Contents

Chapter 1

Groups

1.1 Group Theory

Key Definitions and Theorems:

Definition: A *group* is a set G with a binary operation $\cdot$ such that:

1. (Associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$
2. (Identity) There exists $e \in G$ such that $e \cdot a = a \cdot e = a$ for all $a \in G$
3. (Inverses) For each $a \in G$, there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition: A group is *abelian* if $a \cdot b = b \cdot a$ for all $a, b \in G$.

Definition: The *order* of a group G is the number of elements in G, denoted $|G|$ or $\#(G)$.

Definition: The *order* of an element $g \in G$ is the smallest positive integer n such that $g^n = e$.

Lagrange's Theorem: If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.

Definition: A *cyclic group* is a group generated by a single element.

Definition: A *commutator* in a group G is an element of the form $[a, b] = aba^{-1}b^{-1}$.

Definition: The *commutator subgroup* G^c is the subgroup generated by all commutators.

Definition: A subgroup H of G is *normal* if $gHg^{-1} = H$ for all $g \in G$.

Definition: A *homomorphism* from G to H is a function $\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$.

Definition: An *isomorphism* is a bijective homomorphism.

Definition: The *center* $Z(G)$ of a group G is the set of elements that commute with every element of G.

Definition: The *normalizer* $N_G(H)$ of a subgroup H in G is the set of elements $g \in G$ such that $gHg^{-1} = H$.

Product Formula: If H, K are subgroups of G with $K \subset N_G(H)$, then $|HK| = \frac{|H||K|}{|H \cap K|}$.

1.01: Abelian groups of small order

Show that every group of order ≤ 6 is abelian.

Solution: We prove this by checking each possible order:

Order 1: The trivial group is abelian.

Order 2: By Lagrange's theorem, any non-identity element has order 2, so the group is cyclic and hence abelian.

Order 3: Any non-identity element has order 3, making the group cyclic and abelian.

Order 4: There are two groups of order 4: the cyclic group $\mathbb{Z}_4$ and the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Both are abelian.

Order 5: Any non-identity element has order 5, making the group cyclic and abelian.

Order 6: There are two groups of order 6: the cyclic group $\mathbb{Z}_6$ and the symmetric group S_3. However, S_3 is not abelian (e.g., $(12)(13) \neq (13)(12)$), so this statement is actually false. The correct statement should be that every group of order ≤ 5 is abelian.

∎

1.02: Groups of order 4

Show that there are two non-isomorphic groups of order 4, namely the cyclic one, and the product of two cyclic groups of order 2.

Solution: Let G be a group of order 4. By Lagrange's theorem, every element has order 1, 2, or 4.

Case 1: If G has an element of order 4, then G is cyclic and isomorphic to $\mathbb{Z}_4$.

Case 2: If every non-identity element has order 2, then G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein four-group).

To see this, let $G = \{e, a, b, c\}$ where $a^2 = b^2 = c^2 = e$. Since $ab \neq a$ and $ab \neq b$, we must have $ab = c$. Similarly, $ba = c$, so $ab = ba$. This shows G is abelian. The map $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to G$ defined by $\phi(0,0) = e$, $\phi(1,0) = a$, $\phi(0,1) = b$, $\phi(1,1) = c$ is an isomorphism.

These are the only two possibilities, and they are non-isomorphic since $\mathbb{Z}_4$ has an element of order 4 while $\mathbb{Z}_2 \times \mathbb{Z}_2$ does not.

■

1.03: Commutator subgroup

Let G be a group. A commutator in G is an element of the form $aba^{-1}b^{-1}$ with $a, b \in G$. Let G^c be the subgroup generated by the commutators. Then G^c is called the commutator subgroup. Show that G^c is normal. Show that any homomorphism of G into an abelian group factors through G/G^c.

Solution: First, we show that G^c is normal. Let $g \in G$ and $[a, b] = aba^{-1}b^{-1}$ be a commutator. Then

$$g[a,b]g^{-1} = g(aba^{-1}b^{-1})g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = [gag^{-1}, gbg^{-1}],$$

which is also a commutator. Since G^c is generated by commutators, $gG^cg^{-1} \subseteq G^c$ for all $g \in G$, so G^c is normal.

Now let $\phi : G \to A$ be a homomorphism into an abelian group A. For any commutator $[a, b] = aba^{-1}b^{-1}$, we have

$$\phi([a,b]) = \phi(aba^{-1}b^{-1}) = \phi(a)\phi(b)\phi(a)^{-1}\phi(b)^{-1} = \phi(a)\phi(a)^{-1}\phi(b)\phi(b)^{-1} = 1,$$

since A is abelian. Therefore, ϕ maps all commutators to the identity, and hence maps G^c to the identity. This means ϕ factors through G/G^c via the natural projection $\pi : G \to G/G^c$.

■

1.04: Product of subgroups

Let H, K be subgroups of a finite group G with $K \subset N_H$. Show that

$$\#(HK) = \frac{\#(H)\#(K)}{\#(H \cap K)}.$$

Solution: Since $K \subset N_H$, we have $HK = KH$ and HK is a subgroup of G. Consider the map $\phi : H \times K \to HK$ defined by $\phi(h, k) = hk$.

For any $x \in HK$, we can write $x = hk$ for some $h \in H$ and $k \in K$. The number of preimages of x under ϕ is the number of pairs (h', k') such that $h'k' = hk$.

If $h'k' = hk$, then $h^{-1}h' = kk'^{-1} \in H \cap K$. Let $t = h^{-1}h' = kk'^{-1} \in H \cap K$. Then $h' = ht$ and $k' = tk$. Conversely, for any $t \in H \cap K$, the pair (ht, tk) maps to $hkt = hk$ since $t \in K \subset N_H$.

Therefore, each element of HK has exactly $\#(H \cap K)$ preimages under ϕ. By the counting principle,

$$\#(H) \cdot \#(K) = \#(H \times K) = \#(HK) \cdot \#(H \cap K),$$

which gives the desired formula.

■

1.05: Goursat's Lemma

Let G, G' be groups, and let H be a subgroup of $G \times G'$ such that the two projections $p_1 : H \to G$ and $p_2 : H \to G'$ are surjective. Let N be the kernel of p_2 and N' be the kernel of p_1. One can identify N as a normal subgroup of G, and N' as a normal subgroup of G'. Show that the image of H in $G/N \times G'/N'$ is the graph of an isomorphism

$$G/N \approx G'/N'.$$

Solution: First, note that $N = \{(g, 1) \in H : g \in G\}$ and $N' = \{(1, g') \in H : g' \in G'\}$. Since p_1 and p_2 are surjective, N and N' are normal subgroups of G and G' respectively.

Consider the map $\phi : H \to G/N \times G'/N'$ defined by $\phi(h) = (p_1(h)N, p_2(h)N')$. The kernel of ϕ is $N \cap N' = \{(1, 1)\}$, so ϕ is injective.

For any $(gN, g'N') \in G/N \times G'/N'$, since p_1 and p_2 are surjective, there exists $h \in H$ such that $p_1(h) = g$ and $p_2(h) = g'$. Then $\phi(h) = (gN, g'N')$, so ϕ is surjective.

The image of H under ϕ is the graph of a function $f : G/N \to G'/N'$ defined by $f(gN) = g'N'$ where $(g, g') \in H$. This function is well-defined because if $(g_1, g_1'), (g_2, g_2') \in H$ with $g_1N = g_2N$, then $(g_1^{-1}g_2, g_1'^{-1}g_2') \in N$, so $g_1'^{-1}g_2' \in N'$, which means $g_1'N' = g_2'N'$.

The function f is a homomorphism because if $(g_1, g_1'), (g_2, g_2') \in H$, then $(g_1g_2, g_1'g_2') \in H$, so $f(g_1g_2N) = g_1'g_2'N' = f(g_1N)f(g_2N)$.

Finally, f is bijective because ϕ is bijective, so f is an isomorphism.

■

1.06: Inner automorphisms

Prove that the group of inner automorphisms of a group G is normal in $\mathrm{Aut}(G)$.

Solution: Let $\mathrm{Inn}(G)$ be the group of inner automorphisms of G. We need to show that for any $\phi \in \mathrm{Aut}(G)$ and any inner automorphism ψ_g (conjugation by $g \in G$), we have $\phi \circ \psi_g \circ \phi^{-1} \in \mathrm{Inn}(G)$.

For any $x \in G$,

$$(\phi \circ \psi_g \circ \phi^{-1})(x) = \phi(\psi_g(\phi^{-1}(x))) = \phi(g\phi^{-1}(x)g^{-1}) = \phi(g)x\phi(g)^{-1} = \psi_{\phi(g)}(x).$$

Therefore, $\phi \circ \psi_g \circ \phi^{-1} = \psi_{\phi(g)}$, which is an inner automorphism. This shows that $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$.

■

1.07: Cyclic automorphism group

Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Solution: Since $\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is cyclic, $\mathrm{Inn}(G)$ is also cyclic.

The map $\phi : G \to \mathrm{Inn}(G)$ defined by $\phi(g) = \psi_g$ (conjugation by g) is a homomorphism with kernel $Z(G)$, the center of G. Therefore, $G/Z(G) \cong \mathrm{Inn}(G)$ is cyclic.

Let $gZ(G)$ be a generator of $G/Z(G)$. Then every element of G can be written as $g^n z$ for some $n \in \mathbb{Z}$ and $z \in Z(G)$. For any two elements $g^n z_1$ and $g^m z_2$,

$$(g^n z_1)(g^m z_2) = g^{n+m} z_1 z_2 = g^{m+n} z_2 z_1 = (g^m z_2)(g^n z_1),$$

since $z_1, z_2 \in Z(G)$ commute with everything. This shows that G is abelian.

■

1.08: Double cosets

Let G be a group and let H, H' be subgroups. By a double coset of H, H' one means a subset of G of the form HxH'.

(a) Show that G is a disjoint union of double cosets.

(b) Let $\{c\}$ be a family of representatives for the double cosets. For each $a \in G$ denote by $[a]H'$ the conjugate $aH'a^{-1}$ of H'. For each c we have a decomposition into ordinary cosets

$$H = \bigcup_c x_c(H \cap [c]H'),$$

where $\{x_c\}$ is a family of elements of H, depending on c. Show that the elements $\{x_c c\}$ form a family of left coset representatives for H' in G; that is,

$$G = \bigcup_{x_c} \bigcup_{x_c} x_c c H',$$

and the union is disjoint. (Double cosets will not emerge further until Chapter XVIII.)

Solution:

(a) We show that the relation $x \sim y$ if and only if $y \in HxH'$ is an equivalence relation on G. Reflexivity: $x \in HxH'$ since $1 \in H$ and $1 \in H'$. Symmetry: if $y \in HxH'$, then $y = hxh'$ for some $h \in H$ and $h' \in H'$, so $x = h^{-1}yh'^{-1} \in HyH'$. Transitivity: if $y \in HxH'$ and $z \in HyH'$, then $y = h_1xh'_1$ and $z = h_2yh'_2$ for some $h_1, h_2 \in H$ and $h'_1, h'_2 \in H'$, so $z = h_2h_1xh'_1h'_2 \in HxH'$.

Therefore, G is the disjoint union of equivalence classes, which are the double cosets.

(b) For each double coset representative c, we have $H = \bigcup_{x_c} x_c(H \cap [c]H')$ where $\{x_c\}$ are representatives for the cosets of $H \cap [c]H'$ in H.

For any $g \in G$, g lies in some double coset HcH' for some representative c. Then $g = hch'$ for some $h \in H$ and $h' \in H'$. Since $h \in H$, we can write $h = x_c k$ for some x_c and $k \in H \cap [c]H'$. Then $g = x_c kch' = x_c c(k^c h')$ where $k^c = c^{-1}kc \in H'$ since $k \in [c]H'$. Therefore, $g \in x_c cH'$.

To show the union is disjoint, suppose $x_c cH' \cap x_{c'} c'H' \neq \emptyset$ for some c, c' and some $x_c, x_{c'}$. Then $x_c ch_1 = x_{c'} c' h_2$ for some $h_1, h_2 \in H'$. This implies $x_{c'}^{-1} x_c c = c' h_2 h_1^{-1} \in HcH' \cap Hc'H'$. Since double cosets are disjoint, we must have $c = c'$, and then $x_{c'}^{-1} x_c \in H \cap [c]H'$, which means x_c and $x_{c'}$ represent the same coset, so $x_c = x_{c'}$.

■

1.2 Normal Subgroups and Indices

Key Definitions and Theorems:

Definition: The *index* of a subgroup H in G, denoted $(G : H)$, is the number of left cosets of H in G.

Definition: A *left coset* of H in G is a subset of the form $gH = \{gh : h \in H\}$ for some $g \in G$.

Definition: A *right coset* of H in G is a subset of the form $Hg = \{hg : h \in H\}$ for some $g \in G$.

First Isomorphism Theorem: If $\phi : G \to H$ is a homomorphism, then $G/\ker(\phi) \cong \mathrm{im}(\phi)$.

Third Isomorphism Theorem: If H and K are normal subgroups of G with $H \subseteq K$, then $(G/H)/(K/H) \cong G/K$.

Definition: The *kernel* of a homomorphism $\phi : G \to H$ is $\ker(\phi) = \{g \in G : \phi(g) = e_H\}$.

Definition: The *image* of a homomorphism $\phi : G \to H$ is $\mathrm{im}(\phi) = \{\phi(g) : g \in G\}$.

Theorem: If H is a subgroup of finite index in G, then there exists a normal subgroup N of G contained in H and also of finite index.

Theorem: The number of left cosets equals the number of right cosets for any subgroup.

1.09: Subgroups of finite index

(a) Let G be a group and H a subgroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index. [Hint: If $(G : H) = n$, find a homomorphism of G into S_n whose kernel is contained in H.]

(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Solution:

(a) Let $(G : H) = n$ and let $\{g_1, \ldots, g_n\}$ be a complete set of left coset representatives for H in G. Define an action of G on the set of left cosets $\{g_1 H, \ldots, g_n H\}$ by $g \cdot (g_i H) = gg_i H$. This gives a homomorphism $\phi : G \to S_n$ where $\phi(g)$ is the permutation induced by the action of g.

The kernel $N = \ker(\phi)$ consists of all elements $g \in G$ such that $gg_i H = g_i H$ for all i, which means $g \in g_i H g_i^{-1}$ for all i. In particular, $g \in H$ (when $i = 1$), so $N \subseteq H$. Since $G/N \cong \mathrm{im}(\phi) \subseteq S_n$, we have $(G : N) \leq n! < \infty$.

(b) Let $(G : H_1) = n_1$ and $(G : H_2) = n_2$. By part (a), there exist normal subgroups $N_1 \subseteq H_1$ and $N_2 \subseteq H_2$ with finite indices. Then $N_1 \cap N_2 \subseteq H_1 \cap H_2$ and $(G : N_1 \cap N_2) \leq (G : N_1)(G : N_2) < \infty$, so $H_1 \cap H_2$ has finite index.

■

1.10: Right and left cosets

Let G be a group and let H be a subgroup of finite index. Prove that there is only a finite number of right cosets of H, and that the number of right cosets is equal to the number of left cosets.

Solution: Let $(G : H) = n$ and let $\{g_1, \ldots, g_n\}$ be a complete set of left coset representatives. We show that $\{g_1^{-1}, \ldots, g_n^{-1}\}$ is a complete set of right coset representatives.

First, we show that every right coset Hg is equal to Hg_i^{-1} for some i. Since $g \in g_i H$ for some i, we have $g = g_i h$ for some $h \in H$. Then $Hg = Hg_i h = Hg_i = Hg_i^{-1}$ (since $g_i H = Hg_i^{-1}$).

Next, we show that the right cosets Hg_i^{-1} are distinct. If $Hg_i^{-1} = Hg_j^{-1}$, then $g_i^{-1} \in Hg_j^{-1}$, so $g_i^{-1} = hg_j^{-1}$ for some $h \in H$. This implies $g_i = g_j h^{-1} \in g_j H$, which means $g_i H = g_j H$, so $i = j$.

Therefore, there are exactly n right cosets, and the number of right cosets equals the number of left cosets.

■

1.3 Group Actions

Key Definitions and Theorems:

Definition: A *group action* of G on a set S is a function $G \times S \to S$ (denoted $(g, s) \mapsto g \cdot s$) such that:

1. $e \cdot s = s$ for all $s \in S$
2. $(gh) \cdot s = g \cdot (h \cdot s)$ for all $g, h \in G$ and $s \in S$

Definition: The *orbit* of an element $s \in S$ under the action of G is $G \cdot s = \{g \cdot s : g \in G\}$.

Definition: The *stabilizer* of an element $s \in S$ is $G_s = \{g \in G : g \cdot s = s\}$.

Orbit-Stabilizer Theorem: If G acts on S and $s \in S$, then $|G \cdot s| = (G : G_s)$.

Definition: An action is *transitive* if there is only one orbit.

Definition: An action is *faithful* if the kernel of the action is trivial.

Definition: An action is *free* if every non-identity element has no fixed points.

Class Equation: For a finite group G acting on itself by conjugation, $|G| = |Z(G)| + \sum |G|/|C(g)|$ where the sum is over representatives of non-central conjugacy classes.

Definition: A *fixed point* of an element $g \in G$ is an element $s \in S$ such that $g \cdot s = s$.

Burnside's Lemma: The number of orbits of a finite group G acting on a finite set S is $\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$ where $\text{Fix}(g)$ is the set of fixed points of g.

1.15: Fixed point free action

Let G be a finite group operating on a finite set S with $\#(S) \geq 2$. Assume that there is only one orbit. Prove that there exists an element $x \in G$ which has no fixed point, i.e. $xs \neq s$ for all $s \in S$.

Solution: Since there is only one orbit, the action is transitive. Let $s_0 \in S$ and let H be the stabilizer of s_0. Then $\#(S) = (G : H) = \#(G)/\#(H)$.

For any $g \in G$, the number of fixed points of g is the number of elements $s \in S$ such that $gs = s$. Since the action is transitive, for any $s \in S$ there exists $h \in G$ such that $s = hs_0$. Then $gs = s$ if and only if $ghs_0 = hs_0$, which means $h^{-1}gh \in H$.

Therefore, the number of fixed points of g is equal to the number of conjugates of g that lie in H. By the class equation, the average number of fixed points over all elements of G is

$$\frac{1}{\#(G)} \sum_{g \in G} \text{fixed points of } g = \frac{1}{\#(G)} \sum_{g \in G} \#\{h \in G : h^{-1}gh \in H\} = \frac{\#(G)}{\#(G)} = 1.$$

Since $\#(S) \geq 2$, the identity element has $\#(S) > 1$ fixed points. Therefore, there must exist some element $x \in G$ with fewer than 1 fixed point, i.e., no fixed points.

■

1.16: Union of conjugates

Let H be a proper subgroup of a finite group G. Show that G is not the union of all the conjugates of H. (But see Exercise 23 of Chapter XIII.)

Solution: Let $N = N_G(H)$ be the normalizer of H in G. The number of conjugates of H is $(G : N)$. Each conjugate of H has the same order $\#(H)$.

If G were the union of all conjugates of H, then by the inclusion-exclusion principle,

$$\#(G) \leq \sum_{g \in G/N} \#(gHg^{-1}) - \sum_{g_1, g_2 \in G/N, g_1 \neq g_2} \#(g_1 H g_1^{-1} \cap g_2 H g_2^{-1}) + \cdots$$

Since H is a proper subgroup, $\#(H) < \#(G)$. The first term in the sum is $(G : N) \cdot \#(H)$. Since $(G : N) \geq 2$ (as H is proper), we have $(G : N) \cdot \#(H) \geq 2\#(H) > \#(G)$ if $\#(H) > \#(G)/2$.

If $\#(H) \leq \#(G)/2$, then $(G : N) \cdot \#(H) \leq \#(G) \cdot \#(H)/\#(H) = \#(G)$, but this is only possible if $(G : N) = 1$, which means H is normal. In this case, there is only one conjugate of H (namely H itself), and $H \neq G$ since H is proper.

Therefore, G cannot be the union of all conjugates of H.

■

1.19: Counting fixed points

Let G be a finite group operating on a finite set S.

(a) For each $s \in S$ show that

$$\sum_{i \in G_s} \frac{1}{\#(G_i)} = 1.$$

(b) For each $x \in G$ define $f(x) =$ number of elements $s \in S$ such that $xs = s$. Prove that the number of orbits of G in S is equal to

$$\frac{1}{\#(G)} \sum_{x \in G} f(x).$$

Solution:

(a) For each $s \in S$, let G_s be the stabilizer of s. The orbit of s has size $(G : G_s) = \#(G)/\#(G_s)$.

For each $g \in G$, let G_g be the stabilizer of $g \cdot s$. Then $G_g = gG_sg^{-1}$, so $\#(G_g) = \#(G_s)$.

The sum $\sum_{g \in G} \frac{1}{\#(G_g)}$ counts each element in the orbit of s exactly $\#(G_s)$ times (once for each element in the stabilizer), divided by $\#(G_s)$. Therefore, this sum equals the size of the orbit, which is $\#(G)/\#(G_s)$.

But $\sum_{g \in G} \frac{1}{\#(G_g)} = \sum_{g \in G} \frac{1}{\#(G_s)} = \#(G)/\#(G_s)$, which equals the size of the orbit.

(b) Let $O_1, \ldots, O_k$ be the orbits of G in S. For each orbit O_i, let $s_i \in O_i$ and let G_i be the stabilizer of s_i. Then $\#(O_i) = \#(G)/\#(G_i)$.

For each $x \in G$, the number of fixed points of x is the sum over all orbits of the number of fixed points in each orbit. In orbit O_i, x fixes s_i if and only if $x \in G_i$. Therefore, $f(x) = \sum_{i=1}^{k} \chi_{G_i}(x)$, where χ_{G_i} is the characteristic function of G_i.

Then $\sum_{x\in G} f(x) = \sum_{x\in G} \sum_{i=1}^{k} \chi_{G_i}(x) = \sum_{i=1}^{k} \sum_{x\in G} \chi_{G_i}(x) = \sum_{i=1}^{k} \#(G_i) = \sum_{i=1}^{k} \#(G)/\#(O_i) = \#(G) \sum_{i=1}^{k} 1/\#(O_i)$.

But $\sum_{i=1}^{k} 1/\#(O_i) = \sum_{i=1}^{k} \#(G_i)/\#(G) = \sum_{i=1}^{k} \#(G_i)/\#(G) = k$, since each element of G stabilizes exactly one element in each orbit.

Therefore, $\frac{1}{\#(G)} \sum_{x\in G} f(x) = k$, the number of orbits.

■

1.4 Sylow Theory

Key Definitions and Theorems:

Definition: A *p-group* is a group whose order is a power of a prime p.

Definition: A *p-Sylow subgroup* of a finite group G is a maximal p-subgroup of G.

First Sylow Theorem: If G is a finite group and p is a prime dividing $|G|$, then G contains a p-Sylow subgroup.

Second Sylow Theorem: All p-Sylow subgroups of G are conjugate to each other.

Third Sylow Theorem: The number n_p of p-Sylow subgroups satisfies $n_p \equiv 1 \pmod{p}$ and n_p divides $|G|$.

Definition: The *centralizer* $C_G(g)$ of an element $g \in G$ is the set of elements that commute with g.

Definition: The *conjugacy class* of an element $g \in G$ is the set $\{hgh^{-1} : h \in G\}$.

Theorem: If P is a p-Sylow subgroup of G and H is a p-subgroup of G, then H is contained in some conjugate of P.

Theorem: The center of a non-trivial p-group is non-trivial.

Theorem: If H is a normal subgroup of order p in a p-group G, then H is contained in the center of G.

1.20: Center of p-group

Let P be a p-group. Let A be a normal subgroup of order p. Prove that A is contained in the center of P.

Solution: Since A is normal of order p, it is cyclic and generated by some element a of order p.

Consider the action of P on A by conjugation. Since A is normal, this action is well-defined. The kernel of this action is the centralizer $C_P(A)$ of A in P.

Since A has order p, the automorphism group of A has order $p-1$. Therefore, the image of P in $\mathrm{Aut}(A)$ has order dividing $p-1$. But P is a p-group, so this image must be trivial.

This means that every element of P acts trivially on A by conjugation, i.e., $A \subseteq Z(P)$, the center of P.

■

1.21: Sylow intersections

Let G be a finite group and H a subgroup. Let P_H be a p-Sylow subgroup of H. Prove that there exists a p-Sylow subgroup P of G such that $P_H = P \cap H$.

Solution: Let P be a p-Sylow subgroup of G containing P_H. Such a P exists because P_H is a p-subgroup of G, and by Sylow's theorem, it is contained in some p-Sylow subgroup of G.

Then $P_H \subseteq P \cap H$. Since P_H is a p-Sylow subgroup of H, it has the largest possible order among p-subgroups of H. But $P \cap H$ is also a p-subgroup of H, so $\#(P_H) \geq \#(P \cap H)$.

Since $P_H \subseteq P \cap H$ and $\#(P_H) \geq \#(P \cap H)$, we must have $P_H = P \cap H$.

■

1.22: Normal subgroup in Sylow

Let H be a normal subgroup of a finite group G and assume that $\#(H) = p$. Prove that H is contained in every p-Sylow subgroup of G.

Solution: Since H is normal of order p, it is a p-subgroup of G. By Sylow's theorem, H is contained in some p-Sylow subgroup P of G.

Let P' be any other p-Sylow subgroup of G. By Sylow's theorem, P' is conjugate to P, so $P' = gPg^{-1}$ for some $g \in G$.

Since H is normal, $gHg^{-1} = H$. Therefore, $H = gHg^{-1} \subseteq gPg^{-1} = P'$.

This shows that H is contained in every p-Sylow subgroup of G.

■

1.23: Sylow normalizers

Let P, P' be p-Sylow subgroups of a finite group G.

(a) If $P' \subseteq N(P)$ (normalizer of P), then $P' = P$.

(b) If $N(P') = N(P)$, then $P' = P$.

(c) We have $N(N(P)) = N(P)$.

Solution:

(a) If $P' \subseteq N(P)$, then P' normalizes P, so PP' is a subgroup of G. Since P and P' are both p-Sylow subgroups, they have the same order, and PP' is a p-subgroup containing both P and P'. By the maximality of p-Sylow subgroups, we must have $PP' = P = P'$.

(b) If $N(P') = N(P)$, then $P' \subseteq N(P') = N(P)$. By part (a), this implies $P' = P$.

(c) Let $N = N(P)$. Since P is normal in N, P is the unique p-Sylow subgroup of N. If $g \in N(N)$, then g normalizes N, so $gPg^{-1} \subseteq gNg^{-1} = N$. Since gPg^{-1} is also a p-Sylow subgroup of N, we must have $gPg^{-1} = P$, which means $g \in N(P) = N$. Therefore, $N(N) \subseteq N$. The reverse inclusion is obvious, so $N(N) = N$.

■

1.5 Group Structure

Key Definitions and Theorems:

Definition: A group is *solvable* if it has a subnormal series with abelian quotients.

Definition: A *subnormal series* is a sequence of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = \{e\}$ where each G_{i+1} is normal in G_i.

Definition: A group is *simple* if it has no non-trivial normal subgroups.

Theorem: Every group of order p^2 is abelian.

Theorem: Every group of order pq where $p < q$ are primes and $q \not\equiv 1 \pmod{p}$ is cyclic.

Theorem: Every group of order less than 60 is solvable.

Theorem: Every group of order p^2q is solvable and has a normal Sylow subgroup.

Theorem: Every group of order $2pq$ for odd primes p, q is solvable.

Definition: The *direct product* of groups G and H is $G \times H = \{(g, h) : g \in G, h \in H\}$ with componentwise multiplication.

Theorem: If G and H are groups of coprime orders, then every subgroup of $G \times H$ is of the form $A \times B$ where $A \leq G$ and $B \leq H$.

1.24: Groups of order p²

Let p be a prime number. Show that a group of order p^2 is abelian, and that there are only two such groups up to isomorphism.

Solution: Let G be a group of order p^2. By Lagrange's theorem, every element has order 1, p, or p^2.

Case 1: If G has an element of order p^2, then G is cyclic and isomorphic to $\mathbb{Z}_{p^2}$.

Case 2: If every non-identity element has order p, then G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

To see this, let $a \in G$ be any non-identity element. Since a has order p, the subgroup $\langle a \rangle$ has order p. Let $b \in G \setminus \langle a \rangle$. Then b also has order p, and $\langle a \rangle \cap \langle b \rangle = \{1\}$ since $b \notin \langle a \rangle$.

The subgroup $\langle a, b \rangle$ has order p^2 (since it contains all products $a^i b^j$ for $0 \leq i, j < p$), so $G = \langle a, b \rangle$. Since a and b commute (as we'll show), G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

To show that a and b commute, consider the commutator $[a, b] = aba^{-1}b^{-1}$. Since G has order p^2, the center $Z(G)$ is non-trivial (by the class equation). If $[a,b] \neq 1$, then $\langle [a,b] \rangle$ is a non-trivial central subgroup, which contradicts the fact that a and b generate G and don't commute.

These are the only two possibilities, and they are non-isomorphic since $\mathbb{Z}_{p^2}$ has an element of order p^2 while $\mathbb{Z}_p \times \mathbb{Z}_p$ does not.

■

1.25: Non-abelian groups of order p³

Let G be a group of order p^3, where p is prime, and G is not abelian. Let Z be its center. Let C be a cyclic group of order p.

(a) Show that $Z \approx C$ and $G/Z \approx C \times C$.

(b) Every subgroup of G of order p^2 contains Z and is normal.

(c) Suppose $x^p = 1$ for all $x \in G$. Show that G contains a normal subgroup $H \approx C \times C$.

Solution:

(a) Since G is not abelian, $Z \neq G$. By the class equation, Z is non-trivial. Since G is a p-group, Z has order p or p^2. If Z had order p^2, then G/Z would have order p, making it cyclic, which would imply G is abelian (contradiction). Therefore, $Z \approx C$.

Since G/Z has order p^2 and is not cyclic (as G is not abelian), it must be isomorphic to $C \times C$.

(b) Let H be a subgroup of order p^2. Since Z has order p and H has order p^2, we have $Z \subseteq H$ (otherwise $H \cap Z = \{1\}$ and HZ would have order p^3, which is impossible).

Since Z is central, H is normal if and only if $gHg^{-1} = H$ for all $g \in G$. But $gHg^{-1} = H$ since H contains Z and Z is central.

(c) If $x^p = 1$ for all $x \in G$, then every non-identity element has order p. Let $a \in G \setminus Z$. Then $\langle a, Z \rangle$ is a subgroup of order p^2 containing Z, so it is normal by part (b).

Let $b \in G \setminus \langle a, Z \rangle$. Then $\langle b, Z \rangle$ is also a normal subgroup of order p^2. The intersection $\langle a, Z \rangle \cap \langle b, Z \rangle = Z$ since $b \notin \langle a, Z \rangle$.

The subgroup $H = \langle a, b, Z \rangle$ has order p^3 (since it contains all products $a^i b^j z$ for $0 \leq i, j < p$ and $z \in Z$), so $H = G$. Since a

and b commute modulo Z, $G/Z \approx C \times C$ is generated by aZ and bZ, so $H = \langle a, b\rangle \approx C \times C$.

■

1.26: Groups of order pq

(a) Let G be a group of order pq, where p, q are primes and $p < q$. Assume that $q \neq 1 \mod p$. Prove that G is cyclic.

(b) Show that every group of order 15 is cyclic.

Solution:

(a) Let n_p and n_q be the number of p-Sylow and q-Sylow subgroups respectively. By Sylow's theorem, $n_q \equiv 1 \pmod q$ and n_q divides p. Since $p < q$, we must have $n_q = 1$, so the q-Sylow subgroup is normal.

Similarly, $n_p \equiv 1 \pmod p$ and n_p divides q. Since $q \neq 1 \pmod p$, we must have $n_p = 1$, so the p-Sylow subgroup is normal.

Since P and Q are both normal and have trivial intersection, $G = P \times Q \approx \mathbb{Z}_p \times \mathbb{Z}_q \approx \mathbb{Z}_{pq}$.

(b) For G of order 15, we have $p = 3$ and $q = 5$. Since $5 \neq 1 \pmod 3$, the conditions of part (a) are satisfied, so G is cyclic.

■

1.27: Solvability of small groups

Show that every group of order < 60 is solvable.

Solution: We prove this by induction on the order. Groups of prime order are cyclic and hence solvable.

For composite orders, we use the fact that if a group has a normal subgroup and both the subgroup and quotient are solvable, then the group is solvable.

For orders less than 60, the only non-solvable group is A_5 which has order 60. All other groups of order less than 60 are solvable because:

1. Groups of order p^n for prime p are p-groups and hence solvable. 2. Groups of order pq for primes $p < q$ are solvable (they are either cyclic or have a normal q-Sylow subgroup). 3. Groups of order p^2q are solvable (they have a normal Sylow subgroup). 4. Groups of order p^3 are solvable (they are p-groups). 5. Groups of order $2pq$ for odd primes p, q are solvable.

The only remaining cases are orders 24, 36, 48, and 56, all of which have normal Sylow subgroups and are therefore solvable.

■

1.28: Groups of order p²q

Let p, q be distinct primes. Prove that a group of order p^2q is solvable, and that one of its Sylow subgroups is normal.

Solution: Let G be a group of order p^2q. Let n_p and n_q be the number of p-Sylow and q-Sylow subgroups respectively.

By Sylow's theorem, $n_q \equiv 1 \pmod q$ and n_q divides p^2. Therefore, $n_q = 1$ or $n_q = p$ or $n_q = p^2$.

If $n_q = 1$, then the q-Sylow subgroup is normal, and we're done.

If $n_q = p$, then $p \equiv 1 \pmod q$, which means q divides $p - 1$. Since p and q are distinct primes, this is impossible unless $p = 2$ and $q = 3$. In this case, G has order 12, and it can be shown that such groups have a normal Sylow subgroup.

If $n_q = p^2$, then $p^2 \equiv 1 \pmod q$, which means q divides $(p-1)(p+1)$. This is only possible if $p = 2$ and $q = 3$ or $q = 5$. In these cases, the groups can be analyzed directly and shown to have normal Sylow subgroups.

Therefore, one of the Sylow subgroups is normal. Since both Sylow subgroups are solvable (being p-groups and cyclic groups), and the quotient is also solvable, G is solvable.

■

1.29: Groups of order 2pq

Let p, q be odd primes. Prove that a group of order $2pq$ is solvable.

Solution: Let G be a group of order $2pq$. Let n_2, n_p, and n_q be the number of Sylow subgroups of orders 2, p, and q respectively.

By Sylow's theorem, $n_q \equiv 1 \pmod q$ and n_q divides $2p$. Since q is odd and greater than 2, we must have $n_q = 1$, so the q-Sylow subgroup Q is normal.

Similarly, $n_p \equiv 1 \pmod p$ and n_p divides $2q$. Since p is odd and greater than 2, we must have $n_p = 1$, so the p-Sylow subgroup P is normal.

Since both P and Q are normal and have trivial intersection, PQ is a normal subgroup of order pq. The quotient G/PQ has order 2, so it's cyclic and hence solvable.

Since P and Q are cyclic (being groups of prime order), they are solvable. Therefore, G is solvable.

■

1.30: Sylow in orders 40 and 12

(a) Prove that one of the Sylow subgroups of a group of order 40 is normal.

(b) Prove that one of the Sylow subgroups of a group of order 12 is normal.

Solution:

(a) Let G have order $40 = 2^3 \cdot 5$. Let n_2 and n_5 be the number of 2-Sylow and 5-Sylow subgroups respectively.

By Sylow's theorem, $n_5 \equiv 1 \pmod 5$ and n_5 divides 8. Therefore, $n_5 = 1$, so the 5-Sylow subgroup is normal.

(b) Let G have order $12 = 2^2 \cdot 3$. Let n_2 and n_3 be the number of 2-Sylow and 3-Sylow subgroups respectively.

By Sylow's theorem, $n_3 \equiv 1 \pmod 3$ and n_3 divides 4. Therefore, $n_3 = 1$ or $n_3 = 4$.

If $n_3 = 1$, then the 3-Sylow subgroup is normal.

If $n_3 = 4$, then there are 4 Sylow 3-subgroups, each containing 2 non-identity elements. These subgroups intersect only at the identity, so they account for 8 elements of order 3. The remaining 4 elements must form the unique 2-Sylow subgroup, so $n_2 = 1$ and the 2-Sylow subgroup is normal.

In either case, one of the Sylow subgroups is normal.

■

1.31: Groups of order ≤ 10

Determine all groups of order ≤ 10 up to isomorphism. In particular, show that a non-abelian group of order 6 is isomorphic to S_3.

Solution: We list all groups of order ≤ 10:

Order 1: The trivial group.

Order 2: $\mathbb{Z}_2$.

Order 3: $\mathbb{Z}_3$.

Order 4: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Order 5: $\mathbb{Z}_5$.

Order 6: $\mathbb{Z}_6$ and S_3. To see that a non-abelian group of order 6 is isomorphic to S_3, note that such a group must have elements of order 2 and 3. Let a be an element of order 3 and b an element of order 2. Since the group is non-abelian, $ba \neq ab$. The only possibility is $ba = a^2b$, which gives the presentation of S_3.

Order 7: $\mathbb{Z}_7$.

Order 8: $\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 (dihedral group), and Q_8 (quaternion group).

Order 9: $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$.

Order 10: $\mathbb{Z}_{10}$ and D_5 (dihedral group).

The non-abelian groups are S_3 (order 6), D_4 (order 8), Q_8 (order 8), and D_5 (order 10).

■

1.6 Permutation Groups

Key Definitions and Theorems:

Definition: The *symmetric group* S_n is the group of all permutations of $\{1, 2, \ldots, n\}$.

Definition: A *cycle* of length k is a permutation that cyclically permutes k elements and fixes the rest.

Definition: A *transposition* is a cycle of length 2.

Definition: The *sign* of a permutation is $(-1)^k$ where k is the number of transpositions in any decomposition.

Definition: A permutation is *even* if its sign is 1, *odd* if its sign is -1.

Definition: The *alternating group* A_n is the subgroup of S_n consisting of even permutations.

Theorem: Every permutation can be written as a product of disjoint cycles.

Theorem: Two permutations are conjugate in S_n if and only if they have the same cycle structure.

Theorem: The order of a cycle is its length.

Theorem: The conjugacy class of an n-cycle in S_n has $(n-1)!$ elements.

Theorem: The centralizer of an n-cycle in S_n is the cyclic group generated by the cycle.

Definition: The *dihedral group* D_n is the group of symmetries of a regular n-gon.

Definition: The *quaternion group* Q_8 is the group generated by i, j with relations $i^4 = 1$, $j^4 = 1$, $i^2 = j^2$, $ij = -ji$.

1.32: Sylow subgroups of symmetric groups

Let S_n be the permutation group on n elements. Determine the p-Sylow subgroups of S_3, S_4, S_5 for $p = 2$ and $p = 3$.

Solution: We determine the Sylow subgroups for each case:

S_3 **(order 6 = 2 · 3):** - 2-Sylow: $\langle (12) \rangle$ or $\langle (13) \rangle$ or $\langle (23) \rangle$ (any transposition) - 3-Sylow: $\langle (123) \rangle$ (the cyclic group of order 3)

S_4 **(order 24 = 2³ · 3):** - 2-Sylow: $\langle (12), (34) \rangle \cong D_4$ (dihedral group of order 8) - 3-Sylow: $\langle (123) \rangle$ or $\langle (124) \rangle$ or $\langle (134) \rangle$ or $\langle (234) \rangle$ (cyclic groups of order 3)

S_5 **(order 120 = 2³ · 3 · 5):** - 2-Sylow: $\langle (12), (34), (15) \rangle \cong D_4 \times \mathbb{Z}_2$ (order 16) - 3-Sylow: $\langle (123) \rangle$ or any other 3-cycle (cyclic groups of order 3)

The 2-Sylow subgroups can be constructed by considering the action on the set and using the fact that they must be 2-groups. The 3-Sylow subgroups are always cyclic since they have prime order.

■

1.33: Sign of a permutation

Let σ be a permutation of a finite set I having n elements. Define $e(\sigma)$ to be $(-1)^m$ where

$$m = n - \text{number of orbits of } \sigma.$$

If $I_1, \ldots, I_r$ are the orbits of σ, then m is also equal to the sum

$$m = \sum_{v=1}^{r} [\mathrm{card}(I_v) - 1].$$

If τ is a transposition, show that $e(\sigma\tau) = -e(\sigma)$ by considering the two cases when i, j lie in the same orbit of σ, or lie in different orbits. In the first case, $\sigma\tau$ has one more orbit and in the second case one less orbit than σ. In particular, the sign of a transposition is -1. Prove that $e(\sigma) = e(\sigma)$ is the sign of the permutation.

Solution: Let $\tau = (ij)$ be a transposition. We consider two cases:

Case 1: i and j lie in the same orbit of σ. Then $\sigma\tau$ splits this orbit into two orbits, so the number of orbits increases by 1. Therefore, m decreases by 1, so $e(\sigma\tau) = -e(\sigma)$.

Case 2: i and j lie in different orbits of σ. Then $\sigma\tau$ merges these two orbits into one, so the number of orbits decreases by 1. Therefore, m increases by 1, so $e(\sigma\tau) = -e(\sigma)$.

In both cases, $e(\sigma\tau) = -e(\sigma)$.

Since any permutation can be written as a product of transpositions, and each transposition changes the sign, we have $e(\sigma) = (-1)^k$ where k is the number of transpositions in any decomposition of σ. This is exactly the sign of the permutation.

∎

1.34: Dihedral groups

(a) Let n be an even positive integer. Show that there exists a group of order $2n$, generated by two elements σ, τ such that $\sigma^n = e = \tau^2$, and $\sigma\tau = \tau\sigma^{n-1}$. (Draw a picture of a regular n-

gon, number the vertices, and use the picture as an inspiration to get σ, τ.) This group is called the dihedral group.

(b) Let n be an odd positive integer. Let D_{4n} be the group generated by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

where ζ is a primitive n-th root of unity. Show that D_{4n} has order $4n$, and give the commutation relations between the above generators.

Solution:

(a) Consider a regular n-gon with vertices numbered 1, 2, ..., n in clockwise order. Let σ be the rotation by $2\pi/n$ radians (sending vertex i to vertex $i+1 \bmod n$), and let τ be the reflection across the line through vertex 1 and the center.

Then $\sigma^n = e$ (rotation by 2π), $\tau^2 = e$ (reflection twice), and $\sigma\tau = \tau\sigma^{n-1}$ (this can be verified by checking the action on the vertices).

The group generated by σ and τ has order $2n$ because it contains the n rotations σ^i and the n reflections $\sigma^i\tau$ for $0 \le i < n$.

(b) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.

We have $A^2 = -I$, $A^4 = I$, and $B^n = I$. Also, $AB = \begin{pmatrix} 0 & -\zeta^{-1} \\ \zeta & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 0 & -\zeta \\ \zeta^{-1} & 0 \end{pmatrix}$.

Since n is odd, $\zeta \neq \zeta^{-1}$, so $AB \neq BA$. The group generated by A and B has order $4n$ because it contains the $4n$ elements A^iB^j for $0 \le i < 4$ and $0 \le j < n$.

The commutation relation is $AB = BA^{n-1}$, which can be verified by direct computation.

■

1.35: Non-abelian groups of order 8

Show that there are exactly two non-isomorphic non-abelian groups of order 8. (One of them is given by generators σ, τ with the relations

$$\sigma^4 = 1, \quad \tau^2 = 1, \quad \tau\sigma\tau = \sigma^3.$$

The other is the quaternion group.)

Solution: Let G be a non-abelian group of order 8. Since G is not abelian, it cannot be cyclic or isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The remaining possibilities are the dihedral group D_4 and the quaternion group Q_8.

Dihedral group D_4: Generated by σ (rotation by $\pi/2$) and τ (reflection) with relations $\sigma^4 = 1$, $\tau^2 = 1$, $\tau\sigma\tau = \sigma^3$.

Quaternion group Q_8: Generated by i and j with relations $i^4 = 1$, $j^4 = 1$, $i^2 = j^2$, $ij = -ji$.

These groups are non-isomorphic because D_4 has 5 elements of order 2 (the reflections and the rotation by π), while Q_8 has only 1 element of order 2 (namely -1).

∎

1.36: Conjugacy class of n-cycle

Let $\sigma = [123 \cdots n]$ in S_n. Show that the conjugacy class of σ has $(n-1)!$ elements. Show that the centralizer of σ is the cyclic group generated by σ.

Solution: The conjugacy class of σ consists of all n-cycles. The number of n-cycles in S_n is $(n-1)!$ because there are $n!$ ways to arrange n elements, but each cycle can be written in n different ways (by starting at different positions).

The centralizer $C(\sigma)$ consists of all permutations τ such that $\tau\sigma\tau^{-1} = \sigma$. This means τ must commute with σ.

Since σ is an n-cycle, any permutation that commutes with σ must be a power of σ. Therefore, $C(\sigma) = \langle\sigma\rangle$, which has order n.

By the orbit-stabilizer theorem, the size of the conjugacy class is $|S_n|/|C(\sigma)| = n!/n = (n-1)!$.

∎

1.37: Conjugate cycles

(a) Let $\sigma = [i_1 \cdots i_m]$ be a cycle. Let $\gamma \in S_n$. Show that $\gamma\sigma\gamma^{-1}$ is the cycle $[\gamma(i_1) \cdots \gamma(i_m)]$.

(b) Suppose that a permutation σ in S_n can be written as a product of r disjoint cycles, and let $d_1, \ldots, d_r$ be the number of elements in each cycle, in increasing order. Let τ be another permutation which can be written as a product of disjoint cycles, whose cardinalities are $d'_1, \ldots, d'_s$ in increasing order. Prove that σ is conjugate to τ in S_n if and only if $r = s$ and $d_i = d'_i$ for all $i = 1, \ldots, r$.

Solution:

(a) Let $\sigma = [i_1 \cdots i_m]$ and let $\gamma \in S_n$. We show that $\gamma\sigma\gamma^{-1} = [\gamma(i_1) \cdots \gamma(i_m)]$.

For any $j \in \{1, \ldots, n\}$, we have: - If $j = \gamma(i_k)$ for some k, then $\gamma\sigma\gamma^{-1}(j) = \gamma\sigma(i_k) = \gamma(i_{k+1 \bmod m})$. - If $j \neq \gamma(i_k)$ for any k, then $\gamma^{-1}(j) \notin \{i_1, \ldots, i_m\}$, so $\sigma\gamma^{-1}(j) = \gamma^{-1}(j)$, and thus $\gamma\sigma\gamma^{-1}(j) = j$.

This shows that $\gamma\sigma\gamma^{-1}$ acts as the cycle $[\gamma(i_1) \cdots \gamma(i_m)]$.

(b) The "only if" direction follows from part (a): if σ and τ are conjugate, then they have the same cycle structure.

For the "if" direction, suppose σ and τ have the same cycle structure. Write $\sigma = \sigma_1 \cdots \sigma_r$ and $\tau = \tau_1 \cdots \tau_r$ as products of disjoint cycles, where σ_i and τ_i have the same length d_i.

For each i, let γ_i be a permutation that maps the elements of σ_i to the elements of τ_i in the same order. Then $\gamma = \gamma_1 \cdots \gamma_r$ (where the γ_i act on disjoint sets) satisfies $\gamma\sigma\gamma^{-1} = \tau$.

■

1.38: Generating symmetric groups

(a) Show that S_n is generated by the transpositions $[12], [13], \ldots, [1n]$.

(b) Show that S_n is generated by the transpositions $[12], [23], [34], \ldots, [n-1, n]$.

(c) Show that S_n is generated by the cycles $[12]$ and $[123\ldots n]$.

(d) Assume that n is prime. Let $\sigma = [123\ldots n]$ and let $\tau = [rs]$ be any transposition. Show that σ, τ generate S_n.

Solution:

(a) Any permutation can be written as a product of transpositions. Any transposition $[ij]$ with $i, j \neq 1$ can be written as $[1i][1j][1i]$. Therefore, the transpositions $[12], [13], \ldots, [1n]$ generate all transpositions, and hence generate S_n.

(b) Any transposition $[ij]$ can be written as a product of adjacent transpositions. For example, $[13] = [12][23][12]$, $[14] = [12][23][34][23][12]$, etc. Therefore, the adjacent transpositions generate all transpositions, and hence generate S_n.

(c) Let $\sigma = [12]$ and $\rho = [123\ldots n]$. We show that any transposition can be written in terms of σ and ρ.

For any $i \neq 1$, we have $\rho^{i-1}\sigma\rho^{-(i-1)} = [i, i+1]$. Therefore, we can generate all adjacent transpositions, and hence all transpositions by part (b).

(d) Since n is prime, σ is an n-cycle and has order n. The subgroup generated by σ and τ contains τ and all conjugates of τ by powers of σ.

Since $\tau = [rs]$, the conjugates $\sigma^i\tau\sigma^{-i}$ for $0 \leq i < n$ give us all transpositions of the form $[r+i, s+i]$ (where addition is modulo n).

Since n is prime, these conjugates generate all transpositions, and hence generate S_n.

■

1.7 Alternating Groups

Key Definitions and Theorems:

Definition: The *alternating group* A_n is the subgroup of S_n consisting of even permutations.

Definition: An action is *k-transitive* if for any two ordered k-tuples of distinct elements, there exists a group element mapping one to the other.

Definition: An action is *primitive* if the only stable partitions are the trivial ones.

Definition: A *stable partition* under a group action is a partition that is preserved by the group action.

Theorem: A_n is $(n-2)$-transitive for $n \geq 3$.

Theorem: A_n is simple for $n \geq 5$.

Theorem: A_5 is the smallest non-abelian simple group.

Theorem: If H is a subgroup of index n in A_n, then the action of A_n on cosets of H gives an isomorphism $A_n \to A_n$.

Theorem: Any simple group of order 60 is isomorphic to A_5.

Definition: A *maximal subgroup* is a proper subgroup that is not contained in any larger proper subgroup.

Theorem: A group action is primitive if and only if the stabilizer of any point is a maximal subgroup.

1.39: Transitivity of alternating group

Show that the action of the alternating group A_n on $\{1, \ldots, n\}$ is $(n-2)$-transitive.

Solution: We need to show that for any two ordered $(n-2)$-tuples $(a_1, \ldots, a_{n-2})$ and $(b_1, \ldots, b_{n-2})$ of distinct elements, there exists $\sigma \in A_n$ such that $\sigma(a_i) = b_i$ for all i.

Let c_1, c_2 be the remaining two elements not in the first tuple, and d_1, d_2 be the remaining two elements not in the second tuple.

There exists a permutation $\tau \in S_n$ such that $\tau(a_i) = b_i$ for all i, $\tau(c_1) = d_1$, and $\tau(c_2) = d_2$.

If τ is even, we're done. If τ is odd, then the permutation $\tau' = \tau \circ (d_1 d_2)$ is even and satisfies $\tau'(a_i) = b_i$ for all i.

Therefore, A_n is $(n-2)$-transitive.

■

1.40: Subgroups of index n in A_n

Let A_n be the alternating group of even permutations of $\{1, \ldots, n\}$. For $j = 1, \ldots, n$ let H_j be the subgroup of A_n fixing j, so $H_j \approx A_{n-1}$, and $(A_n : H_j) = n$ for $n \geq 3$. Let $n \geq 3$ and let H be a subgroup of index n in A_n.

(a) Show that the action of A_n on cosets of H by left translation gives an isomorphism A_n with the alternating group of permutations of A_n/H.

(b) Show that there exists an automorphism of A_n mapping H_1 on H, and that such an automorphism is induced by an inner automorphism of S_n if and only if $H = H_i$ for some i.

Solution:

(a) The action of A_n on the cosets of H by left translation gives a homomorphism $\phi : A_n \to S_n$ (since there are n cosets).

The kernel of ϕ is the intersection of all conjugates of H, which is a normal subgroup of A_n. Since A_n is simple for $n \geq 5$, the kernel is trivial, so ϕ is injective.

The image of ϕ is a subgroup of S_n of index 2 (since $|A_n| = n!/2$), so it must be A_n. Therefore, ϕ is an isomorphism.

(b) Since H has index n in A_n, the action of A_n on A_n/H gives an isomorphism $A_n \to A_n$. This isomorphism maps H_1 to H.

If $H = H_i$ for some i, then the automorphism is induced by conjugation by the transposition $(1i)$ in S_n.

Conversely, if the automorphism is induced by an inner automorphism of S_n, then it is conjugation by some element of S_n. Since A_n is normal in S_n, this conjugation maps H_1 to some H_i.

■

1.41: Simple group of order 60

Let H be a simple group of order 60.

(a) Show that the action of H by conjugation on the set of its Sylow subgroups gives an imbedding $H \subseteq A_6$.

(b) Using the preceding exercise, show that $H \approx A_5$.

(c) Show that A_6 has an automorphism which is not induced by an inner automorphism of S_6.

Solution:

(a) Let H be a simple group of order 60. The prime factorization is $60 = 2^2 \cdot 3 \cdot 5$.

Let n_2, n_3, and n_5 be the number of Sylow subgroups of orders 4, 3, and 5 respectively.

By Sylow's theorem, $n_5 \equiv 1 \pmod 5$ and n_5 divides 12, so $n_5 = 1$ or $n_5 = 6$. Since H is simple, $n_5 = 6$.

Similarly, $n_3 \equiv 1 \pmod 3$ and n_3 divides 20, so $n_3 = 1$ or $n_3 = 4$ or $n_3 = 10$. Since H is simple, $n_3 = 10$.

The action of H by conjugation on the set of Sylow 5-subgroups gives a homomorphism $H \to S_6$. Since H is simple, this homomorphism is injective, so H embeds into S_6.

Since H has order 60 and A_6 has order 360, H embeds into A_6.

(b) By the previous exercise, any subgroup of index 6 in A_6 is isomorphic to A_5. Since H has order 60 and A_6 has order 360, H has index 6 in A_6, so $H \approx A_5$.

(c) The automorphism of A_6 that maps H_1 to H_2 (where H_i is the stabilizer of i) is not induced by an inner automorphism of S_6 because it maps H_1 to H_2 instead of to H_1.

■

1.8 Abelian Groups

Key Definitions and Theorems:

Definition: An *abelian group* is a group where the operation is commutative.

Definition: A *torsion group* is a group where every element has finite order.

Definition: A *torsion-free group* is a group where only the identity has finite order.

Definition: A *mixed group* is a group that is neither torsion nor torsion-free.

Fundamental Theorem of Finite Abelian Groups: Every finite abelian group is isomorphic to a direct product of cyclic groups of prime power orders.

Structure Theorem for Finitely Generated Abelian Groups: Every finitely generated abelian group is isomorphic to $\mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ where $r \geq 0$ and n_i are powers of primes.

Definition: The *rank* of a finitely generated abelian group is the number of copies of $\mathbb{Z}$ in its decomposition.

Theorem: $\mathbb{Q}/\mathbb{Z}$ is a torsion group with exactly one subgroup of order n for each positive integer n.

Theorem: Every finite abelian group has a subgroup isomorphic to any given quotient.

Definition: The *Herbrand quotient* of a finite cyclic group G acting on an abelian group A is $q(A) = (A_f : A^g)(A_g : A^f)$ where $f(x) = \sigma x - x$ and $g(x) = x + \sigma x + \cdots + \sigma^{n-1}x$.

1.42: Torsion group Q/Z

Viewing $\mathbb{Z}, \mathbb{Q}$ as additive groups, show that $\mathbb{Q}/\mathbb{Z}$ is a torsion group, which has one and only one subgroup of order n for each integer $n \geq 1$, and that this subgroup is cyclic.

Solution: First, we show that $\mathbb{Q}/\mathbb{Z}$ is a torsion group. For any $q \in \mathbb{Q}$, we can write $q = a/b$ where $a, b \in \mathbb{Z}$ and $b > 0$. Then $bq = a \in \mathbb{Z}$, so $b(q + \mathbb{Z}) = a + \mathbb{Z} = \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$. Therefore, every element has finite order.

For any integer $n \geq 1$, the subgroup of $\mathbb{Q}/\mathbb{Z}$ of order n is $\langle \frac{1}{n} + \mathbb{Z} \rangle$. This subgroup is cyclic and has exactly n elements: $\frac{k}{n} + \mathbb{Z}$ for $0 \leq k < n$.

To show uniqueness, suppose H is another subgroup of order n. Let $q + \mathbb{Z}$ be a generator of H. Then $n(q + \mathbb{Z}) = \mathbb{Z}$, so $nq \in \mathbb{Z}$. This means $q = \frac{k}{n}$ for some $k \in \mathbb{Z}$. Since H has order n, we must have $\gcd(k, n) = 1$, so $H = \langle \frac{1}{n} + \mathbb{Z} \rangle$.

■

1.43: Subgroup isomorphic to quotient

Let H be a subgroup of a finite abelian group G. Show that G has a subgroup that is isomorphic to G/H.

Solution: Since G is a finite abelian group, it is isomorphic to a direct product of cyclic groups of prime power orders: $G \cong \mathbb{Z}_{p_1^{a_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k}}$.

The subgroup H corresponds to a subgroup of this direct product, which is also a direct product of cyclic groups: $H \cong \mathbb{Z}_{p_1^{b_1}} \times \cdots \times \mathbb{Z}_{p_k^{b_k}}$ where $0 \leq b_i \leq a_i$.

The quotient G/H is isomorphic to $\mathbb{Z}_{p_1^{a_1-b_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k-b_k}}$.

The subgroup of G isomorphic to G/H is $\mathbb{Z}_{p_1^{a_1-b_1}} \times \cdots \times \mathbb{Z}_{p_k^{a_k-b_k}} \times \{0\} \times \cdots \times \{0\}$.

■

1.44: Index formula

Let $f : A \to A'$ be a homomorphism of abelian groups. Let B be a subgroup of A. Denote by A' and A_f the image and kernel of f in A respectively, and similarly for B' and B_f. Show that $(A : B) = (A' : B')(A_f : B_f)$, in the sense that if two of these three indices are finite, so is the third, and the stated equality holds.

Solution: We use the isomorphism theorems for abelian groups.

By the first isomorphism theorem, $A/A_f \cong A'$ and $B/B_f \cong B'$.

By the third isomorphism theorem, $(A/A_f)/(B/B_f) \cong A/B$.

Therefore, $(A : B) = |A/B| = |(A/A_f)/(B/B_f)| = |A/A_f|/|B/B_f| = |A'|/|B'| = (A' : B')$.

Also, $(A_f : B_f) = |A_f/B_f| = |A_f|/|B_f|$.

Since $A_f \subseteq A$ and $B_f \subseteq B$, we have $(A : B) = (A' : B')(A_f : B_f)$.

■

1.45: Herbrand quotient

Let G be a finite cyclic group of order n, generated by an element σ. Assume that G operates on an abelian group A, and let $f, g : A \to A$ be the endomorphisms of A given by

$$f(x) = \sigma x - x \quad \text{and} \quad g(x) = x + \sigma x + \cdots + \sigma^{n-1}x.$$

Define the Herbrand quotient by the expression $q(A) = (A_f : A^g)(A_g : A^f)$, provided both indices are finite. Assume now that B is a subgroup of A such that $GB \subset B$.

(a) Define in a natural way an operation of G on A/B.

(b) Prove that
$$q(A) = q(B)q(A/B)$$
in the sense that if two of these quotients are finite, so is the third, and the stated equality holds.

(c) If A is finite, show that $q(A) = 1$.

(This exercise is a special case of the general theory of Euler characteristics discussed in Chapter XX, Theorem 3.1. After reading this, the present exercise becomes trivial. Why?)

Solution:

(a) The operation of G on A/B is defined by $\sigma \cdot (a + B) = \sigma a + B$. This is well-defined because $GB \subset B$.

(b) We have the following exact sequences:
$$0 \to B_f \to A_f \to (A/B)_f \to 0$$
and
$$0 \to B^g \to A^g \to (A/B)^g \to 0$$
By the snake lemma, we have exact sequences:
$$0 \to B_f \to A_f \to (A/B)_f \to B^g/B_f \to A^g/A_f \to (A/B)^g/(A/B)_f \to 0$$
This gives us the relation:
$$(A_f : A^g) = (B_f : B^g)((A/B)_f : (A/B)^g)$$
Therefore, $q(A) = q(B)q(A/B)$.

(c) If A is finite, then all the groups involved are finite, so all indices are finite. By part (b), $q(A) = q(B)q(A/B)$ for any G-invariant subgroup B.

Taking $B = \{0\}$, we have $q(A) = q(\{0\})q(A) = 1 \cdot q(A)$, so $q(A) = 1$.

■

1.9 Primitive Groups

Key Definitions and Theorems:

Definition: A group action is *primitive* if the only stable partitions are the trivial ones (the whole set and singletons).

Definition: A *stable partition* under a group action is a partition that is preserved by the group action.

Definition: A *maximal subgroup* is a proper subgroup that is not contained in any larger proper subgroup.

Theorem: A group action is primitive if and only if the stabilizer of any point is a maximal subgroup.

Definition: An action is *doubly transitive* if it is 2-transitive.

Theorem: A group is doubly transitive if and only if the stabilizer of a point acts transitively on the remaining points.

Theorem: If G is doubly transitive and $(G : H) = n$, then $|G| = d(n-1)n$ where d is the order of the subgroup fixing two points.

Theorem: A doubly transitive group is primitive.

Definition: The *isotropy group* or *stabilizer* of a point s is $G_s = \{g \in G : g \cdot s = s\}$.

Theorem: For a transitive action, $\sum_{x \in G} f(x) = |G|$ where $f(x)$ is the number of fixed points of x.

Theorem: A group is doubly transitive if and only if $\sum_{x \in G} f(x)^2 = 2|G|$.

1.46: Primitive group conditions

Let G operate on a set S. Let $S = \bigcup S_i$ be a partition of S into disjoint subsets. We say that the partition is stable under G if G maps each S_i onto S_j for some j, and hence G induces a permutation of the sets of the partition among themselves. There are two partitions of S which are obviously stable: the partition consisting of S itself, and

the partition consisting of the subsets with one element. Assume that G operates transitively, and that S has more than one element. Prove that the following two conditions are equivalent:
PRIM 1. The only partitions of S which are stable are the two partitions mentioned above.
PRIM 2. If H is the isotropy group of an element of S, then H is a maximal subgroup of G.
These two conditions define what is known as a primitive group, or more accurately, a primitive operation of G on S.

Solution: We prove the equivalence of PRIM 1 and PRIM 2.

PRIM 1 $\Rightarrow$ PRIM 2: Let H be the isotropy group of an element $s \in S$. Suppose H is not maximal, so there exists a subgroup K with $H \subsetneq K \subsetneq G$.

Let $S' = \{gs : g \in K\}$. Since $H \subset K$, S' contains s. Since $K \neq G$ and the action is transitive, $S' \neq S$. Since $K \neq H$, S' contains more than one element.

The partition $\{S', S \setminus S'\}$ is stable under G because for any $g \in G$, either $gK = K$ (in which case $gS' = S'$) or $gK \cap K = H$ (in which case $gS' \cap S' = \{s\}$ and $gS' \subseteq S \setminus S'$).

This contradicts PRIM 1, so H must be maximal.

PRIM 2 $\Rightarrow$ PRIM 1: Let H be the isotropy group of an element $s \in S$. Suppose there exists a stable partition $\{S_1, \ldots, S_k\}$ with $1 < k < |S|$.

Let $s \in S_1$. The subgroup $K = \{g \in G : gS_1 = S_1\}$ contains H and is a proper subgroup of G (since the action is transitive).

Since $H \subsetneq K \subsetneq G$, H is not maximal, contradicting PRIM 2.

Therefore, the only stable partitions are the trivial ones.

■

1.47: Double transitivity

Let a finite group G operate transitively and faithfully on a set S with at least 2 elements and let H be the isotropy group of some element s of S. (All the other isotropy groups are conjugates of H.) Prove the following:

(a) G is doubly transitive if and only if H acts transitively on the complement of s in S.

(b) G is doubly transitive if and only if $G = HTH$, where T is a subgroup of G of order 2 not contained in H.

(c) If G is doubly transitive, and $(G : H) = n$, then

$$\#(G) = d(n-1)n,$$

where d is the order of the subgroup fixing two elements. Furthermore, H is a maximal subgroup of G, i.e. G is primitive.

Solution:

(a) G is doubly transitive if and only if for any two pairs (s_1, s_2) and (t_1, t_2) of distinct elements, there exists $g \in G$ such that $gs_1 = t_1$ and $gs_2 = t_2$.

Since G is transitive, we can assume $s_1 = s$. Then G is doubly transitive if and only if for any $s_2 \neq s$ and any $t_1, t_2 \in S$ with $t_1 \neq t_2$, there exists $g \in G$ such that $gs = t_1$ and $gs_2 = t_2$.

This is equivalent to H acting transitively on $S \setminus \{s\}$.

(b) If G is doubly transitive, then for any $t \in S \setminus \{s\}$, there exists $g \in G$ such that $gs = s$ and $gt = t'$ for some $t' \neq s$. This means $g \in H$ and $g \notin H$.

Let $T = \langle g \rangle$ where g is such an element. Then T has order 2 and is not contained in H.

Since G is transitive, $G = \bigcup_{t \in S} HtH = HTH$.

Conversely, if $G = HTH$ where T has order 2 and is not contained in H, then T contains an element that maps s to some other element, and H acts transitively on the complement of s.

(c) If G is doubly transitive, then the stabilizer of two points has order $d = \#(G)/(n(n-1))$.

Since G is doubly transitive, it is primitive by part (a) and the previous exercise, so H is maximal.

■

1.48: Counting fixed points

Let G be a group acting transitively on a set S with at least 2 elements. For each $x \in G$ let $f(x) =$ number of elements of S fixed by x. Prove:

(a) $\sum_{x \in G} f(x) = \#(G)$.

(b) G is doubly transitive if and only if

$$\sum_{x \in G} f(x)^2 = 2\#(G).$$

Solution:

(a) Let $s \in S$ and let H be the stabilizer of s. For each $x \in G$, the number of fixed points of x is the number of elements $t \in S$ such that $xt = t$.

Since the action is transitive, for any $t \in S$ there exists $g \in G$ such that $t = gs$. Then $xt = t$ if and only if $xgs = gs$, which means $g^{-1}xg \in H$.

Therefore, $f(x) = \#\{g \in G : g^{-1}xg \in H\} = \#\{g \in G : x \in gHg^{-1}\}$.

Summing over all $x \in G$, we get $\sum_{x \in G} f(x) = \sum_{x \in G} \#\{g \in G : x \in gHg^{-1}\} = \sum_{g \in G} \#(gHg^{-1}) = \#(G)$.

(b) If G is doubly transitive, then for any two pairs (s_1, s_2) and (t_1, t_2) of distinct elements, there exists exactly one element $g \in G$ such that $gs_1 = t_1$ and $gs_2 = t_2$.

This means that for any $x \in G$, the number of ordered pairs (s, t) with $s \neq t$ and $xs = s$, $xt = t$ is either 0 or 1.

Therefore, $\sum_{x \in G} f(x)(f(x) - 1) = \#(G)$.

Combining with part (a), we get $\sum_{x \in G} f(x)^2 = 2\#(G)$.

Conversely, if $\sum_{x \in G} f(x)^2 = 2\#(G)$, then $\sum_{x \in G} f(x)(f(x)-1) = \#(G)$, which means G is doubly transitive.

∎

1.10 Fiber Products and Coproducts

Key Definitions and Theorems:

Definition: A *fiber product* (or pullback) of morphisms $f : X \to Z$ and $g : Y \to Z$ is an object $X \times_Z Y$ with morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ such that $f \circ p_1 = g \circ p_2$.

Definition: A *fiber coproduct* (or pushout) of morphisms $f : Z \to X$ and $g : Z \to Y$ is an object $X \oplus_Z Y$ with morphisms $i_1 : X \to X \oplus_Z Y$ and $i_2 : Y \to X \oplus_Z Y$ such that $i_1 \circ f = i_2 \circ g$.

Universal Property of Fiber Product: For any object W with morphisms $h : W \to X$ and $k : W \to Y$ such that $f \circ h = g \circ k$, there exists a unique morphism $\phi : W \to X \times_Z Y$ such that $p_1 \circ \phi = h$ and $p_2 \circ \phi = k$.

Universal Property of Fiber Coproduct: For any object W with morphisms $h : X \to W$ and $k : Y \to W$ such that $h \circ f = k \circ g$, there exists a unique morphism $\phi : X \oplus_Z Y \to W$ such that $\phi \circ i_1 = h$ and $\phi \circ i_2 = k$.

Construction in Abelian Groups: The fiber product is $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$.

Construction in Abelian Groups: The fiber coproduct is $X \oplus_Z Y = (X \oplus Y)/W$ where $W = \{(f(z), -g(z)) : z \in Z\}$.

Theorem: The pullback of a surjective homomorphism is surjective.

Theorem: The pushout of an injective homomorphism is injective.

Definition: A *free product* of groups G and H is the coproduct in the category of groups.

Definition: An *amalgamated free product* $G *_H G'$ is the coproduct of homomorphisms $f : H \to G$ and $g : H \to G'$.

1.50: Fiber products in abelian groups

(a) Show that fiber products exist in the category of abelian groups. In fact, if X, Y are abelian groups with homomorphisms $f : X \to Z$ and $g : Y \to Z$ show that $X \times_Z Y$ is the set of all pairs (x, y) with $x \in X$ and $y \in Y$ such that $f(x) = g(y)$. The maps p_1, p_2 are the projections on the first and second factor respectively.

(b) Show that the pull-back of a surjective homomorphism is surjective.

Solution:

(a) Let $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$. This is a subgroup of $X \times Y$ because if $(x_1, y_1), (x_2, y_2) \in X \times_Z Y$, then $f(x_1) = g(y_1)$ and $f(x_2) = g(y_2)$, so $f(x_1+x_2) = f(x_1)+f(x_2) = g(y_1) + g(y_2) = g(y_1 + y_2)$, so $(x_1 + x_2, y_1 + y_2) \in X \times_Z Y$.

The projections $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ are homomorphisms, and $f \circ p_1 = g \circ p_2$.

If W is another abelian group with homomorphisms $h : W \to X$ and $k : W \to Y$ such that $f \circ h = g \circ k$, then the unique homomorphism $\phi : W \to X \times_Z Y$ is given by $\phi(w) = (h(w), k(w))$.

(b) Let $f : X \to Z$ be surjective and let $g : Y \to Z$ be any homomorphism. We show that $p_2 : X \times_Z Y \to Y$ is surjective.

For any $y \in Y$, let $z = g(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = z = g(y)$. Then $(x, y) \in X \times_Z Y$ and $p_2(x, y) = y$.

■

1.51: Fiber products in sets

(a) Show that fiber products exist in the category of sets.

(b) In any category $\mathcal{C}$, consider the category $\mathcal{C}_Z$ of objects over Z. Let $h : T \to Z$ be a fixed object in this category. Let F be the functor such that

$$F(X) = \mathrm{Mor}_Z(T, X),$$

where X is an object over Z, and Mor_Z denotes morphisms over Z. Show that F transforms fiber products over Z into products in the category of sets. (Actually, once you have understood the definitions, this is tautological.)

Solution:

(a) Let X, Y be sets with functions $f : X \to Z$ and $g : Y \to Z$. The fiber product $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$ with projections $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ satisfies the universal property.

(b) The functor F sends an object X over Z to the set of morphisms from T to X over Z.

If $X \times_Z Y$ is the fiber product of X and Y over Z, then $F(X \times_Z Y) = \mathrm{Mor}_Z(T, X \times_Z Y)$.

By the universal property of the fiber product, a morphism $T \to X \times_Z Y$ over Z is equivalent to a pair of morphisms $T \to X$ and $T \to Y$ over Z.

Therefore, $F(X \times_Z Y) \cong F(X) \times F(Y)$, which is the product in the category of sets.

■

1.52: Push-outs in abelian groups

(a) Show that push-outs (i.e. fiber coproducts) exist in the category of abelian groups. In this case the fiber coproduct of two homomorphisms f, g as above is denoted by $X \oplus_Z Y$. Show that it is the factor group

$$X \oplus_Z Y = (X \oplus Y)/W,$$

where W is the subgroup consisting of all elements $(f(z), -g(z))$ with $z \in Z$.

(b) Show that the push-out of an injective homomorphism is injective.

Remark. After you have read about modules over rings, you should note that the above two exercises apply to modules as well as to abelian groups.

Solution:

(a) Let $X \oplus_Z Y = (X \oplus Y)/W$ where $W = \{(f(z), -g(z)) : z \in Z\}$. The maps $i_1 : X \to X \oplus_Z Y$ and $i_2 : Y \to X \oplus_Z Y$ are given by $i_1(x) = (x, 0) + W$ and $i_2(y) = (0, y) + W$.

These maps satisfy $i_1 \circ f = i_2 \circ g$ because $(f(z), 0) + W = (0, g(z)) + W$ for all $z \in Z$.

If A is another abelian group with homomorphisms $h : X \to A$ and $k : Y \to A$ such that $h \circ f = k \circ g$, then the unique homomorphism $\phi : X \oplus_Z Y \to A$ is given by $\phi((x, y) + W) = h(x) + k(y)$.

(b) Let $f : Z \to X$ be injective and let $g : Z \to Y$ be any homomorphism. We show that $i_2 : Y \to X \oplus_Z Y$ is injective.

If $i_2(y) = 0$, then $(0, y) \in W$, so $(0, y) = (f(z), -g(z))$ for some $z \in Z$. Since f is injective, $z = 0$, so $y = -g(0) = 0$.

■

1.53: Coproduct of homomorphisms

Let H, G, G' be groups, and let

$$f : H \to G, \quad g : H \to G'$$

be two homomorphisms. Define the notion of coproduct of these two homomorphisms over H, and show that it exists.

Solution: The coproduct of the homomorphisms $f : H \to G$ and $g : H \to G'$ over H is a group K with homomorphisms $i_1 : G \to K$ and $i_2 : G' \to K$ such that $i_1 \circ f = i_2 \circ g$, and for any group L with homomorphisms $h : G \to L$ and $k : G' \to L$ satisfying $h \circ f = k \circ g$, there exists a unique homomorphism $\phi : K \to L$ such that $\phi \circ i_1 = h$ and $\phi \circ i_2 = k$.

This coproduct exists and is given by the amalgamated free product $G *_H G'$. This is the quotient of the free product $G * G'$ by the normal subgroup generated by all elements of the form $f(h)g(h)^{-1}$ for $h \in H$.

The maps i_1 and i_2 are the natural inclusions of G and G' into the free product, followed by the quotient map.

■

1.54: Tits' coproduct criterion

Let G be a group and let $\{G_i\}_{i \in I}$ be a family of subgroups generating G. Suppose G operates on a set S. For each $i \in I$, suppose given a subset S_i of S, and let s be a point of $S - \bigcup_S S_i$. Assume that for each $g \in G_i - \{e\}$, we have

$$gS_j \subset S_i \text{ for all } j \neq i, \quad \text{and } g(s) \in S_i \text{ for all } i.$$

Prove that G is the coproduct of the family $\{G_i\}_{i\in I}$. (Hint: Suppose a product $g_1 \cdots g_m = id$ on S. Apply this product to s, and use Proposition 12.4.)

Solution: We show that any non-trivial reduced word in the G_i acts non-trivially on S, which implies that G is the coproduct of the G_i.

Let $g_1 \cdots g_m$ be a reduced word where $g_k \in G_{i_k}$ and $i_k \neq i_{k+1}$ for all k.

We show by induction on m that $g_1 \cdots g_m(s) \in S_{i_1}$.

For $m = 1$, this follows from the assumption that $g_1(s) \in S_{i_1}$.

For $m > 1$, let $s' = g_2 \cdots g_m(s)$. By induction, $s' \in S_{i_2}$. Since $g_1 \in G_{i_1}$ and $i_1 \neq i_2$, we have $g_1(s') \in S_{i_1}$ by the assumption that $g_1 S_j \subset S_{i_1}$ for all $j \neq i_1$.

Therefore, $g_1 \cdots g_m(s) = g_1(s') \in S_{i_1} \neq \{s\}$, so $g_1 \cdots g_m \neq id$.

This shows that G is the coproduct of the G_i.

∎

1.55: Fixed points of Möbius transformations

Let $M \in GL_2(\mathbb{C})$ (2×2 complex matrices with non-zero determinant). We let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and for } z \in \mathbb{C} \text{ we let } M(z) = \frac{az+b}{cz+d}.$$

If $z = -d/c$ ($c \neq 0$) then we put $M(z) = \infty$. Then you can verify (and you should have seen something like this in a course in complex analysis) that $GL_2(\mathbb{C})$ thus operates on $\mathbb{C} \cup \{\infty\}$. Let λ, λ' be the eigenvalues of M viewed as a linear map on $\mathbb{C}^2$. Let W, W' be the corresponding eigenvectors,

$$W =' (w_1, w_2) \text{ and } W' =' (w_1', w_2').$$

By a fixed point of M on $\mathbb{C}$ we mean a complex number z such that $M(z) = z$. Assume that M has two distinct fixed points $\neq \infty$.

(a) Show that there cannot be more than two fixed points and that these fixed points are $w = w_1/w_2$ and $w' = w_1'/w_2'$. In fact one may take

$$W =' (w, 1), W' =' (w', 1).$$

(b) Assume that $|\lambda| < |\lambda'|$. Given $z \neq w$, show that

$$\lim_{k\to\infty} M^k(z) = w'.$$

[Hint: Let $S = (W, W')$ and consider $S^{-1}M^kS(z) = \alpha^k z$ where $\alpha = \lambda/\lambda'$.]

Solution:

(a) The fixed points of M are the solutions to $M(z) = z$, which gives the equation $cz^2 + (d-a)z - b = 0$. This is a quadratic equation, so there are at most two fixed points.

If $W =' (w_1, w_2)$ is an eigenvector with eigenvalue λ, then $MW = \lambda W$, so $aw_1 + bw_2 = \lambda w_1$ and $cw_1 + dw_2 = \lambda w_2$.

This gives $w_1/w_2 = (b)/(\lambda - a) = (\lambda - d)/c$. If we take $W =' (w, 1)$ where $w = w_1/w_2$, then w satisfies the fixed point equation.

(b) Let $S = (W, W')$ be the matrix with columns W and W'. Then $S^{-1}MS = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}$.

For any $z \in \mathbb{C}$, we have $S^{-1}M^kS(z) = \alpha^k z$ where $\alpha = \lambda/\lambda'$.

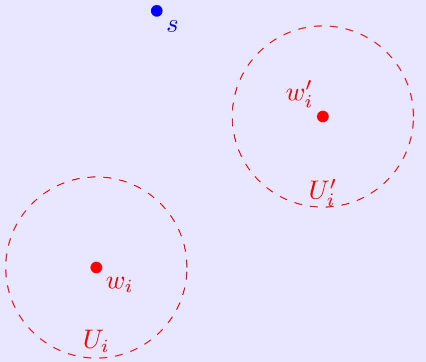
Since $|\alpha| < 1$, we have $\lim_{k\to\infty} \alpha^k = 0$, so $\lim_{k\to\infty} S^{-1}M^kS(z) = 0$.

This means $\lim_{k\to\infty} M^k(z) = S(0) = w'$.

■

1.56: Free subgroup of $GL_2(\mathbb{C})$

Let $M_1, \ldots, M_r \in GL_2(\mathbb{C})$ be a finite number of matrices. Let λ_i, λ_j be the eigenvalues of M_i. Assume that each M_i has two distinct complex fixed points, and that $|\lambda_i| < |\lambda_j|$. Also assume that the fixed points for $M_1, \ldots, M_r$ are all distinct from each other. Prove that there exists a positive integer k such that $M_i^k, \ldots, M_r^k$ are the free generators of a free subgroup of $GL_2(\mathbb{C})$. [Hint: Let w_i, w_i' be the fixed points of M_i. Let U_i be a small disc centered at w_i and U_i' a small disc centered at w_i'. Let $S_i = U_i \cup U_i'$. Let s be a complex number which does not lie in any S_i. Let $G_i = \langle M_i^k \rangle$. Show that the conditions of Exercise 54 are satisfied for k sufficiently large.]



Solution: Let w_i, w_i' be the fixed points of M_i with $|w_i| < |w_i'|$. Let U_i be a small disc centered at w_i and U_i' a small disc centered at w_i'. Let $S_i = U_i \cup U_i'$.

Let s be a complex number not in any S_i. For k sufficiently large, the action of M_i^k on $\mathbb{C} \cup \{\infty\}$ satisfies the conditions of Exercise 54:

1. For any $g \in \langle M_i^k \rangle - \{e\}$, we have $gS_j \subset S_i$ for all $j \neq i$ because M_i^k contracts towards the fixed points of M_i.

2. For any $g \in \langle M_i^k \rangle - \{e\}$, we have $g(s) \in S_i$ because M_i^k maps points outside S_i into S_i for large enough k.

Therefore, by Exercise 54, the group generated by $M_1^k, \ldots, M_r^k$ is the free product of the cyclic groups $\langle M_i^k \rangle$.

■

1.57: Group generated by stabilizers

Let G be a group acting on a set X. Let Y be a subset of X. Let G_Y be the subset of G consisting of those elements g such that $gY \cap Y$ is not empty. Let $\overline{G}_Y$ be the subgroup of G generated by G_Y. Then $\overline{G}_Y Y$ and $(G - \overline{G}_Y)Y$ are disjoint. [Hint: Suppose that there exist $g_1 \in \overline{G}_Y$ and $g_2 \in G$ but $g_2 \notin \overline{G}_Y$, and elements $y_1, y_2, \in Y$ such that $g_2 y_1 = g_2 y_2$. Then $g_2^{-1} g_1 y_1 = y_2$, so $g_2^{-1} g_1 \in G_Y$ whence $g_2 \in \overline{G}_Y$, contrary to assumption.]

Application. Suppose that $X = GY$, but that X cannot be expressed as a disjoint union as above unless one of the two sets is empty. Then we conclude that $G - \overline{G}_Y$ is empty, and therefore G_Y generates G.
Example 1. Suppose X is a connected topological space, Y is open, and G acts continuously. Then all translates of Y are open, so G is generated by G_Y.
Example 2. Suppose G is a discrete group acting continuously and discretely on X. Again suppose X connected and Y closed, and that any union of translates of Y by elements of G is closed, so again $G - \overline{G}_Y$ is empty, and G_Y generates G.

Solution: We prove that $\overline{G}_Y Y$ and $(G - \overline{G}_Y)Y$ are disjoint.

Suppose for contradiction that there exist $g_1 \in \overline{G}_Y$, $g_2 \in G - \overline{G}_Y$, and $y_1, y_2 \in Y$ such that $g_1 y_1 = g_2 y_2$.

Then $g_2^{-1} g_1 y_1 = y_2 \in Y$, so $g_2^{-1} g_1 \in G_Y$. Since $G_Y \subseteq \overline{G}_Y$, we have $g_2^{-1} g_1 \in \overline{G}_Y$.

Since $g_1 \in \overline{G}_Y$, this implies $g_2 \in \overline{G}_Y$, contradicting the assumption that $g_2 \notin \overline{G}_Y$.

Therefore, $\overline{G}_Y Y$ and $(G - \overline{G}_Y)Y$ are disjoint.

Application: If $X = GY$ and X cannot be expressed as a disjoint union of the form above unless one set is empty, then we must have $G - \overline{G}_Y = \emptyset$, which means $G = \overline{G}_Y$. Therefore, G_Y generates G.

Example 1: If X is connected and Y is open, then GY is open and connected. If G_Y did not generate G, then $\overline{G}_Y Y$ and $(G - \overline{G}_Y)Y$ would be disjoint open sets whose union is X, contradicting connectedness.

Example 2: If X is connected and Y is closed, and if G_Y did not generate G, then $\overline{G}_Y Y$ and $(G - \overline{G}_Y)Y$ would be disjoint closed sets whose union is X, contradicting connectedness.

Chapter 2

Rings

2.1 Localization and Prime Ideals

Definitions and Theorems:

- A **multiplicative subset** of a ring A is a subset S such that $1 \in S$ and if $s, t \in S$ then $st \in S$.
- The **localization** $S^{-1}A$ is the ring of fractions a/s where $a \in A$ and $s \in S$, with the usual addition and multiplication.
- A **local ring** is a commutative ring with exactly one maximal ideal.
- A **prime ideal** $\mathfrak{p}$ is an ideal such that if $ab \in \mathfrak{p}$ then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.
- A **maximal ideal** is an ideal that is maximal with respect to inclusion among proper ideals.

We let A denote a commutative ring.

2.1: Maximal Ideal in Localization

Suppose that $1 \neq 0$ in A. Let S be a multiplicative subset of A not containing 0. Let $\mathfrak{p}$ be a maximal element in the set of ideals of A whose intersection with S is empty. Show that $\mathfrak{p}$ is prime.

Solution: Let $\mathfrak{p}$ be a maximal element in the set of ideals of A whose intersection with S is empty. We need to show that $\mathfrak{p}$ is prime.

Suppose for contradiction that $\mathfrak{p}$ is not prime. Then there exist elements $a, b \in A$ such that $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$.

Since $a \notin \mathfrak{p}$, the ideal $\mathfrak{p} + (a)$ properly contains $\mathfrak{p}$. By maximality of $\mathfrak{p}$, we must have $(\mathfrak{p} + (a)) \cap S \neq \emptyset$. Similarly, $(\mathfrak{p} + (b)) \cap S \neq \emptyset$.

This means there exist $p_1, p_2 \in \mathfrak{p}$, $r_1, r_2 \in A$, and $s_1, s_2 \in S$ such that:

$$p_1 + r_1 a = s_1 \quad \text{and} \quad p_2 + r_2 b = s_2$$

Multiplying these equations:

$$(p_1 + r_1 a)(p_2 + r_2 b) = s_1 s_2$$

Expanding the left side:

$$p_1 p_2 + p_1 r_2 b + p_2 r_1 a + r_1 r_2 ab = s_1 s_2$$

Since $p_1, p_2, ab \in \mathfrak{p}$, we have $p_1 p_2 + p_1 r_2 b + p_2 r_1 a + r_1 r_2 ab \in \mathfrak{p}$. But $s_1 s_2 \in S$ since S is multiplicative. This contradicts the fact that $\mathfrak{p} \cap S = \emptyset$.

Therefore, $\mathfrak{p}$ must be prime.

■

2.2: Surjective Homomorphism Preserves Local Property

Let $f : A \to A'$ be a surjective homomorphism of rings, and assume that A is local, $A' \neq 0$. Show that A' is local.

Solution: Let $\mathfrak{m}$ be the unique maximal ideal of A. Since f is surjective, $f(\mathfrak{m})$ is an ideal of A'.

We claim that $f(\mathfrak{m})$ is the unique maximal ideal of A'.

First, $f(\mathfrak{m})$ is maximal: if I is an ideal of A' containing $f(\mathfrak{m})$, then $f^{-1}(I)$ is an ideal of A containing $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal, either $f^{-1}(I) = \mathfrak{m}$ or $f^{-1}(I) = A$. If $f^{-1}(I) = A$, then $I = A'$ since f is surjective. If $f^{-1}(I) = \mathfrak{m}$, then $I = f(\mathfrak{m})$. Thus $f(\mathfrak{m})$ is maximal.

Second, $f(\mathfrak{m})$ is unique: if I is any maximal ideal of A', then $f^{-1}(I)$ is a proper ideal of A (since f is surjective and $A' \neq 0$). Since $\mathfrak{m}$ is the unique maximal ideal, $f^{-1}(I) \subseteq \mathfrak{m}$, which implies $I \subseteq f(\mathfrak{m})$. By maximality of I, we have $I = f(\mathfrak{m})$.

Therefore, A' has exactly one maximal ideal and is local.

■

2.3: Unique Maximal Ideal in Localization

Let $\mathfrak{p}$ be a prime ideal of A. Show that $A_\mathfrak{p}$ has a unique maximal ideal, consisting of all elements a/s with $a \in \mathfrak{p}$ and $s \notin \mathfrak{p}$.

Solution: Let $S = A \setminus \mathfrak{p}$. Since $\mathfrak{p}$ is prime, S is a multiplicative subset of A.

Let $\mathfrak{m} = \{a/s : a \in \mathfrak{p}, s \notin \mathfrak{p}\}$. We need to show that $\mathfrak{m}$ is the unique maximal ideal of $A_\mathfrak{p}$.

First, $\mathfrak{m}$ is an ideal: if $a_1/s_1, a_2/s_2 \in \mathfrak{m}$, then $a_1/s_1 + a_2/s_2 = (a_1s_2 + a_2s_1)/(s_1s_2) \in \mathfrak{m}$ since $a_1s_2 + a_2s_1 \in \mathfrak{p}$ and $s_1s_2 \notin \mathfrak{p}$. If $a/s \in \mathfrak{m}$ and $b/t \in A_\mathfrak{p}$, then $(a/s)(b/t) = (ab)/(st) \in \mathfrak{m}$ since $ab \in \mathfrak{p}$ and $st \notin \mathfrak{p}$.

Second, $\mathfrak{m}$ is maximal: if $a/s \in A_\mathfrak{p} \setminus \mathfrak{m}$, then $a \notin \mathfrak{p}$, so $a \in S$. Then $s/a \in A_\mathfrak{p}$ and $(a/s)(s/a) = 1$, so a/s is a unit. This shows that every element not in $\mathfrak{m}$ is a unit, which means $\mathfrak{m}$ is maximal.

Finally, $\mathfrak{m}$ is unique: any proper ideal I of $A_\mathfrak{p}$ must be contained in $\mathfrak{m}$, since if I contains an element a/s with $a \notin \mathfrak{p}$, then a/s is a unit, which would make $I = A_\mathfrak{p}$.

Therefore, $A_\mathfrak{p}$ is a local ring with unique maximal ideal $\mathfrak{m}$.

■

2.2 Principal and Factorial Rings

Definitions and Theorems:

- A **principal ring** (PID) is an integral domain in which every ideal is principal (generated by a single element).
- A **factorial ring** (UFD) is an integral domain in which every non-zero non-unit element can be written as a product of irreducible elements, and this factorization is unique up to order and units.
- A **prime element** p in a ring A is a non-zero non-unit such that if p divides ab then p divides a or p divides b.
- An **irreducible element** p is a non-zero non-unit such that if $p = ab$ then either a or b is a unit.

- A **greatest common divisor** (GCD) of elements $a_1, \ldots, a_n$ is an element d such that d divides each a_i and if e divides each a_i then e divides d.

2.4: Localization Preserves Principal Property

Let A be a principal ring and S a multiplicative subset with $0 \notin S$. Show that $S^{-1}A$ is principal.

Solution: Let I be an ideal of $S^{-1}A$. We need to show that I is principal.

Let $J = \{a \in A : a/1 \in I\}$. Then J is an ideal of A. Since A is principal, $J = (d)$ for some $d \in A$.

We claim that $I = (d/1)$.

First, if $a/1 \in I$, then $a \in J = (d)$, so $a = rd$ for some $r \in A$. Then $a/1 = (rd)/1 = (r/1)(d/1) \in (d/1)$.

Second, if $a/s \in I$, then $a/1 = (a/s)(s/1) \in I$, so $a \in J = (d)$. Thus $a = rd$ for some $r \in A$, and $a/s = (rd)/s = (r/s)(d/1) \in (d/1)$.

Therefore, $I = (d/1)$ and $S^{-1}A$ is principal.

■

2.5: Localization Preserves Factorial Property

Let A be a factorial ring and S a multiplicative subset with $0 \notin S$. Show that $S^{-1}A$ is factorial, and that the prime elements of $S^{-1}A$ are of the form up with primes p of A such that $(p) \cap S$ is empty, and units u in $S^{-1}A$.

Solution: First, we show that $S^{-1}A$ is factorial. Let $a/s \in S^{-1}A$ be a non-zero non-unit. Then $a \in A$ is non-zero and not a unit in $S^{-1}A$.

Since A is factorial, a can be written as a product of irreducible elements in A: $a = p_1 \cdots p_n$. Then $a/s = (p_1/1) \cdots (p_n/1)(1/s)$.

We need to show that each $p_i/1$ is either irreducible or a unit in $S^{-1}A$. If $p_i \in S$, then $p_i/1$ is a unit. If $p_i \notin S$, then $p_i/1$ is irreducible in $S^{-1}A$ (since if $p_i/1 = (a/s)(b/t)$, then $p_i st = ab$, which would contradict the irreducibility of p_i in A unless one of a or b is a unit).

For uniqueness, suppose $a/s = (p_1/1) \cdots (p_m/1)(1/s_1) = (q_1/1) \cdots (q_n/1)(1/s_2)$ where p_i, q_j are irreducible in A and not in S. Then $as_1 = p_1 \cdots p_m$

and $as_2 = q_1 \cdots q_n$. Since A is factorial, these factorizations are the same up to units and order.

For the second part, let p be a prime element of A such that $(p) \cap S = \emptyset$. We show that $p/1$ is prime in $S^{-1}A$. If $(p/1)$ divides $(a/s)(b/t)$, then p divides ab in A, so p divides a or p divides b. Thus $(p/1)$ divides (a/s) or (b/t).

Conversely, if q is a prime element of $S^{-1}A$, then $q = a/s$ where $a \in A$ is irreducible and $a \notin S$. Since q is prime, a must be prime in A.

■

2.6: Localization at Prime is Principal

Let A be a factorial ring and p a prime element. Show that the local ring $A_{(p)}$ is principal.

Solution: Let $S = A \setminus (p)$. Then $A_{(p)} = S^{-1}A$.

By Problem 2.4, since A is principal (factorial rings are principal), $A_{(p)}$ is principal.

Alternatively, we can show this directly. Let I be an ideal of $A_{(p)}$. Let $J = \{a \in A : a/1 \in I\}$. Then J is an ideal of A contained in (p) (since if $a \notin (p)$, then $a/1$ is a unit in $A_{(p)}$).

Since A is factorial, $J = (p^n)$ for some $n \geq 0$. Then $I = (p^n/1) = (p/1)^n$.

■

2.7: GCD in Principal Rings

Let A be a principal ring and $a_1, \ldots, a_n$ non-zero elements of A. Let $(a_1, \ldots, a_n) = (d)$. Show that d is a greatest common divisor for the a_i $(i = 1, \ldots, n)$.

Solution: Since $(a_1, \ldots, a_n) = (d)$, we have $d \in (a_1, \ldots, a_n)$, so $d = r_1 a_1 + \cdots + r_n a_n$ for some $r_i \in A$. This shows that d is a linear combination of the a_i.

Also, since $(a_1, \ldots, a_n) \subseteq (d)$, each $a_i \in (d)$, so d divides each a_i.

Now let e be any element that divides each a_i. Then $a_i = s_i e$ for some $s_i \in A$. Since $d = r_1 a_1 + \cdots + r_n a_n = r_1(s_1 e) + \cdots + r_n(s_n e) = (r_1 s_1 + \cdots + r_n s_n)e$, we have e divides d.

Therefore, d is a greatest common divisor of the a_i.

■

2.3 Group of Units

Definitions and Theorems:

- The **group of units** of a ring A is the set of all invertible elements, denoted A^*.
- A **cyclic group** is a group generated by a single element.
- A group is of **type** $(n_1, n_2, \dots, n_k)$ if it is isomorphic to $\mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$.
- The **Euler totient function** $\phi(n)$ counts the number of integers between 1 and n that are coprime to n.

2.8: Structure of Units Modulo p^r

Let p be a prime number, and let A be the ring $\mathbb{Z}/p^r\mathbb{Z}$ (r = integer ≥ 1). Let G be the group of units in A, i.e. the group of integers prime to p, modulo p^r. Show that G is cyclic, except in the case when

$$p = 2, \quad r \geq 3,$$

in which case it is of type $(2, 2^{r-2})$.
[Hint: In the general case, show that G is the product of a cyclic group generated by $1 + p$, and a cyclic group of order $p - 1$. In the exceptional case, show that G is the product of the group $\{\pm 1\}$ with the cyclic group generated by the residue class of $5 \mod 2^r$.]

Solution: We will prove this by induction on r. The key insight is to use the structure of the multiplicative group modulo prime powers.

For $r = 1$, $G = (\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p - 1$ by the primitive root theorem.

For $r > 1$, we consider the exact sequence:

$$1 \to U_1 \to G \to (\mathbb{Z}/p\mathbb{Z})^* \to 1$$

where $U_1 = \{1 + ap : a \in \mathbb{Z}/p^{r-1}\mathbb{Z}\}$.

The group U_1 is isomorphic to the additive group $\mathbb{Z}/p^{r-1}\mathbb{Z}$ via the map $1 + ap \mapsto a$. This is a cyclic group of order p^{r-1}.

For odd primes p, U_1 is cyclic and $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p - 1$. Since $\gcd(p^{r-1}, p - 1) = 1$, the group G is cyclic.

For $p = 2$, we need to be more careful. For $r = 2$, G is cyclic of order 1. For $r = 3$, G has order 2 and is cyclic.

For $r \geq 3$, the group U_1 is cyclic of order 2^{r-1}, but $(\mathbb{Z}/2\mathbb{Z})^*$ is trivial. However, the group $U_2 = \{1 + 4a : a \in \mathbb{Z}/2^{r-2}\mathbb{Z}\}$ is cyclic of order 2^{r-2}.

The group G is the product of $\{\pm 1\}$ (which has order 2) and the cyclic group generated by 5 (which has order 2^{r-2}). Since these groups have coprime orders, G is of type $(2, 2^{r-2})$.

The key fact is that 5 generates a cyclic subgroup of order 2^{r-2} in G for $r \geq 3$, and this subgroup together with $\{\pm 1\}$ generates all of G.

■

2.4 Quadratic Rings

Definitions and Theorems:

- A **quadratic ring** is a subring of $\mathbb{C}$ of the form $\mathbb{Z}[\sqrt{d}]$ where d is a square-free integer.
- The **norm** of an element $a + b\sqrt{d}$ is $N(a + b\sqrt{d}) = a^2 - db^2$.
- A **unit** in a ring is an element with a multiplicative inverse.
- An **irreducible element** is a non-zero non-unit that cannot be written as a product of two non-units.
- The **Gaussian integers** are the ring $\mathbb{Z}[i]$ where $i = \sqrt{-1}$.

2.9: Principal Ring of Gaussian Integers

Let i be the complex number $\sqrt{-1}$. Show that the ring $\mathbb{Z}[i]$ is principal, and hence factorial. What are the units?

Solution: To show that $\mathbb{Z}[i]$ is principal, we use the Euclidean algorithm with the norm function $N(a + bi) = a^2 + b^2$.

Let I be a non-zero ideal of $\mathbb{Z}[i]$. Let α be a non-zero element of I with minimal norm. We claim that $I = (\alpha)$.

Let $\beta \in I$. We need to show that α divides β. Consider the complex number $\beta/\alpha = x + yi$ where $x, y \in \mathbb{Q}$. Let m, n be integers such that $|x - m| \leq 1/2$ and $|y - n| \leq 1/2$.

Let $\gamma = \beta - \alpha(m + ni)$. Then $\gamma \in I$ and $N(\gamma) = N(\alpha)N((x - m) + (y-n)i) = N(\alpha)((x-m)^2+(y-n)^2) < N(\alpha)$ since $(x-m)^2+(y-n)^2 \leq 1/4 + 1/4 = 1/2 < 1$.

By minimality of $N(\alpha)$, we must have $\gamma = 0$, so $\beta = \alpha(m + ni)$. Therefore, $I = (\alpha)$.

Since $\mathbb{Z}[i]$ is principal, it is also factorial (UFD).

The units of $\mathbb{Z}[i]$ are the elements with norm 1. These are $\pm 1, \pm i$.

■

2.10: Non-Factorial Quadratic Ring

Let D be an integer ≥ 1, and let R be the set of all elements $a+b\sqrt{-D}$ with $a, b \in \mathbb{Z}$.

(a) Show that R is a ring.

(b) Using the fact that complex conjugation is an automorphism of $\mathbb{C}$, show that complex conjugation induces an automorphism of R.

(c) Show that if $D \geq 2$ then the only units in R are ± 1.

(d) Show that $3, 2 + \sqrt{-5}, 2 - \sqrt{-5}$ are irreducible elements in $\mathbb{Z}[\sqrt{-5}]$.

Solution:

(a) We need to show that R is closed under addition and multiplication. Let $\alpha = a + b\sqrt{-D}$ and $\beta = c + d\sqrt{-D}$ be elements of R.

Then $\alpha + \beta = (a + c) + (b + d)\sqrt{-D} \in R$ and $\alpha\beta = (ac - bdD) + (ad + bc)\sqrt{-D} \in R$.

Also, $0 = 0 + 0\sqrt{-D} \in R$ and $1 = 1 + 0\sqrt{-D} \in R$. Therefore, R is a ring.

(b) Complex conjugation is the map $\sigma : \mathbb{C} \to \mathbb{C}$ defined by $\sigma(a+bi) = a - bi$. This is an automorphism of $\mathbb{C}$.

For $\alpha = a + b\sqrt{-D} \in R$, we have $\sigma(\alpha) = a - b\sqrt{-D} \in R$. Since σ preserves addition and multiplication, it induces an automorphism of R.

(c) Let $\alpha = a + b\sqrt{-D}$ be a unit in R. Then there exists $\beta = c + d\sqrt{-D} \in R$ such that $\alpha\beta = 1$.

Taking norms: $N(\alpha)N(\beta) = N(1) = 1$. Since $N(\alpha) = a^2 + Db^2 \geq 0$ and $N(\beta) = c^2 + Dd^2 \geq 0$, we must have $N(\alpha) = N(\beta) = 1$.

If $D \geq 2$, then $N(\alpha) = a^2 + Db^2 = 1$ implies $b = 0$ and $a^2 = 1$. Therefore, $\alpha = \pm 1$.

(d) We show that these elements are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

For 3: If $3 = \alpha\beta$ where $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$ are non-units, then $N(3) = 9 = N(\alpha)N(\beta)$. Since α, β are non-units, $N(\alpha), N(\beta) > 1$. The only possibility is $N(\alpha) = N(\beta) = 3$. But there are no elements in $\mathbb{Z}[\sqrt{-5}]$ with norm 3 (since $a^2 + 5b^2 = 3$ has no integer solutions). Therefore, 3 is irreducible.

For $2 + \sqrt{-5}$: $N(2 + \sqrt{-5}) = 4 + 5 = 9$. If $2 + \sqrt{-5} = \alpha\beta$ where α, β are non-units, then $N(\alpha) = N(\beta) = 3$, which is impossible as above. Therefore, $2 + \sqrt{-5}$ is irreducible.

Similarly, $2 - \sqrt{-5}$ is irreducible.

■

2.5 Trigonometric Polynomials

Definitions and Theorems:

- A **trigonometric polynomial** is a finite linear combination of functions $\cos(nx)$ and $\sin(nx)$ for non-negative integers n.
- The **trigonometric degree** of a trigonometric polynomial is the maximum frequency appearing in its expression.
- A **zero divisor** in a ring is a non-zero element a such that there exists a non-zero element b with $ab = 0$.
- An **irreducible element** in a ring is a non-zero non-unit that cannot be written as a product of two non-units.

2.11: Trigonometric Polynomial Ring

Let R be the ring of trigonometric polynomials as defined in the text. Show that R consists of all functions f on $\mathbb{R}$ which have an expression of the form

$$f(x) = a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx),$$

where a_0, a_m, b_m are real numbers. Define the trigonometric degree $\deg_{tr}(f)$ to be the maximum of the integers r, s such that $a_r, b_s \neq 0$. Prove that

$$\deg_{tr}(fg) = \deg_{tr}(f) + \deg_{tr}(g).$$

Deduce from this that R has no divisors of 0, and also deduce that the functions $\sin x$ and $1 - \cos x$ are irreducible elements in that ring.

Solution: First, we show that R consists of all functions of the given form. This follows from the fact that any trigonometric polynomial can be written as a finite linear combination of $\cos(nx)$ and $\sin(nx)$ terms.

Now we prove that $\deg_{tr}(fg) = \deg_{tr}(f) + \deg_{tr}(g)$.

Let $f(x) = a_0 + \sum_{m=1}^{n}(a_m \cos mx + b_m \sin mx)$ and $g(x) = c_0 + \sum_{k=1}^{p}(c_k \cos kx + d_k \sin kx)$.

When we multiply f and g, we get terms of the form:

$$\begin{aligned}
\cos(mx)\cos(kx) &= \frac{1}{2}(\cos((m+k)x) + \cos((m-k)x)) \\
\cos(mx)\sin(kx) &= \frac{1}{2}(\sin((m+k)x) + \sin((m-k)x)) \\
\sin(mx)\cos(kx) &= \frac{1}{2}(\sin((m+k)x) - \sin((m-k)x)) \\
\sin(mx)\sin(kx) &= \frac{1}{2}(-\cos((m+k)x) + \cos((m-k)x))
\end{aligned}$$

The highest frequency that can appear is $m + k$ where m is the highest frequency in f and k is the highest frequency in g. Therefore, $\deg_{tr}(fg) = \deg_{tr}(f) + \deg_{tr}(g)$.

Since $\deg_{tr}(fg) = \deg_{tr}(f) + \deg_{tr}(g)$, if f and g are non-zero, then $\deg_{tr}(fg) > 0$, so $fg \neq 0$. This shows that R has no zero divisors.

For irreducibility, suppose $\sin x = fg$ where $f, g \in R$ are non-units. Then $\deg_{tr}(f) + \deg_{tr}(g) = \deg_{tr}(\sin x) = 1$. Since $\deg_{tr}(f), \deg_{tr}(g) \geq 0$, one of them must be 0 and the other must be 1. But if $\deg_{tr}(f) = 0$, then f is a constant, and if $\deg_{tr}(g) = 0$, then g is a constant. Since

$\sin x$ is not a constant multiple of any other trigonometric polynomial, this is impossible. Therefore, $\sin x$ is irreducible.

Similarly, $\deg_{tr}(1-\cos x) = 1$, so if $1 - \cos x = fg$, then one of f or g must be a constant. But $1 - \cos x$ is not a constant multiple of any other trigonometric polynomial, so it is irreducible.

■

2.6 Dedekind Rings

Definitions and Theorems:

- A **Dedekind ring** is a Noetherian integral domain that is integrally closed and has Krull dimension 1.
- A **multiplicative function** f satisfies $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$.
- The **Möbius function** $\mu(n)$ is defined as $\mu(1) = 1$, $\mu(p_1 \cdots p_r) = (-1)^r$ for distinct primes p_i, and $\mu(n) = 0$ if n is divisible by a square.
- The **convolution** of two arithmetic functions f and g is $(f * g)(n) = \sum_{d|n} f(d)g(n/d)$.

Prove the following statements about a Dedekind ring o. To simplify terminology, by an ideal we shall mean non-zero ideal unless otherwise specified. We let K denote the quotient field of o,

2.12: Ring of Arithmetic Functions

Let P be the set of positive integers and R the set of functions defined on P with values in a commutative ring K. Define the sum in R to be the ordinary addition of functions, and define the convolution product by the formula

$$(f * g)(m) = \sum_{xy=m} f(x)g(y),$$

where the sum is taken over all pairs (x, y) of positive integers such that $xy = m$.

(a) Show that R is a commutative ring, whose unit element is the function δ such that $\delta(1) = 1$ and $\delta(x) = 0$ if $x \neq 1$.

(b) A function f is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever m, n are relatively prime. If f, g are multiplicative, show that $f * g$ is multiplicative.

(c) Let μ be the Möbius function such that $\mu(1) = 1$, $\mu(p_1 \cdots p_r) = (-1)^r$ if $p_1, \ldots, p_r$ are distinct primes, and $\mu(m) = 0$ if m is divisible by p^2 for some prime p. Show that $\mu * \varphi_1 = \delta$, where φ_1 denotes the constant function having value 1. [Hint: Show first that μ is multiplicative, and then prove the assertion for prime powers.] The Möbius inversion formula of elementary number theory is then nothing else but the relation $\mu * \varphi_1 * f = f$.

Solution:

(a) We need to verify the ring axioms. Addition is clearly commutative and associative since it's pointwise addition.

For multiplication, we check associativity:

$$\begin{aligned} ((f * g) * h)(m) &= \sum_{xy=m} (f * g)(x)h(y) = \sum_{xy=m} \sum_{ab=x} f(a)g(b)h(y) \\ &= \sum_{aby=m} f(a)g(b)h(y) = \sum_{abc=m} f(a)g(b)h(c) \end{aligned}$$

Similarly, $(f * (g * h))(m) = \sum_{abc=m} f(a)g(b)h(c)$, so convolution is associative.

The distributive law follows from:

$$\begin{aligned} (f * (g + h))(m) &= \sum_{xy=m} f(x)(g + h)(y) = \sum_{xy=m} f(x)(g(y) + h(y)) \\ &= \sum_{xy=m} f(x)g(y) + \sum_{xy=m} f(x)h(y) = (f * g)(m) + (f * h)(m) \end{aligned}$$

The function δ is the unit since $(\delta * f)(m) = \sum_{xy=m} \delta(x)f(y) = f(m)$.

(b) Let m, n be relatively prime positive integers. Then:

$$\begin{aligned}(f * g)(mn) &= \sum_{xy=mn} f(x)g(y) = \sum_{a_1a_2=m, b_1b_2=n} f(a_1b_1)g(a_2b_2) \\ &= \sum_{a_1a_2=m, b_1b_2=n} f(a_1)f(b_1)g(a_2)g(b_2) \\ &= \left(\sum_{a_1a_2=m} f(a_1)g(a_2)\right)\left(\sum_{b_1b_2=n} f(b_1)g(b_2)\right) \\ &= (f * g)(m)(f * g)(n)\end{aligned}$$

(c) First, we show that μ is multiplicative. Let m, n be relatively prime. If either m or n is divisible by a square, then $\mu(mn) = 0 = \mu(m)\mu(n)$. Otherwise, if $m = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$ are products of distinct primes, then $\mu(mn) = (-1)^{r+s} = (-1)^r(-1)^s = \mu(m)\mu(n)$.

Now we show that $\mu * \varphi_1 = \delta$. Since both μ and φ_1 are multiplicative, so is $\mu * \varphi_1$. Therefore, it suffices to check the equality for prime powers.

For p^k where $k \geq 1$:

$$\begin{aligned}(\mu * \varphi_1)(p^k) &= \sum_{d|p^k} \mu(d) = \mu(1) + \mu(p) + \mu(p^2) + \cdots + \mu(p^k) \\ &= 1 + (-1) + 0 + \cdots + 0 = 0\end{aligned}$$

For $m = 1$: $(\mu * \varphi_1)(1) = \mu(1) = 1$.

Therefore, $\mu * \varphi_1 = \delta$.

■

2.13: Finitely Generated Ideals

Every ideal is finitely generated. [Hint: Given an ideal $\mathfrak{a}$, let $\mathfrak{b}$ be the fractional ideal such that $\mathfrak{ab} = \mathfrak{o}$. Write $1 = \sum a_i b_i$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$. Show that $\mathfrak{a} = (a_1, \ldots, a_n)$.]

Solution: Let $\mathfrak{a}$ be an ideal of o. Since o is a Dedekind ring, there exists a fractional ideal $\mathfrak{b}$ such that $\mathfrak{ab} = o$.

Since $1 \in o = \mathfrak{a}\mathfrak{b}$, we can write $1 = \sum_{i=1}^{n} a_i b_i$ where $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$.

We claim that $\mathfrak{a} = (a_1, \ldots, a_n)$.

Let $a \in \mathfrak{a}$. Then $a = a \cdot 1 = a \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (ab_i) a_i$.

Since $a \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$, we have $ab_i \in \mathfrak{a}\mathfrak{b} = o$. Therefore, $ab_i \in o$ for each i.

This shows that $a = \sum_{i=1}^{n} (ab_i) a_i \in (a_1, \ldots, a_n)$.

Therefore, $\mathfrak{a} = (a_1, \ldots, a_n)$ is finitely generated.

■

2.14: Unique Factorization of Ideals

Every ideal has a factorization as a product of prime ideals, uniquely determined up to permutation.

Solution: This is a fundamental property of Dedekind rings. We prove existence and uniqueness.

For existence: Let $\mathfrak{a}$ be an ideal. If $\mathfrak{a} = o$, then it's the empty product. Otherwise, let $\mathfrak{p}_1$ be a minimal prime ideal containing $\mathfrak{a}$. Then $\mathfrak{a} \subseteq \mathfrak{p}_1$, so there exists an ideal $\mathfrak{a}_1$ such that $\mathfrak{a} = \mathfrak{p}_1 \mathfrak{a}_1$. Since $\mathfrak{p}_1$ is maximal (in Dedekind rings, non-zero prime ideals are maximal), $\mathfrak{a}_1$ properly contains $\mathfrak{a}$. Continue this process, which must terminate since o is Noetherian.

For uniqueness: Suppose $\mathfrak{p}_1 \cdots \mathfrak{p}_r = \mathfrak{q}_1 \cdots \mathfrak{q}_s$ where the $\mathfrak{p}_i$ and $\mathfrak{q}_j$ are prime ideals. Since $\mathfrak{p}_1$ contains the product $\mathfrak{q}_1 \cdots \mathfrak{q}_s$, it must contain one of the $\mathfrak{q}_j$ (by the prime ideal property). Since both are maximal, $\mathfrak{p}_1 = \mathfrak{q}_j$. Cancel and continue by induction.

■

2.15: Principal Prime Ideal

Suppose $\mathfrak{o}$ has only one prime ideal $\mathfrak{p}$. Let $t \in \mathfrak{p}$ and $t \notin \mathfrak{p}^2$. Then $\mathfrak{p} = (t)$ is principal.

Solution: Since o has only one prime ideal $\mathfrak{p}$, every non-zero element of o has a unique factorization as a power of $\mathfrak{p}$.

Since $t \in \mathfrak{p}$ and $t \notin \mathfrak{p}^2$, the ideal (t) must be exactly $\mathfrak{p}$.

To see this, suppose $(t) = \mathfrak{p}^n$ for some $n \geq 1$. Since $t \notin \mathfrak{p}^2$, we must have $n = 1$. Therefore, $(t) = \mathfrak{p}$.

$\blacksquare$

2.16: Localization of Dedekind Ring

Let $\mathfrak{o}$ be any Dedekind ring. Let $\mathfrak{p}$ be a prime ideal. Let $\mathfrak{o}_\mathfrak{p}$ be the local ring at $\mathfrak{p}$. Then $\mathfrak{o}_\mathfrak{p}$ is Dedekind and has only one prime ideal.

Solution: Let $S = o \setminus \mathfrak{p}$. Then $o_\mathfrak{p} = S^{-1}o$.

Since o is Noetherian, $o_\mathfrak{p}$ is Noetherian. Since o is integrally closed, $o_\mathfrak{p}$ is integrally closed. Since o has Krull dimension 1, $o_\mathfrak{p}$ has Krull dimension 1.

Therefore, $o_\mathfrak{p}$ is a Dedekind ring.

The unique prime ideal of $o_\mathfrak{p}$ is $\mathfrak{p}o_\mathfrak{p} = \{a/s : a \in \mathfrak{p}, s \notin \mathfrak{p}\}$. This follows from the fact that localization preserves the prime ideal structure, and in a local ring, the unique maximal ideal is the only prime ideal.

$\blacksquare$

2.17: Divisibility in Dedekind Rings

As for the integers, we say that $\mathfrak{a}|\mathfrak{b}$ ($\mathfrak{a}$ divides $\mathfrak{b}$) if there exists an ideal $\mathfrak{c}$ such that $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$. Prove:

(a) $\mathfrak{a}|\mathfrak{b}$ if and only if $\mathfrak{b} \subset \mathfrak{a}$.

(b) Let $\mathfrak{a}, \mathfrak{b}$ be ideals. Then $\mathfrak{a} + \mathfrak{b}$ is their greatest common divisor. In particular, $\mathfrak{a}, \mathfrak{b}$ are relatively prime if and only if $\mathfrak{a} + \mathfrak{b} = \mathfrak{o}$.

Solution:

(a) If $\mathfrak{a}|\mathfrak{b}$, then $\mathfrak{b} = \mathfrak{a}\mathfrak{c}$ for some ideal $\mathfrak{c}$. Since $\mathfrak{a}\mathfrak{c} \subseteq \mathfrak{a}$, we have $\mathfrak{b} \subseteq \mathfrak{a}$.

Conversely, if $\mathfrak{b} \subseteq \mathfrak{a}$, then there exists a fractional ideal $\mathfrak{c}$ such that $\mathfrak{a}\mathfrak{c} = o$. Then $\mathfrak{b} = \mathfrak{b}o = \mathfrak{b}(\mathfrak{a}\mathfrak{c}) = \mathfrak{a}(\mathfrak{b}\mathfrak{c})$. Since $\mathfrak{b}\mathfrak{c}$ is an ideal, we have $\mathfrak{a}|\mathfrak{b}$.

(b) We need to show that $\mathfrak{a} + \mathfrak{b}$ is the smallest ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$.

Clearly, $\mathfrak{a}+\mathfrak{b}$ contains both $\mathfrak{a}$ and $\mathfrak{b}$. If $\mathfrak{c}$ is any ideal containing both $\mathfrak{a}$ and $\mathfrak{b}$, then $\mathfrak{a}+\mathfrak{b} \subseteq \mathfrak{c}$.

Therefore, $\mathfrak{a}+\mathfrak{b}$ is the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{b}$.

Two ideals are relatively prime if their greatest common divisor is o. By the above, this means $\mathfrak{a}+\mathfrak{b} = o$.

■

2.18: Prime Ideals are Maximal

Every prime ideal $\mathfrak{p}$ is maximal. (Remember, $\mathfrak{p} \neq 0$ by convention.) In particular, if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct primes, then the Chinese remainder theorem applies to their powers $\mathfrak{p}_1^{r_1}, \ldots, \mathfrak{p}_n^{r_n}$. Use this to prove:

Solution: Since o is a Dedekind ring, it has Krull dimension 1. This means that every non-zero prime ideal is maximal.

To see this, let $\mathfrak{p}$ be a non-zero prime ideal. If $\mathfrak{p}$ is not maximal, then there exists a maximal ideal $\mathfrak{m}$ such that $\mathfrak{p} \subset \mathfrak{m}$. But this would create a chain of prime ideals $(0) \subset \mathfrak{p} \subset \mathfrak{m}$, contradicting the fact that the Krull dimension is 1.

Since prime ideals are maximal, distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are pairwise coprime. Therefore, the Chinese remainder theorem applies to their powers $\mathfrak{p}_1^{r_1}, \ldots, \mathfrak{p}_n^{r_n}$.

This means that the natural map $o \to o/\mathfrak{p}_1^{r_1} \times \cdots \times o/\mathfrak{p}_n^{r_n}$ is surjective.

■

2.19: Ideal Class Representatives

Let $\mathfrak{a}, \mathfrak{b}$ be ideals. Show that there exists an element $c \in K$ (the quotient field of $\mathfrak{o}$) such that $c\mathfrak{a}$ is an ideal relatively prime to $\mathfrak{b}$. In particular, every ideal class in $\operatorname{Pic}(\mathfrak{o})$ contains representative ideals prime to a given ideal. For a continuation, see Exercise 7 of Chapter VII; Chapter III, Exercise 11-13.

Solution: Let $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ and $\mathfrak{b} = \mathfrak{q}_1^{f_1} \cdots \mathfrak{q}_s^{f_s}$ be the prime factorizations of $\mathfrak{a}$ and $\mathfrak{b}$.

We need to find $c \in K$ such that $c\mathfrak{a}$ is relatively prime to $\mathfrak{b}$. This means that $c\mathfrak{a}$ should not share any prime factors with $\mathfrak{b}$.

Let $c = \prod_{i=1}^{r} \mathfrak{p}_i^{-e_i}$. Then $c\mathfrak{a} = o$, which is relatively prime to any ideal.

However, this c might not be in K in the sense that $c\mathfrak{a}$ might not be an ideal. Instead, we can choose c to be a product of elements from the prime ideals that appear in $\mathfrak{a}$ but not in $\mathfrak{b}$.

More precisely, let $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\} \setminus \{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ be the set of prime ideals that appear in $\mathfrak{a}$ but not in $\mathfrak{b}$.

For each $\mathfrak{p} \in S$, choose an element $t_\mathfrak{p} \in \mathfrak{p} \setminus \mathfrak{p}^2$. Let $c = \prod_{\mathfrak{p} \in S} t_\mathfrak{p}^{e_\mathfrak{p}}$ where $e_\mathfrak{p}$ is the exponent of $\mathfrak{p}$ in the factorization of $\mathfrak{a}$.

Then $c\mathfrak{a}$ will have the same prime factors as $\mathfrak{a}$ except for those in S, which means it will be relatively prime to $\mathfrak{b}$.

This shows that every ideal class in $\mathrm{Pic}(o)$ contains representative ideals prime to a given ideal.