Chapter 1

The Real and Complex Number Systems

1.1 Integers

1.1: No Largest Prime

Prove that there is no largest prime. (A proof was known to Euclid.)

Solution: We will prove this by contradiction. Assume there exists a largest prime number, call it p.

Consider the number $N = p! + 1$, where $p!$ is the factorial of p.

Since $p!$ is divisible by all integers from 1 to p, the number $N = p! + 1$ is not divisible by any prime number less than or equal to p.

Now, N is either prime or composite:

- If N is prime, then $N > p$, contradicting our assumption that p is the largest prime.
- If N is composite, then N has a prime factor q. Since N is not divisible by any prime $\leq p$, we must have $q > p$. This again contradicts our assumption that p is the largest prime.

In both cases, we reach a contradiction. Therefore, our assumption that there exists a largest prime is false, and there must be infinitely many prime numbers.

1.2: Algebraic Identity

If n is a positive integer, prove the algebraic identity:

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Solution: We can prove this identity by expanding the right-hand side and showing it equals the left-hand side.

Let's expand the sum:

$$\begin{aligned}(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} &= (a-b)(a^0b^{n-1} + a^1b^{n-2} + a^2b^{n-3} + \cdots + a^{n-1}b^0)\\ &= (a-b)(b^{n-1} + ab^{n-2} + a^2b^{n-3} + \cdots + a^{n-1})\end{aligned}$$

Now distribute $(a-b)$:

$$\begin{aligned}&= a\cdot b^{n-1} + a^2b^{n-2} + a^3b^{n-3} + \cdots + a^n\\ &\quad - b\cdot b^{n-1} - ab^{n-1} - a^2b^{n-2} - \cdots - a^{n-1}b\end{aligned}$$

Notice that most terms cancel out:

$$\begin{aligned}&= a^n - b^n + (\text{canceling terms})\\ &= a^n - b^n\end{aligned}$$

Alternatively, we can use the geometric series formula. Let $r = \frac{a}{b}$. Then:

$$\begin{aligned}\sum_{k=0}^{n-1} a^k b^{n-1-k} &= b^{n-1}\sum_{k=0}^{n-1}\left(\frac{a}{b}\right)^k\\ &= b^{n-1}\cdot\frac{1-\left(\frac{a}{b}\right)^n}{1-\frac{a}{b}}\\ &= b^{n-1}\cdot\frac{b^n - a^n}{b^n(b-a)}\\ &= \frac{a^n - b^n}{a-b}\end{aligned}$$

Therefore, $(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} = a^n - b^n$.

1.3: Mersenne Primes

If $2^n - 1$ is prime, prove that n is prime. A prime of the form $2^p - 1$, where p is prime, is called a *Mersenne prime*.

Solution: We will prove the contrapositive: if n is composite, then $2^n - 1$ is composite.

Let $n = ab$ where $a, b > 1$ are integers. Then:

$$\begin{aligned} 2^n - 1 &= 2^{ab} - 1 \\ &= (2^a)^b - 1 \end{aligned}$$

Using the identity from Problem 1.2 with $x = 2^a$ and $y = 1$:

$$(2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1)$$

Since $a > 1$, we have $2^a - 1 > 1$. Also, since $b > 1$, the second factor is greater than 1. Therefore, $2^n - 1$ is the product of two integers greater than 1, making it composite.

This proves that if $2^n - 1$ is prime, then n must be prime.

1.4: Fermat Primes

If $2^n + 1$ is prime, prove that n is a power of 2. A prime of the form $2^{2^n} + 1$ is called a *Fermat prime*. Hint: Use Exercise 1.2.

Solution: We will prove the contrapositive: if n is not a power of 2, then $2^n + 1$ is composite.

If n is not a power of 2, then n has an odd factor greater than 1. Let $n = 2^k \cdot m$ where $m > 1$ is odd and $k \geq 0$.

Then:

$$\begin{aligned} 2^n + 1 &= 2^{2^k \cdot m} + 1 \\ &= (2^{2^k})^m + 1 \end{aligned}$$

Since m is odd, we can use the identity from Problem 1.2 with $a = 2^{2^k}$ and $b = -1$:

$$(2^{2^k})^m - (-1)^m = (2^{2^k} - (-1))((2^{2^k})^{m-1} + (2^{2^k})^{m-2}(-1) + \cdots + (-1)^{m-1})$$

Since m is odd, $(-1)^m = -1$, so:

$$(2^{2^k})^m + 1 = (2^{2^k} + 1)((2^{2^k})^{m-1} - (2^{2^k})^{m-2} + \cdots + 1)$$

Since $m > 1$, both factors are greater than 1, making $2^n + 1$ composite.

Therefore, if $2^n + 1$ is prime, then n must be a power of 2.

1.5: Fibonacci Numbers Formula

The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $x_n = \frac{a^n - b^n}{a - b}$, where a and b are the roots of the equation $x^2 - x - 1 = 0$.

Solution: Let the proposition be $P(n) : x_n = \frac{a^n - b^n}{a-b}$. The roots of $x^2 - x - 1 = 0$ are $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. A key property of these roots is that they satisfy $a^2 = a + 1$ and $b^2 = b + 1$.

Base cases: For $n = 1$:

$$\frac{a^1 - b^1}{a - b} = 1 = x_1.$$

For $n = 2$:

$$\frac{a^2 - b^2}{a - b} = \frac{(a-b)(a+b)}{a-b} = a + b = \left(\frac{1+\sqrt{5}}{2}\right) + \left(\frac{1-\sqrt{5}}{2}\right) = 1 = x_2.$$

The base cases hold.

Inductive step: Assume $P(k)$ is true for all integers $k \leq n$, where $n \geq 2$. We will show $P(n+1)$ is true. By the definition of the Fibonacci sequence, $x_{n+1} = x_n + x_{n-1}$. Using the inductive hypothesis for x_n and x_{n-1}:

$$\begin{aligned} x_{n+1} &= \left(\frac{a^n - b^n}{a-b}\right) + \left(\frac{a^{n-1} - b^{n-1}}{a-b}\right) \\ &= \frac{(a^n + a^{n-1}) - (b^n + b^{n-1})}{a-b} \\ &= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{a-b} \end{aligned}$$

Using the property that $a + 1 = a^2$ and $b + 1 = b^2$:

$$\begin{aligned} x_{n+1} &= \frac{a^{n-1}(a^2) - b^{n-1}(b^2)}{a-b} \\ &= \frac{a^{n+1} - b^{n+1}}{a-b} \end{aligned}$$

This is $P(n+1)$. By the principle of strong induction, the formula is true for all positive integers n.

1.6: Well-Ordering Principle

Prove that every nonempty set of positive integers contains a smallest member. This is called the *well-ordering principle.*

Solution: We will prove this by contradiction. Let S be a nonempty set of positive integers that has no smallest member. Let $P(n)$ be the proposition that the integer n is not in S. We will use induction to show that $P(n)$ is true for all positive integers n.

Base case: For $n = 1$: If $1 \in S$, then 1 would be the smallest member of S (since S contains only positive integers). But we assumed S has no smallest member. So 1 cannot be in S. Thus, $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true for all positive integers $k < n$. This means that none of the integers $1, 2, \ldots, n-1$ are in S. Now consider the integer n. If $n \in S$, then from our inductive hypothesis (that $1, 2, \ldots, n-1$ are not in S), n would be the smallest member of S. This contradicts our initial assumption that S has no smallest member. Therefore, n cannot be in S. Thus, $P(n)$ is true.

By the principle of strong induction, $P(n)$ is true for all positive integers n. This means that no positive integer is in S, which implies that S is an empty set. This contradicts our initial assumption that S is a nonempty set. Therefore, the original assumption must be false, and every nonempty set of positive integers must contain a smallest member.

1.2 Rational and Irrational Numbers

1.7: Decimal Expansion to Rational

Find the rational number whose decimal expansion is $0.334444\ldots$.

Solution: We can use an algebraic method to find the equivalent fraction. Let x be the rational number.

$$x = 0.334444\ldots$$

The goal is to manipulate the equation to eliminate the repeating decimal part. The repeating digit '4' begins at the third decimal place.

First, multiply by 100 to move the non-repeating part to the left of the decimal point:

$$100x = 33.4444\ldots$$

Next, multiply by 1000 to shift the decimal point past the first repeating digit:

$$1000x = 334.4444\ldots$$

Now, subtract the first equation from the second. This will cancel the infinite repeating tail.

$$\begin{array}{rr} & 1000x = 334.4444\ldots \\ - & 100x = 33.4444\ldots \\ \hline & 900x = 301 \end{array}$$

Finally, solve for x:

$$x = \frac{301}{900}$$

Therefore, the rational number is $\frac{301}{900}$.

1.8: Decimal Expansion Ending in Zeroes

Prove that the decimal expansion of x will end in zeroes (or in nines) if and only if x is a rational number whose denominator is of the form 2^m5^n, where m and n are nonnegative integers.

Solution: We need to prove both directions of this statement.

Forward direction: If x is rational with denominator of the form 2^m5^n, then its decimal expansion terminates.

Let $x = \frac{p}{q}$ where $q = 2^m5^n$ for some nonnegative integers m, n.

We can write $x = \frac{p}{2^m5^n} = \frac{p \cdot 2^n5^m}{2^m5^n \cdot 2^n5^m} = \frac{p \cdot 2^n5^m}{10^{m+n}}$

This shows that x can be written as a fraction with denominator a power of 10, which means its decimal expansion terminates.

Reverse direction: If the decimal expansion of x terminates, then x is rational with denominator of the form 2^m5^n.

Let x have a terminating decimal expansion. Then x can be written as $x = \frac{N}{10^k}$ for some integer N and nonnegative integer k.

Since $10 = 2 \cdot 5$, we have $10^k = 2^k \cdot 5^k$, which is of the required form.

Note about ending in nines: If a decimal expansion ends in nines (e.g., $0.999\ldots$), this is equivalent to the next terminating decimal. For example, $0.999\ldots = 1.000\ldots$. This is because $0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = \frac{9/10}{1-1/10} = 1$.

Therefore, both terminating decimals and those ending in nines correspond to rational numbers with denominators of the form 2^m5^n.

1.9: Irrationality of $\sqrt{2} + \sqrt{3}$

Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution: We will prove this by contradiction. Assume that $\sqrt{2} + \sqrt{3}$ is rational, say $\sqrt{2} + \sqrt{3} = \frac{p}{q}$ where p, q are integers with no common factors.

Then:

$$\begin{aligned}
\sqrt{2}+\sqrt{3} &= \frac{p}{q} \\
(\sqrt{2}+\sqrt{3})^2 &= \left(\frac{p}{q}\right)^2 \\
2+2\sqrt{6}+3 &= \frac{p^2}{q^2} \\
5+2\sqrt{6} &= \frac{p^2}{q^2} \\
2\sqrt{6} &= \frac{p^2}{q^2}-5 \\
\sqrt{6} &= \frac{p^2-5q^2}{2q^2}
\end{aligned}$$

This shows that $\sqrt{6}$ is rational, which is a contradiction since $\sqrt{6}$ is irrational.

To see why $\sqrt{6}$ is irrational, suppose $\sqrt{6} = \frac{a}{b}$ where a, b are integers with no common factors. Then:

$$\begin{aligned}
6 &= \frac{a^2}{b^2} \\
6b^2 &= a^2
\end{aligned}$$

This means a^2 is divisible by 6, so a must be divisible by 6. Let $a = 6k$. Then:

$$\begin{aligned}
6b^2 &= (6k)^2 = 36k^2 \\
b^2 &= 6k^2
\end{aligned}$$

This means b^2 is divisible by 6, so b must also be divisible by 6. But this contradicts our assumption that a and b have no common factors.

Therefore, $\sqrt{6}$ is irrational, and consequently $\sqrt{2}+\sqrt{3}$ is irrational.

1.10: Rational Functions of Irrational Numbers

If a, b, c, d are rational and if x is irrational, prove that $\frac{ax+b}{cx+d}$ is usually irrational. When do exceptions occur?

Solution: We need to analyze when $\frac{ax+b}{cx+d}$ is rational, given that x is irrational and a, b, c, d are rational.

Let's assume that $\frac{ax+b}{cx+d} = \frac{p}{q}$ where p, q are integers with no common factors.

Then:

$$\begin{aligned}
\frac{ax+b}{cx+d} &= \frac{p}{q} \\
q(ax+b) &= p(cx+d) \\
qax+qb &= pcx+pd \\
(qa-pc)x &= pd-qb
\end{aligned}$$

Since x is irrational and the right-hand side is rational, we must have $qa - pc = 0$ and $pd - qb = 0$.

This gives us:

$$\begin{aligned}
qa &= pc \\
pd &= qb
\end{aligned}$$

From the first equation: $a = \frac{pc}{q}$ From the second equation: $b = \frac{pd}{q}$
Therefore, we have:

$$\begin{aligned}
\frac{a}{c} &= \frac{p}{q} \\
\frac{b}{d} &= \frac{p}{q}
\end{aligned}$$

This means $\frac{a}{c} = \frac{b}{d}$, or equivalently, $ad = bc$.

Conclusion: The expression $\frac{ax+b}{cx+d}$ is rational if and only if $ad = bc$.

Exceptions occur when: $ad = bc$, which means the numerator and denominator are proportional, making the fraction rational regardless of the value of x.

Examples:

- If $a = 2, b = 1, c = 4, d = 2$, then $ad = 4 = bc = 4$, so $\frac{2x+1}{4x+2} = \frac{1}{2}$ for all x.
- If $a = 1, b = 0, c = 1, d = 0$, then $ad = 0 = bc = 0$, so $\frac{x}{x} = 1$ for all $x \neq 0$.

1.11: Irrational Numbers Between 0 and x

Given any real $x > 0$, prove that there is an irrational number between 0 and x.

Solution: We will construct an irrational number between 0 and x for any positive real number x.

Case 1: If x is irrational, then $\frac{x}{2}$ is irrational and lies between 0 and x.

To see why $\frac{x}{2}$ is irrational, suppose it were rational. Then $\frac{x}{2} = \frac{p}{q}$ for some integers p, q, which would mean $x = \frac{2p}{q}$, making x rational, a contradiction.

Case 2: If x is rational, let $x = \frac{p}{q}$ where p, q are positive integers.

Consider the number $y = \frac{x}{\sqrt{2}} = \frac{p}{q\sqrt{2}}$.

Since $\sqrt{2}$ is irrational, y is irrational (if y were rational, then $\sqrt{2} = \frac{p}{qy}$ would be rational, a contradiction).

Also, since $\sqrt{2} > 1$, we have $y < x$.

Therefore, y is an irrational number between 0 and x.

Alternative construction: For any positive real x, we can also use $y = \frac{x}{\pi}$. Since π is irrational and greater than 1, we have $0 < y < x$, and y is irrational.

1.12: Fraction Between Two Fractions

If $\frac{a}{b} < \frac{c}{d}$ with $b > 0, d > 0$, prove that $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Solution: We need to prove that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Let's prove both inequalities:

First inequality: $\frac{a}{b} < \frac{a+c}{b+d}$

Cross-multiplying:

$$\begin{aligned} a(b+d) &< b(a+c) \\ ab + ad &< ab + bc \\ ad &< bc \end{aligned}$$

Since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$, so this inequality holds.

Second inequality: $\frac{a+c}{b+d} < \frac{c}{d}$

Cross-multiplying:

$$\begin{aligned} d(a+c) &< c(b+d) \\ ad + cd &< bc + cd \\ ad &< bc \end{aligned}$$

Again, since $\frac{a}{b} < \frac{c}{d}$, we have $ad < bc$, so this inequality also holds.

Therefore, $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

Geometric interpretation: This result is known as the "mediant" of two fractions. If we think of fractions as points on a line, the mediant $\frac{a+c}{b+d}$ lies between the two original fractions $\frac{a}{b}$ and $\frac{c}{d}$.

1.13: $\sqrt{2}$ Between Fractions

Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions $\frac{a}{b}$ and $\frac{a+2b}{a+b}$. Which fraction is closer to $\sqrt{2}$?

Solution: Let's first establish the ordering of the two fractions by examining their difference:

$$\frac{a+2b}{a+b} - \frac{a}{b} = \frac{b(a+2b) - a(a+b)}{b(a+b)} = \frac{ab + 2b^2 - a^2 - ab}{b(a+b)} = \frac{2b^2 - a^2}{b(a+b)}$$

The sign of this difference depends on the sign of $2b^2 - a^2$, which relates $\frac{a}{b}$ to $\sqrt{2}$.

Case 1: $\frac{a}{b} < \sqrt{2}$. This means $a < b\sqrt{2}$, so $a^2 < 2b^2$, and $2b^2 - a^2 > 0$. Thus, $\frac{a}{b} < \frac{a+2b}{a+b}$. We need to show that $\frac{a+2b}{a+b} > \sqrt{2}$.

$$\frac{a+2b}{a+b} > \sqrt{2} \iff a+2b > \sqrt{2}(a+b) \iff b(2-\sqrt{2}) > a(\sqrt{2}-1) \iff \frac{2-\sqrt{2}}{\sqrt{2}-1} > \frac{a}{b}$$

Since $\frac{2-\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}(\sqrt{2}-1)}{\sqrt{2}-1} = \sqrt{2}$, this simplifies to $\sqrt{2} > \frac{a}{b}$, which is true by our case assumption. Thus, $\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b}$.

Case 2: $\frac{a}{b} > \sqrt{2}$. This means $a^2 > 2b^2$, and $2b^2 - a^2 < 0$. Thus, $\frac{a}{b} > \frac{a+2b}{a+b}$. A similar calculation shows that $\frac{a+2b}{a+b} < \sqrt{2}$. Therefore, $\frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}$. In both cases, $\sqrt{2}$ lies between the two fractions.

Which fraction is closer to $\sqrt{2}$? We compare the absolute distances:

- Distance 1: $\left|\frac{a}{b} - \sqrt{2}\right| = \frac{|a-b\sqrt{2}|}{b}$
- Distance 2: $\left|\frac{a+2b}{a+b} - \sqrt{2}\right| = \left|\frac{a+2b-a\sqrt{2}-b\sqrt{2}}{a+b}\right| = \left|\frac{a(1-\sqrt{2})-b(\sqrt{2}-2)}{a+b}\right| = \frac{|a-b\sqrt{2}|(\sqrt{2}-1)}{a+b}$

To see which distance is smaller, we compare $\frac{1}{b}$ with $\frac{\sqrt{2}-1}{a+b}$. This is equivalent to comparing $a+b$ with $b(\sqrt{2}-1) = b\sqrt{2}-b$, which is equivalent to comparing $a+2b$ with $b\sqrt{2}$, or $\frac{a}{b}+2$ with $\sqrt{2}$. Since a, b are positive integers, $\frac{a}{b} > 0$, so $\frac{a}{b}+2 > 2 > \sqrt{2}$. This implies $\frac{1}{b} > \frac{\sqrt{2}-1}{a+b}$. Therefore, Distance 1 is always greater than Distance 2. The new fraction $\frac{a+2b}{a+b}$ is **always** closer to $\sqrt{2}$.

1.3 Inequalities

1.14: Irrationality of $\sqrt{n-1}+\sqrt{n+1}$

Prove that $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

Solution: Assume $\sqrt{n-1}+\sqrt{n+1} = \frac{p}{q}$, where p, q are integers, $q \neq 0$, $\gcd(p, q) = 1$.

Square both sides:

$$(n-1) + 2\sqrt{(n-1)(n+1)} + (n+1) = \frac{p^2}{q^2} \implies 2n + 2\sqrt{n^2-1} = \frac{p^2}{q^2}.$$

Thus:

$$\sqrt{n^2-1} = \frac{p^2-2nq^2}{2q^2}.$$

Suppose $\sqrt{n^2-1}$ is rational, say $\frac{a}{b}$, $\gcd(a,b)=1$. Then:

$$n^2 - 1 = \frac{a^2}{b^2} \implies a^2 = (n^2-1)b^2.$$

For $n=1$, $\sqrt{0}+\sqrt{2}=\sqrt{2}$, irrational. For $n \geq 2$, $n^2-1=(n-1)(n+1)$ is not a perfect square (since $(n-1)^2 < n^2-1 < n^2$). If $a^2=(n^2-1)b^2$, n^2-1 must be a perfect square, a contradiction for $n \geq 2$. Thus, $\sqrt{n^2-1}$ is irrational, so $\sqrt{n-1}+\sqrt{n+1}$ is irrational.

1.15: Approximation by Rational Numbers

Given a real x and an integer $N > 1$, prove that there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. Hint. Consider the $N+1$ numbers $tx - [tx]$ for $t = 0, 1, 2, \ldots, N$ and show that some pair differs by at most $1/N$.

Solution: We will use the pigeonhole principle to prove this result.

Consider the $N+1$ numbers: $0, x, 2x, 3x, \ldots, Nx$.

Let's look at the fractional parts of these numbers. The fractional part of a number y is $y - \lfloor y \rfloor$, where $\lfloor y \rfloor$ is the greatest integer less than or equal to y.

The fractional parts of $0, x, 2x, \ldots, Nx$ all lie in the interval $[0,1)$.

Divide the interval $[0,1)$ into N equal subintervals: $[0, 1/N), [1/N, 2/N), \ldots, [(N-1)/N, 1)$

By the pigeonhole principle, since we have $N+1$ numbers and only N subintervals, at least two of these numbers must fall into the same subinterval.

Let's say ix and jx (where $0 \leq i < j \leq N$) have fractional parts in the same subinterval. Then:

$$|(jx - \lfloor jx \rfloor) - (ix - \lfloor ix \rfloor)| < \frac{1}{N}$$
$$|(j-i)x - (\lfloor jx \rfloor - \lfloor ix \rfloor)| < \frac{1}{N}$$

Let $k = j - i$ and $h = \lfloor jx \rfloor - \lfloor ix \rfloor$. Then:

$$|kx - h| < \frac{1}{N}$$

Since $0 < i < j \leq N$, we have $0 < k \leq N$, and h is an integer.

Therefore, we have found integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$.

Example: If $x = \pi$ and $N = 5$, we might find that $3\pi \approx 9.4248$ and $5\pi \approx 15.7080$ have fractional parts in the same subinterval, giving us $|2\pi - 6| < 1/5$.

1.16: Infinitely Many Rational Approximations

If x is irrational, prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$.

Solution: We will construct an infinite sequence of distinct rational numbers satisfying the condition.

From Problem 1.15 (Dirichlet's Approximation Theorem), for any integer $N > 1$, there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. Dividing by k, we get:

$$\left|x - \frac{h}{k}\right| < \frac{1}{Nk}$$

Since $k \leq N$, we have $\frac{1}{N} \leq \frac{1}{k}$, which implies $\frac{1}{Nk} \leq \frac{1}{k^2}$. Thus, for any integer $N > 1$, we can find a rational number h/k such that:

$$\left|x - \frac{h}{k}\right| < \frac{1}{k^2}$$

Now we must show that this process can generate infinitely many distinct fractions. Assume, for the sake of contradiction, that there are only a finite number of such rational approximations, say $\{h_1/k_1, h_2/k_2, \ldots, h_m/k_m\}$. Since x is irrational, for any rational number h_i/k_i, the distance $|x - h_i/k_i|$ is non-zero. Let ϵ be the smallest of these non-zero distances:

$$\epsilon = \min_{i=1,\ldots,m} \left|x - \frac{h_i}{k_i}\right| > 0.$$

Now, choose an integer N large enough such that $1/N < \epsilon$. By the result from Problem 1.15, there exist integers h' and k' with $0 < k' \leq N$ such that:

$$|k'x - h'| < \frac{1}{N}$$

This implies $|x - h'/k'| < \frac{1}{Nk'} \leq \frac{1}{N}$. So we have found a new rational approximation h'/k' such that:

$$\left|x - \frac{h'}{k'}\right| < \frac{1}{N} < \epsilon$$

Since the approximation error of h'/k' is smaller than ϵ, h'/k' cannot be one of the fractions in our finite list $\{h_1/k_1, \ldots, h_m/k_m\}$. This contradicts our assumption that we had a complete list of all such approximations. Therefore, there must be infinitely many such rational numbers.

1.17: Factorial Representation of Rationals (Precise Form)

Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is a nonnegative integer with $a_k \le k-1$ for $k \ge 2$ and $a_n > 0$. Let $[x]$ denote the greatest integer less than or equal to x. Prove that $a_1 = [x]$, that

$$a_k = [k!x] - k[(k-1)!x] \quad \text{for } k = 2, \ldots, n,$$

and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Solution: Let $x = \sum_{k=1}^{n} \frac{a_k}{k!}$ with the given conditions on a_k.

1. Proof that $a_1 = [x]$: The sum can be written as $x = a_1 + \sum_{k=2}^{n} \frac{a_k}{k!}$. We must show the summation part is a positive fraction less than 1. Since $a_n > 0$, the sum is positive. We can bound the sum using the property $a_k \le k-1$:

$$\sum_{k=2}^{n} \frac{a_k}{k!} \le \sum_{k=2}^{n} \frac{k-1}{k!} < \sum_{k=2}^{\infty} \frac{k-1}{k!}$$

The infinite sum is a known identity: $\sum_{k=2}^{\infty} \frac{k-1}{k!} = \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)!} - \frac{1}{k!} \right)$. This is a telescoping series whose sum is the first term, $1/(2-1)! = 1$. Thus, $0 < \sum_{k=2}^{n} \frac{a_k}{k!} < 1$. This means $a_1 < x < a_1 + 1$, so by definition, $a_1 = [x]$.

2. Formula for a_k: Define $x_1 = x - a_1 = \sum_{k=2}^{n} \frac{a_k}{k!}$. Then $k!x_1$ is an integer for $k \ge n$. Consider the expression $k!x - k((k-1)!x) = k!(a_1 + x_1) - k((k-1)!(a_1 + x_1)) = ka_1k!/k!...$ this gets complicated. Let's use the given formula. Let $x_k = k!x - \sum_{j=1}^{k} a_j \frac{k!}{j!} = \sum_{j=k+1}^{n} a_j \frac{k!}{j!} = \frac{a_{k+1}}{k+1} + \frac{a_{k+2}}{(k+1)(k+2)} + \ldots$. From part (1), we know $0 \le x_k < 1$. So $[k!x] = \sum_{j=1}^{k} a_j \frac{k!}{j!}$. Let's test the formula: $a_k = [k!x] - k[(k-1)!x]$. We have $[k!x] = k! \sum_{j=1}^{k} \frac{a_j}{j!}$ and $[(k-1)!x] = (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!}$. So, $[k!x] - k[(k-1)!x] = \sum_{j=1}^{k} a_j \frac{k!}{j!} - k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!} = \left(a_k + k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!} \right) - k \sum_{j=1}^{k-1} a_j \frac{(k-1)!}{j!} = a_k$. This proves the formula for a_k.

3. Minimality of n: Multiplying x by $n!$ gives $n!x = \sum_{k=1}^{n} a_k \frac{n!}{k!}$. Since $k \le n$, each term $\frac{n!}{k!}$ is an integer, so $n!x$ is an integer. For any $m < n$, when we compute $m!x$, the term corresponding to $k = n$ is $m! \frac{a_n}{n!} = \frac{a_n}{n(n-1)\ldots(m+1)}$. Since $0 < a_n \le n-1$, this term is a non-integer fraction. Because all other terms for $k > m$ are also fractions and terms for $k \le m$ are integers, $m!x$ cannot be an integer. Thus, n is the smallest such integer.

4. Converse (Uniqueness): Suppose a positive rational number x has two different representations:

$$x = \sum_{k=1}^{n} \frac{a_k}{k!} = \sum_{k=1}^{m} \frac{b_k}{k!}$$

with the conditions $0 \leq a_k \leq k-1$ for $k \geq 2$, $a_n > 0$, and similarly for b_k. From part (3), n is the smallest integer such that $n!x$ is an integer, and m is the smallest integer such that $m!x$ is an integer. This implies $n = m$.

Let j be the largest index for which the coefficients differ, so $a_j \neq b_j$. Assume, without loss of generality, that $a_j > b_j$. Since $a_k = b_k$ for $k > j$, we can subtract the sums:

$$\sum_{k=1}^{j} \frac{a_k}{k!} = \sum_{k=1}^{j} \frac{b_k}{k!}$$

Rearranging the terms, we get:

$$\frac{a_j - b_j}{j!} = \sum_{k=1}^{j-1} \frac{b_k - a_k}{k!}$$

Multiply both sides by $(j-1)!$:

$$\frac{a_j - b_j}{j} = \sum_{k=1}^{j-1} (b_k - a_k) \frac{(j-1)!}{k!}$$

The right-hand side is an integer, because for each $k \in \{1, \ldots, j-1\}$, $k!$ divides $(j-1)!$. Let's analyze the left-hand side. Since a_j and b_j are integers and $a_j > b_j$, we have $a_j - b_j \geq 1$. From the conditions on the coefficients, $a_j \leq j-1$ (for $j \geq 2$) and $b_j \geq 0$. Therefore, $1 \leq a_j - b_j \leq j-1$. This implies that for $j \geq 2$, the left-hand side $\frac{a_j - b_j}{j}$ is a non-integer fraction, since the numerator is an integer between 1 and $j-1$, and the denominator is j. This creates a contradiction: the left-hand side cannot be an integer, while the right-hand side must be an integer. For the case $j = 1$, the equation becomes $a_1 - b_1 = 0$, which contradicts $a_1 \neq b_1$. Thus, our assumption that there is a largest index j where $a_j \neq b_j$ must be false. All coefficients must be identical. The representation is unique.

5. *Uniqueness:* Suppose x has two different representations, $\sum \frac{a_k}{k!} = \sum \frac{b_k}{k!}$. Let j be the largest index where $a_j \neq b_j$. Assume $a_j > b_j$. Then $\frac{a_j - b_j}{j!} = \sum_{k=1}^{j-1} \frac{b_k - a_k}{k!}$. The left side is $\geq 1/j!$. The right side is bounded above by $\sum_{k=1}^{j-1} \frac{k-1}{k!} < 1/j!$, a contradiction. Thus, all coefficients must be the same.

1.4 Upper Bounds

1.18: Uniqueness of Supremum and Infimum

Show that the sup and inf of a set are uniquely determined whenever they exist.

Solution: We will prove that if a set has a supremum, it is unique. The proof for infimum is similar.

Proof by contradiction: Suppose a set S has two different suprema, say s_1 and s_2, with $s_1 < s_2$.

Since s_1 is a supremum of S: 1. s_1 is an upper bound of S (every element of S is $\leq s_1$) 2. s_1 is the least upper bound (no number less than s_1 is an upper bound)

Since s_2 is also a supremum of S: 1. s_2 is an upper bound of S (every element of S is $\leq s_2$) 2. s_2 is the least upper bound (no number less than s_2 is an upper bound)

But since $s_1 < s_2$, the number s_1 is less than s_2 and is also an upper bound of S. This contradicts the fact that s_2 is the least upper bound.

Therefore, our assumption that there are two different suprema is false, and the supremum must be unique.

Alternative proof: Let s_1 and s_2 both be suprema of S. Then: - s_1 is an upper bound, so $s_2 \leq s_1$ (since s_2 is the least upper bound) - s_2 is an upper bound, so $s_1 \leq s_2$ (since s_1 is the least upper bound)

Therefore, $s_1 = s_2$.

For infimum: The same argument applies to infimum. If a set has two infima i_1 and i_2, then: - i_1 is a lower bound, so $i_1 \leq i_2$ (since i_1 is the greatest lower bound) - i_2 is a lower bound, so $i_2 \leq i_1$ (since i_2 is the greatest lower bound)

Therefore, $i_1 = i_2$.

1.19: Finding Supremum and Infimum

Find the sup and inf of each of the following sets:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$ for positive integers p, q, r.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$.

(c) $S = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$ where $a < b < c < d$.

Solution:

1. Numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$:

Let's analyze the range of each term: - 2^{-p} ranges from $\frac{1}{2}$ (when $p = 1$) to 0 (as $p \to \infty$) - 3^{-q} ranges from $\frac{1}{3}$ (when $q = 1$) to 0 (as $q \to \infty$) - 5^{-r} ranges from $\frac{1}{5}$ (when $r = 1$) to 0 (as $r \to \infty$)

Therefore: - The maximum value occurs when $p = q = r = 1$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} = \frac{15+10+6}{30} = \frac{31}{30}$ - The minimum value occurs as $p, q, r \to \infty$: $0 + 0 + 0 = 0$

So $\sup = \frac{31}{30}$ and $\inf = 0$.

2. Set $S = \{x : 3x^2 - 10x + 3 < 0\}$:

First, let's find the roots of $3x^2 - 10x + 3 = 0$:

$$\begin{aligned} x &= \frac{10 \pm \sqrt{100 - 36}}{6} \\ &= \frac{10 \pm \sqrt{64}}{6} \\ &= \frac{10 \pm 8}{6} \\ &= \frac{18}{6} = 3 \quad \text{or} \quad \frac{2}{6} = \frac{1}{3} \end{aligned}$$

Since the coefficient of x^2 is positive, the parabola opens upward. The inequality $3x^2 - 10x + 3 < 0$ holds between the roots.

Therefore, $S = (\frac{1}{3}, 3)$, so $\sup = 3$ and $\inf = \frac{1}{3}$.

3. Set $S = \{x : (x - a)(x - b)(x - c)(x - d) < 0\}$ where $a < b < c < d$:

The expression $(x-a)(x-b)(x-c)(x-d)$ changes sign at each root a, b, c, d.

Starting from $-\infty$: - For $x < a$: all factors are negative, so the product is positive - For $a < x < b$: one factor is positive, three negative, so product is negative - For $b < x < c$: two factors positive, two negative, so product is positive - For $c < x < d$: three factors positive, one negative, so product is negative - For $x > d$: all factors are positive, so product is positive

Therefore, $S = (a, b) \cup (c, d)$.

So $\sup = d$ and $\inf = a$.

1.20: Comparison Property for Suprema

Let S and T be nonempty subsets of $\mathbb{R}$ such that $s \le t$ for all $s \in S$ and $t \in T$. Suppose T has a supremum. Then S has a supremum and

$$\sup S \le \sup T.$$

Solution: Let S and T be nonempty subsets of $\mathbb{R}$ with the property that for every $s \in S$ and $t \in T$, we have $s \le t$.

1. **Existence of** $\sup S$: Since T is nonempty, we can pick an arbitrary element $t_0 \in T$. By the given property, for every $s \in S$, we have $s \le t_0$. This shows that S is bounded above (by any element of T). Since S is also nonempty and bounded above, the completeness axiom of $\mathbb{R}$ guarantees that $\sup S$ exists. Let's call it $\alpha = \sup S$.

2. **Proof that** $\sup S \le \sup T$: Let $\alpha = \sup S$ and $\beta = \sup T$. From step 1, we know that any element $t \in T$ is an upper bound for the set S. Since α is the *least* upper bound of S, it must be less than or equal to any other upper

bound of S. Therefore, for any $t \in T$, we must have:

$$\alpha \leq t$$

This inequality shows that α is a lower bound for the set T. Now, by definition, $\beta = \sup T$ is the least upper bound of T. As an upper bound for T, β must be greater than or equal to every element of T. More importantly, it must be greater than or equal to any *lower bound* of T. Since we have established that α is a lower bound for T, it must follow that:

$$\alpha \leq \beta$$

Substituting the definitions of α and β, we get:

$$\sup S \leq \sup T$$

This completes the proof.

1.21: Product of Suprema

Let A and B be two sets of positive real numbers, each bounded above. Let $a = \sup A$, $b = \sup B$. Define

$$C = \{xy : x \in A, y \in B\}.$$

Prove that

$$\sup C = ab.$$

Proof:

Since A and B are sets of positive real numbers bounded above, their suprema $a = \sup A$ and $b = \sup B$ exist and are finite.

We are to prove that:

$$\sup C = ab.$$

Step 1: Show that ab is an upper bound for C.

Let $x \in A$, $y \in B$. Since $x \leq a$ and $y \leq b$, we have:

$$xy \leq ab.$$

Therefore, every element $c \in C$ satisfies $c \leq ab$, so ab is an upper bound for C.

Step 2: Show that ab is the least upper bound.

Let $\varepsilon > 0$. Since $a = \sup A$, there exists $x_\varepsilon \in A$ such that:

$$x_\varepsilon > a - \frac{\varepsilon}{2b}.$$

Similarly, since $b = \sup B$, there exists $y_\varepsilon \in B$ such that:

$$y_\varepsilon > b - \frac{\varepsilon}{2a}.$$

Now consider:

$$x_\varepsilon y_\varepsilon > \left(a - \frac{\varepsilon}{2b}\right)\left(b - \frac{\varepsilon}{2a}\right) = ab - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4ab}.$$

Since $\frac{\varepsilon^2}{4ab} > 0$, we have:

$$x_\varepsilon y_\varepsilon > ab - \varepsilon.$$

Therefore, for every $\varepsilon > 0$, there exists $c \in C$ such that $c > ab - \varepsilon$. Hence, ab is the least upper bound of C.

$$\boxed{\sup C = ab}$$

1.22: Representation of Rationals in Base k

Given $x \geq 0$ and an integer $k \geq 2$, let a_0 denote the largest integer $\leq x$, and, assuming that $a_0, a_1, \ldots, a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots + \frac{a_n}{k^n} \leq x.$$

(a) Prove that $0 \leq a_i \leq k-1$ for each $i = 1, 2, \ldots$.

(b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + \cdots + a_n k^{-n}$ and show that $x = \sup\{r_n\}$, the supremum of the set of rational numbers $r_1, r_2, \ldots$.

Solution: Let $r_n = \sum_{i=0}^{n} \frac{a_i}{k^i}$. By definition, a_n is the largest integer such that $r_n \leq x$.

(a) Show $0 \leq a_i \leq k-1$: Since a_n is the largest integer satisfying the condition, choosing $a_n + 1$ would violate it:

$$r_{n-1} + \frac{a_n + 1}{k^n} > x$$

From the definition of a_{n-1}, we know it was the largest integer such that $r_{n-1} \leq x$. This implies $x - r_{n-1} < \frac{1}{k^{n-1}}$. Now, from the definition of a_n, we have $r_{n-1} + \frac{a_n}{k^n} \leq x$, which implies $a_n \leq k^n(x - r_{n-1})$. Combining these facts:

$$a_n \leq k^n(x - r_{n-1}) < k^n\left(\frac{1}{k^{n-1}}\right) = k.$$

Since a_n is an integer and $a_n < k$, we must have $a_n \leq k-1$. Also, a_n must be non-negative, otherwise we could choose $a_n = 0$ to get a larger (or equal) sum r_n that is still less than or equal to x, contradicting the "largest integer" definition if the original a_n were negative. Thus, $0 \leq a_n \leq k-1$.

(b) Show that $x = \sup\{r_n\}$: The sequence $\{r_n\}$ is non-decreasing by construction, since $a_n \geq 0$. It is also bounded above by x. Therefore, its supremum exists; let $r = \sup\{r_n\}$. We know $r \leq x$. We will prove $r = x$ by contradiction. Assume $r < x$. Let $\delta = x - r > 0$. By the Archimedean property, we can choose an integer N large enough such that $\frac{1}{k^N} < \delta$. From the definition of a_N, we know $r_N = r_{N-1} + \frac{a_N}{k^N} \leq x$ and $r_{N-1} + \frac{a_N+1}{k^N} > x$. The second inequality rearranges to $x - r_N < \frac{1}{k^N}$. Since $r = \sup\{r_n\}$, we know $r_N \leq r$. Therefore, $x - r \leq x - r_N < \frac{1}{k^N}$. Substituting $\delta = x - r$, we get $\delta < \frac{1}{k^N}$. But we chose N such that $\frac{1}{k^N} < \delta$. This gives $\delta < \frac{1}{k^N} < \delta$, a contradiction. Thus, our assumption must be false, and $x = r = \sup\{r_n\}$.

1.5 Inequalities and Identities

1.23: Lagrange's Identity

Prove Lagrange's identity for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2.$$

Proof: We will prove the equivalent identity:

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \left(\sum_{k=1}^{n} a_k b_k\right)^2 + \sum_{1\leq k<j\leq n} (a_k b_j - a_j b_k)^2.$$

Let's expand the terms. The left-hand side (LHS) is:

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{j=1}^{n} b_j^2\right) &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 b_j^2 \\ &= \sum_{k=j} a_k^2 b_j^2 + \sum_{k\neq j} a_k^2 b_j^2 \\ &= \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{1\leq k<j\leq n} (a_k^2 b_j^2 + a_j^2 b_k^2)\end{aligned}$$

Now we expand the terms on the right-hand side (RHS). The first term is:

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{j=1}^{n} a_j b_j\right) &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k b_k a_j b_j \\ &= \sum_{k=j} a_k b_k a_j b_j + \sum_{k\neq j} a_k b_k a_j b_j \\ &= \sum_{k=1}^{n} a_k^2 b_k^2 + 2\sum_{1\leq k<j\leq n} a_k b_k a_j b_j\end{aligned}$$

The second term on the RHS is:

$$\sum_{1\le k<j\le n}(a_kb_j - a_jb_k)^2 = \sum_{1\le k<j\le n}(a_k^2b_j^2 - 2a_kb_ja_jb_k + a_j^2b_k^2)$$

Adding the two terms on the RHS gives:

$$\begin{aligned}\text{RHS} &= \left(\sum_{k=1}^n a_k^2b_k^2 + 2\sum_{1\le k<j\le n}a_kb_ka_jb_j\right) + \left(\sum_{1\le k<j\le n}(a_k^2b_j^2 - 2a_ka_jb_kb_j + a_j^2b_k^2)\right)\\ &= \sum_{k=1}^n a_k^2b_k^2 + \sum_{1\le k<j\le n}(a_k^2b_j^2 + a_j^2b_k^2)\end{aligned}$$

The RHS now matches our expansion of the LHS, which completes the proof.

1.24: A Holder-type Inequality

Prove that for arbitrary real numbers a_k, b_k, c_k we have

$$\left(\sum_{k=1}^n a_kb_kc_k\right)^4 \le \left(\sum_{k=1}^n a_k^4\right)\left(\sum_{k=1}^n b_k^2\right)^2\left(\sum_{k=1}^n c_k^4\right).$$

Proof: We will prove this inequality by applying the Cauchy-Schwarz inequality twice. First, group the terms as (a_kc_k) and b_k. Applying the Cauchy-Schwarz inequality to the sequences $\{a_kc_k\}$ and $\{b_k\}$ gives:

$$\left(\sum_{k=1}^n (a_kc_k)b_k\right)^2 \le \left(\sum_{k=1}^n (a_kc_k)^2\right)\left(\sum_{k=1}^n b_k^2\right) = \left(\sum_{k=1}^n a_k^2c_k^2\right)\left(\sum_{k=1}^n b_k^2\right).$$

Next, we apply the Cauchy-Schwarz inequality to the term $\sum_{k=1}^n a_k^2c_k^2$, treating it as the dot product of sequences $\{a_k^2\}$ and $\{c_k^2\}$:

$$\left(\sum_{k=1}^n a_k^2c_k^2\right)^2 \le \left(\sum_{k=1}^n (a_k^2)^2\right)\left(\sum_{k=1}^n (c_k^2)^2\right) = \left(\sum_{k=1}^n a_k^4\right)\left(\sum_{k=1}^n c_k^4\right).$$

This implies:

$$\sum_{k=1}^n a_k^2c_k^2 \le \left(\sum_{k=1}^n a_k^4\right)^{1/2}\left(\sum_{k=1}^n c_k^4\right)^{1/2}.$$

Now, substitute this result back into our first inequality:

$$\left(\sum_{k=1}^n a_kb_kc_k\right)^2 \le \left(\sum_{k=1}^n a_k^4\right)^{1/2}\left(\sum_{k=1}^n c_k^4\right)^{1/2}\left(\sum_{k=1}^n b_k^2\right).$$

Finally, squaring both sides gives the desired result:

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \leq \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} c_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2.$$

1.25: Minkowski's Inequality

Prove Minkowski's inequality:

$$\left(\sum_{k=1}^{n}(a_k + b_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

Proof:
Let $A = \left(\sum a_k^2\right)^{1/2}$, $B = \left(\sum b_k^2\right)^{1/2}$, and expand the square:

$$\sum(a_k + b_k)^2 = \sum a_k^2 + 2\sum a_k b_k + \sum b_k^2 = A^2 + 2\sum a_k b_k + B^2.$$

Apply Cauchy–Schwarz:

$$\sum a_k b_k \leq AB.$$

Thus,

$$\sum(a_k + b_k)^2 \leq A^2 + 2AB + B^2 = (A + B)^2.$$

Taking square roots:

$$\left(\sum(a_k + b_k)^2\right)^{1/2} \leq A + B.$$

1.26: Chebyshev's Sum Inequality

If $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$, prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \leq n\sum_{k=1}^{n} a_k b_k.$$

Proof: Consider the double summation

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j).$$

Since the sequences $\{a_k\}$ and $\{b_k\}$ are sorted in the same order (both non-increasing), the terms $(a_i - a_j)$ and $(b_i - b_j)$ always have the same sign. If $i > j$, then $a_i \leq a_j$ and $b_i \leq b_j$, so both differences are non-positive. If $i < j$, both are non-negative. Therefore, their product is always non-negative:

$$(a_i - a_j)(b_i - b_j) \geq 0.$$

This implies that the total sum S must be non-negative, $S \geq 0$.

Now, let's expand the sum:

$$\begin{aligned} S &= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_i - a_ib_j - a_jb_i + a_jb_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_i - \sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j - \sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_i + \sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_j \end{aligned}$$

We evaluate each double summation:

- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_i = \sum_{i=1}^{n} (n \cdot a_ib_i) = n\sum_{i=1}^{n} a_ib_i$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_j = (\sum_{i=1}^{n} a_i)\left(\sum_{j=1}^{n} b_j\right)$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_i = \left(\sum_{j=1}^{n} a_j\right)(\sum_{i=1}^{n} b_i)$
- $\sum_{i=1}^{n}\sum_{j=1}^{n} a_jb_j = \sum_{j=1}^{n} (n \cdot a_jb_j) = n\sum_{j=1}^{n} a_jb_j$

Substituting these back into the expression for S:

$$S = n\sum a_kb_k - \left(\sum a_k\right)\left(\sum b_k\right) - \left(\sum a_k\right)\left(\sum b_k\right) + n\sum a_kb_k$$

$$S = 2n\sum_{k=1}^{n} a_kb_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

Since we established that $S \geq 0$:

$$2n\sum_{k=1}^{n} a_kb_k - 2\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \geq 0$$

Dividing by 2 and rearranging gives the desired inequality:

$$n\sum_{k=1}^{n} a_kb_k \geq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right).$$

1.27: Express Complex Numbers in $a + bi$ Form

Express the following complex numbers in the form $a + bi$:

(a) $(1+i)^3$

(b) $\frac{2+3i}{3-4i}$

(c) $i^5 + i^{16}$

(d) $\frac{1}{2}(1+i)(1+i^{-8})$

Solution:

(a) $(1+i)^3 = (1+i)^2(1+i) = (2i)(1+i) = 2i + 2i^2 = 2i - 2 = -2 + 2i$

(b) Rationalize the denominator:

$$\frac{2+3i}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{(2+3i)(3+4i)}{9+16} = \frac{6+8i+9i+12i^2}{25} = \frac{-6+17i}{25} = -\frac{6}{25}+\frac{17}{25}i$$

(c) $i^5 = i$, since $i^4 = 1$, and $i^{16} = (i^4)^4 = 1$, so:

$$i^5 + i^{16} = i + 1 = 1 + i$$

(d) $\frac{1}{2}(1+i)(1+i^{-8})$, note that $i^{-8} = (i^4)^{-2} = 1^{-2} = 1$, so:

$$\frac{1}{2}(1+i)(1+1) = \frac{1}{2}(1+i)(2) = \frac{1}{2}(2+2i) = 1+i$$

1.28: Solve Complex Equations

In each case, determine all real x and y which satisfy the given relation:

(a) $x + iy = |x - iy|$

(b) $x + iy = (x - iy)^2$

(c) $\sum_{k=0}^{100} i^k = x + iy$

Solution:

(a) RHS is real and nonnegative. LHS is complex. For equality, imaginary part must vanish:

$$\mathrm{Im}(x+iy) = y = 0, \quad \text{and } x = |x| \Rightarrow x \geq 0.$$

So solution: $y = 0$, $x \geq 0$

(b) Compute RHS:

$$(x-iy)^2 = x^2 - 2ixy - y^2 = (x^2 - y^2) - 2ixy.$$

Set equal to $x + iy$, equate real and imaginary parts:

$$x = x^2 - y^2, \quad y = -2xy.$$

From second equation: $y = -2xy \Rightarrow y(1+2x) = 0 \Rightarrow y = 0$ or $x = -\frac{1}{2}$

If $y = 0$, then first equation: $x = x^2 \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$

If $x = -\frac{1}{2}$, then first equation:

$$x = x^2 - y^2 \Rightarrow -\frac{1}{2} = \frac{1}{4} - y^2 \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

So all solutions:

$$(x, y) = (0,0), (1,0), \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$$

(c) The powers of i cycle every 4: $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$

There are 101 terms, which form 25 full cycles and one leftover term $i^{100} \equiv i^0 = 1$

Each full cycle sums to 0. So total sum:

$$\sum_{k=0}^{100} i^k = 25 \cdot 0 + 1 = 1 \Rightarrow x = 1, y = 0.$$

1.29: Basic Identities for Complex Conjugates

If $z = x + iy$, where x and y are real, the complex conjugate of z is $\overline{z} = x - iy$. Prove the following:

a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$,

c) $z \cdot \overline{z} = |z|^2$,

d) $z + \overline{z}$ is twice the real part of z,

e) $\frac{z - \overline{z}}{i}$ is twice the imaginary part of z.

Solution: Let $z = x + iy$ and $w = u + iv$ be two complex numbers.

a) **Conjugate of a sum:**

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = \overline{z_1} + \overline{z_2}.$$

b) **Conjugate of a product:**

$$\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{(x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1))} = x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1) = \overline{z_1} \cdot \overline{z_2}.$$

c) **Modulus squared:**

$$z \cdot \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

d) **Twice the real part:**

$$z + \overline{z} = (x + iy) + (x - iy) = 2x = 2\Re(z).$$

e) **Twice the imaginary part:**

$$\frac{z - \overline{z}}{i} = \frac{(x + iy) - (x - iy)}{i} = \frac{2iy}{i} = 2y = 2\Im(z).$$

1.30: Geometric Descriptions of Complex Sets

Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

a) $|z| = 1$,

b) $|z| < 1$,

c) $|z| \leq 1$,

d) $z + \overline{z} = 1$,

e) $z - \overline{z} = i$,

f) $\overline{z} + z = |z|^2$.

Solution:

a) The unit circle centered at the origin.

b) The open unit disk centered at the origin.

c) The closed unit disk centered at the origin.

d) $2\Re(z) = 1 \Rightarrow \Re(z) = \frac{1}{2}$: a vertical line in the complex plane.

e) $2i\Im(z) = i \Rightarrow \Im(z) = \frac{1}{2}$: a horizontal line.

f) Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then:

$$z + \overline{z} = (x + iy) + (x - iy) = 2x,$$
$$|z|^2 = x^2 + y^2.$$

So the equation becomes:

$$2x = x^2 + y^2.$$

Rewriting this:

$$x^2 - 2x + y^2 = 0.$$

We now complete the square on the x-terms:

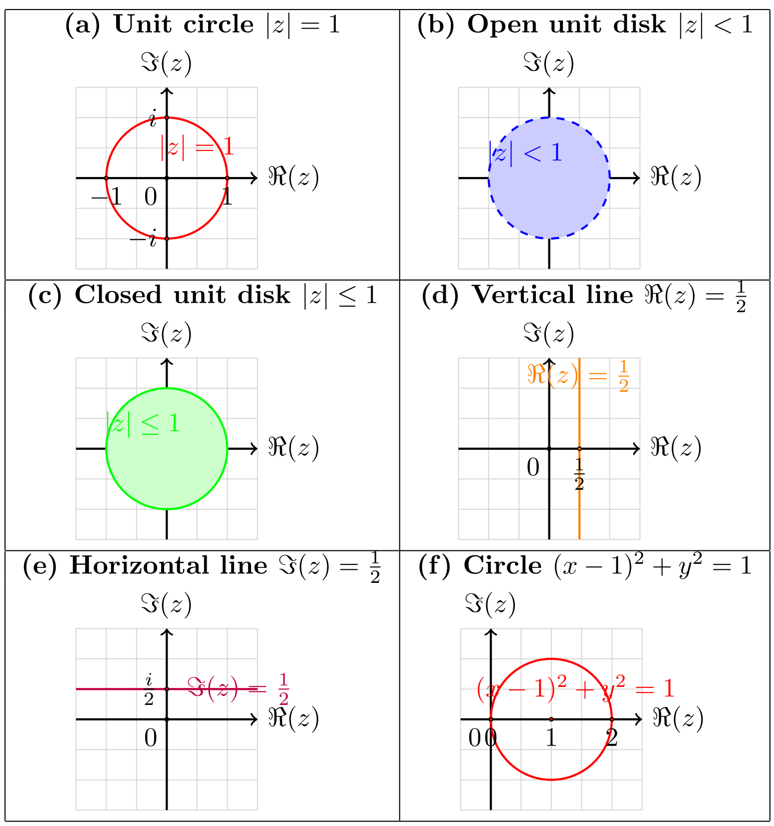
$$x^2 - 2x = (x - 1)^2 - 1,$$

which gives:

$$(x - 1)^2 - 1 + y^2 = 0 \quad \Rightarrow \quad (x - 1)^2 + y^2 = 1.$$

This is the standard equation of a circle with center at $(1, 0)$ and radius 1 in the complex plane.

Visualizations:



1.31: Equilateral Triangle on the Unit Circle

Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Solution: Since $|z_i| = 1$, each $z_i = e^{i\theta_i}$ lies on the unit circle. Given $z_1 + z_2 + z_3 = 0$, we need to show they form an equilateral triangle. Consider the angles $\theta_1, \theta_2, \theta_3$. The sum condition implies:

$$e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3} = 0.$$

For three points on the unit circle to form an equilateral triangle, their arguments must differ by $120^\circ = \frac{2\pi}{3}$. Assume:

$$z_1 = e^{i\theta}, \quad z_2 = e^{i(\theta + \frac{2\pi}{3})}, \quad z_3 = e^{i(\theta + \frac{4\pi}{3})}.$$

Check the sum:

$$e^{i\theta} + e^{i(\theta + \frac{2\pi}{3})} + e^{i(\theta + \frac{4\pi}{3})} = e^{i\theta}\left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}}\right).$$

Since $e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, we have:

$$1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}} = 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0.$$

The angles $\theta, \theta + \frac{2\pi}{3}, \theta + \frac{4\pi}{3}$ are spaced $\frac{2\pi}{3}$ apart, forming an equilateral triangle. Any three points with $|z_i| = 1$ and sum zero are rotations of the cube roots of unity, ensuring an equilateral triangle.

1.32: Inequality with Complex Numbers

If a and b are complex numbers, prove:

a) $|a - b|^2 \leq (1 + |a|^2)(1 + |b|^2)$,

b) If $a \neq 0$, then $|a+b| = |a|+|b|$ if and only if $\frac{b}{a}$ is real and nonnegative.

Solution:

a) Compute:

$$|a - b|^2 = (a - b)\overline{(a - b)} = |a|^2 + |b|^2 - a\overline{b} - \overline{a}b.$$

Consider the right-hand side:

$$(1 + |a|^2)(1 + |b|^2) = 1 + |a|^2 + |b|^2 + |a|^2|b|^2.$$

Evaluate:

$$(1+|a|^2)(1+|b|^2)-|a-b|^2 = 1+|ab|^2+a\bar{b}+\bar{a}b = 1+|ab|^2+2\Re(a\bar{b}).$$

Since $|ab|^2 \geq 0$, $\Re(a\bar{b}) \geq -|ab|$:

$$1+|ab|^2+2\Re(a\bar{b}) \geq 1+|ab|^2-2|ab| = (1-|ab|)^2 \geq 0.$$

Thus, $|a-b|^2 \leq (1+|a|^2)(1+|b|^2)$.

b) For $|a+b| = |a|+|b|$, the triangle inequality requires a, b collinear in the same direction. Let $b = ka$, $k \in \mathbb{R}_{\geq 0}$:

$$|a+b| = |a+ka| = |a|(1+k) = |a|+|b|.$$

Thus, $\frac{b}{a} = k \geq 0$. Conversely, if $|a+b| = |a|+|b|$, then $a\bar{b}+\bar{a}b = 2|a||b|$, so $\frac{b}{a}$ is real and nonnegative.

1.33: Equality Condition for Complex Difference

If a and b are complex numbers, prove that

$$|a-b| = |1-\bar{a}b|$$

if and only if $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a-b| < |1-\bar{a}b|$ valid?

Solution: Let $|a| = r$, $|b| = s$. Compute:

$$|a-b|^2 = r^2+s^2-a\bar{b}-\bar{a}b, \quad |1-\bar{a}b|^2 = 1+r^2s^2-a\bar{b}-\bar{a}b.$$

Thus:

$$|a-b|^2-|1-\bar{a}b|^2 = r^2+s^2-1-r^2s^2 = (r^2-1)(s^2-1).$$

Equality holds when:

$$(r^2-1)(s^2-1) = 0 \implies r = 1 \text{ or } s = 1.$$

For the inequality:

$$(r^2-1)(s^2-1) < 0 \implies (r^2 < 1 \text{ and } s^2 > 1) \text{ or } (r^2 > 1 \text{ and } s^2 < 1).$$

Thus, equality holds if $|a| = 1$ or $|b| = 1$; the inequality holds when one modulus is less than 1 and the other is greater than 1.

1.34: Complex Circle in the Plane

If a and c are real constants, b is complex, show that the equation

$$az\bar{z}+bz+\bar{b}\bar{z}+c = 0 \qquad (a \neq 0, z = x+iy)$$

represents a circle in the xy-plane.

Solution: Let $z = x + iy$, $\overline{z} = x - iy$, then $z\overline{z} = x^2 + y^2$, $bz + \overline{b}\overline{z} = 2\Re(bz)$. Hence the equation becomes:

$$a(x^2 + y^2) + 2\Re(bz) + c = 0.$$

This is the general form of a circle in $\mathbb{R}^2$.

1.35: Argument of a Complex Number via Arctangent

Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ satisfying:

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \text{and} \quad \tan\theta = t.$$

If $z = x + iy$, show that:

a) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$, if $x > 0$,

b) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, if $x < 0$, $y \geq 0$,

c) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$,

d) $\arg(z) = \frac{\pi}{2}$, if $x = 0, y > 0$; $\quad \arg(z) = -\frac{\pi}{2}$, if $x = 0, y < 0$.

Solution: For $z = x+iy$, $\arg(z)$ is the angle $\theta \in (-\pi, \pi]$ such that $z = |z|e^{i\theta}$.

a) If $x > 0$, z is in Quadrant I or IV, and $\tan\theta = \frac{y}{x}$, so $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

b) If $x < 0$, $y \geq 0$, z is in Quadrant II. $\tan^{-1}\left(\frac{y}{x}\right) \in (-\frac{\pi}{2}, 0]$, so add π to get $\theta \in (\frac{\pi}{2}, \pi]$.

c) If $x < 0$, $y < 0$, z is in Quadrant III. $\tan^{-1}\left(\frac{y}{x}\right) \in (0, \frac{\pi}{2}]$, so subtract π to get $\theta \in (-\pi, -\frac{\pi}{2}]$.

d) If $x = 0$, $z = iy$. If $y > 0$, $\theta = \frac{\pi}{2}$; if $y < 0$, $\theta = -\frac{\pi}{2}$.

- **Axiom 6 (Trichotomy):** For any $z_1, z_2 \in \mathbb{C}$, we can compare their moduli. Exactly one of $|z_1| < |z_2|$, $|z_1| > |z_2|$, or $|z_1| = |z_2|$ holds. If $|z_1| = |z_2|$, we compare their principal arguments, for which trichotomy holds on $(-\pi, \pi]$. Thus, exactly one of $z_1 < z_2$, $z_2 < z_1$, or $z_1 = z_2$ is true. This axiom is **satisfied**.

- **Axiom 9 (Transitivity):** If $z_1 < z_2$ and $z_2 < z_3$, the transitivity of the $<$ relation on the real numbers for both the moduli and the arguments ensures that $z_1 < z_3$. This axiom is **satisfied**.

- **Axiom 7 (Translation Invariance):** This axiom states that if $z_1 < z_2$, then $z_1 + z < z_2 + z$ for any $z \in \mathbb{C}$. This axiom is **not satisfied**.

Counterexample: Let $z_1 = 1$ and $z_2 = 2$. According to the ordering, $z_1 < z_2$ because $|z_1| = 1 < |z_2| = 2$. Now, let $z = -2$. Then $z_1 + z = 1+(-2) = -1$. And $z_2+z = 2+(-2) = 0$. We must compare $z_1+z = -1$ and $z_2 + z = 0$. We have $|-1| = 1$ and $|0| = 0$. Since $|0| < |-1|$, we have $0 < -1$ in this pseudo-ordering. So, $z_2 + z < z_1 + z$. The order relation was reversed, which violates the axiom.

- **Axiom 8 (Multiplication):** This axiom states that if $z_1 < z_2$ and $z > 0$, then $z_1 z < z_2 z$. Let us define $z > 0$ to mean $0 < z$. This holds for any $z \neq 0$. This axiom is also **not satisfied**.

  Counterexample: Let $z_1 = e^{i\pi} = -1$ and $z_2 = e^{-i\pi/2} = -i$. We have $|z_1| = |z_2| = 1$. The arguments are $\arg(z_1) = \pi$ and $\arg(z_2) = -\pi/2$. Since $-\pi/2 < \pi$, we have $z_2 < z_1$. Now, let $z = i$. Since $i \neq 0$, z is a "positive" number under this definition. Then $z_1 z = (-1)(i) = -i$. And $z_2 z = (-i)(i) = 1$. We must compare $z_1 z = -i$ and $z_2 z = 1$. We have $|-i| = 1$ and $|1| = 1$. The arguments are $\arg(-i) = -\pi/2$ and $\arg(1) = 0$. Since $-\pi/2 < 0$, we have $-i < 1$. So, $z_1 z < z_2 z$. The order relation was reversed from $z_2 < z_1$ to $z_1 z < z_2 z$. The axiom is violated.

Conclusion: Axioms 6 and 9 are satisfied; Axiom 7 and 8 is not applicable.

1.37: Order Axioms and Lexicographic Ordering on $\mathbb{R}^2$

Define a pseudo-ordering on ordered pairs $(x_1, y_1) < (x_2, y_2)$ if either

(i) $x_1 < x_2$, or

(ii) $x_1 = x_2$ and $y_1 < y_2$.

Which of Axioms 6, 7, 8, 9 are satisfied by this relation?

Solution:

- **Axiom 6: Trichotomy.** For any $(x_1, y_1), (x_2, y_2)$, if $x_1 < x_2$, then $(x_1, y_1) < (x_2, y_2)$; if $x_1 > x_2$, then $(x_2, y_2) < (x_1, y_1)$; if $x_1 = x_2$, compare y_1, y_2. Exactly one holds. Satisfied.
- **Axiom 7: Translation Invariance.** If $(x_1, y_1) < (x_2, y_2)$, add (u, v): if $x_1 < x_2$, then $x_1+u < x_2+u$; if $x_1 = x_2$, then $y_1 < y_2 \implies y_1+v < y_2+v$. Satisfied.
- **Axiom 8: Multiplication.** Not applicable, as $\mathbb{R}^2$ lacks scalar multiplication.
- **Axiom 9: Transitivity.** If $(x_1, y_1) < (x_2, y_2)$, $(x_2, y_2) < (x_3, y_3)$, lexicographic order ensures $(x_1, y_1) < (x_3, y_3)$. Satisfied.

Conclusion: Axioms 6, 7, and 9 are satisfied; Axiom 8 is not applicable.

1.38: Argument of a Quotient Using Theorem 1.48

State and prove a theorem analogous to Theorem 1.48, expressing $\arg\left(\frac{z_1}{z_2}\right)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Solution: Theorem: If $z_1, z_2 \neq 0$, then:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2^{-1}),$$

where $n(z_1, z_2^{-1})$ adjusts the argument to $(-\pi, \pi]$.

Proof: Since $\frac{z_1}{z_2} = z_1 z_2^{-1}$, and $\arg(z_2^{-1}) = -\arg(z_2)$, apply Theorem 1.48:

$$\arg(z_1 z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) + 2\pi n(z_1, z_2^{-1}) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2^{-1}).$$

1.39: Logarithm of a Quotient Using Theorem 1.54

State and prove a theorem analogous to Theorem 1.54, expressing $\log\left(\frac{z_1}{z_2}\right)$ in terms of $\log(z_1)$ and $\log(z_2)$.

Solution: Theorem: If $z_1, z_2 \neq 0$, then:

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 + 2\pi i n(z_1, z_2^{-1}).$$

Proof: Since $\frac{z_1}{z_2} = z_1 z_2^{-1}$, apply Theorem 1.54:

$$\log(z_1 z_2^{-1}) = \log z_1 + \log(z_2^{-1}) + 2\pi i n(z_1, z_2^{-1}) = \log z_1 - \log z_2 + 2\pi i n(z_1, z_2^{-1}).$$

1.40: Roots of Unity and Polynomial Identity

Prove that the nth roots of 1 are given by $\alpha, \alpha^2, \ldots, \alpha^n$, where $\alpha = e^{2\pi i/n}$, and that these roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + \cdots + x^{n-1} = 0.$$

Solution: Let $\alpha = e^{2\pi i/n}$. Then $\alpha^n = 1$, so it's a root of $x^n - 1 = 0$. Also,

$$\frac{1-\alpha^n}{1-\alpha} = 0 \Rightarrow 1 + \alpha + \cdots + \alpha^{n-1} = 0 \quad \text{for } \alpha \neq 1.$$

1.41: Inequalities and Boundedness of $\cos z$

a) Prove that $|z^i| < e^{\pi}$ for all complex $z \neq 0$.

b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Solution:

a) For $z = re^{i\theta}$, $z^i = e^{i(\ln r + i\theta)} = e^{-\theta}e^{i\ln r}$, so $|z^i| = e^{-\theta}$. Since $\theta \in (-\pi, \pi]$, $|z^i| \leq e^{\pi}$, strict unless $\theta = -\pi$.

b) For $z = iy$, $\cos(iy) = \cosh y$, which is unbounded as $|y| \to \infty$. Thus, no $M > 0$ exists.

1.42: Complex Exponential via Real and Imaginary Parts

If $w = u + iv$, where u and v are real, show that

$$z^w = e^{u\log|z| - v\arg(z)} \cdot e^{i[v\log|z| + u\arg(z)]}.$$

Solution: For $z^w = e^{w\log z}$, where $\log z = \log|z| + i\arg z$:

$$w\log z = (u+iv)(\log|z| + i\arg z) = (u\log|z| - v\arg z) + i(v\log|z| + u\arg z).$$

Thus:

$$z^w = e^{u\log|z| - v\arg z}e^{i(v\log|z| + u\arg z)}.$$

1.43: Logarithmic Identities for Complex Powers

a) Prove that $\log(z^w) = w\log z + 2\pi in$, where n is an integer.

b) Prove that $(z^w)^{\alpha} = z^{w\alpha}e^{2\pi in\alpha}$, where n is an integer.

Solution:

a) Since $z^w = e^{w\log z}$:

$$\log(z^w) = \log(e^{w\log z}) = w\log z + 2\pi in.$$

b) Compute:

$$(z^w)^{\alpha} = e^{\alpha\log(z^w)} = e^{\alpha(w\log z + 2\pi in)} = z^{w\alpha}e^{2\pi in\alpha}.$$

1.44: Conditions for De Moivre's Formula

i) If θ and a are real numbers, $-\pi < \theta \leq +\pi$, prove that

$$(\cos\theta + i\sin\theta)^a = \cos(a\theta) + i\sin(a\theta).$$

ii) Show that, in general, the restriction $-\pi < \theta \leq +\pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as De Moivre's theorem.

Solution:

i) Since $\cos\theta + i\sin\theta = e^{i\theta}$:

$$(\cos\theta + i\sin\theta)^a = (e^{i\theta})^a = e^{ia\theta} = \cos(a\theta) + i\sin(a\theta).$$

ii) For $\theta = -\pi$, $a = \frac{1}{2}$:

$$(-1)^{1/2} = i, \quad \text{but} \quad \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right) = -i.$$

The restriction ensures the principal branch.

iii) For integer a, $(e^{i\theta})^a = e^{ia\theta}$, and multiples of 2π cancel, so the formula holds for all θ.

1.45: Deriving Trigonometric Identities from De Moivre's Theorem

Use De Moivre's theorem (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta,$$
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

Solution: By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

Expand:

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Equate parts:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta, \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

These hold for complex θ, as $\cos z$ and $\sin z$ are analytic.

1.46: Tangent of Complex Numbers

Define $\tan z = \frac{\sin z}{\cos z}$, and show that for $z = x + iy$,

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

Solution: For $z = x + iy$:

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \quad \cos z = \cos x \cosh y - i \sin x \sinh y.$$

Compute:

$$\tan z = \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}.$$

Multiply by the conjugate of the denominator:

$$N = (\sin x \cosh y + i \cos x \sinh y)(\cos x \cosh y + i \sin x \sinh y) = \sin 2x + i \sinh 2y,$$

$$D = (\cos x \cosh y)^2 + (\sin x \sinh y)^2 = \frac{1}{2}(\cos 2x + \cosh 2y).$$

Thus:

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

1.47: Solving Cosine Equation

Let w be a complex number. If $w \neq \pm 1$, show that there exist two values $z = x + iy$ with $\cos z = w$ and $-\pi < x \leq \pi$. Find such z when $w = i$ and $w = 2$.

Solution: For $z = x + iy$, $\cos z = \cos x \cosh y - i \sin x \sinh y = w = u + iv$. Solve:

$$\cos x \cosh y = u, \quad -\sin x \sinh y = v.$$

Square and add:

$$\sin^2 x = \sinh^2 y + 1 - u^2 - v^2.$$

Since $w \neq \pm 1$, solutions exist, with two x in $(-\pi, \pi]$.

Case 1: $w = i$. $u = 0$, $v = 1$:

$$\cos x \cosh y = 0 \implies x = \pm \frac{\pi}{2}.$$

For $x = \frac{\pi}{2}$, $\sinh y = -1 \implies y = -\ln(1 + \sqrt{2})$. For $x = -\frac{\pi}{2}$, $\sinh y = 1 \implies y = \ln(1 + \sqrt{2})$. Solutions: $z_1 = \frac{\pi}{2} - i \ln(1 + \sqrt{2})$, $z_2 = -\frac{\pi}{2} + i \ln(1 + \sqrt{2})$.

Case 2: $w = 2$**.** $u = 2$, $v = 0$:

$$\cos x \cosh y = 2, \quad \sin x \sinh y = 0.$$

Thus, $x = 0$, $\cosh y = 2 \implies y = \pm \ln(2+\sqrt{3})$. Solutions: $z_1 = i\ln(2+\sqrt{3})$, $z_2 = -i\ln(2+\sqrt{3})$.

1.48: Lagrange's Identity and the Cauchy–Schwarz Inequality

Prove Lagrange's identity for complex numbers:

$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 = \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right) - \sum_{1\le k<j\le n} |a_k\overline{b_j} - a_j\overline{b_k}|^2.$$

Use this to deduce a Cauchy–Schwarz inequality for complex numbers.

Solution: We want to prove the identity:

$$\left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 = \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right) - \sum_{1\le k<j\le n} |a_k\overline{b_j} - a_j\overline{b_k}|^2.$$

It is easier to prove the equivalent formulation:

$$\left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{j=1}^{n} |b_j|^2\right) = \left|\sum_{k=1}^{n} a_k \overline{b_k}\right|^2 + \sum_{1\le k<j\le n} |a_k\overline{b_j} - a_j\overline{b_k}|^2.$$

Let's expand the left-hand side (LHS):

$$\begin{aligned}
\text{LHS} &= \left(\sum_{k=1}^{n} a_k\overline{a_k}\right)\left(\sum_{j=1}^{n} b_j\overline{b_j}\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} a_k\overline{a_k}b_j\overline{b_j} \\
&= \sum_{k=1}^{n} |a_k|^2|b_k|^2 + \sum_{k\ne j} |a_k|^2|b_j|^2 \\
&= \sum_{k=1}^{n} |a_k|^2|b_k|^2 + \sum_{1\le k<j\le n} (|a_k|^2|b_j|^2 + |a_j|^2|b_k|^2)
\end{aligned}$$

Now, let's expand the right-hand side (RHS). The first term is:

$$\begin{aligned}\left|\sum_{k=1}^{n} a_k\overline{b_k}\right|^2 &= \left(\sum_{k=1}^{n} a_k\overline{b_k}\right)\overline{\left(\sum_{j=1}^{n} a_j\overline{b_j}\right)} = \left(\sum_{k=1}^{n} a_k\overline{b_k}\right)\left(\sum_{j=1}^{n} \overline{a_j}b_j\right)\\ &= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k\overline{b_k}\overline{a_j}b_j\\ &= \sum_{k=1}^{n}|a_k|^2|b_k|^2 + \sum_{k\neq j} a_k\overline{b_k}\overline{a_j}b_j\\ &= \sum_{k=1}^{n}|a_k|^2|b_k|^2 + \sum_{1\le k<j\le n}(a_k\overline{b_k}\overline{a_j}b_j + a_j\overline{b_j}\overline{a_k}b_k)\end{aligned}$$

The second term on the RHS is:

$$\begin{aligned}\sum_{1\le k<j\le n}|a_k\overline{b_j} - a_j\overline{b_k}|^2 &= \sum_{1\le k<j\le n}(a_k\overline{b_j} - a_j\overline{b_k})\overline{(a_k\overline{b_j} - a_j\overline{b_k})}\\ &= \sum_{1\le k<j\le n}(a_k\overline{b_j} - a_j\overline{b_k})(\overline{a_k}b_j - \overline{a_j}b_k)\\ &= \sum_{1\le k<j\le n}(a_k\overline{b_j}\overline{a_k}b_j - a_k\overline{b_j}\overline{a_j}b_k - a_j\overline{b_k}\overline{a_k}b_j + a_j\overline{b_k}\overline{a_j}b_k)\\ &= \sum_{1\le k<j\le n}(|a_k|^2|b_j|^2 - a_kb_k\overline{a_j}\overline{b_j} - \overline{a_k}\overline{b_k}a_jb_j + |a_j|^2|b_k|^2)\end{aligned}$$

Adding the two expanded terms of the RHS:

$$\begin{aligned}\text{RHS} &= \left(\sum_{k=1}^{n}|a_k|^2|b_k|^2 + \sum_{1\le k<j\le n}(a_k\overline{b_k}\overline{a_j}b_j + a_j\overline{b_j}\overline{a_k}b_k)\right)\\ &\quad + \left(\sum_{1\le k<j\le n}(|a_k|^2|b_j|^2 - a_kb_k\overline{a_j}\overline{b_j} - \overline{a_k}\overline{b_k}a_jb_j + |a_j|^2|b_k|^2)\right)\\ &= \sum_{k=1}^{n}|a_k|^2|b_k|^2 + \sum_{1\le k<j\le n}(|a_k|^2|b_j|^2 + |a_j|^2|b_k|^2)\end{aligned}$$

The cross terms cancel perfectly. Comparing the final expressions for the LHS and RHS, we see they are identical. This proves Lagrange's identity.

To deduce the Cauchy-Schwarz inequality, note that the term $\sum_{1\le k<j\le n}|a_k\overline{b_j} - a_j\overline{b_k}|^2$ is a sum of squares of absolute values, so it must be non-negative. From the original identity, this implies:

$$\left|\sum_{k=1}^{n} a_k\overline{b_k}\right|^2 \le \left(\sum_{k=1}^{n}|a_k|^2\right)\left(\sum_{k=1}^{n}|b_k|^2\right).$$

1.49: Polynomial Identity via DeMoivre's Theorem

(a) By equating imaginary parts in DeMoivre's formula, prove that

$$\sin(n\theta) = \sin\theta\left(\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - + \cdots\right).$$

(b) If $0 < \theta < \pi/2$, prove that

$$\sin((2m+1)\theta) = \sin^{2m+1}\theta \cdot P_m(\cot^2\theta),$$

where P_m is a polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - + \cdots.$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2\left(\frac{\pi k}{2m+1}\right)$ for $k = 1, 2, \ldots, m$.

(c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)}{3},$$

and that the sum of their squares is given by

$$\sum_{k=1}^{m} \cot^4\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

Note. These identities can be used to prove that

$$\sum_{n=1}^{\infty} n^2 = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} n^4 = \frac{\pi^4}{90}.$$

(See Exercises 8.46 and 8.47.)

Solution:

(a) By De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k}\theta (i\sin\theta)^k.$$

Imaginary part:

$$\sin(n\theta) = \sin\theta \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{2j+1} \cot^{n-(2j+1)}\theta.$$

(b) For $n = 2m+1$:

$$\sin((2m+1)\theta) = \sin^{2m+1}\theta \sum_{j=0}^{m} (-1)^j \binom{2m+1}{2j+1} \cot^{2(m-j)}\theta = \sin^{2m+1}\theta P_m(\cot^2\theta).$$

Zeros at $\sin((2m+1)\theta) = 0$, i.e., $\theta_k = \frac{\pi k}{2m+1}$, so $x_k = \cot^2\left(\frac{\pi k}{2m+1}\right)$.

(c) Sum of roots:

$$\frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{m(2m-1)}{3}.$$

Sum of squares uses trigonometric identities, yielding:

$$\sum_{k=1}^{m} \cot^4\left(\frac{\pi k}{2m+1}\right) = \frac{m(2m-1)(4m^2+10m-9)}{45}.$$

1.50: Product Formula for $\sin$

Prove that

$$z^n - 1 = \prod_{k=1}^{n-1} \left(z - e^{2\pi i k/n}\right)$$

for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \quad \text{for } n \geq 2.$$

Solution: The roots of $z^n - 1 = 0$ are $e^{2\pi i k/n}$, $k = 0, \ldots, n-1$. Excluding $z = 1$:

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (z - e^{2\pi i k/n}).$$

At $z = 1$, the left-hand side is n, and:

$$|1 - e^{2\pi i k/n}| = 2\sin\left(\frac{\pi k}{n}\right).$$

Thus:

$$n = 2^{n-1} \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) \implies \prod_{k=1}^{n-1} \sin\left(\frac{\pi k}{n}\right) = \frac{n}{2^{n-1}}.$$