solution. *(a)* Let A be as in the statement. Given $\varepsilon > 0$, pick δ so that $\|P\| < \delta$ implies $|S(P, f, \alpha) - A| < \varepsilon$ for every choice of tags. For such P, taking in each subinterval tags attaining $M_k(f)$ and $m_k(f)$ gives

$$L(P, f, \alpha) \leq A + \varepsilon \quad \text{and} \quad U(P, f, \alpha) \geq A - \varepsilon.$$

Thus the lower integral $\geq A - \varepsilon$ and the upper integral $\leq A + \varepsilon$ for all $\varepsilon > 0$, so both equal A and $f \in R(\alpha)$ with integral A by Definition 7.1.

(b) With f and α as given (jump at c), choose partitions P that contain c as a partition point. Then the only nonzero increment $\Delta\alpha$ occurs on an interval of the form $[x_{k-1}, c]$, where $f \equiv 0$; hence $U(P, f, \alpha) = L(P, f, \alpha) = 0$. Therefore $\int_a^b f\,d\alpha = 0$ by Definition 7.1. In the alternative definition, for partitions not containing c, the unique subinterval containing c yields $\Delta\alpha = 1$ while $f(t_k)$ can be 0 (if $t_k \leq c$) or 1 (if $t_k > c$). As the mesh tends to 0, the sums can be forced arbitrarily close to 0 or to 1 depending on tag choices, so there is no A satisfying the uniform tag condition. Hence the second definition fails.

7.4: Equivalence of Integral Definitions

If $f \in R$ according to Definition 7.1, prove that $\int_a^b f(x)dx$ also exists according to the definition of Exercise 7.3. [Contrast with Exercise 7.3(b).] Hint. Let $I = \int_a^b f(x)dx$, $M = \sup\{|f(x)| : x \in [a,b]\}$. Given $\epsilon > 0$, choose P_ϵ so that $U(P_\epsilon, f) < I + \epsilon/2$ (notation of Section 7.11). Let N be the number of subdivision points in P_ϵ and let $\delta = \epsilon/(2MN)$. If $\|P\| < \delta$, write

$$U(P, f) = \sum M_k(f)\Delta x_k = S_1 + S_2,$$

where S_1 is the sum of terms arising from those subintervals of P containing no points of P_ϵ and S_2 is the sum of the remaining terms. Then

$$S_1 \le U(P_\epsilon, f) < I + \epsilon/2 \quad \text{and} \quad S_2 \le NM\|P\| < NM\delta = \epsilon/2,$$

and hence $U(P, f) < I + \epsilon$. Similarly,

$$L(P, f) > I - \epsilon \text{ if } \|P\| < \delta' \quad \text{for some } \delta'.$$

Hence $|S(P, f) - I| < \epsilon$ if $\|P\| < \min(\delta, \delta')$.

Solution. Let $I = \int_a^b f\,dx$, $M = \sup_{[a,b]} |f|$. Using the hint, choose P_ε with $U(P_\varepsilon, f) < I + \varepsilon/2$, let N be its number of subintervals and set $\delta = \varepsilon/(2MN)$. If $\|P\| < \delta$, write $U(P, f) = S_1 + S_2$ as indicated, so $U(P, f) < I + \varepsilon$. Similarly, $L(P, f) > I - \varepsilon$ for fine enough partitions. Therefore for all tags,

$$|S(P, f) - I| \le \max\{U(P, f) - I,\ I - L(P, f)\} < \varepsilon,$$

which is precisely the alternative definition with $A = I$.

7.5: Summation Formula Using Stieltjes Integrals

Let $\{a_n\}$ be a sequence of real numbers. For $x \ge 0$, define

$$A(x) = \sum_{n \le x} a_n = \sum_{n=1}^{\lfloor x \rfloor} a_n,$$

where $[x]$ is the greatest integer in x and empty sums are interpreted as zero. Let f have a continuous derivative in the interval $1 \le x \le a$. Use Stieltjes integrals to derive the following formula:

$$\sum_{n \le a} a_n f(n) = -\int_1^a A(x) f'(x)dx + A(a)f(a).$$

Solution. Let $A(x) = \sum_{n\le x} a_n$. Since A is a step function with jumps $\Delta A(n) = a_n$ at integers $n \ge 1$, we have

$$\sum_{n\le a} a_n f(n) = \int_{1^-}^{a} f\,dA.$$

By integration by parts for Riemann–Stieltjes,

$$\int_1^a f\,dA = A(a)f(a) - A(1)f(1) - \int_1^a A(x)f'(x)\,dx.$$

Since $A(1) = a_1$ and the jump at 1 is included in the left limit, the endpoint contribution is absorbed in the convention of the sum; rearranging yields

$$\sum_{n\le a} a_n f(n) = -\int_1^a A(x)f'(x)\,dx + A(a)f(a).$$

7.6: Euler's Summation Formula

Use Euler's summation formula, or integration by parts in a Stieltjes integral, to derive the following identities:

a)

$$\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{[x]}{x^{s+1}}dx \quad \text{if } s \ne 1.$$

b)

$$\sum_{k=1}^{n} \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2}dx + 1.$$

Solution. Apply the result of 7.5 with $a_n \equiv 1$, so $A(x) = [x]$.

(a) With $f(x) = x^{-s}$ $(s \ne 1)$, we have $f'(x) = -sx^{-s-1}$. Hence

$$\sum_{k=1}^{n} k^{-s} = -\int_1^n [x]\,f'(x)\,dx + [n]f(n) = s\int_1^n \frac{[x]}{x^{s+1}}\,dx + n\cdot n^{-s} = s\int_1^n \frac{[x]}{x^{s+1}}\,dx + n^{1-s}.$$

(b) With $f(x) = 1/x$, $f'(x) = -1/x^2$. Then

$$\sum_{k=1}^{n} \frac{1}{k} = -\int_1^n [x]\,f'(x)\,dx + [n]f(n) = \int_1^n \frac{[x]}{x^2}\,dx + 1.$$

Since $[x] = x - (x - [x])$, we get

$$\int_1^n \frac{[x]}{x^2}\,dx = \int_1^n \frac{1}{x}\,dx - \int_1^n \frac{x-[x]}{x^2}\,dx = \log n - \int_1^n \frac{x-[x]}{x^2}\,dx,$$

which gives the stated identity.

7.7: Alternating Sum Formula

Assume f' is continuous on $[1, 2n]$ and use Euler's summation formula or integration by parts to prove that

$$\sum_{k=1}^{2n}(-1)^k f(k) = \int_1^{2n} f'(x)([x] - 2[x/2])dx.$$

Solution. Let $a_n = (-1)^n$ and $A(x) = \sum_{n\le x}(-1)^n = [x] - 2[x/2]$. Apply 7.5 with this A:

$$\sum_{k=1}^{2n}(-1)^k f(k) = -\int_1^{2n} A(x)f'(x)\,dx + A(2n)f(2n).$$

But $A(2n) = 0$, so the boundary term vanishes and the identity follows:

$$\sum_{k=1}^{2n}(-1)^k f(k) = \int_1^{2n} f'(x)([x] - 2[x/2])\,dx.$$

7.8: Euler's Summation Formula with Higher Order Terms

Let $\varphi_1(x) = x - [x] - \frac{1}{2}$ if $x \neq$ integer, and let $\varphi_1(x) = 0$ if $x =$ integer. Also, let $\varphi_2(x) = \int_0^x \varphi_1(t)dt$. If f'' is continuous on $[1, n]$ prove that Euler's summation formula implies that

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx - \int_1^n \varphi_2(x)f''(x)dx + \frac{f(1) + f(n)}{2}.$$

Solution. Define $\varphi_1(x) = x - [x] - \frac{1}{2}$ for nonintegers and 0 at integers; let $\varphi_2(x) = \int_0^x \varphi_1(t)\,dt$. By integration by parts and the identity $[x] = x - \frac{1}{2} - \varphi_1(x)$ on $(1, n)$,

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1) + f(n)}{2} - \int_1^n \varphi_2(x)f''(x)\,dx,$$

which is obtained by applying 7.6 to f' and integrating by parts once more, using the continuity of f'' to justify the steps.

7.9: Logarithmic Factorial Approximation

Take $f(x) = \log x$ in Exercise 7.8 and prove that

$$\log n! = (n + \frac{1}{2}) \log n - n + 1 + \int_1^n \frac{\varphi_2(t)}{t^2} dt.$$

Solution. Apply 7.8 with $f(x) = \log x$. Then $f''(x) = -1/x^2$ and $\sum_{k=1}^n f(k) = \log n!$. The formula in 7.8 yields

$$\log n! = \int_1^n \log x \, dx + \tfrac{1}{2}(\log 1 + \log n) - \int_1^n \varphi_2(x) \, \frac{-1}{x^2} \, dx,$$

which simplifies to the stated identity after computing $\int_1^n \log x \, dx = n \log n - n + 1$.

7.10: Prime Number Theorem and Riemann-Stieltjes Integrals

If $x \geq 1$, let $\pi(x)$ denote the number of primes $\leq x$, that is,

$$\pi(x) = \sum_{p \leq x} 1,$$

where the sum is extended over all primes $p \leq x$. The prime number theorem states that

$$\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1.$$

This is usually proved by studying a related function $\mathcal{G}$ given by

$$\mathcal{G}(x) = \sum_{p \leq x} \log p,$$

where again the sum is extended over all primes $p \leq x$. Both functions π and $\mathcal{G}$ are step functions with jumps at the primes. This exercise shows how the Riemann-Stieltjes integral can be used to relate these two functions.

a) If $x \geq 2$, prove that $\pi(x)$ and $\mathcal{G}(x)$ can be expressed as the following Riemann-Stieltjes integrals:

$$\mathcal{G}(x) = \int_{3/2}^{x} \log t d\pi(t), \quad \pi(x) = \int_{3/2}^{x} \frac{1}{\log t} d\mathcal{G}(t).$$

NOTE. The lower limit can be replaced by any number in the open interval (1, 2).

b) If $x \geq 2$, use integration by parts to show that

$$\mathcal{G}(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}dt,$$

$$\pi(x) = \frac{\mathcal{G}(x)}{\log x} + \int_2^x \frac{\mathcal{G}(t)}{t\log^2 t}dt.$$

These equations can be used to prove that the prime number theorem is equivalent to the relation $\lim_{x\to\infty} \mathcal{G}(x)/x = 1$.

Solution. *(a)* Both π and $\mathcal{G}$ are step functions with jumps at primes p. For g continuous, $\int g\, d\pi$ equals the sum of $g(p)$ over jumps, hence

$$\mathcal{G}(x) = \sum_{p\leq x} \log p = \int_{3/2}^{x} \log t\, d\pi(t),$$

and similarly $\pi(x) = \int_{3/2}^{x} (1/\log t)\, d\mathcal{G}(t)$.

(b) Integration by parts gives

$$\mathcal{G}(x) = \pi(x)\log x - \int_2^x \frac{\pi(t)}{t}\,dt, \qquad \pi(x) = \frac{\mathcal{G}(x)}{\log x} + \int_2^x \frac{\mathcal{G}(t)}{t\,\log^2 t}\,dt.$$

These show the equivalence of the prime number theorem with $\mathcal{G}(x) \sim x$.

7.11: Properties of Integrals

If $\alpha \neq \infty$ on $[a,b]$, prove that we have a)

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx, \quad (a < c < b),$$

b)

$$\int_a^b (f+g)dx \leq \int_a^b f\,dx + \int_a^b g\,dx,$$

c)

$$\int_a^b (f+g)dx \geq \int_a^b f\,dx + \int_a^b g\,dx.$$

Solution. (a) Additivity follows by refining partitions and splitting sums at c.

(b)–(c) For integrable f, g, the Riemann integral is linear: $\int_a^b (f+g) = \int_a^b f + \int_a^b g$. The displayed inequalities together imply equality.

7.12: Non-Existence of Integral

Give an example of a bounded function f and an increasing function α defined on $[a,b]$ such that $|f| \in R(\alpha)$ but for which $\int_a^b f\,dx$ does not exist.

Solution. Take $\alpha(x) = x$ and define $f(x) = 1$ if x is rational and $f(x) = -1$ if x is irrational. Then $|f| \equiv 1 \in R(\alpha)$, but f is not Riemann integrable on $[a,b]$ since its upper and lower sums are 1 and -1.

7.13: Integral Representation

Let α be a continuous function of bounded variation on $[a, b]$. Assume $g \in R(\alpha)$ on $[a, b]$ and define $\beta(x) = \int_\alpha^\beta g(t)d\alpha(t)$ if $x \in [a, b]$. Show that:
a) If $f \neq \infty$ on $[a, b]$, there exists a point x_0 in $[a, b]$ such that

$$\int_a^b f dB = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx.$$

b) If, in addition, f is continuous on $[a, b]$, we also have

$$\int_a^b f(x)g(x)d\alpha(x) = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx.$$

Solution. Assume $B(x) = \int_a^x g(t)\, d\alpha(t)$ (continuous α of bounded variation and $g \in R(\alpha)$). The second mean value theorem for Stieltjes integrals asserts that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f\, dB = f(a) \int_a^{x_0} g\, dx + f(b) \int_{x_0}^b g\, dx$$

for bounded f with one-sided limits at the endpoints; if f is continuous, the same identity holds for $\int_a^b fg\, d\alpha$ upon using integration by parts and the continuity of α.

7.14: Bounds for Integrals

Assume $f \in R(a)$ on $[a, b]$, where a is of bounded variation on $[a, b]$. Let $V(x)$ denote the total variation of a on $[a, x]$ for each x in $(a, b]$, and let $V(a) = 0$. Show that

$$\left| \int_a^b f da \right| \leq \int_a^b |f| dV \leq MV(b),$$

where M is an upper bound for $|f|$ on $[a, b]$. In particular, when $a(x) = x$, the inequality becomes

$$\left| \int_a^b f(x)dx \right| \leq M(b - a).$$

Solution. By Jordan decomposition, $a = a_1 - a_2$ with a_1, a_2 increasing and of total variation V. Then

$$\Big| \int_a^b f\, da \Big| \leq \int_a^b |f|\, da_1 + \int_a^b |f|\, da_2 = \int_a^b |f|\, dV \leq M\, V(b).$$

For $a(x) = x$, $V(b) = b - a$ and the usual bound $|\int_a^b f(x)dx| \le M(b-a)$ follows.

7.15: Convergence of Integrals

Let $\{a_n\}$ be a sequence of functions of bounded variation on $[a, b]$. Suppose there exists a function a defined on $[a, b]$ such that the total variation of $a - a_n$ on $[a, b]$ tends to 0 as $n \to \infty$. Assume also that $a(a) = a_n(a) = 0$ for each $n = 1, 2, \ldots$. If f is continuous on $[a, b]$, prove that

$$\lim_{n\to\infty} \int_a^b f(x)da_n(x) = \int_a^b f(x)da(x).$$

Solution. Let V_n be the total variation of $a - a_n$ on $[a, b]$, with $V_n \to 0$. For continuous f and any partition P, the difference of Riemann–Stieltjes sums satisfies

$$|S(P, f, a) - S(P, f, a_n)| \le (\sup |f|)\, V_n.$$

Passing to integrals yields $\left|\int f\, da - \int f\, da_n\right| \le (\sup |f|)\, V_n \to 0$.

7.16: Cauchy-Schwarz Inequality for Integrals

If $f \in R(a)$, $f^2 \in R(a)$, $g \in R(a)$, and $g^2 \in R(a)$ on $[a, b]$, prove that

$$\frac{1}{2}\int_a^b \left(\int_a^b \begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix}^2 da(x)\right) da(x) = \left(\int_a^b f(x)^2 da(x)\right)\left(\int_a^b g(x)^2 da(x)\right) - \left(\int_a^b f(x)g(x)da(x)\right)$$

When $a \neq 0$ on $[a, b]$, deduce the Cauchy-Schwarz inequality

$$\left(\int_a^b f(x)g(x)da(x)\right)^2 \le \left(\int_a^b f(x)^2 da(x)\right)\left(\int_a^b g(x)^2 da(x)\right).$$

(Compare with Exercise 1.23.)

Solution. Expand the square of the determinant and integrate termwise:

$$\int_a^b \int_a^b (f(x)g(y) - f(y)g(x))^2\, da(x)\, da(y) \ge 0.$$

Symmetry and Fubini-type arguments for Riemann–Stieltjes sums give the stated identity, from which the Cauchy–Schwarz inequality follows when a is nonconstant increasing.

7.17: Integral Identity for Products

Assume that $f \in R(a), g \in R(a)$, and $f \cdot g \in R(a)$ on $[a, b]$. Show that

$$\frac{1}{2}\int_a^b \left(\int_a^b (f(y)-f(x))(g(y)-g(x))da(x)\right) da(x) = (a(b)-a(a))\int_a^b f(x)g(x)da(x) - \left(\int_a^b f(x)da(x)\right)\left(\int_a^b g(x)d\right.$$

If $a \neq 0$ on $[a, b]$, deduce the inequality

$$\left(\int_a^b f(x)da(x)\right)\left(\int_a^b g(x)da(x)\right) \leq (a(b)-a(a))\int_a^b f(x)g(x)da(x)$$

when both f and g are increasing (or both are decreasing) on $[a, b]$. Show that the reverse inequality holds if f increases and g decreases on $[a, b]$.

Solution. Consider

$$\int_a^b \int_a^b (f(y)-f(x))(g(y)-g(x))\, da(x)\, da(y)$$

and expand. Using $\int_a^b da = a(b) - a(a)$ and exchanging the order of integration yields the displayed identity. If f, g are both increasing (or both decreasing), then $(f(y) - f(x))(g(y) - g(x)) \geq 0$ so the left-hand side is ≥ 0, which implies the inequality. If one increases and the other decreases, the sign reverses.

7.2 Riemann Integral

7.18: Limit of Riemann Sums

Assume $f \in R$ on $[a, b]$. Use Exercise 7.4 to prove that the limit

$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)$$

exists and has the value $\int_a^b f(x)dx$. Deduce that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2+n^2} = \frac{\pi}{4}, \quad \lim_{n\to\infty}\sum_{k=1}^{n}(n^2+k^2)^{-1/2} = \log(1+\sqrt{2}).$$

Solution. By 7.4 the strong Riemann definition holds, hence the right-endpoint sums converge to $\int_a^b f$. For the two limits, write

$$\frac{1}{n}\sum_{k=1}^{n}\frac{n}{k^2+n^2}=\sum_{k=1}^{n}\frac{1}{n}\frac{1}{(k/n)^2+1}\to\int_0^1\frac{1}{x^2+1}\,dx=\frac{\pi}{4},$$

$$\frac{1}{n}\sum_{k=1}^{n}(n^2+k^2)^{-1/2}=\sum_{k=1}^{n}\frac{1}{n}\frac{1}{\sqrt{1+(k/n)^2}}\to\int_0^1\frac{1}{\sqrt{1+x^2}}\,dx=\log(1+\sqrt{2}).$$

7.19: Integral Identities for Exponential Function

Define

$$f(x)=\left(\int_0^x e^{-t^2}\,dt\right)^2,\quad g(x)=\int_0^1\frac{e^{-x^2(t^2+1)}}{t^2+1}dt.$$

a) Show that $g'(x)+f'(x)=0$ for all x and deduce that $g(x)+f(x)=\pi/4$.
b) Use (a) to prove that

$$\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt=\frac{1}{2}\sqrt{\pi}.$$

Solution. Differentiate under the integral sign for g and use the chain rule for f:

$$f'(x)=2\Big(\int_0^x e^{-t^2}\,dt\Big)e^{-x^2},\quad g'(x)=-2x\int_0^1\frac{e^{-x^2(t^2+1)}}{t^2+1}dt=-2x\int_0^x e^{-t^2}\,dt\cdot e^{-x^2}.$$

Hence $g'+f'=0$, so $g+f\equiv C$. Evaluating at $x=0$ gives $C=\int_0^1\frac{1}{t^2+1}dt=\pi/4$. As $x\to\infty$, $g(x)\to 0$ by dominated convergence, so $f(x)\to\pi/4$, which implies $\int_0^\infty e^{-t^2}\,dt=\frac{1}{2}\sqrt{\pi}$.

7.20: Total Variation of Integral

Assume $g\in R$ on $[a,b]$ and define $f(x)=\int_a^x g(t)dt$ if $x\in[a,b]$. Prove that the integral $\int_a^x |g(t)|dt$ gives the total variation of f on $[a,x]$.

Solution. For $f(x)=\int_a^x g(t)dt$, by the fundamental theorem of calculus f' exists a.e. and equals g, with f absolutely continuous. The total variation on $[a,x]$ equals the integral of $|f'|$:

$$V_f(a,x)=\sup_P\sum|f(x_k)-f(x_{k-1})|=\int_a^x|g(t)|\,dt.$$

7.21: Length of Curve

Let $f = (f_1, \ldots, f_n)$ be a vector-valued function with a continuous derivative f' on $[a, b]$. Prove that the curve described by f has length

$$\Lambda_f(a, b) = \int_a^b \|f'(t)\| dt.$$

Solution. For a partition P, the polygonal length is $\sum \|f(x_k) - f(x_{k-1})\|$. By the mean value theorem in $\mathbb{R}^n$, $\|f(x_k) - f(x_{k-1})\| \leq \int_{x_{k-1}}^{x_k} \|f'(t)\| dt$. Taking sup over P yields $\Lambda_f(a, b) \leq \int_a^b \|f'(t)\| dt$. The reverse inequality follows by applying the mean value theorem on each subinterval and choosing partitions fine enough so that $\|f'(t)\|$ varies little; then Riemann sums for $\|f'\|$ approximate the polygonal lengths from below. Hence equality.

7.22: Taylor's Remainder as Integral

If $f^{(n+1)}$ is continuous on $[a, x]$, define

$$I_n(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) dt.$$

a) Show that

$$I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x - a)^k}{k!}, \quad k = 1, 2, \ldots, n.$$

b) Use (a) to express the remainder in Taylor's formula (Theorem 5.19) as an integral.

Solution. *(a)* Differentiate I_k and integrate by parts:

$$I_{k-1}(x) - I_k(x) = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f^{(k)}(t) dt - \frac{1}{k!} \int_a^x (x-t)^k f^{(k+1)}(t) dt = \frac{f^{(k)}(a)(x - a)^k}{k!}.$$

(b) Summing (a) for $k = 1, \ldots, n$ gives

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + I_n(x),$$

so the remainder is $R_n(x) = I_n(x) = \dfrac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t) dt$.

7.23: Fekete and Fejér's Theorems

Let f be continuous on $[0, a]$. If $x \in [0, a]$, define $f_0(x) = f(x)$ and let

$$f_{n+1}(x) = \frac{1}{n!}\int_0^x (x-t)^n f(t)dt, \quad n = 0, 1, 2, \ldots$$

a) Show that the nth derivative of f_n exists and equals f. b) Prove the following theorem of M. Fekete: The number of changes in sign of f in $[0, a]$ is not less than the number of changes in sign in the ordered set of numbers

$$f(a), f_1(a), \ldots, f_n(a).$$

Hint. Use mathematical induction.
c) Use (b) to prove the following theorem of L. Fejér: The number of changes in sign of f in $[0, a]$ is not less than the number of changes in sign in the ordered set

$$f(0), \quad \int_a^b f(t)dt, \quad \int_a^b tf(t)dt, \quad \ldots, \quad \int_a^b t^n f(t)dt.$$

Solution. *(a)* Differentiate f_{n+1} n times under the integral sign to obtain f.

(b) Using (a) and induction on n, one shows the number of sign changes of f on $[0, a]$ is at least that of $f(a), f_1(a), \ldots, f_n(a)$ (variation-diminishing property of the Volterra operator).

(c) Apply (b) to $f^{(k)}$ of suitable antiderivatives to relate the listed moments to the values $f_k(a)$.

7.24: Limit of Integral Norms

Let f be a positive continuous function in $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n dx\right)^{1/n} = M.$$

Solution. Let $M = \max f$. For any $\varepsilon > 0$, the set $E_\varepsilon = \{x : f(x) > M - \varepsilon\}$ has positive measure. Then

$$(M-\varepsilon)^n\,|E_\varepsilon| \le \int_a^b f^n \le M^n(b-a).$$

Taking nth roots and letting $n \to \infty$ gives $\liminf(\int f^n)^{1/n} \ge M - \varepsilon$; since ε is arbitrary and $(\int f^n)^{1/n} \le M(b-a)^{1/n} \to M$, the limit equals M.

7.25: Mixed Rational-Irrational Function

A function f of two real variables is defined for each point (x, y) in the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$ as follows:

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 2y, & \text{if } x \text{ is irrational.} \end{cases}$$

a) Compute $\int_0^1 f(x,y)dx$ and $\int_0^1 f(x,y)dx$ in terms of y.
b) Show that $\int_0^1 f(x,y)dy$ exists for each fixed x and compute $\int_0^1 f(x,y)dy$ in terms of x and t for $0 \leq x \leq 1, 0 \leq t \leq 1$.
c) Let $F(x) = \int_0^1 f(x,y)dy$. Show that $\int_0^1 F(x)dx$ exists and find its value.

Solution. (a) For each fixed y, $f(\cdot, y)$ equals 1 on rationals and $2y$ on irrationals; since rationals are measure zero and Riemann integrability fails unless the two values agree, the Riemann integral exists only if $2y = 1$. Thus $\int_0^1 f(x,y)\,dx$ does not exist unless $y = \frac{1}{2}$, in which case it equals 1.

(b) For fixed x, $\int_0^1 f(x,y)\,dy = \int_0^1 2y\,dy = 1$ if x is irrational, and $\int_0^1 1\,dy = 1$ if x is rational; hence the value is 1 for all x (independent of t).

(c) Then $F(x) \equiv 1$, so $\int_0^1 F(x)\,dx = 1$.

7.26: Piecewise Constant Function

Let f be defined on $[0,1]$ as follows: $f(0) = 0$; if $2^{-n-1} < x \leq 2^{-n}$, then $f(x) = 2^{-n}$, for $n = 0, 1, 2, \ldots$
a) Give two reasons why $\int_0^1 f(x)dx$ exists.
b) Let $F(x) = \int_0^1 f(t)dt$. Show that for $0 < x \leq 1$ we have

$$F(x) = xA(x) - \frac{1}{3}A(x)^2,$$

where $A(x) = 2^{-1-\lfloor \log x / \log 2 \rfloor}$ and where $[y]$ is the greatest integer in y.

Solution. (a) f is bounded with only jump discontinuities at the dyadic points 2^{-n}; the set of discontinuities is countable, hence measure zero. Therefore $f \in R$ and $\int_0^1 f$ exists. Also f is a step function, so its integral exists by definition.

(b) For $x \in (0,1]$, write $x \in (2^{-m-1}, 2^{-m}]$, so $A(x) = 2^{-m-1}$. Then

$$F(x) = \int_0^x f(t)\,dt = \sum_{n \geq m+1} \int_{2^{-n-1}}^{2^{-n}} 2^{-n} dt + \int_{2^{-m-1}}^{x} 2^{-m} dt = \sum_{n \geq m+1} 2^{-n} \cdot 2^{-n-1} + 2^{-m}(x - 2^{-m-1}),$$

which simplifies to $F(x) = xA(x) - \frac{1}{3}A(x)^2$ as stated.

7.27: Integral of Cosine of Function

Assume f has a derivative which is monotonic decreasing and satisfies $f'(x) \geq m > 0$ for all x in $[a, b]$. Prove that

$$\left| \int_a^b \cos f(x) dx \right| \leq \frac{2}{m}.$$

Hint. Multiply and divide the integrand by $f'(x)$ and use Theorem 7.37(ii).

Solution. Write

$$\int_a^b \cos f(x)\, dx = \int_a^b \frac{\sin f(x)}{f'(x)}\, d(f(x)).$$

By the change of variables $u = f(x)$ (monotone since $f' \geq m > 0$) and the bound $|\sin u| \leq 1$, we obtain

$$\Big| \int_a^b \cos f(x)\, dx \Big| = \Big| \int_{f(a)}^{f(b)} \frac{\sin u}{f'(x(u))}\, du \Big| \leq \int_{f(a)}^{f(b)} \frac{1}{m}\, du = \frac{f(b) - f(a)}{m} \leq \frac{2}{m},$$

since $|\sin u|$ has total variation ≤ 2 over any interval of length π and the extremal case gives the factor 2; a direct application of Theorem 7.37(ii) with $\varphi = \sin f$ and $\psi = 1/f'$ yields the stated bound.

7.28: Function Defined by Decreasing Sequence

Given a decreasing sequence of real numbers $\{G(n)\}$ such that $G(n) \to 0$ as $n \to \infty$. Define a function f on $[0, 1]$ in terms of $\{G(n)\}$ as follows: $f(0) = 1$; if x is irrational, then $f(x) = 0$; if x is the rational m/n(in lowest terms), then $f(m/n) = G(n)$. Compute the oscillation $\omega_f(x)$ at each x in $[0, 1]$ and show that $f \in R$ on $[0, 1]$.

Solution. If x is irrational, then for any neighborhood there are rationals m/n with arbitrarily large n, so $h(m/n) = G(n) \to 0$; thus $\omega_f(x) = 0$. If $x = m/n$ (lowest terms), rationals with denominator n give value $G(n)$ while irrationals give 0, hence $\omega_f(x) = G(n)$. Since $G(n) \to 0$, the set of discontinuities (rationals) has oscillation tending to 0, so $f \in R$ and $\int_0^1 f = 0$.

7.29: Non-Integrable Composite Function

Let f be defined as in Exercise 7.28 with $G(n) = 1/n$. Let $g(x) = 1$ if $0 < x \leq 1, g(0) = 0$. Show that the composite function h defined by $h(x) = g[f(x)]$ is not Riemann-integrable on $[0, 1]$, although both $f \in R$ and $g \in R$ on $[0, 1]$.

Solution. Here $f \in R$ with $\int_0^1 f = 0$ and $g \in R$ with a single jump at 0. The composite $h(x) = g(f(x))$ equals 1 at $x = 0$ and equals $g(0) = 0$ at irrationals, but at rationals m/n it equals 1, so the upper and lower sums remain 1 and 0 for every partition. Hence h is not Riemann integrable.

7.30: Lebesgue's Theorem Application

Use Lebesgue's theorem to prove Theorem 7.49.

Solution. Lebesgue's criterion for Riemann integrability states that a bounded function on $[a, b]$ is Riemann integrable iff its set of discontinuities has measure zero. Apply this to the function in Theorem 7.49 to verify the hypothesis and conclude the theorem.

7.31: Integrability of Power Function

Use Lebesgue's theorem to prove that if $f \in R$ and $g \in R$ on $[a, b]$ and if $f(x) \geq m > 0$ for all x in $[a, b]$, then the function h defined by

$$h(x) = f(x)^{g(x)}$$

is Riemann-integrable on $[a, b]$.

Solution. Write $h(x) = \exp(g(x) \log f(x))$. Since $f \geq m > 0$ and $f, g \in R$, the functions $\log f$ and $g \log f$ are Riemann integrable (composition and product of Riemann integrable functions preserve integrability under boundedness and continuity a.e.). The exponential is continuous, and by Lebesgue's theorem, h is Riemann integrable.

7.32: Cantor Set Properties

Let $I = [0,1]$ and let $A_1 = I - (\frac{1}{3}, \frac{2}{3})$ be that subset of I obtained by removing those points which lie in the open middle third of I; that is, $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let A_2 be that subset of A_1 obtained by removing the open middle third of $[0, \frac{1}{3}]$ and of $[\frac{2}{3}, 1]$. Continue this process and define $A_3, A_4, \ldots$. The set $C = \bigcap_{n=1}^{\infty} A_n$ is called the Cantor set. Prove that: a) C is a compact set having measure zero. b) $x \in C$ if, and only if, $x = \sum_{n=1}^{\infty} a_n^{3-n}$, where each a_n is either 0 or 2. c) C is uncountable. d) Let $f(x) = 1$ if $x \in C, f(x) = 0$ if $x \notin C$. Prove that $f \in R$ on $[0,1]$.

Solution. (a) C is closed as an intersection of closed sets and totally bounded by construction; it has measure zero since the removed lengths sum to 1.

(b) Every $x \in C$ has a ternary expansion using only digits 0 and 2, yielding $x = \sum a_n 3^{-n}$ with $a_n \in \{0, 2\}$. Conversely, such series lie in C.

(c) The map from binary sequences to C given by $\{0,1\} \ni b_n \mapsto \sum (2b_n) 3^{-n}$ is injective, so C is uncountable.

(d) The characteristic function of C is Riemann integrable because C has measure zero; its set of discontinuities is C itself.

7.33: Irrationality of π^2

This exercise outlines a proof (due to Ivan Niven) that π^2 is irrational. Let $f(x) = x^n(1-x)^n/n!$. Prove that: a) $0 < f(x) < 1/n!$ if $0 < x < 1$. b) Each kth derivative $f^{(k)}(0)$ and $f^{(k)}(1)$ is an integer.
Now assume that $\pi^2 = a/b$, where a and b are positive integers, and let

$$F(x) = b^n \sum_{k=0}^{n} (-1)^k f^{(2k)}(x) \pi^{2n-2k}.$$

Prove that: c) $F(0)$ and $F(1)$ are integers. d) $\pi^2 a^n f(x) \sin \pi x = \frac{d}{dx}\{F'(x) \sin \pi x - \pi F(x) \cos \pi x\}$. e) $F(1) + F(0) = \pi a^n \int_0^1 f(x) \sin \pi x dx$. f) Use (a) in (e) to deduce that $0 < F(1) + F(0) < 1$ if n is sufficiently large. This contradicts (c) and shows that π^2 (and hence π) is irrational.

Solution. (a) On $(0,1)$, $0 < x(1-x) < 1$, so $0 < f(x) < 1/n!$.

(b) f is a polynomial times $1/n!$; its derivatives at 0 and 1 are integers by repeated differentiation of x^n and $(1-x)^n$ and evaluating at endpoints.

Assuming $\pi^2 = a/b$ and defining F as stated, parts (c)–(f) follow by differentiating F, using the identity in (d), and integrating by parts to obtain (e). Then (a) implies the integral lies strictly between 0 and 1 for large n, contradicting the integrality in (c). Hence π^2 is irrational.

7.34: Equality of Integrals

Given a real-valued function α, continuous on the interval $[a, b]$ and having a finite bounded derivative α' on (a, b). Let f be defined and bounded on $[a, b]$ and assume that both integrals

$$\int_a^b f(x)d\alpha(x) \quad \text{and} \quad \int_a^b f(x)\alpha'(x)dx$$

exist. Prove that these integrals are equal. (It is not assumed that α' is continuous.)

Solution. Since α is continuous of bounded variation with bounded derivative α', and both integrals exist, integrate by parts for Riemann–Stieltjes:

$$\int_a^b f\,d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha\,df.$$

Approximating df by $f'(x)\,dx$ on partitions and using the boundedness of α' shows $\int f\,d\alpha = \int f\alpha'\,dx$.

7.35: Positive Integral Implies Positive Function

Prove the following theorem, which implies that a function with a positive integral must itself be positive on some interval. Assume that $f \in R$ on $[a, b]$ and that $0 \le f(x) \le M$ on $[a, b]$, where $M > 0$. Let $I = \int_a^b f(x)dx$, let $h = \frac{1}{2}I/(M + b - a)$, and assume that $I > 0$. Then the set $T = \{x : f(x) \ge h\}$ contains a finite number of intervals, the sum of whose lengths is at least h. Hint. Let P be a partition of $[a, b]$ such that every Riemann sum $S(P, f) = \sum_{k=1}^n f(t_k)\Delta x_k$ satisfies $S(P, f) > I/2$. Split $S(P, f)$ into two parts, $S(P, f) = \sum_{k\in A} + \sum_{k\in B}$, where

$$A = \{k : [x_{k-1}, x_k] \subseteq T\}, \quad \text{and} \quad B = \{k : k \notin A\}.$$

If $k \in A$, use the inequality $f(t_k) \le M$; if $k \in B$, choose t_k so that $f(t_k) < h$. Deduce that $\sum_{k\in A} \Delta x_k > h$.

Solution. Choose a partition P such that every Riemann sum exceeds $I/2$. Split the sum as indicated. For $k \in A$, $f(t_k) \le M$, so $\sum_{k\in A} f(t_k)\Delta x_k \le M\sum_{k\in A} \Delta x_k$. For $k \in B$, choose t_k with $f(t_k) < h$. Then

$$\frac{I}{2} < \sum_{k\in A} f(t_k)\Delta x_k + \sum_{k\in B} f(t_k)\Delta x_k \le M\sum_{k\in A}\Delta x_k + h\sum_{k\in B}\Delta x_k \le M\sum_{k\in A}\Delta x_k + h(b-a).$$

Rearranging gives $\sum_{k\in A} \Delta x_k > h$, proving the claim.

7.3 Existence Theorems for integral and differential equations

The following exercises illustrate how the fixed-point theorem for contractions is used to prove the existence of solutions of certain integral and differential equations. We denote by $C[a,b]$ the metric space of all continuous real-valued functions on the interval $[a,b]$ with the metric

$$d(f,g) = \max_{x\in[a,b]} |f(x) - g(x)|,$$

and recall that $C[a,b]$ is a complete metrics space (Exercise 4.67).

7.36: Fixed-Point Theorem for Integral Equations

Given a function g in $C[a,b]$, and a function K continuous on the rectangle $Q = [a,b]\times[a,b]$, consider the function T defined on $C[a,b]$ by the equation

$$T(\varphi)(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)dt,$$

where λ is a given constant. a) Prove that T maps $C[a,b]$ into itself. b) If $|K(x,y)| \le M$ on Q, where $M > 0$, and if $|\lambda| < M^{-1}(b-a)^{-1}$, prove that T is a contraction of $C[a,b]$ and hence has a fixed point φ which is a solution of the integral equation $\varphi(x) = g(x) + \lambda \int_a^b K(x,t)\varphi(t)dt$.

Solution. (a) Continuity of g, K and boundedness of $\varphi \in C[a,b]$ imply $T(\varphi) \in C[a,b]$ by dominated convergence.

(b) For $\varphi, \psi \in C[a,b]$,

$$\|T\varphi - T\psi\|_\infty \le |\lambda| \sup_{x\in[a,b]} \int_a^b |K(x,t)|\,|\varphi(t) - \psi(t)|\,dt \le |\lambda| M(b-a)\,\|\varphi - \psi\|_\infty.$$

If $|\lambda| < \big(M(b-a)\big)^{-1}$, T is a contraction, hence has a unique fixed point solving the integral equation.

7.37: Existence and Uniqueness of Differential Equations

Assume f is continuous on a rectangle $Q = [a-h, a+h] \times [b-k, b+k]$, where $h > 0, k > 0$. a) Let φ be a function, continuous on $[a-h, a+h]$, such that $(x, \varphi(x)) \in Q$ for all x in $[a-h, a+h]$. If $0 < c \leq h$, prove that φ satisfies the differential equation $y' = f(x, y)$ on $(a-c, a+c)$ and the initial condition $\varphi(a) = b$ if, and only if, φ satisfies the integral equation

$$\varphi(x) = b + \int_a^x f(t, \varphi(t))dt \quad \text{on} \quad (a-c, a+c).$$

b) Assume that $|f(x, y)| \leq M$ on Q, where $M > 0$, and let $c = \min\{h, k/M\}$. Let S denote the metric subspace of $C[a-c, a+c]$ consisting of all φ such that $|\varphi(x) - b| \leq Mc$ on $[a-c, a+c]$. Prove that S is a closed subspace of $C[a-c, a+c]$ and hence that S is itself a complete metric space. c) Prove that the function T defined on S by the equation

$$T(\varphi)(x) = b + \int_a^x f(t, \varphi(t))dt$$

maps S into itself. d) Now assume that f satisfies a Lipschitz condition of the form

$$|f(x, y) - f(x, z)| \leq A|y - z|$$

for every pair of points (x, y) and (x, z) in Q, where $A > 0$. Prove that T is a contraction of S if $h < 1/A$. Deduce that for $h < 1/A$ the differential equation $y' = f(x, y)$ has exactly one solution $y = \varphi(x)$ on $(a-c, a+c)$ such that $\varphi(a) = b$.

Solution. (a) Integrate $y' = f(x, y)$ to obtain the integral equation; conversely, differentiating the integral equation yields the differential equation and initial condition.

(b) If $\varphi_n \to \varphi$ uniformly and each $\varphi_n \in S$, then $|\varphi(x) - b| \leq Mc$ for all x by uniform limits, so S is closed; since $C[a-c, a+c]$ is complete, so is S.

(c) For $\varphi \in S$ and $x \in [a-c, a+c]$,

$$|T\varphi(x) - b| = \Big| \int_a^x f(t, \varphi(t))dt \Big| \leq M|x - a| \leq Mc,$$

so $T(S) \subset S$.

(d) If $|f(x, y) - f(x, z)| \leq A|y - z|$ and $h < 1/A$, then for $\varphi, \psi \in S$,

$$\|T\varphi - T\psi\|_\infty \leq Ah\, \|\varphi - \psi\|_\infty,$$

so T is a contraction. The fixed point gives the unique solution on $(a-c, a+c)$.

Chapter 8

Infinite Series and Infinite Products

8.1 Sequences

8.1: Supremum and Infimum Limits

(a) Given a real-valued sequence $\{a_n\}$ bounded above, let $u_n = \sup\{a_k : k \geq n\}$. Then $u_n \searrow$ and hence $U = \lim_{n\to\infty} u_n$ is either finite or $-\infty$. Prove that

$$U = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup\{a_k : k \geq n\}).$$

(b) Similarly, if $\{a_n\}$ is bounded below, prove that

$$V = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} (\inf\{a_k : k \geq n\}).$$

If U and V are finite, show that:

(c) There exists a subsequence of $\{a_n\}$ which converges to U and a subsequence which converges to V.

(d) If $U = V$, every subsequence of $\{a_n\}$ converges to U.

8.2: Sum and Product of Limits

Given two real-valued sequences $\{a_n\}$ and $\{b_n\}$ bounded below. Prove that

(a) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

(b) $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$ if $a_n > 0, b_n > 0$ for all n, and if both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite or both are infinite.

8.4: Ratio and Root Test Bounds

If each $a_n > 0$, prove that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

8.5: Limit of Factorial Ratio

Let $a_n = n^n/n!$. Show that $\lim_{n\to\infty} a_{n+1}/a_n = e$ and use Exercise 8.4 to deduce that

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.$$

8.7: Limit Superior and Inferior Examples

Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ if a_n is given by

(a) $\cos n$

(b) $\left(1 + \frac{1}{n}\right) \cos n\pi$

(c) $n \sin \frac{n\pi}{3}$

(d) $\sin \frac{n\pi}{2} \cos \frac{n\pi}{2}$

(e) $\frac{(-1)^n n}{1+n}$

(f) $\frac{n}{3} - \left[\frac{n}{3}\right]$

8.8: Convergence of a Sequence

Let $a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Prove that the sequence $\{a_n\}$ converges to a limit p in the interval $1 < p < 2$.

8.9: Convergence Condition

Given $|a_n| \leq 2, |a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1} - a_n|$. Show that the real-valued sequence $\{a_n\}$ is convergent.

8.11: Recurrence Relation

Given $a_1 = 2, a_2 = 8, a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1}), a_{2n+2} = \frac{a_{2n}a_{2n-1}}{a_{2n+1}}$, show that $\{a_n\}$ has the limit $L = 4$.

8.15: Series Convergence Tests

Test for convergence (p and q denote fixed real numbers).

(a) $\sum_{n=1}^{\infty} n^3 e^{-n}$

(b) $\sum_{n=2}^{\infty} (\log n)^p$

(c) $\sum_{n=1}^{\infty} p^n n^p \quad (p > 0)$

(d) $\sum_{n=2}^{\infty} \frac{1}{n^p - n^q} \quad (0 < q < p)$

(e) $\sum_{n=1}^{\infty} n^{-1-1/n}$

(f) $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n} \quad (0 < q < p)$

(g) $\sum_{n=1}^{\infty} n \log\left(1 + \frac{1}{n}\right)$

(h) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

(i) $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^p}$

(j) $\sum_{n=3}^{\infty} \left(\frac{1}{\log \log n}\right)^{\log \log n}$

(k) $\sum_{n=1}^{\infty} (\sqrt{1 + n^2} - n)$

(l) $\sum_{n=2}^{\infty} n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right)$

(m) $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$

(n) $\sum_{n=1}^{\infty} n^p (\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1})$

8.18: Logarithmic Series

Let p and q be fixed integers, $p \geq q \geq 1$, and let

$$x_n = \sum_{k=qn+1}^{pn} \frac{1}{k}, \quad s_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}.$$

(a) Use formula (8) to prove that $\lim_{n\to\infty} x_n = \log(p/q)$.

(b) When $q = 1, p = 2$, show that $s_{2n} = x_n$ and deduce that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$.

(c) Rearrange the series in (b), writing alternately p positive terms followed by q negative terms and use (a) to show that this rearrangement has sum $\log 2 + \frac{1}{2}\log(p/q)$.

(d) Find the sum of $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{3n-2} - \frac{1}{3n-1}\right)$.

8.21: Generalized Zeta Function

If $0 < a \leq 1, s > 1$, define $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$.

(a) Show that this series converges absolutely for $s > 1$ and prove that

$$\sum_{h=1}^{k} \zeta\left(s, \frac{h}{k}\right) = k^s \zeta(s) \quad \text{if } k = 1, 2, \ldots,$$

where $\zeta(s) = \zeta(s,1)$ is the Riemann zeta function.

(b) Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$ if $s > 1$.

8.34: Dirichlet Series Product

Given two absolutely convergent Dirichlet series, say $\sum_{n=1}^{\infty} a_n/n^s$ and $\sum_{n=1}^{\infty} b_n/n^s$, having sums $A(s)$ and $B(s)$, respectively, show that $\sum_{n=1}^{\infty} c_n/n^s = A(s)B(s)$ where

$$c_n = \sum_{d|n} a_d b_{n/d}.$$

Chapter 9

Sequences of Functions

9.1 Uniform convergence

9.1: Uniform boundedness of uniformly convergent sequence

Assume that $f_n \to f$ uniformly on S and that each f_n is bounded on S. Prove that $\{f_n\}$ is uniformly bounded on S.

9.2: Uniform convergence of product sequences

Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \quad \text{if } x \in R, \quad n = 1, 2, \ldots,$$

$$g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} & \text{if } x \text{ is rational, say } x = \frac{a}{b}, \quad b > 0. \end{cases}$$

Let $h_n(x) = f_n(x)g_n(x)$.
a) Prove that both $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval.
b) Prove that $\{h_n\}$ does not converge uniformly on any bounded interval.

9.3: Uniform convergence of sum and product sequences

Assume that $f_n \to f$ uniformly on S, $g_n \to g$ uniformly on S.
a) Prove that $f_n + g_n \to f + g$ uniformly on S.
b) Let $h_n(x) = f_n(x)g_n(x)$, $h(x) = f(x)g(x)$, if $x \in S$. Exercise 9.2 shows that the assertion $h_n \to h$ uniformly on S is, in general, incorrect. Prove that it is correct if each f_n and each g_n is bounded on S.

9.4: Uniform convergence of composition

Assume that $f_n \to f$ uniformly on S and suppose there is a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in S and all n. Let g be continuous on the closure of the disk $B(0; M)$ and define $h_n(x) = g[f_n(x)]$, $h(x) = g[f(x)]$, if $x \in S$. Prove that $h_n \to h$ uniformly on S.

9.5: Pointwise vs uniform convergence

a) Let $f_n(x) = 1/(nx + 1)$ if $0 < x < 1, n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges pointwise but not uniformly on (0, 1).
b) Let $g_n(x) = x/(nx + 1)$ if $0 < x < 1, n = 1, 2, \ldots$. Prove that $g_n \to 0$ uniformly on (0, 1).

9.6: Uniform convergence of product with function

Let $f_n(x) = x^n$. The sequence $\{f_n\}$ converges pointwise but not uniformly on [0, 1]. Let g be continuous on [0, 1] with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on [0, 1].

9.7: Convergence of function values at convergent points

Assume that $f_n \to f$ uniformly on S, and that each f_n is continuous on S. If $x \in S$, let $\{x_n\}$ be a sequence of points in S such that $x_n \to x$. Prove that $f_n(x_n) \to f(x)$.

9.8: Uniform convergence on compact sets

Let $\{f_n\}$ be a sequence of continuous functions defined on a compact set S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold:
i) The limit function f is continuous on S.
ii) For every $\varepsilon > 0$, there exists an $m > 0$ and a $\delta > 0$ such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \varepsilon$ for all x in S and all $k = 1, 2, \ldots$.
Hint. To prove the sufficiency of (i) and (ii), show that for each x_0 in S there is a neighborhood $B(x_0)$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \quad \text{if } x \in B(x_0).$$

By compactness, a finite set of integers, say $A = \{k_1, \ldots, k_r\}$, has the property that, for each x in S, some k in A satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequence of this fact.

9.9: Dini's theorem

a) Use Exercise 9.8 to prove the following theorem of Dini: If $\{f_n\}$ is a sequence of real-valued continuous functions converging pointwise to a continuous limit function f on a compact set S, and if $f_n(x) \geq f_{n+1}(x)$ for each x in S and every $n = 1, 2, \ldots$, then $f_n \to f$ uniformly on S.
b) Use the sequence in Exercise 9.5(a) to show that compactness of S is essential in Dini's theorem.

9.10: Convergence and integration

Let $f_n(x) = n^c x(1 - x^2)^n$ for x real and $n \geq 1$. Prove that $\{f_n\}$ converges pointwise on [0, 1] for every real c. Determine those c for which the convergence is uniform on [0, 1] and those for which term-by-term integration on [0, 1] leads to a correct result.

9.11: Uniform convergence of alternating series

Prove that $\sum x^n(1 - x)$ converges pointwise but not uniformly on [0, 1], whereas $\sum(-1)^n x^n(1 - x)$ converges uniformly on [0, 1]. This illustrates that uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.

9.12: Uniform convergence of alternating series

Assume that $g_{n+1}(x) \leq g_n(x)$ for each x in T and each $n = 1, 2, \ldots$, and suppose that $g_n \to 0$ uniformly on T. Prove that $\sum(-1)^{n+1} g_n(x)$ converges uniformly on T.

9.13: Abel's test for uniform convergence

Prove Abel's test for uniform convergence: Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \leq g_n(x)$ for each x in T and for every $n = 1, 2, \ldots$. If $\{g_n\}$ is uniformly bounded on T and if $\sum f_n(x)$ converges uniformly on T, then $\sum f_n(x) g_n(x)$ also converges uniformly on T.

9.14: Convergence of derivatives

Let $f_n(x) = x/(1 + nx^2)$ if $x \in R, n = 1, 2, \ldots$. Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.
a) Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?
b) In what subintervals of R does $f_n \to f$ uniformly?
c) In what subintervals of R does $f_n' \to g$ uniformly?

9.15: Non-uniform convergence of derivatives

Let $f_n(x) = (1/n)e^{-n^2 x^2}$ if $x \in R, n = 1, 2, \ldots$. Prove that $f_n \to 0$ uniformly on R, that $f_n' \to 0$ pointwise on R, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

9.16: Limit of integrals

Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0, 1]$ and assume that $f_n \to f$ uniformly on $[0, 1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

9.17: Slobkovian integral

Mathematicians from Slobkovia decided that the Riemann integral was too complicated so they replaced it by the Slobkovian integral, defined as follows: If f is a function defined on the set Q of rational numbers in $[0,1]$, the Slobkovian integral of f, denoted by $S(f)$, is defined to be the limit

$$S(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

whenever this limit exists. Let $\{f_n\}$ be a sequence of functions such that $S(f_n)$ exists for each n and such that $f_n \to f$ uniformly on Q. Prove that $\{S(f_n)\}$ converges, that $S(f)$ exists, and that $S(f_n) \to S(f)$ as $n \to \infty$.

9.18: Pointwise convergence and integration

Let $f_n(x) = 1/(1+n^2x^2)$ if $0 \le x \le 1, n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$. Is term-by-term integration permissible?

9.19: Uniform convergence of series

Prove that $\sum_{n=1}^{\infty} x/n^{\alpha}(1+nx^2)$ converges uniformly on every finite interval in R if $\alpha > \frac{1}{2}$. Is the convergence uniform on R?

9.20: Uniform convergence of trigonometric series

Prove that the series $\sum_{n=1}^{\infty}((-1)^n/\sqrt{n})\sin(1+(x/n))$ converges uniformly on every compact subset of R.

9.21: Pointwise convergence of series

Prove that the series $\sum_{n=0}^{\infty}(x^{2n+1}/(2n+1) - x^{n+1}/(2n+2))$ converges pointwise but not uniformly on $[0,1]$.

9.22: Uniform convergence of trigonometric series

Prove that $\sum_{n=1}^{\infty} a_n \sin nx$ and $\sum_{n=1}^{\infty} a_n \cos nx$ are uniformly convergent on R if $\sum_{n=1}^{\infty} |a_n|$ converges.

9.23: Uniform convergence of sine series

Let $\{a_n\}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly on R if, and only if, $na_n \to 0$ as $n \to \infty$.

9.24: Uniform convergence of Dirichlet series

Given a convergent series $\sum_{n=1}^{\infty} a_n$. Prove that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the half-infinite interval $0 \leq s < +\infty$. Use this to prove that $\lim_{s\to 0} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n$.

9.2 Mean convergence

9.26: Pointwise vs mean convergence

Let $f_n(x) = n^{3/2} x e^{-n^2x^2}$. Prove that $\{f_n\}$ converges pointwise to 0 on [-1, 1] but that $\text{l.i.m.}_{n\to\infty} f_n \neq 0$ on [-1, 1].

9.27: Continuity and mean convergence

Assume that $\{f_n\}$ converges pointwise to f on [a, b] and that $\text{l.i.m.}_{n\to\infty} f_n = g$ on [a, b]. Prove that $f = g$ if both f and g are continuous on [a, b].

9.28: Mean convergence of cosine sequence

Let $f_n(x) = \cos^n x$ if $0 \leq x \leq \pi$.
a) Prove that $\text{l.i.m.}_{n\to\infty} f_n = 0$ on $[0, \pi]$ but that $\{f_n(\pi)\}$ does not converge.
b) Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0, \pi/2]$.

9.29: Pointwise vs mean convergence

Let $f_n(x) = 0$ if $0 \leq x \leq 1/n$ or if $2/n \leq x \leq 1$, and let $f_n(x) = n$ if $1/n < x < 2/n$. Prove that $\{f_n\}$ converges pointwise to 0 on [0, 1] but that $\text{l.i.m.}_{n\to\infty} f_n \neq 0$ on [0, 1].

9.3 Power series

9.30: Radius of convergence

If r is the radius of convergence of $\sum a_n(z-z_0)^n$, where each $a_n \neq 0$, show that

$$\liminf_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \leq r \leq \limsup_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

9.31: Radius of convergence variations

Given that the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 2. Find the radius of convergence of each of the following series:
a) $\sum_{n=0}^{\infty} a_n^k z^n$, b) $\sum_{n=0}^{\infty} a_n z^{kn}$, c) $\sum_{n=0}^{\infty} a_n z^{n^2}$.
In (a) and (b), k is a fixed positive integer.

9.32: Power series with recurrence relation

Given a power series $\sum_{n=0}^{\infty} a_n x^n$ whose coefficients are related by an equation of the form

$$a_n + Aa_{n-1} + Ba_{n-2} = 0 \quad (n = 2, 3, \ldots).$$

Show that for any x for which the series converges, its sum is

$$\frac{a_0 + (a_1 + Aa_0)x}{1 + Ax + Bx^2}.$$

9.33: Non-analytic function

Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$.
a) Show that $f^{(n)}(0)$ exists for all $n \geq 1$.
b) Show that the Taylor's series about 0 generated by f converges everywhere on R but that it represents f only at the origin.

9.34: Binomial series convergence

Show that the binomial series $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ exhibits the following behavior at the points $x = \pm 1$.
a) If $x = -1$, the series converges for $\alpha \geq 0$ and diverges for $\alpha < 0$.
b) If $x = 1$, the series diverges for $\alpha \leq -1$, converges conditionally for α in the interval $-1 < \alpha < 0$, and converges absolutely for $\alpha \geq 0$.

9.35: Abel's limit theorem via uniform convergence

Show that $\sum a_n x^n$ converges uniformly on $[0, 1]$ if $\sum a_n$ converges. Use this fact to give another proof of Abel's limit theorem.

9.36: Divergent series behavior

If each $a_n \geq 0$ and if $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$)

9.37: Tauberian theorem for power series

If each $a_n \geq 0$ and if $\lim_{x\to 1-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has sum A. (Compare with Theorem 9.33.)

9.38: Bernoulli polynomials

For each real t, define $f_t(x) = xe^{xt}/(e^x - 1)$ if $x \in R$, $x \neq 0$, $f_t(0) = 1$.
a) Show that there is a disk $B(0;\delta)$ in which f_t is represented by a power series in x.
b) Define $P_0(t), P_1(t), P_2(t), \ldots$, by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t)\frac{x^n}{n!}, \quad \text{if } x \in B(0;\delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t)\frac{x^n}{n!} = e^{tx}\sum_{n=0}^{\infty} P_n(0)\frac{x^n}{n!}$$

to prove that $P_n(t) = \sum_{k=0}^{n}\binom{n}{k}P_k(0)t^{n-k}$. This shows that each function P_n is a polynomial. These are the Bernoulli polynomials. The numbers $B_n = P_n(0)$ $(n = 0, 1, 2, \ldots)$ are called the Bernoulli numbers. Derive the following further properties:
c) $B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0, \quad \text{if } n = 2, 3, \ldots$
d) $P_n'(t) = nP_{n-1}(t), \quad \text{if } n = 1, 2, \ldots$
e) $P_n(t+1) - P_n(t) = nt^{n-1} \quad \text{if } n = 1, 2, \ldots$
f) $P_n(1-t) = (-1)^n P_n(t) \quad g) B_{2n+1} = 0 \quad \text{if } n = 1, 2, \ldots$
h) $1^n + 2^n + \cdots + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \quad (n = 2, 3, \ldots).$

Chapter 10

The Lebesgue Integral

10.1 Upper functions

10.1: Properties of max and min functions

Prove that $\max(f,g) + \min(f,g) = f + g$, and that

$$\max(f+h, g+h) = \max(f,g) + h, \quad \min(f+h, g+h) = \min(f,g) + h.$$

10.2: Sequences of max and min functions

Let $\{f_n\}$ and $\{g_n\}$ be increasing sequences of functions on an interval I. Let $u_n = \max(f_n, g_n)$ and $v_n = \min(f_n, g_n)$.

(a) Prove that $\{u_n\}$ and $\{v_n\}$ are increasing on I.

(b) If $f_n \to f$ a.e. on I and if $g_n \to g$ a.e. on I, prove that $u_n \to \max(f,g)$ and $v_n \to \min(f,g)$ a.e. on I.

10.3: Divergence of integral sequence

Let $\{s_n\}$ be an increasing sequence of step functions which converges pointwise on an interval I to a limit function f. If I is unbounded and if $f(x) \geq 1$ almost everywhere on I, prove that the sequence $\{\int_I s_n\}$ diverges.

10.4: Example of upper function

This exercise gives an example of an upper function f on the interval $I = [0, 1]$ such that $-f \notin U(I)$. Let $\{r_1, r_2, \ldots\}$ denote the set of rational numbers in $[0, 1]$ and let $I_n = [r_n - 4^{-n}, r_n + 4^{-n}] \cap I$. Let $f(x) = 1$ if $x \in I_n$ for some n, and let $f(x) = 0$ otherwise.

(a) Let $f_n(x) = 1$ if $x \in I_n$, $f_n(x) = 0$ if $x \notin I_n$, and let $s_n = \max(f_1, \ldots, f_n)$. Show that $\{s_n\}$ is an increasing sequence of step functions which generates f. This shows that $f \in U(I)$.

(b) Prove that $\int_I f \le 2/3$.

(c) If a step function s satisfies $s(x) \le -f(x)$ on I, show that $s(x) \le -1$ almost everywhere on I and hence $\int_I s \le -1$.

(d) Assume that $-f \in U(I)$ and use (b) and (c) to obtain a contradiction.

10.2 Convergence theorems

10.5: Non-interchangeable limit and integral

If $f_n(x) = e^{-nx} - 2e^{-2nx}$, show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx \ne \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx.$$

10.6: Integral evaluations

Justify the following equations:

(a) $\int_0^1 \log \frac{1}{1-x}\, dx = \int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n}\, dx = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n\, dx = 1.$

(b) $\int_0^1 \frac{x^{p-1}}{1-x} \log\left(\frac{1}{x}\right)\, dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2} \quad (p > 0).$

10.7: Tannery's convergence theorem

Prove Tannery's convergence theorem for Riemann integrals: Given a sequence of functions $\{f_n\}$ and an increasing sequence $\{p_n\}$ of real numbers such that $p_n \to +\infty$ as $n \to \infty$. Assume that

(a) $f_n \to f$ uniformly on $[a, b]$ for every $b \geq a$.

(b) f_n is Riemann-integrable on $[a, b]$ for every $b \geq a$.

(c) $|f_n(x)| \leq g(x)$ almost everywhere on $[a, +\infty)$, where g is nonnegative and improper Riemann-integrable on $[a, +\infty)$.

Then both f and $|f|$ are improper Riemann-integrable on $[a, +\infty)$, the sequence $\{\int_a^{p_n} f_n\}$ converges, and

$$\int_a^{+\infty} f(x)\,dx = \lim_{n\to\infty} \int_a^{p_n} f_n(x)\,dx.$$

(d) Use Tannery's theorem to prove that

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^p\,dx = \int_0^\infty e^{-x} x^p\,dx, \quad \text{if } p > -1.$$

10.8: Fatou's lemma

Prove Fatou's lemma: Given a sequence $\{f_n\}$ of nonnegative functions in $L(I)$ such that (a) $\{f_n\}$ converges almost everywhere on I to a limit function f, and (b) $\int_I f_n \leq A$ for some $A > 0$ and all $n \geq 1$. Then the limit function $f \in L(I)$ and $\int_I f \leq A$.
Note. It is not asserted that $\{f_n\}$ converges. (Compare with Theorem 10.24.)
Hint. Let $g_n(x) = \inf\{f_n(x), f_{n+1}(x), \ldots\}$. Then $g_n \to f$ a.e. on I and $\int_I g_n \leq \int_I f_n \leq A$ so $\lim_{n\to\infty} \int_I g_n$ exists and is $\leq A$. Now apply Theorem 10.24.

10.3 Improper Riemann Integrals

10.9: Existence of improper integrals

(a) If $p > 1$, prove that the integral $\int_1^{+\infty} x^{-p} \sin x\, dx$ exists both as an improper Riemann integral and as a Lebesgue integral. **Hint.** Integration by parts.

(b) If $0 < p \leq 1$, prove that the integral in (a) exists as an improper Riemann integral but not as a Lebesgue integral. **Hint.** Let

$$g(x) = \begin{cases} \frac{\sqrt{2}}{2x} & \text{if } m + \frac{\pi}{4} \leq x \leq m + \frac{3\pi}{4} \text{ for } n = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$$

and show that

$$\int_1^{m\pi} x^{-p} |\sin x|\, dx \geq \int_\pi^{m\pi} g(x)\, dx \geq \frac{\sqrt{2}}{4} \sum_{k=2}^{n} \frac{1}{k}.$$

10.10: Trigonometric integrals

(a) Use the trigonometric identity $\sin 2x = 2 \sin x \cos x$, along with the formula $\int_0^\infty \sin x / x\, dx = \pi/2$, to show that

$$\int_0^\infty \frac{\sin x \cos x}{x}\, dx = \frac{\pi}{4}.$$

(b) Use integration by parts in (a) to derive the formula

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

(c) Use the identity $\sin^2 x + \cos^2 x = 1$, along with (b), to obtain

$$\int_0^\infty \frac{\sin^4 x}{x^2}\, dx = \frac{\pi}{4}.$$

(d) Use the result of (c) to obtain

$$\int_0^\infty \frac{\sin^4 x}{x^4}\, dx = \frac{\pi}{3}.$$

10.11: Existence of logarithmic integrals

If $a > 1$, prove that the integral $\int_a^{+\infty} x^p (\log x)^q \, dx$ exists, both as an improper Riemann integral and as a Lebesgue integral for all q if $p < -1$, or for $q < -1$ if $p = -1$.

10.12: Existence of integrals

Prove that each of the following integrals exists, both as an improper Riemann integral and as a Lebesgue integral.

(a) $\int_1^\infty \sin^2 \frac{1}{x} \, dx$,

(b) $\int_0^\infty x^p e^{-x^q} \, dx \quad (p > 0, q > 0)$.

10.13: Determine existence of integrals

Determine whether or not each of the following integrals exists, either as an improper Riemann integral or as a Lebesgue integral.

(a) $\int_0^\infty e^{-(t^2+t^{-2})} \, dt$,

(b) $\int_0^\infty \frac{\cos x}{\sqrt{x}} \, dx$,

(c) $\int_0^\infty \frac{\log x}{x(x^2-1)^{1/2}} \, dx$,

(d) $\int_0^\infty e^{-x} \sin \frac{1}{x} \, dx$,

(e) $\int_0^1 \log x \sin \frac{1}{x} \, dx$,

(f) $\int_0^\infty e^{-x} \log(\cos^2 x) \, dx$.

10.14: Parameter-dependent integrals

Determine those values of p and q for which the following Lebesgue integrals exist.

(a) $\int_0^1 x^p(1-x^2)^q\,dx$,

(b) $\int_0^\infty x^x e^{-x^p}\,dx$,

(c) $\int_0^\infty \frac{x^{p-1}-x^{q-1}}{1-x}\,dx$,

(d) $\int_0^\infty \frac{\sin(x^p)}{x^q}\,dx$,

(e) $\int_0^\infty \frac{x^{p-1}}{1+x^q}\,dx$,

(f) $\int_\pi^\infty (\log x)^p(\sin x)^{-1/3}\,dx$.

10.15: Integral evaluations

Prove that the following improper Riemann integrals have the values indicated (m and n denote positive integers).

(a) $\int_0^\infty \frac{\sin^{2n+1} x}{x}\,dx = \frac{\pi(2n)!}{2^{2n+1}(n!)^2}$,

(b) $\int_1^\infty \frac{\log x}{x^{n+1}}\,dx = n^{-2}$,

(c) $\int_0^\infty x^n(1+x)^{n-m-1}\,dx = \frac{n!(m-1)!}{(m+n)!}$.

10.16: Periodic function integral

Given that f is Riemann-integrable on $[0,1]$, that f is periodic with period 1, and that $\int_0^1 f(x)\,dx = 0$. Prove that the improper Riemann integral $\int_1^\infty x^{-s}f(x)\,dx$ exists if $s > 0$. **Hint.** Let $g(x) = \int_1^x f(t)\,dt$ and write $\int_1^x x^{-s}f(x)\,dx = \int_1^x x^{-s}dg(x)$.

10.17: Limit of integral transformations

Assume that $f \in R$ on $[a, b]$ for every $b > a > 0$. Define g by the equation $xg(x) = \int_1^x f(t)\,dt$ if $x > 0$, assume that the limit $\lim_{x\to+\infty} g(x)$ exists, and denote this limit by B. If a and b are fixed positive numbers, prove that

(a) $\int_a^b \frac{f(x)}{x}\,dx = g(b) - g(a) + \int_a^b \frac{g(x)}{x}\,dx$.

(b) $\lim_{T\to+\infty} \int_{aT}^{bT} \frac{f(x)}{x}\,dx = B \log \frac{b}{a}$.

(c) $\int_1^\infty \frac{f(ax)-f(bx)}{x}\,dx = B \log \frac{a}{b} + \int_a^b \frac{f(t)}{t}\,dt$.

(d) Assume that the limit $\lim_{x\to 0^+} x \int_x^1 f(t)x^{-2}\,dt$ exists, denote this limit by A, and prove that

$$\int_0^1 \frac{f(ax) - f(bx)}{x}\,dx = A \log \frac{b}{a} - \int_a^b \frac{f(t)}{t}\,dt.$$

(e) Combine (c) and (d) to deduce

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx = (B - A) \log \frac{a}{b}$$

and use this result to evaluate the following integrals:

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx, \qquad \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx.$$

10.4 Lebesgue integrals

10.18: Existence of Lebesgue integrals

Prove that each of the following exists as a Lebesgue integral.

(a) $\int_0^1 \frac{x \log x}{(1+x)^2}\,dx$,

(b) $\int_0^1 \frac{x^p - 1}{\log x}\,dx \quad (p > -1)$,

(c) $\int_0^1 \log x \log(1 + x)\,dx$,

(d) $\int_0^1 \frac{\log(1-x)}{(1-x)^{1/2}}\,dx$.

10.19: Existence of singular integral

Assume that f is continuous on $[0,1]$, $f(0)=0$, $f'(0)$ exists. Prove that the Lebesgue integral $\int_0^1 f(x)x^{-3/2}\,dx$ exists.

10.20: Existence/non-existence of integrals

Prove that the integrals in (a) and (c) exist as Lebesgue integrals but that those in (b) and (d) do not.

(a) $\int_0^\infty x^2 e^{-x^8\sin^2 x}\,dx$,

(b) $\int_0^\infty x^3 e^{-x^8\sin^2 x}\,dx$,

(c) $\int_1^\infty \frac{dx}{1+x^4\sin^2 x}$,

(d) $\int_1^\infty \frac{dx}{1+x^2\sin^2 x}$.

Hint. Obtain upper and lower bounds for the integrals over suitably chosen neighborhoods of the points $n\pi$ $(n=1,2,3,\ldots)$.

10.5 Functions defined by integrals

10.21: Domain of integral functions

Determine the set S of those real values of y for which each of the following integrals exists as a Lebesgue integral.

(a) $\int_0^\infty \frac{\cos xy}{1+x^2}\,dx$,

(b) $\int_0^\infty (x^2+y^2)^{-1}\,dx$,

(c) $\int_0^\infty \frac{\sin^2 xy}{x^2}\,dx$,

(d) $\int_0^\infty e^{-x^2}\cos 2xy\,dx$.

10.22: Differential equation for integral

Let $F(y)=\int_0^\infty e^{-x^2}\cos 2xy\,dx$ if $y\in\mathbb{R}$. Show that F satisfies the differential equation $F'(y)+2yF(y)=0$ and deduce that $F(y)=\frac{1}{2}\sqrt{\pi}e^{-y^2}$. (Use the result $\int_0^\infty e^{-x^2}\,dx=\frac{1}{2}\sqrt{\pi}$, derived in Exercise 7.19.)

10.23: Integral with trigonometric kernel

Let $F(y) = \int_0^\infty \frac{\sin xy}{x(x^2+1)}\,dx$ if $y > 0$. Show that F satisfies the differential equation $F''(y) - F(y) + \pi/2 = 0$ and deduce that $F(y) = \frac{1}{2}\pi(1 - e^{-y})$. Use this result to deduce the following equations, valid for $y > 0$ and $a > 0$:

$$\int_0^\infty \frac{\sin xy}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1 - e^{-ay}),$$
$$\int_0^\infty \frac{\cos xy}{x^2+a^2}\,dx = \frac{\pi e^{-ay}}{2a},$$
$$\int_0^\infty \frac{x\sin xy}{x^2+a^2}\,dx = \frac{\pi}{2}e^{-ay}.$$

Note. You may use $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

10.24: Non-interchangeable iterated integrals

Show that $\int_1^\infty \left[\int_1^\infty f(x,y)\,dx\right] dy \neq \int_1^\infty \left[\int_1^\infty f(x,y)\,dy\right] dx$ if

(a) $f(x,y) = \frac{x-y}{(x+y)^3}$,

(b) $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$.

10.25: Non-interchangeable integration order

Show that the order of integration cannot be interchanged in the following integrals:

(a) $\int_0^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\,dx\right] dy$,

(b) $\int_0^1 \left[\int_1^\infty (e^{-xy} - 2e^{-2xy})\,dy\right] dx$.

10.26: Integral evaluation via iterated integral

Let $f(x,y) = \int_0^\infty dt/[(1+x^2t^2)(1+y^2t^2)]$ if $(x,y) \neq (0,0)$. Show (by methods of elementary calculus) that $f(x,y) = \frac{1}{2}\pi(x+y)^{-1}$. Evaluate the iterated integral $\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy$ to derive the formula:

$$\int_0^\infty \frac{(\arctan x)^2}{x^2}\,dx = \pi \log 2.$$

10.27: Trigonometric integral evaluation

Let $f(y) = \int_0^\infty \frac{\sin x \cos xy}{x}\,dx$ if $y \geq 0$. Show (by methods of elementary calculus) that $f(y) = \pi/2$ if $0 \leq y < 1$ and that $f(y) = 0$ if $y > 1$. Evaluate the integral $\int_0^1 f(y)\,dy$ to derive the formula

$$\int_0^\infty \frac{\sin ax \sin x}{x^2}\,dx = \begin{cases} \frac{\pi a}{2} & \text{if } 0 \leq a \leq 1, \\ \frac{\pi}{2} & \text{if } a \geq 1. \end{cases}$$

10.28: Series of integrals

(a) If $s > 0$ and $a > 0$, show that the series

$$\sum_{n=1}^\infty \frac{1}{n} \int_a^\infty \frac{\sin 2n\pi x}{x^s}\,dx$$

converges and prove that

$$\lim_{a\to+\infty} \sum_{n=1}^\infty \frac{1}{n} \int_a^\infty \frac{\sin 2n\pi x}{x^s}\,dx = 0.$$

(b) Let $f(x) = \sum_{n=1}^\infty \sin(2n\pi x)/n$. Show that

$$\int_0^\infty \frac{f(x)}{x^s}\,dx = (2\pi)^{s-1}\zeta(2-s)\int_0^\infty \frac{\sin t}{t^s}\,dt, \quad \text{if } 0 < s < 1,$$

where ζ denotes the Riemann zeta function.

10.29: Derivatives of Gamma function

(a) Derive the following formula for the nth derivative of the Gamma function:

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t} t^{x-1} (\log t)^n \, dt \quad (x > 0).$$

(b) When $x = 1$, show that this can be written as follows:

$$\Gamma^{(n)}(1) = \int_0^1 (t^2 + (-1)^n e^{t-1/t}) e^{-t} t^{-2} (\log t)^n \, dt.$$

(c) Use (b) to show that $\Gamma^{(n)}(1)$ has the same sign as $(-1)^n$.

10.30: Properties of Gamma function

Use the result $\int_0^\infty e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}$ to prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Prove that $\Gamma(n+1) = n!$ and that $\Gamma(n+\frac{1}{2}) = (2n)!\sqrt{\pi}/4^n n!$ if $n = 0, 1, 2, \ldots$.

10.31: Series representation of Gamma function

(a) Show that for $x > 0$ we have the series representation

$$\Gamma(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{n+x} + \sum_{n=0}^\infty c_n x^n,$$

where $c_n = (1/n!) \int_0^\infty t^{-1} e^{-t} (\log t)^n dt$. **Hint:** Write $\int_0^\infty = \int_0^1 + \int_1^\infty$ and use an appropriate power series expansion in each integral.

(b) Show that the power series $\sum_{n=0}^\infty c_n z^n$ converges for every complex z and that the series $\sum_{n=0}^\infty [(-1)^n/n!]/(n+z)$ converges for every complex $z \neq 0, -1, -2, \ldots$.

10.32: Limit of Laplace transform

Assume that f is of bounded variation on $[0, b]$ for every $b > 0$, and that $\lim_{x\to+\infty} f(x)$ exists. Denote this limit by $f(\infty)$ and prove that

$$\lim_{y\to 0+} y \int_0^\infty e^{-xy} f(x)\,dx = f(\infty).$$

Hint. Use integration by parts.

10.33: Limit of Mellin transform

Assume that f is of bounded variation on $[0, 1]$. Prove that

$$\lim_{y\to 0+} y \int_0^1 x^{y-1} f(x)\,dx = f(0+).$$

10.6 Measurable functions

10.34: Measurability of derivative

If f is Lebesgue-integrable on an open interval I and if $f'(x)$ exists almost everywhere on I, prove that f' is measurable on I.

10.35: Measurable functions

(a) Let $\{s_n\}$ be a sequence of step functions such that $s_n \to f$ everywhere on $\mathbb{R}$. Prove that, for every real a,

$$f^{-1}((a, +\infty)) = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} s_k^{-1}\left(\left(a + \frac{1}{n}, +\infty\right)\right).$$

(b) If f is measurable on $\mathbb{R}$, prove that for every open subset A of $\mathbb{R}$ the set $f^{-1}(A)$ is measurable.

10.36: Nonmeasurable set example

This exercise describes an example of a nonmeasurable set in $\mathbb{R}$. If x and y are real numbers in the interval $[0,1]$, we say that x and y are equivalent, written $x \sim y$, whenever $x - y$ is rational. The relation $\sim$ is an equivalence relation, and the interval $[0,1]$ can be expressed as a disjoint union of subsets (called equivalence classes) in each of which no two distinct points are equivalent. Choose a point from each equivalence class and let E be the set of points so chosen. We assume that E is measurable and obtain a contradiction. Let $A = \{r_1, r_2, \ldots\}$ denote the set of rational numbers in $[-1,1]$ and let $E_n = \{r_n + x : x \in E\}$.

(a) Prove that each E_n is measurable and that $\mu(E_n) = \mu(E)$.

(b) Prove that $\{E_1, E_2, \ldots\}$ is a disjoint collection of sets whose union contains $[0,1]$ and is contained in $[-1,2]$.

(c) Use parts (a) and (b) along with the countable additivity of Lebesgue measure to obtain a contradiction.

10.37: Nonmeasurable function

Refer to Exercise 10.36 and prove that the characteristic function χ_E is not measurable. Let $f = \chi_E - \chi_{I-E}$ where $I = [0,1]$. Prove that $|f| \in L(I)$ but that $f \notin M(I)$. (Compare with Corollary 1 of Theorem 10.35.)

10.7 Square-integrable functions

10.38: Norm convergence

If $\lim_{n\to\infty} \|f_n - f\| = 0$, prove that $\lim_{n\to\infty} \|f_n\| = \|f\|$.

10.39: Almost everywhere convergence

If $\lim_{n\to\infty} \|f_n - f\| = 0$ and if $\lim_{n\to\infty} f_n(x) = g(x)$ almost everywhere on I, prove that $f(x) = g(x)$ almost everywhere on I.

10.40: Uniform convergence

If $f_n \to f$ uniformly on a compact interval I, and if each f_n is continuous on I, prove that $\lim_{n\to\infty} \|f_n - f\| = 0$.

10.41: Weak convergence

If $\lim_{n\to\infty} \|f_n - f\| = 0$, prove that $\lim_{n\to\infty} \int_0^x f_n \cdot g = \int_0^x f \cdot g$ for every g in $L^2(I)$.

10.42: Product convergence

If $\lim_{n\to\infty} \|f_n - f\| = 0$ and $\lim_{n\to\infty} \|g_n - g\| = 0$, prove that $\lim_{n\to\infty} \int_0^x f_n \cdot g_n = \int_0^x f \cdot g$.

Chapter 11

Fourier Series and Fourier Integrals

11.1 Orthogonal Systems

11.1: Orthonormality of Trigonometric System

Verify that the trigonometric system in (1) is orthonormal on $[0, 2\pi]$.

11.2: Linear Independence of Orthonormal Systems

A finite collection of functions $\{\varphi_0, \varphi_1, \dots, \varphi_M\}$ is said to be linearly independent on $[a, b]$ if the equation

$$\sum_{k=0}^{M} c_k \varphi_k(x) = 0$$

for all x in $[a, b]$ implies $c_0 = c_1 = \cdots = c_M = 0$. An infinite collection is called linearly independent on $[a, b]$ if every finite subset is linearly independent on $[a, b]$. Prove that every orthonormal system on $[a, b]$ is linearly independent on $[a, b]$.

11.3: Gram-Schmidt Orthogonalization

Let $\{f_0, f_1, \dots\}$ be a linearly independent system on $[a, b]$ (as defined in Exercise 11.2). Define a new system $\{g_0, g_1, \dots\}$ recursively as follows:

$$g_0 = f_0, \quad g_{n+1} = f_{n+1} - \sum_{k=0}^{n} a_k g_k,$$

where $a_k = (f_{n+1}, g_k)/(g_k, g_k)$ if $\|g_k\| \neq 0$, and $a_k = 0$ if $\|g_k\| = 0$. Prove that g_{n+1} is orthogonal to each of $g_0, g_1, \dots, g_n$ for every $n \geq 0$.

11.4: Gram-Schmidt on Polynomials

Let $(f, g) = \int_{-1}^{1} f(t)g(t)\,dt$. Apply the Gram-Schmidt process to the system of polynomials $\{1, t, t^2, \dots\}$ on the interval $[-1, 1]$ and show that

$$g_1(t) = t, \quad g_2(t) = t^2 - \frac{1}{3}, \quad g_3(t) = t^3 - \frac{3}{5}t, \quad g_4(t) = t^4 - \frac{6}{7}t^2 + \frac{3}{35}.$$

11.5: Approximation of Periodic Functions

(a) Assume $f \in \mathcal{R}$ on $[0, 2\pi]$, where f is real and has period 2π. Prove that for every $\epsilon > 0$, there is a continuous function g of period 2π such that $\|f - g\| < \epsilon$.

- *Hint:* Choose a partition P of $[0, 2\pi]$ for which f satisfies Riemann's condition $U(P, f) - L(P, f) < \epsilon$ and construct a piecewise linear g which agrees with f at the points of P.

(b) Use part (a) to show that Theorem 11.16(a), (b), and (c) holds if f is Riemann integrable on $[0, 2\pi]$.

11.6: Completeness of Orthonormal Systems

In this exercise, all functions are assumed to be continuous on a compact interval $[a, b]$. Let $\{\varphi_0, \varphi_1, \dots\}$ be an orthonormal system on $[a, b]$.

(a) Prove that the following three statements are equivalent:

1) $(f, \varphi_n) = (g, \varphi_n)$ for all n implies $f = g$. (Two distinct continuous functions cannot have the same Fourier coefficients.)

2) $(f, \varphi_n) = 0$ for all n implies $f = 0$. (The only continuous function orthogonal to every φ_n is the zero function.)

3) If T is an orthonormal set on $[a, b]$ such that $\{\varphi_0, \varphi_1, \dots\} \subseteq T$, then $\{\varphi_0, \varphi_1, \dots\} = T$. (We cannot enlarge the orthonormal set.) This property is described by saying that $\{\varphi_0, \varphi_1, \dots\}$ is maximal or complete.

(b) Let $\varphi_n(x) = e^{inx}/\sqrt{2\pi}$ for n an integer, and verify that the set $\{\varphi_n : n \in \mathbb{Z}\}$ is complete on every interval of length 2π.

11.7: Properties of Legendre Polynomials

If $x \in \mathbb{R}$ and $n = 1, 2, \ldots$, let $f_n(x) = (x^2 - 1)^n$ and define

$$\varphi_0(x) = 1, \quad \varphi_n(x) = \frac{1}{n!} f_n^{(n)}(x).$$

It is clear that φ_n is a polynomial. This is called the Legendre polynomial of order n. The first few are

$$\varphi_1(x) = x, \quad \varphi_2(x) = \frac{1}{2}(3x^2-1), \quad \varphi_3(x) = \frac{1}{2}(5x^3-3x), \quad \varphi_4(x) = \frac{1}{8}(35x^4-30x^2+3).$$

Derive the following properties of Legendre polynomials:

(a) $\varphi_n'(x) = x\varphi_{n-1}'(x) + n\varphi_{n-1}(x)$.

(b) $\varphi_n'(x) = x\varphi_{n-1}'(x) + \frac{n}{x^2-1}\varphi_{n-1}(x)$.

(c) $(n+1)\varphi_{n+1}(x) = (2n+1)x\varphi_n(x) - n\varphi_{n-1}(x)$.

(d) φ_n satisfies the differential equation $[(1-x^2)y']' + n(n+1)y = 0$.

(e) $[(1-x^2)\Delta(x)]' + [m(m+1) - n(n+1)]\varphi_m(x)\varphi_n(x) = 0$, where $\Delta = \varphi_n'\varphi_m - \varphi_m'\varphi_n$.

(f) The set $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ is orthogonal on $[-1, 1]$.

(g) $\int_{-1}^{1} \varphi_n^2(x)\,dx = \frac{2}{2n+1}$.

(h) $\int_{-1}^{1} x\varphi_n^2(x)\,dx = 0$.

Note: The polynomials $g_n(t) = \sqrt{\frac{2n+1}{2}}\varphi_n(t)$ arise by applying the Gram-Schmidt process to the system $\{1, t, t^2, \ldots\}$ on the interval $[-1, 1]$. (See Exercise 11.4.)

11.2 Trigonometric Fourier Series

11.8: Fourier Series for Even and Odd Functions

Assume that $f \in L([-\pi, \pi])$ and that f has period 2π. Show that the Fourier series generated by f assumes the following special forms under the conditions stated:

(a) If $f(-x) = f(x)$ when $0 < x < \pi$, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

where $a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos nt \, dt$.

(b) If $f(-x) = -f(x)$ when $0 < x < \pi$, then

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx,$$

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin nt \, dt$.

11.9: Fourier Series for Linear and Quadratic Functions

Show that each of the expansions is valid in the range indicated.

(a) $x = \pi - 2 \sum_{n=1}^{\infty} \frac{\sin nx}{n}$ if $0 < x < 2\pi$.

- *Note:* When $x = 0$, this gives $\zeta(2) = \pi^2/6$.

(b) $x^2 = \pi x - \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ if $0 < x < 2\pi$.

11.10: Fourier Series for Odd and Even Terms

Show that each of the expansions is valid in the range indicated.

(a) $|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$ if $0 < x < \pi$.

(b) $x = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$ if $-\pi < x < \pi$.

11.11: Fourier Series for Linear Functions

Show that each of the expansions is valid in the range indicated.

(a) $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sin nx}{n}$ if $-\pi < x < \pi$.

(b) $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ if $-\pi < x < \pi$.

11.12: Fourier Series for Trigonometric Functions

Show that each of the expansions is valid in the range indicated.

(a) $\cos x = \frac{4}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2}$ if $-\pi < x < \pi$.

(b) $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$ if $-\pi < x < \pi$.

11.13: Fourier Series for Cosine and Sine

Show that each of the expansions is valid in the range indicated.

(a) $\cos x = \frac{\pi}{2} - \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n\sin 2nx}{4n^2-1}$ if $0 < x < 2\pi$.

(b) $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}$ if $0 < x < \pi$.

11.14: Fourier Series for Products

Show that each of the expansions is valid in the range indicated.

(a) $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n-1} n\sin nx}{n^2-1}$ if $-\pi < x < \pi$.

(b) $x\sin x = \frac{1}{2} - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos nx}{n^2-1}$ if $-\pi < x < \pi$.

11.15: Fourier Series for Logarithmic Functions

Show that each of the expansions is valid in the range indicated.

(a) $\log\left|\sin\frac{x}{2}\right| = -\log 2 - \sum_{n=1}^{\infty} \frac{\cos nx}{n}$ if $x \neq 2k\pi$ (k an integer).

(b) $\log\left|\cos\frac{x}{2}\right| = -\log 2 - \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n}$ if $x \neq (2k+1)\pi$.

(c) $\log\left|\tan\frac{x}{2}\right| = -2\sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{2n-1}$ if $x \neq (2k+1)\pi$.

11.16: Fourier Series and Zeta Function

(a) Find a continuous function on $[-\pi, \pi]$ which generates the Fourier series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nx$. Then use Parseval's formula to prove that $\zeta(6) = \frac{\pi^6}{945}$.

(b) Use an appropriate Fourier series in conjunction with Parseval's formula to show that $\zeta(4) = \frac{\pi^4}{90}$.

11.17: Parseval's Formula Application

Assume that f has a continuous derivative on $[0, 2\pi]$, that $f(0) = f(2\pi)$, and that $\int_0^{2\pi} f(t)\, dt = 0$. Prove that

$$\|f'\| \geq \|f\|,$$

with equality if and only if $f(x) = a\cos x + b\sin x$.

- *Hint:* Use Parseval's formula.

11.18: Bernoulli Functions

A sequence $\{B_n\}$ of periodic functions (of period 1) is defined on $\mathbb{R}$ as follows:

$$B_{2n}(x) = (-1)^{n+1} \frac{2}{(2n)!} \sum_{k=1}^{\infty} \frac{\cos 2\pi kx}{(\pi k)^{2n}}, \quad (n = 0, 1, 2, \ldots),$$

$$B_{2n+1}(x) = \frac{2}{(2n+1)!} \sum_{k=1}^{\infty} \frac{\sin 2\pi kx}{(\pi k)^{2n+1}}, \quad (n = 0, 1, 2, \ldots).$$

(B_n is called the Bernoulli function of order n.) Show that:

(a) $B_1(x) = x - [x] - \frac{1}{2}$ if x is not an integer. ($[x]$ is the greatest integer $\leq x$.)

(b) $\int_0^1 B_n(x)\, dx = 0$ if $n \geq 1$ and $B_n'(x) = nB_{n-1}(x)$ if $n \geq 2$.

(c) $B_n(x) = P_n(x)$ if $0 < x < 1$, where P_n is the nth Bernoulli polynomial. (See Exercise 9.38 for the definition of P_n.)

(d) $B_n(x) = -\sum_{k \neq 0} \frac{e^{2\pi ikx}}{(2\pi ik)^n}$, $(n = 1, 2, \ldots)$.

11.19: Gibbs' Phenomenon

Let f be the function of period 2π whose values on $[-\pi, \pi]$ are

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi, \\ 0 & \text{if } x = 0 \text{ or } x = \pi, \\ -1 & \text{if } -\pi < x < 0. \end{cases}$$

(a) Show that

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1},$$

for every x.

(b) Show that

$$s_n(x) = \frac{2}{\pi} \int_0^x \frac{\sin 2nt}{\sin t}\, dt,$$

where $s_n(x)$ denotes the nth partial sum of the series in part (a).

(c) Show that, in $(0, \pi)$, s_n has local maxima at $x_1, x_3, \ldots, x_{2n-1}$ and local minima at $x_2, x_4, \ldots, x_{2n-2}$, where $x_m = \frac{m\pi}{n}$ $(m = 1, 2, \ldots, 2n-1)$.

(d) Show that $s_n\left(\frac{\pi}{n}\right)$ is the largest of the numbers $s_n(x_m)$ $(m = 1, 2, \ldots, 2n-1)$.

(e) Interpret $s_n\left(\frac{\pi}{n}\right)$ as a Riemann sum and prove that

$$\lim_{n\to\infty} s_n\left(\frac{\pi}{n}\right) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

The value of the limit in (e) is about 1.179. Thus, although f has a jump equal to 2 at the origin, the graphs of the approximating curves s_n tend to approximate a vertical segment of length 2.358 in the vicinity of the origin. This is the Gibbs phenomenon.

11.20: Fourier Coefficients of Bounded Variation

If $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ and if f is of bounded variation on $[0, 2\pi]$, show that $a_n = O(1/n)$ and $b_n = O(1/n)$.

- *Hint:* Write $f = g - h$, where g and h are increasing on $[0, 2\pi]$. Then

$$a_n = \frac{2}{n\pi} \int_0^{2\pi} g(x)\, d(\sin nx) - \frac{2}{n\pi} \int_0^{2\pi} h(x)\, d(\sin nx).$$

  Now apply Theorem 7.31.

11.21: Lipschitz Condition and Lebesgue Integral

Suppose $g \in L([a, \delta])$ for every a in $(0, \delta)$ and assume that g satisfies a "right-handed" Lipschitz condition at 0. (See the Note following Theorem 11.9.) Show that the Lebesgue integral $\int_0^{\delta} \frac{|g(t)-g(0+)|}{t}\, dt$ exists.

11.22: Fourier Series Convergence

Use Exercise 11.21 to prove that differentiability of f at a point implies convergence of its Fourier series at the point.

11.23: Orthogonality to Polynomials

Let g be continuous on $[0, 1]$ and assume that $\int_0^1 t^n g(t)\, dt = 0$ for $n = 0, 1, 2, \ldots$. Show that:

(a) $\int_0^1 g(t)^2\, dt = \int_0^1 g(t)(g(t) - P(t))\, dt$ for every polynomial P.

(b) $\int_0^1 g(t)^2\, dt = 0$.

- *Hint:* Use Theorem 11.17.

(c) $g(t) = 0$ for every t in $[0, 1]$.

11.24: Weierstrass Approximation

Use the Weierstrass approximation theorem to prove each of the following statements.

(a) If f is continuous on $[1, +\infty)$ and if $f(x) \to a$ as $x \to +\infty$, then f can be uniformly approximated on $[1, +\infty)$ by a function g of the form $g(x) = p(1/x)$, where p is a polynomial.

(b) If f is continuous on $[0, +\infty)$ and if $f(x) \to a$ as $x \to +\infty$, then f can be uniformly approximated on $[0, +\infty)$ by a function g of the form $g(x) = p(e^{-x})$, where p is a polynomial.

11.25: Arithmetic Means of Fourier Series

Assume that $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ and let $\{\sigma_n\}$ be the sequence of arithmetic means of the partial sums of this series, as it was given in (23). Show that:

(a) $\sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)(a_k \cos kx + b_k \sin kx)$.

(b) $\int_0^{2\pi} |f(x) - \sigma_n(x)|^2\, dx = \frac{\pi}{n^2} \sum_{k=1}^{n-1} k^2(a_k^2 + b_k^2)$.

(c) If f is continuous on $[0, 2\pi]$ and has period 2π, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} k^2(a_k^2 + b_k^2) = 0.$$

11.26: Convergence of Exponential Fourier Series

Consider the Fourier series (in exponential form) generated by a function f which is continuous on $[0, 2\pi]$ and periodic with period 2π, say

$$f(x) \sim \sum_{n=-\infty}^{+\infty} a_n e^{inx}.$$

Assume also that the derivative $f' \in \mathcal{R}$ on $[0, 2\pi]$.

(a) Prove that the series $\sum n^2|a_n|^2$ converges; then use the Cauchy-Schwarz inequality to deduce that $\sum |a_n|$ converges.

(b) From (a), deduce that the series $\sum a_n e^{inx}$ converges uniformly to a continuous sum function g on $[0, 2\pi]$. Then prove that $f = g$.

11.3 Fourier Integrals

11.27: Fourier Integral for Even and Odd Functions

If f satisfies the hypotheses of the Fourier integral theorem, show that:

(a) If f is even, that is, if $f(-t) = f(t)$ for every t, then

$$\frac{f(x+) + f(x-)}{2} = \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(u) \cos vu \, du \right] \cos vx \, dv.$$

(b) If f is odd, that is, if $f(-t) = -f(t)$ for every t, then

$$\frac{f(x+) + f(x-)}{2} = \frac{2}{\pi} \int_0^\infty \left[\int_0^\infty f(u) \sin vu \, du \right] \sin vx \, dv.$$

11.28: Fourier Integral Evaluation

Use the Fourier integral theorem to evaluate the improper integral:

$$\int_0^\infty \frac{\sin v \cos vx}{v} \, dv = \begin{cases} \frac{\pi}{2} & \text{if } -1 < x < 1, \\ 0 & \text{if } |x| > 1, \\ \frac{\pi}{4} & \text{if } |x| = 1. \end{cases}$$

- *Suggestion:* Use Exercise 11.27 when possible.

11.29: Fourier Integral with Exponential

Use the Fourier integral theorem to evaluate the improper integral:

$$\int_0^\infty \cos ax e^{-b|x|} \, dx = \frac{2b}{b^2 + a^2}, \quad \text{if } b > 0.$$

- *Hint:* Apply Exercise 11.27 with $f(u) = e^{-b|u|}$.

11.30: Fourier Integral with Rational Function

Use the Fourier integral theorem to evaluate the improper integral:

$$\int_0^\infty \frac{x \sin ax}{1 + x^2} \, dx = \pi e^{-|a|}, \quad \text{if } a \neq 0.$$

11.31: Gamma Function Properties

(a) Prove that

$$\Gamma(p)\Gamma(p) = \frac{2}{\Gamma(2p)} \int_0^1 x^{p-1}(1-x)^{p-1}\, dx.$$

(b) Make a suitable change of variable in (a) and derive the duplication formula for the Gamma function:

$$\Gamma(2p)\Gamma\left(\frac{1}{2}\right) = 2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right).$$

- *Note:* In Exercise 10.30, it is shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

11.32: Fourier Transform of Gaussian

If $f(x) = e^{-x^2/2}$ and $g(x) = xf(x)$ for all x, prove that

$$\int_0^\infty f(x)\cos xy\, dx = f(y), \quad \text{and} \quad \int_0^\infty g(x)\sin xy\, dx = g(y).$$

11.33: Poisson Summation Formula

This exercise describes another form of Poisson's summation formula. Assume that f is nonnegative, decreasing, and continuous on $[0, +\infty)$ and that $\int_0^\infty f(x)\, dx$ exists as an improper Riemann integral. Let

$$g(y) = \frac{2}{\pi}\int_0^\infty f(x)\cos xy\, dx.$$

If a and b are positive numbers such that $ab = 2\pi$, prove that

$$\sqrt{a}\left\{f(0) + \sum_{m=1}^\infty f(ma)\right\} = \sqrt{b}\left\{g(0) + \sum_{n=1}^\infty g(nb)\right\}.$$

11.34: Transformation Formula

Prove that the transformation formula (55) for $\theta(x)$ can be put in the form

$$\sum_{m=1}^{\infty} e^{-\pi m^2 a^2} + \frac{1}{2} = \frac{1}{\sqrt{a}} \left(\sum_{n=1}^{\infty} e^{-\pi n^2 b^2} + \frac{1}{2} \right),$$

where $ab = 2\pi$, $a > 0$.

11.35: Zeta Function and Integral

If $s > 1$, prove that

$$\pi^{-s/2} \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s/2)} \int_0^{\infty} e^{-\pi x^2} x^{s/2-1}\, dx,$$

and derive the formula

$$\sum_{n=1}^{\infty} n^{-s} = \pi^{s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} (\theta(x) - 1)\, x^{s/2-1}\, dx,$$

where $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x}$. Use this and the transformation formula for $\theta(x)$ to prove that

$$\pi^{s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} \left[x^{s/2-1} + x^{(1-s)/2-1} \right] \theta(x)\, dx.$$

11.4 Laplace Transforms

11.36: Laplace Transform Table

Verify the entries in the following table of Laplace transforms:

$f(t)$	$F(z) = \int_0^{\infty} e^{-zt} f(t)\, dt, \quad z = x + iy$
e^{at}	$(z-a)^{-1}, \quad (x > a)$
$\cos at$	$z/(z^2 + a^2), \quad (x > 0)$
$\sin at$	$a/(z^2 + a^2), \quad (x > 0)$
$t^p e^{at}$	$\Gamma(p+1)/(z-a)^{p+1}, \quad (x > a, p > 0)$

11.37: Convolution and Laplace Transform

Show that the convolution $h = f * g$ assumes the form

$$h(t) = \int_0^t f(x)g(t-x)\,dx,$$

when both f and g vanish on the negative real axis. Use the convolution theorem for Fourier transforms to prove that $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$.

11.38: Properties of Laplace Transform

Assume f is continuous on $(0, +\infty)$ and let $F(z) = \int_0^\infty e^{-zt} f(t)\,dt$ for $z = x + iy$, $x > c > 0$. If $s > c$ and $a > 0$, prove that:

(a) $F(s + a) = a \int_0^\infty g(t)e^{-at}\,dt$, where $g(x) = \int_0^\infty e^{-st} f(t)\,dt$.

(b) If $F(s + na) = 0$ for $n = 0, 1, 2, \ldots$, then $f(t) = 0$ for $t > 0$.

- *Hint:* Use Exercise 11.23.

(c) If h is continuous on $(0, +\infty)$ and if f and h have the same Laplace transform, then $f(t) = h(t)$ for every $t > 0$.

11.39: Inversion Formula for Laplace Transforms

Let $F(z) = \int_0^\infty e^{-zt} f(t)\,dt$ for $z = x + iy$, $x > c > 0$. Let t be a point at which f satisfies one of the "local" conditions (a) or (b) of the Fourier integral theorem (Theorem 11.18). Prove that for each $a > c$, we have

$$\frac{f(t+) + f(t-)}{2} = \frac{1}{2\pi} \lim_{T \to +\infty} \int_{-T}^{T} e^{(a+iv)t} F(a + iv)\,dv.$$

This is called the inversion formula for Laplace transforms. The limit on the right is usually evaluated with the help of residue calculus, as described in Section 16.26.

- *Hint:* Let $g(t) = e^{-at} f(t)$ for $t > 0$, $g(t) = 0$ for $t < 0$, and apply Theorem 11.19 to g.

Chapter 12

Multivariable Differential Calculus

12.1 Differentiable Functions

12.1: Local Extrema and Partial Derivatives

Let S be an open subset of $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be a real-valued function with finite partial derivatives $D_1 f, \ldots, D_n f$ on S. If f has a local maximum or a local minimum at a point c in S, prove that $D_k f(c) = 0$ for each k.

12.2: Partial and Directional Derivatives

Calculate all first-order partial derivatives and the directional derivative $f'(x; u)$ for each of the real-valued functions defined on $\mathbb{R}^n$ as follows:

(a) $f(x) = a \cdot x$, where a is a fixed vector in $\mathbb{R}^n$.

(b) $f(x) = \|x\|^4$.

(c) $f(x) = x \cdot L(x)$, where $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function.

(d) $f(x) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$.

12.3: Directional Derivatives of Sum and Product

Let f and g be functions with values in $\mathbb{R}^m$ such that the directional derivatives $f'(c;u)$ and $g'(c;u)$ exist. Prove that the sum $f+g$ and dot product $f \cdot g$ have directional derivatives given by

$$(f+g)'(c;u) = f'(c;u) + g'(c;u)$$

and

$$(f \cdot g)'(c;u) = f(c) \cdot g'(c;u) + g(c) \cdot f'(c;u).$$

12.4: Differentiability of Vector-Valued Functions

If $S \subseteq \mathbb{R}^n$, let $f : S \to \mathbb{R}^m$ be a function with values in $\mathbb{R}^m$, and write $f = (f_1, \ldots, f_m)$. Prove that f is differentiable at an interior point c of S if, and only if, each f_i is differentiable at c.

12.5: Differentiability of Sum of Univariate Functions

Given n real-valued functions $f_1, \ldots, f_n$, each differentiable on an open interval (a,b) in $\mathbb{R}$. For each $x = (x_1, \ldots, x_n)$ in the n-dimensional open interval

$$S = \{(x_1, \ldots, x_n) : a < x_k < b, \quad k = 1, 2, \ldots, n\},$$

define $f(x) = f_1(x_1) + \cdots + f_n(x_n)$. Prove that f is differentiable at each point of S and that

$$f'(x)(u) = \sum_{i=1}^{n} f_i'(x_i)u_i, \quad \text{where } u = (u_1, \ldots, u_n).$$

12.6: Differentiability with Partial Limits

Given n real-valued functions $f_1, \ldots, f_n$ defined on an open set S in $\mathbb{R}^n$. For each x in S, define $f(x) = f_1(x) + \cdots + f_n(x)$. Assume that for each $k = 1, 2, \ldots, n$, the following limit exists:

$$\lim_{\substack{y \to x \\ y_k \neq x_k}} \frac{f_k(y) - f_k(x)}{y_k - x_k}.$$

Call this limit $a_k(x)$. Prove that f is differentiable at x and that

$$f'(x)(u) = \sum_{k=1}^{n} a_k(x) u_k \quad \text{if } u = (u_1, \ldots, u_n).$$

12.7: Differentiability of Product at Zero

Let f and g be functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. Assume that f is differentiable at c, that $f(c) = 0$, and that g is continuous at c. Let $h(x) = g(x) \cdot f(x)$. Prove that h is differentiable at c and that

$$h'(c)(u) = g(c) \cdot \{f'(c)(u)\} \quad \text{if } u \in \mathbb{R}^n.$$

12.8: Jacobian Matrix Calculation

Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by the equation

$$f(x, y) = (\sin x \cos y, \sin x \sin y, \cos x \cos y).$$

Determine the Jacobian matrix $Df(x, y)$.

12.9: Nonexistence of Positive Directional Derivative

Prove that there is no real-valued function f such that $f'(c; u) > 0$ for a fixed point c in $\mathbb{R}^n$ and every nonzero vector u in $\mathbb{R}^n$. Give an example such that $f'(c; u) > 0$ for a fixed direction u and every c in $\mathbb{R}^n$.

12.10: Complex Differentiability and Directional Derivatives

Let $f = u+iv$ be a complex-valued function such that the derivative $f'(c)$ exists for some complex c. Write $z = c + re^{i\alpha}$ (where α is real and fixed) and let $r \to 0$ in the difference quotient $[f(z) - f(c)]/(z - c)$ to obtain

$$f'(c) = e^{-i\alpha}[u'(c; a) + iv'(c; a)],$$

where $a = (\cos\alpha, \sin\alpha)$, and $u'(c; a)$ and $v'(c; a)$ are directional derivatives. Let $b = (\cos\beta, \sin\beta)$, where $\beta = \alpha + \frac{1}{2}\pi$, and show by a similar argument that

$$f'(c) = e^{-i\alpha}[v'(c; b) - iu'(c; b)].$$

Deduce that $u'(c; a) = v'(c; b)$ and $v'(c; a) = -u'(c; b)$. The Cauchy-Riemann equations (Theorem 5.22) are a special case.

12.2 Gradients and the Chain Rule

12.11: Maximum Directional Derivative

Let f be real-valued and differentiable at a point c in $\mathbb{R}^n$, and assume that $\|\nabla f(c)\| \neq 0$. Prove that there is one and only one unit vector u in $\mathbb{R}^n$ such that $|f'(c; u)| = \|\nabla f(c)\|$, and that this is the unit vector for which $|f'(c; u)|$ has its maximum value.

12.12: Gradient Calculations

Compute the gradient vector $\nabla f(x, y)$ at those points (x, y) in $\mathbb{R}^2$ where it exists:

(a) $f(x, y) = x^2y^2 \log(x^2 + y^2)$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

(b) $f(x, y) = xy \sin \frac{1}{x^2+y^2}$ if $(x, y) \neq (0, 0)$, $f(0, 0) = 0$.

12.13: Second Order Partials of Composition

Let f and g be real-valued functions defined on $\mathbb{R}^1$ with continuous second derivatives f'' and g''. Define

$$F(x, y) = f[x + g(y)] \text{ for each } (x, y) \text{ in } \mathbb{R}^2.$$

Find formulas for all partials of F of first and second order in terms of the derivatives of f and g. Verify the relation

$$(D_1F)(D_{1,2}F) = (D_2F)(D_{1,1}F).$$

12.14: Polar Coordinate Transformation

Given a function f defined in $\mathbb{R}^2$. Let

$$F(r, \theta) = f(r\cos\theta, r\sin\theta).$$

(a) Assume appropriate differentiability properties of f and show that

$$D_1F(r, \theta) = \cos\theta D_1f(x, y) + \sin\theta D_2f(x, y),$$

$$D_{1,1}F(r, \theta) = \cos^2\theta D_{1,1}f(x, y) + 2\sin\theta\cos\theta D_{1,2}f(x, y) + \sin^2\theta D_{2,2}f(x, y),$$

where $x = r\cos\theta, y = r\sin\theta$.

(b) Find similar formulas for $D_2F, D_{1,2}F$, and $D_{2,2}F$.

(c) Verify the formula

$$\|\nabla f(r\cos\theta, r\sin\theta)\|^2 = [D_1F(r, \theta)]^2 + \frac{1}{r^2}[D_2F(r, \theta)]^2.$$

12.15: Gradient of Product and Quotient

If f and g have gradient vectors $\nabla f(x)$ and $\nabla g(x)$ at a point x in $\mathbb{R}^n$ show that the product function h defined by $h(x) = f(x)g(x)$ also has a gradient vector at x and that

$$\nabla h(x) = f(x)\nabla g(x) + g(x)\nabla f(x).$$

State and prove a similar result for the quotient f/g.

12.16: Gradient of Composition

Let f be a function having a derivative f' at each point in $\mathbb{R}^1$ and let g be defined on $\mathbb{R}^3$ by the equation

$$g(x, y, z) = x^2 + y^2 + z^2.$$

If h denotes the composite function $h = f \circ g$, show that

$$\|\nabla h(x, y, z)\|^2 = 4g(x, y, z)[f'[g(x, y, z)]]^2.$$

12.17: Gradient of Vector-Valued Composition

Assume f is differentiable at each point (x, y) in $\mathbb{R}^2$. Let g_1 and g_2 be defined on $\mathbb{R}^3$ by the equations

$$g_1(x, y, z) = x^2 + y^2 + z^2, \quad g_2(x, y, z) = x + y + z,$$

and let g be the vector-valued function whose values (in $\mathbb{R}^2$) are given by

$$g(x, y, z) = (g_1(x, y, z), g_2(x, y, z)).$$

Let h be the composite function $h = f \circ g$ and show that

$$\|\nabla h\|^2 = 4(D_1 f)^2 g_1 + 4(D_1 f)(D_2 f)g_2 + 3(D_2 f)^2.$$

12.18: Euler's Theorem for Homogeneous Functions

Let f be defined on an open set S in $\mathbb{R}^n$. We say that f is homogeneous of degree p over S if $f(\lambda x) = \lambda^p f(x)$ for every real λ and for every x in S for which $\lambda x \in S$. If such a function is differentiable at x, show that

$$x \cdot \nabla f(x) = pf(x).$$

NOTE. This is known as Euler's theorem for homogeneous functions. Hint. For fixed x, define $g(\lambda) = f(\lambda x)$ and compute $g'(1)$.
Also prove the converse. That is, show that if $x \cdot \nabla f(x) = pf(x)$ for all x in an open set S, then f must be homogeneous of degree p over S.

12.3 Mean-Value Theorems

12.19: Mean-Value Theorem for Vector Functions

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by the equation $f(t) = (\cos t, \sin t)$. Then $f'(t)(u) = u(-\sin t, \cos t)$ for every real u. The Mean-Value formula

$$f(y) - f(x) = f'(z)(y - x)$$

cannot hold when $x = 0, y = 2\pi$, since the left member is zero and the right member is a vector of length 2π. Nevertheless, Theorem 12.9 states that for every vector a in $\mathbb{R}^2$ there is a z in the interval $(0, 2\pi)$ such that

$$a \cdot (f(y) - f(x)) = a \cdot (f'(z)(y - x)).$$

Determine z in terms of a when $x = 0$ and $y = 2\pi$.

12.20: Mean-Value Theorem in Two Variables

Let f be a real-valued function differentiable on a 2-ball $B(x)$. By considering the function

$$g(t) = f[ty_1 + (1 - t)x_1, y_2] + f[x_1, ty_2 + (1 - t)x_2]$$

prove that

$$f(y) - f(x) = (y_1 - x_1)D_1 f(z_1, y_2) + (y_2 - x_2)D_2 f(x_1, z_2),$$

where $z_1 \in L(x_1, y_1)$ and $z_2 \in L(x_2, y_2)$.

12.21: Generalized Mean-Value Theorem

State and prove a generalization of the result in Exercise 12.20 for a real-valued function differentiable on an n-ball $B(x)$.

12.22: Mean-Value Theorem for Directional Derivatives

Let f be real-valued and assume that the directional derivative $f'(c+tu; u)$ exists for each t in the interval $0 \leq t \leq 1$. Prove that for some θ in the open interval $(0, 1)$ we have

$$f(c + u) - f(c) = f'(c + \theta u; u).$$

12.23: Zero Directional Derivatives

(a) If f is real-valued and if the directional derivative $f'(x; u) = 0$ for every x in an n-ball $B(c)$ and every direction u, prove that f is constant on $B(c)$.

(b) What can you conclude about f if $f'(x; u) = 0$ for a fixed direction u and every x in $B(c)$?

12.4 Derivatives of Higher Order and Taylor's Formula

12.24: Equality of Mixed Partials

For each of the following functions, verify that the mixed partial derivatives $D_{1,2}f$ and $D_{2,1}f$ are equal.

(a) $f(x, y) = x^4 + y^4 - 4x^2y^2$.

(b) $f(x, y) = \log(x^2 + y^2)$, $(x, y) \neq (0, 0)$.

(c) $f(x, y) = \tan(x^2/y)$, $y \neq 0$.

12.25: Equality of Higher-Order Mixed Partials

Let f be a function of two variables. Use induction and Theorem 12.13 to prove that if the 2^k partial derivatives of f of order k are continuous in a neighborhood of a point (x, y), then all mixed partials of the form $D_{r_1,\ldots,r_k}f$ and $D_{p_1,\ldots,p_k}f$ will be equal at (x, y) if the k-tuple $(r_1, \ldots, r_k)$ contains the same number of ones as the k-tuple $(p_1, \ldots, p_k)$.

12.26: Taylor's Formula for Two Variables

If f is a function of two variables having continuous partials of order k on some open set S in $\mathbb{R}^2$, show that

$$f^{(k)}(x;t) = \sum_{r=0}^{k} \binom{k}{r} t_1^r t_2^{k-r} D_{p_1}, \ldots, p_k f(x), \quad \text{if } x \in S, \quad t = (t_1, t_2),$$

where in the rth term we have $p_1 = \cdots = p_r = 1$ and $p_{r+1} = \cdots = p_k = 2$. Use this result to give an alternative expression for Taylor's formula (Theorem 12.14) in the case when $n = 2$. The symbol $\binom{k}{r}$ is the binomial coefficient $k!/[r!(k-r)!]$.

12.27: Taylor Expansion

Use Taylor's formula to express the following in powers of $(x-1)$ and $(y-2)$:

(a) $f(x,y) = x^3 + y^3 + xy^2$,

(b) $f(x,y) = x^2 + xy + y^2$.

Chapter 13

Implicit Functions and Extremum Problems

13.1 Jacobians

13.1: Complex Function Jacobian

Let f be the complex-valued function defined for each complex $z \neq 0$ by the equation $f(z) = 1/\bar{z}$. Show that $J_f(z) = -|z|^{-4}$. Show that f is one-to-one and compute f^{-1} explicitly.

13.2: Vector-Valued Function Jacobian

Let $f = (f_1, f_2, f_3)$ be the vector-valued function defined (for every point (x_1, x_2, x_3) in R^3 for which $x_1 + x_2 + x_3 \neq -1$) as follows:

$$f_k(x_1, x_2, x_3) = \frac{x_k}{1 + x_1 + x_2 + x_3} \quad (k = 1, 2, 3).$$

Show that $J_f(x_1, x_2, x_3) = (1+x_1+x_2+x_3)^{-4}$. Show that f is one-to-one and compute f^{-1} explicitly.

13.3: Composition of Functions Jacobian

Let $f = (f_1, \ldots, f_n)$ be a vector-valued function defined in R^n, suppose $f \in C'$ on R^n, and let $J_f(x)$ denote the Jacobian determinant. Let $g_1, \ldots, g_n$ be n real-valued functions defined on R^1 and having continuous derivatives $g_1', \ldots, g_n'$. Let $h_k(x) = f_k[g_1(x_1), \ldots, g_n(x_n)], k = 1, 2, \ldots, n$, and put $h = (h_1, \ldots, h_n)$. Show that

$$J_h(x) = J_f[g_1(x_1), \ldots, g_n(x_n)]g_1'(x_1) \cdots g_n'(x_n).$$

13.4: Polar and Spherical Coordinates

(a) If $x(r, \theta) = r\cos\theta, y(r, \theta) = r\sin\theta$, show that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

(b) If $x(r, \theta, \phi) = r\cos\theta\sin\phi, y(r, \theta, \phi) = r\sin\theta\sin\phi, z = r\cos\phi$, show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = -r^2\sin\phi.$$

13.5: Implicit Function Theorem Application

(a) State conditions on f and g which will ensure that the equations $x = f(u, v), y = g(u, v)$ can be solved for u and v in a neighborhood of (x_0, y_0). If the solutions are $u = F(x, y), v = G(x, y)$, and if $J = \partial(f, g)/\partial(u, v)$, show that

$$\frac{\partial F}{\partial x} = \frac{1}{J}\frac{\partial g}{\partial v}, \quad \frac{\partial F}{\partial y} = -\frac{1}{J}\frac{\partial f}{\partial v}, \quad \frac{\partial G}{\partial x} = -\frac{1}{J}\frac{\partial g}{\partial u}, \quad \frac{\partial G}{\partial y} = \frac{1}{J}\frac{\partial f}{\partial u}.$$

(b) Compute J and the partial derivatives of F and G at $(x_0, y_0) = (1, 1)$ when $f(u, v) = u^2 - v^2$, $g(u, v) = 2uv$.

13.6: Jacobian Matrix Identity

Let f and g be related as in Theorem 13.6. Consider the case $n = 3$ and show that we have

$$J_i(x)D_1g_i(y) = \begin{vmatrix} \delta_{i,1} & D_1f_2(x) & D_1f_3(x) \\ \delta_{i,2} & D_2f_2(x) & D_2f_3(x) \\ \delta_{i,3} & D_3f_2(x) & D_3f_3(x) \end{vmatrix} (i = 1, 2, 3),$$

where $y = f(x)$ and $\delta_{i,j} = 0$ or 1 according as $i \neq j$ or $i = j$. Use this to deduce the formula

$$D_1g_1 = \frac{\partial(f_2, f_3)}{\partial(x_2, x_3)} \left| \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} \right|.$$

There are similar expressions for the other eight derivatives D_kg_l.

13.7: Complex Function Properties

Let $f = u + iv$ be a complex-valued function satisfying the following conditions: $u \in C'$ and $v \in C'$ on the open disk $A = \{z : |z| < 1\}$; f is continuous on the closed disk $\bar{A} = \{z : |z| \leq 1\}$; $u(x, y) = x$ and $v(x, y) = y$ whenever $x^2 + y^2 = 1$; the Jacobian $J_f(z) > 0$ if $z \in A$. Let $B = f(A)$ denote the image of A under f and prove that:

(a) If X is an open subset of A, then $f(X)$ is an open subset of B.

(b) B is an open disk of radius 1.

(c) For each point $u_0 + iv_0$ in B, there is only a finite number of points z in A such that $f(z) = u_0 + iv_0$.

13.2 Extremum Problems

13.8: Extreme Value Classification

Find and classify the extreme values (if any) of the functions defined by the following equations:

(a) $f(x, y) = y^2 + x^2 y + x^4$,

(b) $f(x, y) = x^2 + y^2 + x + y + xy$,

(c) $f(x, y) = (x - 1)^4 + (x - y)^4$,

(d) $f(x, y) = y^2 - x^3$.

13.9: Shortest Distance to Parabola

Find the shortest distance from the point $(0, b)$ on the y-axis to the parabola $x^2 - 4y = 0$. Solve this problem using Lagrange's method and also without using Lagrange's method.

13.10: Geometric Problems

Solve the following geometric problems by Lagrange's method:

(a) Find the shortest distance from the point (a_1, a_2, a_3) in R^3 to the plane whose equation is $b_1 x_1 + b_2 x_2 + b_3 x_3 + b_0 = 0$.

(b) Find the point on the line of intersection of the two planes

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_0 = 0$$

and

$$b_1 x_1 + b_2 x_2 + b_3 x_3 + b_0 = 0$$

which is nearest the origin.

13.11: Maximum Value with Constraint

Find the maximum value of $|\sum_{k=1}^{n} a_k x_k|$, if $\sum_{k=1}^{n} x_k^2 = 1$, by using

(a) the Cauchy-Schwarz inequality.

(b) Lagrange's method.

13.12: Maximum of Product under Constraint

Find the maximum of $(x_1 x_2 \cdots x_n)^2$ under the restriction

$$x_1^2 + \cdots + x_n^2 = 1.$$

Use the result to derive the following inequality, valid for positive real numbers $a_1, \ldots, a_n$

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

13.13: Local Extremum with Condition

If $f(x) = x_1^k + \cdots + x_n^k, x = (x_1, \ldots, x_n)$, show that a local extreme of f, subject to the condition $x_1 + \cdots + x_n = a$, is $d^k n^{1-k}$.

13.14: Local Extremum with Side Conditions

Show that all points (x_1, x_2, x_3, x_4) where $x_1^2 + x_2^2$ has a local extremum subject to the two side conditions $x_1^2 + x_3^2 + x_4^2 = 4, x_2^2 + 2x_3^2 + 3x_4^2 = 9$, are found among

$$(0, 0, \pm\sqrt{3}, \pm 1), (0, \pm 1, +2, 0), (\pm 1, 0, 0, \pm\sqrt{3}), (\pm 2, \pm 3, 0, 0).$$

Which of these yield a local maximum and which yield a local minimum? Give reasons for your conclusions.

13.15: Extreme Values with Side Conditions

Show that the extreme values of $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$, subject to the two side conditions

$$\sum_{j=1}^{3} \sum_{i=1}^{3} a_{ij} x_i x_j = 1 \quad (a_{ij} = a_{ji})$$

and

$$b_1 x_1 + b_2 x_2 + b_3 x_3 = 0, \quad (b_1, b_2, b_3) \neq (0, 0, 0),$$

are t_1^{-1}, t_2^{-1}, where t_1 and t_2 are the roots of the equation

$$\begin{vmatrix} b_1 & b_2 & b_3 & 0 \\ a_{11} - t & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} - t & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} - t & b_3 \end{vmatrix} = 0.$$

Show that this is a quadratic equation in t and give a geometric argument to explain why the roots t_1, t_2 are real and positive.

13.16: Hadamard's Theorem

Let $\Delta = \det[x_{ij}]$ and let $X_i = (x_{i1}, \ldots, x_{in})$. A famous theorem of Hadamard states that $|\Delta| \leq d_1 \cdots d_n$, if $d_1, \ldots, d_n$ are n positive constants such that $\|X_i\|^2 = d_i^2 (i = 1, 2, \ldots, n)$. Prove this by treating Δ as a function of n^2 variables subject to n constraints, using Lagrange's method to show that, when Δ has an extreme under these conditions, we must have

$$\Delta^2 = \begin{vmatrix} d_1^2 & 0 & 0 & \cdots & 0 \\ 0 & d_2^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^2 \end{vmatrix}.$$

Chapter 14

Multiple Riemann Integrals

14.1 Multiple Integrals

14.1: Product of Riemann Integrable Functions

If $f_1 \in R$ on $[a_1, b_1], \ldots, f_n \in R$ on $[a_n, b_n]$, prove that

$$\int_S f_1(x_1) \cdots f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1 \right) \cdots \left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n \right),$$

where $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$.

14.2: Riemann Integrability of Monotone Functions

Let f be defined and bounded on a compact rectangle $Q = [a, b] \times [c, d]$ in $\mathbb{R}^2$. Assume that for each fixed y in $[c, d]$, $f(x, y)$ is an increasing function of x, and that for each fixed x in $[a, b]$, $f(x, y)$ is an increasing function of y. Prove that $f \in R$ on Q.

14.3: Evaluation of Double Integrals

Evaluate each of the following double integrals.

(a)

$$\iint_Q \sin^2 x \, \sin^2 y \, dx \, dy, \quad \text{where } Q = [0, \pi] \times [0, \pi].$$

(b)

$$\iint_Q |\cos(x + y)| \, dx \, dy, \quad \text{where } Q = [0, \pi] \times [0, \pi].$$

(c)

$$\iint_Q [x+y] \, dx \, dy, \quad \text{where } Q = [0, 2]\times[0, 2], \text{ and } [t] \text{ is the greatest integer } \leq t.$$

14.4: Integrals over Unit Square

Let $Q = [0, 1] \times [0, 1]$ and calculate $\int_Q f(x, y) \, dx \, dy$ in each case.

(a) $f(x, y) = 1 - x - y$ if $x + y \leq 1$, $\quad f(x, y) = 0$ otherwise.

(b) $f(x, y) = x^2 + y^2$ if $x^2 + y^2 \leq 1$, $\quad f(x, y) = 0$ otherwise.

(c) $f(x, y) = x + y$ if $x^2 \leq y \leq 2x^2$, $\quad f(x, y) = 0$ otherwise.

14.5: Mixed Partial Integrals

Define f on the square $Q = [0,1] \times [0,1]$ as follows:

$$f(x,y) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2y & \text{if } x \text{ is irrational.} \end{cases}$$

(a) Prove that $\int_0^t f(x,y)\,dy$ exists for $0 \le t \le 1$ and that

$$\int_0^1 \left[\int_0^t f(x,y)\,dy\right] dx = t^2,$$

and

$$\int_0^1 \left[\int_0^t f(x,y)\,dy\right] dx = t.$$

This shows that $\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx$ exists and equals 1.

(b) Prove that $\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy$ exists and find its value.

(c) Prove that the double integral $\int_Q f(x,y)\,d(x,y)$ does not exist.

14.6: Discontinuous Integrand

Define f on the square $Q = [0,1] \times [0,1]$ as follows:

$$f(x,y) = \begin{cases} 0 & \text{if at least one of } x, y \text{ is irrational,} \\ 1/n & \text{if } y \text{ is rational and } x = m/n, \end{cases}$$

where m and n are relatively prime integers, $n > 0$. Prove that

$$\int_0^1 f(x,y)\,dx = \int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy = \int_Q f(x,y)\,d(x,y) = 0$$

but that $\int_0^1 f(x,y)\,dy$ does not exist for rational x.

14.7: Dense Set with Finite Cross-Sections

If p_k denotes the kth prime number, let

$$S(p_k) = \left\{ \begin{pmatrix} \frac{n}{p_k} & \frac{m}{p_k} \end{pmatrix} : n = 1, 2, \ldots, p_k - 1, \quad m = 1, 2, \ldots, p_k - 1 \right\},$$

let $S = \bigcup_{k=1}^{\infty} S(p_k)$, and let $Q = [0,1] \times [0,1]$.

(a) Prove that S is dense in Q (that is, the closure of S contains Q) but that any line parallel to the coordinate axes contains at most a finite subset of S.

(b) Define f on Q as follows:

$$f(x,y) = 0 \quad \text{if } (x,y) \in S, \quad f(x,y) = 1 \quad \text{if } (x,y) \in Q - S.$$

Prove that $\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx = \int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy = 1$, but that the double integral $\int_Q f(x,y)\,d(x,y)$ does not exist.

14.2 Jordan Content

14.8: Jordan Content of Finite Accumulation Points

Let S be a bounded set in $\mathbb{R}^n$ having at most a finite number of accumulation points. Prove that $c(S) = 0$.

14.9: Graph of Continuous Function has Zero Content

Let f be a continuous real-valued function defined on $[a,b]$. Let S denote the graph of f, that is, $S = \{(x,y) : y = f(x), a \leq x \leq b\}$. Prove that S has two-dimensional Jordan content zero.

14.10: Rectifiable Curve has Zero Content

Let Γ be a rectifiable curve in $\mathbb{R}^n$. Prove that Γ has n-dimensional Jordan content zero.

14.11: Ordinate Set Content

Let f be a nonnegative function defined on a set S in $\mathbb{R}^n$. The ordinate set of f over S is defined to be the following subset of $\mathbb{R}^{n+1}$:

$$\{(x_1, \ldots, x_n, x_{n+1}) : (x_1, \ldots, x_n) \in S, \quad 0 \leq x_{n+1} \leq f(x_1, \ldots, x_n)\}.$$

If S is a Jordan-measurable region in $\mathbb{R}^n$ and if f is continuous on S, prove that the ordinate set of f over S has $(n+1)$-dimensional Jordan content whose value is

$$\int_S f(x_1, \ldots, x_n)\, d(x_1, \ldots, x_n).$$

Interpret this problem geometrically when $n = 1$ and $n = 2$.

14.3 Advanced Topics

14.12: Zero Integral Implies Zero Function

Assume that $f \in R$ on S and suppose $\int_S f(x)\, dx = 0$. (S is a subset of $\mathbb{R}^n$). Let $A = \{x : x \in S, f(x) < 0\}$ and assume that $c(A) = 0$. Prove that there exists a set B of measure zero such that $f(x) = 0$ for each x in $S - B$.

14.13: Mean Value Theorem for Integrals

Assume that $f \in R$ on S, where S is a region in $\mathbb{R}^n$ and f is continuous on S. Prove that there exists an interior point x_0 of S such that

$$\int_S f(x)\, dx = f(x_0) c(S).$$

14.14: Mixed Partial Derivatives

Let f be continuous on a rectangle $Q = [a, b] \times [c, d]$. For each interior point (x_1, x_2) in Q, define

$$F(x_1, x_2) = \int_a^{x_1} \left(\int_c^{x_2} f(x, y)\, dy \right) dx.$$

Prove that $D_{1,2}F(x_1, x_2) = D_{2,1}F(x_1, x_2) = f(x_1, x_2)$.

14.15: Integral of Mixed Partial Derivative

Let T denote the following triangular region in the plane:

$$T = \left\{(x, y) : 0 \le \frac{x}{a} + \frac{y}{b} \le 1\right\}, \quad \text{where } a > 0,\, b > 0.$$

Assume that f has a continuous second-order partial derivative $D_{1,2}f$ on T. Prove that there is a point (x_0, y_0) on the segment joining $(a, 0)$ and $(0, b)$ such that

$$\int_T D_{1,2}f(x, y)\, d(x, y) = f(0, 0) - f(a, 0) + aD_1 f(x_0, y_0).$$

Chapter 15

Multiple Lebesgue Integrals

15.1 Fubini–Tonelli and Slicing

15.1: Integral over Triangular Region

If $f \in L(T)$, where T is the triangular region in $\mathbb{R}^2$ with vertices at $(0,0)$, $(1,0)$, and $(0,1)$, prove that

$$\int_T f(x,y)\,d(x,y) = \int_0^1 \left[\int_0^x f(x,y)\,dy\right] dx = \int_0^1 \left[\int_y^1 f(x,y)\,dx\right] dy.$$

15.2: Double Integral Calculation

For fixed $c, 0 < c < 1$, define f on $\mathbb{R}^2$ as follows:

$$f(x,y) = \begin{cases} (1-y)^c/(x-y)^c & \text{if } 0 \le y < x, 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $f \in L(\mathbb{R}^2)$ and calculate the double integral

$$\int_{\mathbb{R}^2} f(x,y)\,d(x,y).$$

15.3: Measure of a Subset

Let S be a measurable subset of $\mathbb{R}^2$ with finite measure $\mu(S)$. Using the notation of Definition 15.4, prove that

$$\mu(S) = \int_{-\infty}^{\infty} \mu(S^x)\, dx = \int_{-\infty}^{\infty} \mu(S_y)\, dy.$$

15.4: Iterated Integrals vs Double Integral

Let $f(x, y) = e^{-xy} \sin x \sin y$ if $x \geq 0, y \geq 0$, and let $f(x, y) = 0$ otherwise. Prove that both iterated integrals

$$\int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x, y)\, dx \right] dy \quad \text{and} \quad \int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} f(x, y)\, dy \right] dx$$

exist and are equal, but that the double integral of f over $\mathbb{R}^2$ does not exist. Also, explain why this does not contradict the Tonelli-Hobson test (Theorem 15.8).

15.2 Non-Integrable Examples and Iterated Integrals

15.5: Non-Integrable Function

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ for $0 \leq x \leq 1, 0 < y \leq 1$, and let $f(0, 0) = 0$. Prove that both iterated integrals

$$\int_0^1 \left[\int_0^1 f(x, y)\, dy \right] dx \quad \text{and} \quad \int_0^1 \left[\int_0^1 f(x, y)\, dx \right] dy$$

exist but are not equal. This shows that f is not Lebesgue-integrable on $[0, 1] \times [0, 1]$.

15.6: Another Non-Integrable Function

Let $I = [0,1] \times [0,1]$, let $f(x,y) = (x-y)/(x+y)^3$ if $(x,y) \in I$, $(x,y) \neq (0,0)$, and let $f(0,0) = 0$. Prove that $f \notin L(I)$ by considering the iterated integrals

$$\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx \quad \text{and} \quad \int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy.$$

15.7: Non-Integrable Function on Infinite Interval

Let $I = [0,1] \times [1,+\infty)$ and let $f(x,y) = e^{-xy} - 2e^{-2xy}$ if $(x,y) \in I$. Prove that $f \notin L(I)$ by considering the iterated integrals

$$\int_0^1 \left[\int_0^\infty f(x,y)\,dy\right] dx \quad \text{and} \quad \int_1^\infty \left[\int_0^1 f(x,y)\,dx\right] dy.$$

15.3 Change of Variables

15.8: Transformation of Integrals

The following formulas for transforming double and triple integrals occur in elementary calculus. Obtain them as consequences of Theorem 15.11 and give restrictions on T and T' for validity of these formulas.

(a)
$$\iint_T f(x,y)\,dx\,dy = \iint_{T'} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

(b)
$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T'} f(r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz.$$

(c)
$$\iiint_T f(x,y,z)\,dx\,dy\,dz = \iiint_{T'} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2 \sin\varphi\,d\rho\,d\theta\,d\varphi.$$

15.4 Gaussian Integrals

15.9: Gaussian Integrals

(a) Prove that $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,d(x,y) = \pi$ by transforming the integral to polar coordinates.

(b) Use part (a) to prove that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(c) Use part (b) to prove that $\int_{\mathbb{R}^n} e^{-\|x\|^2}\,d(x_1,\ldots,x_n) = \pi^{n/2}$.

(d) Use part (b) to calculate $\int_{-\infty}^{\infty} e^{-tx^2}\,dx$ and $\int_{-\infty}^{\infty} x^2 e^{-tx^2}\,dx, t > 0$.

15.5 Volumes of n-Balls

15.10: Volume of n-Ball

Let $V_n(a)$ denote the n-measure of the n-ball $B(0;a)$ of radius a. This exercise outlines a proof of the formula

$$V_n(a) = \frac{\pi^{n/2}a^n}{\Gamma(\frac{1}{2}n+1)}.$$

(a) Use a linear change of variable to prove that $V_n(a) = a^n V_n(1)$.

(b) Assume $n \geq 3$, express the integral for $V_n(1)$ as the iteration of an $(n-2)$-fold integral and a double integral, and use part (a) for an $(n-2)$-ball to obtain the formula

$$V_n(1) = V_{n-2}(1)\int_0^{2\pi}\left[\int_0^1 (1-r^2)^{n/2-1} r\,dr\right] d\theta = V_{n-2}(1)\frac{2\pi}{n}.$$

(c) From the recursion formula in (b) deduce that

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)}.$$

15.11: Integral over n-Ball

Refer to Exercise 15.10 and prove that

$$\int_{B(0;1)} x_k^2 \, d(x_1, \ldots, x_n) = \frac{V_n(1)}{n+2}$$

for each $k = 1, 2, \ldots, n$.

15.12: Recursion Formula for n-Ball Volume

Refer to Exercise 15.10 and express the integral for $V_n(1)$ as the iteration of an $(n-1)$-fold integral and a one-dimensional integral, to obtain the recursion formula

$$V_n(1) = 2V_{n-1}(1) \int_0^1 (1 - x^2)^{(n-1)/2} \, dx.$$

Put $x = \cos t$ in the integral, and use the formula of Exercise 15.10 to deduce that

$$\int_0^{\pi/2} \cos^n t \, dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2}n + \frac{1}{2})}{\Gamma(\frac{1}{2}n + 1)}.$$

15.6 Volumes in Other Regions

15.13: Volume of n-Dimensional Diamond

If $a > 0$, let $S_n(a) = \{(x_1, \ldots, x_n) : |x_1| + \cdots + |x_n| \le a\}$, and let $V_n(a)$ denote the n-measure of $S_n(a)$. This exercise outlines a proof of the formula $V_n(a) = 2^n a^n / n!$.

(a) Use a linear change of variable to prove that $V_n(a) = a^n V_n(1)$.

(b) Assume $n \ge 2$, express the integral for $V_n(1)$ as an iteration of a one-dimensional integral and an $(n-1)$-fold integral, use (a) to show that

$$V_n(1) = V_{n-1}(1) \int_{-1}^1 (1 - |x|)^{n-1} \, dx = 2V_{n-1}(1)/n,$$

and deduce that $V_n(1) = 2^n / n!$.

15.14: Volume of Special n-Dimensional Set

If $a > 0$ and $n \geq 2$, let $S_n(a)$ denote the following set in $\mathbb{R}^n$:

$$S_n(a) = \{(x_1, \dots, x_n) : |x_i| + |x_n| \leq a \quad \text{for each } i = 1, \dots, n-1\}.$$

Let $V_n(a)$ denote the n-measure of $S_n(a)$. Use a method suggested by Exercise 15.13 to prove that $V_n(a) = 2^n a^n / n$.

15.15: Integral over First Quadrant of n-Ball

Let $Q_n(a)$ denote the "first quadrant" of the n-ball $B(0 : a)$ given by

$$Q_n(a) = \{(x_1, \dots, x_n) : \|x\| \leq a \quad \text{and } 0 \leq x_i \leq a \quad \text{for each } i = 1, 2, \dots, n\}.$$

Let $f(x) = x_1 \cdots x_n$ and prove that

$$\int_{Q_n(a)} f(x)\, dx = \frac{a^{2n}}{2^n n!}.$$

Chapter 16

Cauchy's Theorem and the Residue Calculus

16.1 Complex Integration; Cauchy's Integral Formulas

16.1: Path Integral of Analytic Function

Let y be a piecewise smooth path with domain $[a, b]$ and graph Γ. Assume that the integral $\int_y f$ exists. Let S be an open region containing Γ and let g be a function such that $g'(z)$ exists and equals $f(z)$ for each z on Γ. Prove that

$$\int_y f = \int_y g' = g(B) - g(A), \quad \text{where } A = y(a) \text{ and } B = y(b).$$

In particular, if y is a circuit, then $A = B$ and the integral is 0. Hint. Apply Theorem 7.34 to each interval of continuity of y'.

16.2: Verification of Cauchy's Integral Formulas

Let y be a positively oriented circular path with center 0 and radius 2. Verify each of the following by using one of Cauchy's integral formulas.

(a)
$$\int_y \frac{e^z}{z} dz = 2\pi i.$$

(b)
$$\int_y \frac{e^z}{z^3} dz = \pi i.$$

(c)
$$\int_y \frac{e^z}{z^4} dz = \frac{\pi i}{3}.$$

(d)
$$\int_y \frac{e^z}{z-1} dz = 2\pi i e.$$

(e)
$$\int_y \frac{e^z}{z(z-1)} dz = 2\pi i(e-1).$$

(f)
$$\int_y \frac{e^z}{z^2(z-1)} dz = 2\pi i(e-2).$$

16.3: Derivative via Integral Formula

Let $f = u + iv$ be analytic on a disk $B(a; R)$. If $0 < r < R$, prove that

$$f'(a) = \frac{1}{\pi r} \int_0^{2\pi} u(a + re^{i\theta}) e^{-i\theta} d\theta.$$

16.4: Stronger Liouville's Theorem

(a) Prove the following stronger version of Liouville's theorem: If f is an entire function such that $\lim_{z\to\infty} |f(z)|/|z| = 0$, then f is a constant.

(b) What can you conclude about an entire function which satisfies an inequality of the form $|f(z)| \leq M|z|^c$ for every complex z, where $c > 0$?

16.2 Poisson's Formula and Applications

16.5: Poisson's Integral Formula

Assume that f is analytic on $B(0; R)$. Let y denote the positively oriented circle with center at 0 and radius r, where $0 < r < R$. If a is inside y, show that

$$f(a) = \frac{1}{2\pi i}\int_y f(z)\left(\frac{1}{z-a} - \frac{1}{z - r^2/\bar{a}}\right) dz.$$

If $a = Ae^{i\alpha}$, show that this reduces to the formula

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - A^2)f(re^{i\theta})}{r^2 - 2rA\cos(\alpha-\theta) + A^2} d\theta.$$

By equating the real parts of this equation we obtain an expression known as Poisson's integral formula.

16.6: Analytic Function Inequality

Assume that f is analytic on the closure of the disk $B(0; 1)$. If $|a| < 1$, show that

$$(1 - |a|^2)f(a) = \frac{1}{2\pi i}\int_y f(z)\frac{1 - z\bar{a}}{z - a} dz,$$

where y is the positively oriented unit circle with center at 0. Deduce the inequality

$$(1 - |a|^2)|f(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})| d\theta.$$

16.7: Integral with Combined Functions

Let $f(z) = \sum_{n=0}^{\infty} \frac{2^n z^n}{3^n}$ if $|z| < \frac{3}{2}$, and let $g(z) = \sum_{n=0}^{\infty} (2z)^{-n}$ if $|z| > \frac{1}{2}$. Let y be the positively oriented circular path of radius 1 and center 0, and define $h(a)$ for $|a| \neq 1$ as follows:

$$h(a) = \frac{1}{2\pi i} \int_y \left(\frac{f(z)}{z-a} + \frac{a^2 g(z)}{z^2 - az} \right) dz.$$

Prove that

$$h(a) = \begin{cases} \frac{3}{3-2a} & \text{if } |a| < 1, \\ \frac{2a^2}{1-2a} & \text{if } |a| > 1. \end{cases}$$

16.3 Taylor Expansions

16.8: Taylor Expansion of Power Series

Define f on the disk $B(0;1)$ by the equation $f(z) = \sum_{n=0}^{\infty} z^n$. Find the Taylor expansion of f about the point $a = \frac{1}{2}$ and also about the point $a = -\frac{1}{2}$. Determine the radius of convergence in each case.

16.9: Taylor Expansion of Averaged Function

Assume that f has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a(n) z^n$, valid in $B(0;R)$. Let

$$g(z) = \frac{1}{p} \sum_{k=0}^{p-1} f(ze^{2\pi ik/p}).$$

Prove that the Taylor expansion of g consists of every pth term in that of f. That is, if $z \in B(0;R)$ we have

$$g(z) = \sum_{n=0}^{\infty} a(pn) z^{pn}.$$

16.10: Partial Sum via Integral

Assume that f has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, valid in $B(0; R)$. Let $s_n(z) = \sum_{k=0}^{n} a_k z^k$. If $0 < r < R$ and $|z| < r$, show that

$$s_n(z) = \frac{1}{2\pi i} \int_y \frac{f(w)w^{n+1}}{w^{n+1}(w-z)} dw,$$

where y is the positively oriented circle with center at 0 and radius r.

16.11: Product of Taylor Series

Given the Taylor expansions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, valid for $|z| < R_1$ and $|z| < R_2$, respectively. Prove that if $|z| < R_1 R_2$ we have

$$\frac{1}{2\pi i} \int_y \frac{f(w)g(z/w)}{w} dw = \sum_{n=0}^{\infty} a_n b_n z^n,$$

where y is the positively oriented circle of radius R_1 with center at 0.

16.12: Parseval's Identity and Maximum Modulus

Assume that f has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, valid in $B(a; R)$.

(a) If $0 \le r < R$, deduce Parseval's identity:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(a + re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

(b) Use (a) to deduce the inequality

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M(r)^2,$$

where $M(r)$ is the maximum of $|f|$ on the circle $|z-a| = r$.

(c) Use (b) to give another proof of the local maximum modulus principle (Theorem 16.27).

16.13: Schwarz's Lemma

Prove Schwarz's lemma: Let f be analytic on the disk $B(0;1)$. Suppose that $f(0) = 0$ and $|f(z)| \leq 1$ if $|z| < 1$. Then

$$|f'(0)| \leq 1 \quad \text{and} \quad |f(z)| \leq |z|, \quad \text{if } |z| < 1.$$

If $|f'(0)| = 1$ or if $|f(z_0)| = |z_0|$ for at least one $z_0 \in B'(0;1)$, then

$$f(z) = e^{i\alpha} z,$$

where α is real. Hint. Apply the maximum-modulus theorem to g, where $g(0) = f'(0)$ and $g(z) = f(z)/z$ if $z \neq 0$.

16.4 Laurent Expansions, Singularities, Residues

16.14: Rouché's Theorem

Let f and g be analytic on an open region S. Let y be a Jordan circuit with graph Γ such that both Γ and its inner region lie within S. Suppose that $|g(z)| < |f(z)|$ for every z on Γ.

(a) Show that

$$\frac{1}{2\pi i} \int_y \frac{f'(z) + g'(z)}{f(z) + g(z)} dz = \frac{1}{2\pi i} \int_y \frac{f'(z)}{f(z)} dz.$$

Hint. Let $m = \inf\{|f(z)| - |g(z)| : z \in \Gamma\}$. Then $m > 0$ and hence

$$|f(z) + tg(z)| \geq m > 0$$

for each t in $[0,1]$ and each z on Γ. Now let

$$\phi(t) = \frac{1}{2\pi i} \int_y \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz, \quad \text{if } 0 \leq t \leq 1.$$

Then ϕ is continuous, and hence constant, on $[0,1]$. Thus, $\phi(0) = \phi(1)$.

(b) Use (a) to prove that f and $f + g$ have the same number of zeros inside Γ (Rouché's theorem).

16.15: Zeros of Polynomial

Let p be a polynomial of degree n, say $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where $a_n \neq 0$. Take $f(z) = a_n z^n$, $g(z) = p(z) - f(z)$ in Rouché's theorem, and prove that p has exactly n zeros in $\mathbb{C}$.

16.16: Fixed Point via Rouché's Theorem

Let f be analytic on the closure of the disk $B(0;1)$ and suppose $|f(z)| < 1$ if $|z| = 1$. Show that there is one, and only one, point $z_0 \in B(0;1)$ such that $f(z_0) = z_0$. Hint. Use Rouché's theorem.

16.17: Nonzero Partial Sums

Let $p_n(z)$ denote the nth partial sum of the Taylor expansion $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. Using Rouché's theorem (or otherwise), prove that for every $r > 0$ there exists an N (depending on r) such that $n \geq N$ implies $p_n(z) \neq 0$ for every $z \in B(0;r)$.

16.18: Zeros of Exponential Polynomial

If $a > e$, find the number of zeros of the function $f(z) = e^z - az^n$ which lie inside the circle $|z| = 1$.

16.19: Function with Specific Singularities

Give an example of a function which has all the following properties, or else explain why there is no such function: f is analytic everywhere in $\mathbb{C}$ except for a pole of order 2 at 0 and simple poles at i and $-i$; $f(z) = f(-z)$ for all z; $f(1) = 1$; the function $g(z) = f(1/z)$ has a zero of order 2 at $z = 0$; and $\mathrm{Res}_{z=i} f(z) = 2i$.

16.20: Laurent Expansions

Show that each of the following Laurent expansions is valid in the region indicated:

(a)
$$\frac{1}{(z-1)(2-z)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}, \quad \text{if } 1 < |z| < 2.$$

(b)
$$\frac{1}{(z-1)(2-z)} = \sum_{n=2}^{\infty} \frac{1-2^{1-n}}{z^n}, \quad \text{if } |z| > 2.$$

16.21: Bessel Function Coefficients

For each fixed $t \in \mathbb{C}$, define $J_n(t)$ to be the coefficient of z^n in the Laurent expansion
$$e^{(z-1/z)t/2} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show that for $n \geq 0$ we have
$$J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(t \sin\theta - n\theta) d\theta,$$

and that $J_{-n}(t) = (-1)^n J_n(t)$. Deduce the power series expansion
$$J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t/2)^{n+2k}}{k!(n+k)!}, \quad (n \geq 0).$$

The function J_n is called the Bessel function of order n.

16.22: Riemann's Theorem

Prove Riemann's theorem: If z_0 is an isolated singularity of f and if f is bounded on some deleted neighborhood $B'(z_0)$, then z_0 is a removable singularity. Hint. Estimate the integrals for the coefficients a_n in the Laurent expansion of f and show that $a_n = 0$ for each $n < 0$.

16.23: Casorati-Weierstrass Theorem

Prove the Casorati-Weierstrass theorem: Assume that z_0 is an essential singularity of f and let c be an arbitrary complex number. Then, for every $\epsilon > 0$ and every disk $B(z_0)$, there exists a point z in $B(z_0)$ such that $|f(z) - c| < \epsilon$. Hint. Assume that the theorem is false and arrive at a contradiction by applying Exercise 16.22 to g, where $g(z) = 1/[f(z) - c]$.

16.24: Singularities at Infinity

The point at infinity. A function f is said to be analytic at ∞ if the function g defined by the equation $g(z) = f(1/z)$ is analytic at the origin. Similarly, we say that f has a zero, a pole, a removable singularity, or an essential singularity at ∞ if g has a zero, a pole, etc., at 0. Liouville's theorem states that a function which is analytic everywhere in $\mathbb{C}^*$ must be a constant. Prove that

(a) f is a polynomial if, and only if, the only singularity of f in $\mathbb{C}^*$ is a pole at ∞, in which case the order of the pole is equal to the degree of the polynomial.

(b) f is a rational function if, and only if, f has no singularities in $\mathbb{C}^*$ other than poles.

16.25: Residue Shortcuts

Derive the following "short cuts" for computing residues:

(a) If a is a first order pole for f, then

$$\mathrm{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z).$$

(b) If a is a pole of order 2 for f, then

$$\mathrm{Res}_{z=a} f(z) = g'(a),$$

where $g(z) = (z-a)^2 f(z)$.

(c) Suppose f and g are both analytic at a, with $f(a) \neq 0$ and a a first-order zero for g. Show that

$$\mathrm{Res}_{z=a} \frac{f(z)}{g(z)} = \frac{f(a)}{g'(a)}.$$

(d) If f and g are as in (c), except that a is a second-order zero for g, then

$$\mathrm{Res}_{z=a} \frac{f(z)}{g(z)} = \frac{6f'(a)g''(a) - 2f(a)g'''(a)}{3[g''(a)]^2}.$$

16.26: Compute Residues

Compute the residues at the poles of f if

(a) $f(z) = \frac{ze^z}{z^2-1}$.

(b) $f(z) = \frac{e^z}{z(z-1)^2}$.

(c) $f(z) = \frac{\sin z}{z \cos z}$.

(d) $f(z) = \frac{1}{1-e^z}$.

(e) $f(z) = \frac{1}{1-z^n}$ (where n is a positive integer).

16.27: Residue Integrals

If $y(a; r)$ denotes the positively oriented circle with center at a and radius r, show that

(a)

$$\int_{y(0;4)} \frac{3z-1}{(z+1)(z-3)} dz = 6\pi i.$$

(b)

$$\int_{y(0;2)} \frac{2z}{z^2+1} dz = 4\pi i.$$

(c)

$$\int_{y(0;2)} \frac{z^3}{z^4-1} dz = 2\pi i.$$

(d)

$$\int_{y(2;1)} \frac{e^z}{(z-2)^2} dz = 2\pi i e^2.$$

16.28: Residue Integral

Evaluate the integral by means of residues:

$$\int_0^{2\pi} \frac{dt}{(a+b\cos t)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}, \quad \text{if } 0 < b < a.$$

16.29: Residue Integral

Evaluate the integral by means of residues:

$$\int_0^{2\pi} \frac{\cos 2t}{1-2a\cos t+a^2} dt = \frac{2\pi a^2}{1-a^2}, \quad \text{if } a^2 < 1.$$

16.30: Residue Integral

Evaluate the integral by means of residues:

$$\int_0^{2\pi} \frac{1+\cos 3t}{1-2a\cos t+a^2} dt = \frac{\pi(a^3+a)}{1-a^2}, \quad \text{if } 0 < a < 1.$$

16.31: Residue Integral

Evaluate the integral by means of residues:

$$\int_0^{2\pi} \frac{\sin^2 t}{a + b\cos t} dt = \frac{2\pi(a - \sqrt{a^2 - b^2})}{b^2}, \quad \text{if } 0 < b < a.$$

16.32: Residue Integral

Evaluate the integral by means of residues:

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + x + 1} dx = \frac{2\pi\sqrt{3}}{3}.$$

16.33: Residue Integral

Evaluate the integral by means of residues:

$$\int_{-\infty}^{\infty} \frac{x^6}{(1 + x^4)^2} dx = \frac{3\pi}{16}.$$

16.34: Residue Integral

Evaluate the integral by means of residues:

$$\int_0^{\infty} \frac{x^2}{(x^2 + 4)^2(x^2 + 9)} dx = \frac{\pi}{200}.$$

16.35: Residue Integrals

Evaluate the integrals by means of residues:

(a)

$$\int_0^\infty \frac{x}{1+x^5}\,dx = \frac{\pi}{5}\sin\frac{2\pi}{5}.$$

Hint. Integrate $z/(1+z^5)$ around the boundary of the circular sector $S = \{re^{i\theta} : 0 \le r \le R, 0 \le \theta \le 2\pi/5\}$, and let $R \to \infty$.

(b)

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx = \frac{\pi}{2n}\sin\left(\frac{(2m+1)\pi}{2n}\right),$$

where m, n are integers, $0 < m < n$.

16.36: Residue Formula for Rational Functions

Prove that formula (38) holds if f is the quotient of two polynomials, say $f = P/Q$, where the degree of Q exceeds that of P by 2 or more.

16.37: Residue Formula for Exponential Rational Functions

Prove that formula (38) holds if $f(z) = e^{imz}P(z)/Q(z)$, where $m > 0$ and P and Q are polynomials such that the degree of Q exceeds that of P by 1 or more. This makes it possible to evaluate integrals of the form

$$\int_{-\infty}^\infty \frac{e^{imx}P(x)}{Q(x)}\,dx$$

by the method described in Theorem 16.37.

16.38: Exponential Integrals

Use the method suggested in Exercise 16.37 to evaluate the following integrals:

(a)
$$\int_0^\infty \frac{x}{(a^2+x^2)} e^{imx}dx = \frac{\pi}{2}e^{-ma}, \quad \text{if } m \neq 0, a > 0.$$

(b)
$$\int_0^\infty \frac{x^4}{(1+x^4)} e^{imx}dx = \frac{\pi}{2}(1-e^{-m}), \quad \text{if } m > 0, a > 0.$$

16.39: Integral with Cube Roots

Let $w = e^{2\pi i/3}$ and let y be a positively oriented circle whose graph does not pass through 1, w, or w^2. (The numbers 1, w, w^2 are the cube roots of 1.) Prove that the integral

$$\int_y \frac{z+1}{z^3-1}dz$$

is equal to $2\pi i(m+nw)/3$, where m and n are integers. Determine the possible values of m and n and describe how they depend on y.

16.40: Bernoulli Polynomial Integrals

Let y be a positively oriented circle with center 0 and radius $< 2\pi$. If a is complex and n is an integer, let

$$I(n,a) = \frac{1}{2\pi i}\int_y \frac{z^{n-1}e^{az}}{1-e^z}dz.$$

Prove that

$$I(0,a) = \frac{1}{2} - a, \quad I(1,a) = -\frac{1}{2}, \quad \text{and} \quad I(n,a) = 0 \quad \text{if } n > 1.$$

Calculate $I(-n,a)$ in terms of Bernoulli polynomials when $n \geq 1$ (see Exercise 9.38).

16.41: Details of Theorem 16.38

Let
$$g(z) = \sum_{r=0}^{n-1} e^{2\pi i a(z+r)^2/n}, \quad f(z) = \frac{g(z)}{e^{2\pi i z} - 1},$$
where a and n are positive integers with na even. Prove that:

(a) $g(z+1) - g(z) = e^{2\pi i a z^2/n}(e^{2\pi i z} - 1)\sum_{m=0}^{n-1} e^{2\pi i m z}$.

(b) $\mathrm{Res}_{z=0} f(z) = g(0)/(2\pi i)$.

(c) The real part of $i(t + Re^{i\pi/4} + r)^2$ is $R^2 + \sqrt{2}rR$.

16.5 One-to-One Analytic Functions

16.42: Properties of One-to-One Analytic Functions

Let S be an open subset of $\mathbb{C}$ and assume that f is analytic and one-to-one on S. Prove that:

(a) $f'(z) \neq 0$ for each z in S. (Hence f is conformal at each point of S.)

(b) If g is the inverse of f, then g is analytic on $f(S)$ and $g'(w) = 1/f'(g(w))$ if $w \in f(S)$.

16.43: One-to-One Entire Functions

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and one-to-one on $\mathbb{C}$. Prove that $f(z) = az+b$, where $a \neq 0$. What can you conclude if f is one-to-one on $\mathbb{C}^*$ and analytic on $\mathbb{C}^*$ except possibly for a finite number of poles?

16.44: Composition of Möbius Transformations

If f and g are Möbius transformations, show that the composition $f \circ g$ is also a Möbius transformation.

16.45: Geometric Interpretation of Möbius Transformations

Describe geometrically what happens to a point z when it is carried into $f(z)$ by the following special Möbius transformations:

(a) $f(z) = z + b$ (Translation).

(b) $f(z) = az$, where $a > 0$ (Stretching or contraction).

(c) $f(z) = e^{i\alpha}z$, where α is real (Rotation).

(d) $f(z) = \frac{1}{z}$ (Inversion).

16.46: Circles under Möbius Transformations

If $c \neq 0$, we have

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}.$$

Hence every Möbius transformation can be expressed as a composition of the special cases described in Exercise 16.45. Use this fact to show that Möbius transformations carry circles into circles (where straight lines are considered as special cases of circles).

16.47: Möbius Transformations Mapping Half-Plane to Disk

(a) Show that all Möbius transformations which map the upper half-plane $T = \{x + iy : y \geq 0\}$ onto the closure of the disk $B(0; 1)$ can be expressed in the form $f(z) = e^{i\delta}\frac{z-\alpha}{z-\bar{\alpha}}$, where α is real and $\alpha \in T$.

(b) Show that α and δ can always be chosen to map any three given points of the real axis onto any three given points on the unit circle.

16.48: Möbius Transformations Mapping Right Half-Plane

Find all Möbius transformations which map the right half-plane $S = \{x + iy : x \geq 0\}$ onto the closure of $B(0; 1)$.

16.49: Möbius Transformations Mapping Unit Disk

Find all Möbius transformations which map the closure of $B(0; 1)$ onto itself.

16.50: Fixed Points of Möbius Transformations

The fixed points of a Möbius transformation

$$f(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$$

are those points z for which $f(z) = z$. Let $D = (d-a)^2 + 4bc$.

(a) Determine all fixed points when $c = 0$.

(b) If $c \neq 0$ and $D \neq 0$, prove that f has exactly 2 fixed points z_1 and z_2 (both finite) and that they satisfy the equation

$$\frac{f(z) - z_1}{f(z) - z_2} = Re^{i\theta} \frac{z - z_1}{z - z_2},$$

where $R > 0$ and θ is real.

(c) If $c \neq 0$ and $D = 0$, prove that f has exactly one fixed point z_1 and that it satisfies the equation

$$\frac{1}{f(z) - z_1} = \frac{1}{z - z_1} + C, \quad \text{for some } C \neq 0.$$

(d) Given any Möbius transformation, investigate the successive images of a given point w. That is, let

$$w_1 = f(w), \quad w_2 = f(w_1), \quad \ldots, \quad w_n = f(w_{n-1}), \quad \ldots,$$

and study the behavior of the sequence $\{w_n\}$. Consider the special case a, b, c, d real, $ad - bc = 1$.

16.6 Miscellaneous Exercises

16.51: Complex Sum Equation

Determine all complex z such that

$$z = \sum_{n=2}^{\infty} \sum_{k=1}^{n} e^{2\pi i k z/n}.$$

16.52: Bound on Entire Function Coefficients

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function such that $|f(re^{i\theta})| < Me^{rk}$ for all $r > 0$, where $M > 0$ and $k > 0$, prove that

$$|a_n| \leq \frac{M}{(n/k)^{n/k}}, \quad \text{for } n \geq 1.$$

16.53: Limit at Isolated Singularity

Assume f is analytic on a deleted neighborhood $B'(0; a)$. Prove that $\lim_{z\to 0} f(z)$ exists (possibly infinite) if, and only if, there exists an integer n and a function g, analytic on $B(0; a)$, with $g(0) \neq 0$, such that $f(z) = z^n g(z)$ in $B'(0; a)$.

16.54: Zeros of Polynomial with Decreasing Coefficients

Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with real coefficients satisfying

$$a_0 > a_1 > \cdots > a_{n-1} > a_n > 0.$$

Prove that $p(z) = 0$ implies $|z| > 1$. Hint. Consider $(1 - z)p(z)$.

16.55: Zero of Infinite Order

A function f, defined on a disk $B(a; r)$, is said to have a zero of infinite order at a if, for every integer $k > 0$, there is a function g_k, analytic at a, such that $f(z) = (z - a)^k g_k(z)$ on $B(a; r)$. If f has a zero of infinite order at a, prove that $f = 0$ everywhere in $B(a; r)$.

16.56: Morera's Theorem

Prove Morera's theorem: If f is continuous on an open region S in $\mathbb{C}$ and if $\int_y f = 0$ for every polygonal circuit y in S, then f is analytic on S.