

# **Study Guide for Apostol's Mathematical Analysis**

With Very Comprehensive Problem Solutions  
**With Problem Statements and Definitions**

November 16, 2025

Solutions compiled for mathematical analysis students

# Contents

<b>1</b>	<b>The Real and Complex Number Systems</b>	<b>20</b>
1.I	Integers . . . . .	20
1.1:	No Largest Prime . . . . .	20
1.2:	Algebraic Identity . . . . .	21
1.3:	Mersenne Primes . . . . .	22
1.4:	Fermat Primes . . . . .	23
1.5:	Fibonacci Numbers Formula . . . . .	23
1.6:	Well-Ordering Principle . . . . .	25
1.II	Rational and Irrational Numbers . . . . .	25
1.7:	Decimal Expansion to Rational . . . . .	27
1.8:	Decimal Expansion Ending in Zeroes . . . . .	28
1.9:	Irrationality of $\sqrt{2} + \sqrt{3}$ . . . . .	29
1.10:	Rational Functions of Irrational Numbers . . . . .	30
1.11:	Irrational Numbers Between 0 and $x$ . . . . .	31
1.12:	Fraction Between Two Fractions . . . . .	32
1.13:	$\sqrt{2}$ Between Fractions . . . . .	33
1.14:	Irrationality of $\sqrt{n-1} + \sqrt{n+1}$ . . . . .	34
1.15:	Approximation by Rational Numbers . . . . .	35
1.16:	Infinitely Many Rational Approximations . . . . .	36
1.17:	Factorial Representation of Rationals (Precise Form) . . . . .	38
1.III	Upper Bounds . . . . .	40
1.18:	Uniqueness of Supremum and Infimum . . . . .	41
1.19:	Finding Supremum and Infimum . . . . .	42
1.20:	Comparison Property for Suprema . . . . .	45
1.21:	Product of Suprema . . . . .	46
1.22:	Representation of Rationals in Base $k$ . . . . .	47
1.IV	Inequalities and Identities . . . . .	48
1.23:	Lagrange's Identity . . . . .	49
1.24:	A Holder-type Inequality . . . . .	51

1.25: Minkowski's Inequality . . . . .	52
1.26: Chebyshev's Sum Inequality . . . . .	53
1.V Complex Numbers . . . . .	54
1.27: Express Complex Numbers in $a + bi$ Form . . . . .	54
1.28: Solve Complex Equations . . . . .	55
1.29: Basic Identities for Complex Conjugates . . . . .	57
1.30: Geometric Descriptions of Complex Sets . . . . .	58
1.31: Equilateral Triangle on the Unit Circle . . . . .	60
1.32: Inequality with Complex Numbers . . . . .	61
1.33: Equality Condition for Complex Difference . . . . .	62
1.34: Complex Circle in the Plane . . . . .	63
1.35: Argument of a Complex Number via Arctangent . . . . .	64
1.36: Pseudo-Ordering on Complex Numbers . . . . .	65
1.37: Order Axioms and Lexicographic Ordering on $\mathbb{R}^2$ . . . . .	67
1.38: Argument of a Quotient Using Theorem 1.48 . . . . .	68
1.39: Logarithm of a Quotient Using Theorem 1.54 . . . . .	68
1.40: Roots of Unity and Polynomial Identity . . . . .	69
1.41: Inequalities and Boundedness of $\cos z$ . . . . .	69
1.42: Complex Exponential via Real and Imaginary Parts . . . . .	70
1.43: Logarithmic Identities for Complex Powers . . . . .	70
1.44: Conditions for De Moivre's Formula . . . . .	71
1.45: Deriving Trigonometric Identities from De Moivre's Theorem . . . . .	72
1.46: Tangent of Complex Numbers . . . . .	73
1.47: Solving Cosine Equation . . . . .	73
1.48: Lagrange's Identity and the Cauchy-Schwarz In- equality . . . . .	75
1.49: Polynomial Identity via DeMoivre's Theorem . . . . .	77
1.50: Product Formula for $\sin$ . . . . .	79
1.VI Solving and Proving Techniques . . . . .	80
<b>2 Some Basic Notions of Set Theory</b>	<b>85</b>
2.I Ordered Pairs, Relations, and Functions . . . . .	85
2.1: Equality of Ordered Pairs . . . . .	87
2.2: Properties of Relations . . . . .	89
2.3: Composition and Inversion of Functions . . . . .	90
2.4: Associativity of Function Composition . . . . .	93
2.II Set Operations, Images, and Injectivity . . . . .	94
2.5: Set-Theoretic Identities . . . . .	96
2.6: Image of Unions and Intersections . . . . .	98
2.7: Inverse Image Laws . . . . .	99

2.8: Image of Preimage and Surjectivity . . . . .	99
2.9: Equivalent Conditions for Injectivity . . . . .	100
2.10: Subset Transitivity . . . . .	100
2.III Cardinality and Countability . . . . .	101
2.11: Finite Set Bijection Implies Equal Size . . . . .	104
2.12: Infinite Sets Contain Countable Subsets . . . . .	104
2.13: Infinite Set Similar to a Proper Subset . . . . .	105
2.14: Removing Countable from Uncountable . . . . .	106
2.15: Algebraic Numbers are Countable . . . . .	106
2.16: Power Set of Finite Set . . . . .	107
2.17: Real Functions vs Real Numbers . . . . .	107
2.18: Binary Sequences are Uncountable . . . . .	108
2.19: Countability of Specific Sets . . . . .	108
2.20: Countable Support for Real Function . . . . .	109
2.21: Fallacy in Countability of Intervals . . . . .	111
2.IV Additive Set Functions . . . . .	112
2.22: Additive Set Functions . . . . .	112
2.23: Solving for Total Measure from Functional Equations	114
2.V Solving and Proving Techniques . . . . .	115
<b>3 Elements of Point Set Topology</b>	<b>119</b>
3.I Open and Closed Sets in $\mathbb{R}^1$ and $\mathbb{R}^2$ . . . . .	119
3.1: Open and Closed Intervals . . . . .	122
3.2: Accumulation Points and Set Properties . . . . .	123
3.3: Accumulation Points and Set Properties in $\mathbb{R}^2$ . . .	124
3.4: Rational and Irrational Elements in Open Sets . . .	126
3.5: Open and Closed Sets in $\mathbb{R}^1$ and $\mathbb{R}^2$ . . . . .	126
3.6: Closed Sets as Intersection of Open Sets . . . . .	129
3.7: Structure of Bounded Closed Sets in $\mathbb{R}^1$ . . . . .	129
3.II Open and Closed Sets in $\mathbb{R}^n$ . . . . .	130
3.8: Open Balls and Intervals in $\mathbb{R}^n$ . . . . .	131
3.9: Interior of a Set is Open . . . . .	131
3.10: Interior as Union of Open Subsets . . . . .	132
3.11: Interior of Intersection and Union . . . . .	132
3.12: Properties of Derived Set and Closure . . . . .	133
3.13: Closure under Intersection of Sets . . . . .	134
3.14: Properties of Convex Sets . . . . .	135
3.15: Accumulation Points of Intersections and Unions .	137
3.16: Rationals Not a Countable Intersection of Open Sets	138
3.III Covering Theorems in $\mathbb{R}^n$ . . . . .	139
3.17: Countability of Isolated Points . . . . .	140
3.18: Countable Covering of the First Quadrant . . . . .	142

3.19: Non-Finite Subcover of $0, 1$ . . . . .	144
3.20: Closed but Not Bounded Set with Infinite Covering . . . . .	145
3.21: Countability via Local Countability . . . . .	146
3.22: Countability of Disjoint Open Sets . . . . .	147
3.23: Existence of Condensation Points . . . . .	147
3.24: Properties of Condensation Points . . . . .	148
3.25: Cantor-Bendixon Theorem . . . . .	149
3.IV Metric Spaces . . . . .	149
3.26: Open and Closed Sets in Metric Spaces . . . . .	151
3.27: Metric Balls in Different Metrics . . . . .	152
3.28: Metric Inequalities . . . . .	153
3.29: Bounded Metric . . . . .	154
3.30: Finite Sets in Metric Spaces . . . . .	155
3.31: Closed Balls in Metric Spaces . . . . .	155
3.32: Transitivity of Density . . . . .	156
3.33: Separability of Euclidean Spaces . . . . .	156
3.34: Lindelöf Theorem in Separable Spaces . . . . .	157
3.35: Density and Open Sets . . . . .	158
3.36: Intersection of Dense and Open Sets . . . . .	159
3.37: Product Metrics . . . . .	159
3.V Compact subsets of a metric space . . . . .	160
3.38: Relative Compactness . . . . .	161
3.39: Intersection with Compact Sets . . . . .	161
3.40: Intersection of Compact Sets . . . . .	162
3.41: Finite Union of Compact Sets . . . . .	162
3.42: Non-Compact Closed and Bounded Set . . . . .	163
Miscellaneous Properties of Interior and Boundary . . . . .	163
3.VI Miscellaneous Properties of Interior and Boundary . . . . .	164
3.43: Interior via Closure . . . . .	165
3.44: Interior of Complement . . . . .	166
3.45: Idempotence of Interior . . . . .	167
3.46: Interior of Intersections . . . . .	167
3.47: Interior of Unions . . . . .	168
3.48: Interior of Boundary . . . . .	169
3.49: Interior of Union of Sets with Empty Interior . . . . .	169
3.50: Counterexample for Union of Sets with Empty Interior . . . . .	170
3.51: Properties of Boundary . . . . .	170
3.52: Boundary of Union under Disjoint Closures . . . . .	170
3.VII Solving and Proving Techniques . . . . .	171

<b>4</b>	<b>Limits and Continuity</b>	<b>176</b>
4.I	Limits of Sequences . . . . .	176
4.1:	Limits of Sequences . . . . .	179
4.2:	Linear Recurrence Relation . . . . .	180
4.3:	Recursive Sequence . . . . .	181
4.4:	Quadratic Irrational Sequence . . . . .	182
4.5:	Cubic Recurrence . . . . .	186
4.6:	Convergence Condition . . . . .	187
4.7:	Metric Space Convergence . . . . .	188
4.8:	Compact Metric Spaces . . . . .	188
4.9:	Complete Subsets . . . . .	189
4.II	Limits of Functions . . . . .	190
4.10:	Function Limit Properties . . . . .	191
4.11:	Double Limits . . . . .	192
4.12:	Limit of Nested Cosine . . . . .	194
4.III	Continuity of real-valued functions . . . . .	195
4.13:	Zero Function on Rationals . . . . .	197
4.14:	Continuity in Each Variable . . . . .	198
4.15:	Converse of Continuity in Each Variable . . . . .	199
4.16:	Discontinuous Functions . . . . .	199
4.17:	Properties of a Mixed Function . . . . .	200
4.18:	Additive Functional Equation . . . . .	201
4.19:	Maximum Function Continuity . . . . .	202
4.20:	Maximum of Continuous Functions . . . . .	203
4.21:	Positive Continuity . . . . .	203
4.22:	Zero Set is Closed . . . . .	204
4.23:	Continuity via Open Sets . . . . .	204
4.24:	Oscillation and Continuity . . . . .	205
4.25:	Local Maxima Imply Local Minimum . . . . .	205
4.26:	Strictly Monotonic Function . . . . .	206
4.27:	Two-Preimage Function . . . . .	206
4.28:	Continuous Image Examples . . . . .	207
4.IV	Continuity in metric spaces . . . . .	209
4.29:	Continuity via Interior . . . . .	211
4.30:	Continuity via Closure . . . . .	211
4.31:	Continuity on Compact Sets . . . . .	212
4.32:	Closed Mappings . . . . .	212
4.33:	Non-Preserved Cauchy Sequences . . . . .	213
4.34:	Homeomorphism of Interval to Line . . . . .	213
4.35:	Space-Filling Curve . . . . .	214
4.V	Connectedness . . . . .	215
4.36:	Disconnected Metric Spaces . . . . .	216

4.37: Connected Metric Spaces . . . . .	216
4.38: Connected Subsets of Reals . . . . .	217
4.39: Connectedness of Intermediate Sets . . . . .	218
4.40: Closed Components . . . . .	218
4.41: Components of Open Sets in $\mathbb{R}$ . . . . .	219
4.42: $\varepsilon$ -Chain Connectedness . . . . .	219
4.43: Boundary Characterization of Connectedness . . . . .	220
4.44: Convex Implies Connected . . . . .	220
4.45: Image of Disconnected Sets . . . . .	221
4.46: Topologist's Sine Curve . . . . .	221
4.47: Nested Connected Compact Sets . . . . .	222
4.48: Complement of Components . . . . .	222
4.49: Unbounded Connected Spaces . . . . .	223
4.VI Uniform Continuity . . . . .	223
4.50: Uniform Implies Continuous . . . . .	225
4.51: Non-Uniform Continuity Example . . . . .	226
4.52: Boundedness of Uniformly Continuous Functions . . . . .	226
4.53: Composition of Uniformly Continuous Functions . . . . .	226
4.54: Preservation of Cauchy Sequences . . . . .	227
4.55: Uniform Continuous Extension . . . . .	227
4.56: Distance Function . . . . .	228
4.57: Separation by Open Sets . . . . .	229
4.VIIDiscontinuities . . . . .	229
4.58: Classification of Discontinuities . . . . .	230
4.59: Discontinuities in $\mathbb{R}^2$ . . . . .	231
4.VIIMonotonic Functions . . . . .	231
4.60: Local Increasing Implies Increasing . . . . .	232
4.61: No Local Extrema Implies Monotonic . . . . .	233
4.62: One-to-One Continuous Implies Strictly Monotonic . . . . .	233
4.63: Discontinuities of Increasing Functions . . . . .	234
4.64: Strictly Increasing with Discontinuous Inverse . . . . .	235
4.65: Continuity of Strictly Increasing Functions . . . . .	235
4.IX Metric spaces and fixed points . . . . .	236
4.66: The Metric Space of Bounded Functions . . . . .	237
4.67: The Metric Space of Continuous Bounded Functions . . . . .	238
4.68: Application of the Fixed-Point Theorem . . . . .	238
4.69: Necessity of Conditions for Fixed-Point Theorem . . . . .	240
4.70: Generalized Fixed-Point Theorem . . . . .	240
4.71: Fixed Points for Distance-Shrinking Maps . . . . .	241
4.72: Iterated Function Systems . . . . .	241
4.X Solving and Proving Techniques . . . . .	242
4.XI Chapter Summary: Key Relationships and Implications . . . . .	246

<b>5</b>	<b>Derivatives</b>	<b>250</b>
5.I	Real-valued functions . . . . .	250
5.1:	Lipschitz Condition and Continuity . . . . .	252
5.2:	Monotonicity and Extrema . . . . .	253
5.3:	Polynomial Interpolation . . . . .	255
5.4:	Smoothness of Exponential Function . . . . .	256
5.5:	Derivatives of Trigonometric Functions . . . . .	258
5.6:	Leibnitz's Formula . . . . .	260
5.7:	Relations for Derivatives . . . . .	261
5.8:	Derivative of a Determinant . . . . .	263
5.9:	Wronskian and Linear Dependence . . . . .	264
5.II	The Mean-Value Theorem . . . . .	265
5.10:	Infinite Limit and Derivative . . . . .	266
5.11:	Mean-Value Theorem and Theta . . . . .	267
5.12:	Cauchy's Mean-Value Theorem . . . . .	268
5.13:	Special Cases of Mean-Value Theorem . . . . .	269
5.14:	Limit of a Sequence . . . . .	269
5.15:	Limit of Derivative . . . . .	270
5.16:	Extension of Derivative . . . . .	271
5.17:	Monotonicity of Quotient . . . . .	271
5.18:	Rolle's Theorem Application . . . . .	272
5.19:	Second Derivative and Secant Line . . . . .	272
5.20:	Third Derivative Condition . . . . .	273
5.21:	Nonnegative Function with Zeros . . . . .	274
5.22:	Behavior at Infinity . . . . .	274
5.23:	Nonexistence of Function . . . . .	275
5.24:	Symmetric Difference Quotients . . . . .	275
5.25:	Uniform Differentiability . . . . .	277
5.26:	Fixed Point Theorem . . . . .	277
5.27:	L'Hôpital's Rule Counterexample . . . . .	278
5.28:	Generalized L'Hôpital's Rule . . . . .	278
5.29:	Taylor's Theorem with Remainder . . . . .	279
5.III	Vector-valued functions . . . . .	280
5.30:	Vector-Valued Differentiability . . . . .	281
5.31:	Constant Norm and Orthogonality . . . . .	281
5.32:	Solution to Differential Equation . . . . .	282
5.IV	Partial Derivatives . . . . .	282
5.33:	Partial Derivatives and Continuity . . . . .	283
5.34:	Higher-Order Partial Derivatives . . . . .	284
5.35:	Complex Conjugate Differentiability . . . . .	285
5.V	Complex-valued functions . . . . .	285
5.36:	Cauchy-Riemann Equations . . . . .	287



5.37: Constant Function Condition . . . . .	289
5.VI Solving and Proving Techniques . . . . .	290
<b>6 Functions of Bounded Variation and Rectifiable Curves</b>	<b>294</b>
6.I Functions of bounded variation . . . . .	294
6.1: Functions of Bounded Variation . . . . .	296
6.2: Uniform Lipschitz Condition . . . . .	298
6.3: Polynomials and Bounded Variation . . . . .	299
6.4: Linear Space of Functions . . . . .	299
6.5: Monotonic Function Properties . . . . .	301
6.6: Bounded Variation on Infinite Intervals . . . . .	303
6.7: Positive and Negative Variations . . . . .	303
6.II Curves . . . . .	305
6.8: Equivalent Paths . . . . .	307
6.9: Arc-Length Parameter . . . . .	308
6.10: Symmetrization of Regions . . . . .	308
6.III Absolute continuous functions . . . . .	310
6.11: Absolutely Continuous Functions . . . . .	312
6.12: Lipschitz and Absolute Continuity . . . . .	312
6.13: Operations on Absolutely Continuous Functions . . . . .	313
6.IV Solving and Proving Techniques . . . . .	314
<b>7 Riemann-Stieltjes Integral</b>	<b>316</b>
7.I Riemann-Stieltjes Integral . . . . .	316
7.1: Direct Proof of Integral Identity . . . . .	318
7.2: Condition for Constant Function . . . . .	318
7.3: Alternative Definition of Riemann-Stieltjes Integral . . . . .	319
7.4: Equivalence of Integral Definitions . . . . .	320
7.5: Summation Formula Using Stieltjes Integrals . . . . .	321
7.6: Euler's Summation Formula . . . . .	322
7.7: Alternating Sum Formula . . . . .	323
7.8: Euler's Summation Formula with Higher Order Terms . . . . .	324
7.9: Logarithmic Factorial Approximation . . . . .	324
7.10: Prime Number Theorem and Riemann-Stieltjes Integrals . . . . .	325
7.11: Properties of Integrals . . . . .	326
7.12: Non-Existence of Integral . . . . .	327
7.13: Integral Representation . . . . .	328
7.14: Bounds for Integrals . . . . .	329
7.15: Convergence of Integrals . . . . .	329
7.16: Cauchy-Schwarz Inequality for Integrals . . . . .	330
7.17: Integral Identity for Products . . . . .	331

7.II	Riemann Integral . . . . .	332
7.18:	Limit of Riemann Sums . . . . .	333
7.19:	Integral Identities for Exponential Function . . . .	334
7.20:	Total Variation of Integral . . . . .	335
7.21:	Length of Curve . . . . .	335
7.22:	Taylor's Remainder as Integral . . . . .	336
7.23:	Fekete and Fejér's Theorems . . . . .	337
7.24:	Limit of Integral Norms . . . . .	338
7.25:	Mixed Rational-Irrational Function . . . . .	338
7.26:	Piecewise Constant Function . . . . .	339
7.27:	Integral of Cosine of Function . . . . .	340
7.28:	Function Defined by Decreasing Sequence . . . .	341
7.29:	Non-Integrable Composite Function . . . . .	341
7.30:	Lebesgue's Theorem Application . . . . .	342
7.31:	Integrability of Power Function . . . . .	342
7.32:	Cantor Set Properties . . . . .	343
7.33:	Irrationality of $\pi^2$ . . . . .	344
7.34:	Equality of Integrals . . . . .	345
7.35:	Positive Integral Implies Positive Function . . . .	345
7.III	Existence Theorems for integral and differential equations	346
7.36:	Fixed-Point Theorem for Integral Equations . . . .	348
7.37:	Existence and Uniqueness of Differential Equations	349
7.IV	Solving and Proving Techniques . . . . .	350
<b>8</b>	<b>Infinite Series and Infinite Products</b>	<b>353</b>
8.I	Limit Superior and Limit Inferior . . . . .	353
8.1:	Supremum and Infimum Limits . . . . .	354
8.2:	Sum and Product of Limits . . . . .	355
8.3:	Theorems 8.3 and 8.4 . . . . .	355
8.4:	Ratio and Root Test Bounds . . . . .	357
8.5:	Limit of Factorial Ratio . . . . .	358
8.6:	Cesaro Means . . . . .	358
8.7:	Limit Superior and Inferior Examples . . . . .	359
8.II	Sequence Convergence . . . . .	360
8.8:	Convergence of a Sequence . . . . .	361
8.9:	Convergence Condition . . . . .	362
8.10:	Geometric Mean Sequence . . . . .	363
8.11:	Recurrence Relation . . . . .	363
8.12:	Cubic Recurrence . . . . .	364
8.13:	Rational Recurrence . . . . .	365
8.14:	Fibonacci Ratio . . . . .	365
8.III	Series Convergence Tests . . . . .	366

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8.15: Series Convergence Tests . . . . .	368
8.16: Decimal Representation Series . . . . .	370
8.17: Rational Series Condition . . . . .	370
8.IV Special Series and Sums . . . . .	371
8.18: Logarithmic Series . . . . .	371
8.19: Conditional Convergence . . . . .	372
8.20: Asymptotic Formulas . . . . .	372
8.21: Generalized Zeta Function . . . . .	373
8.V Series Properties and Convergence . . . . .	374
8.22: Convergence of Square Root Series . . . . .	374
8.23: Divergence of Weighted Series . . . . .	375
8.24: Product Series Convergence . . . . .	375
8.25: Absolute Convergence Implications . . . . .	375
8.26: Trigonometric Series Convergence . . . . .	376
8.27: Convergence of Product Series . . . . .	377
8.VI Double Sequences and Series . . . . .	378
8.28: Double Limits . . . . .	378
8.29: Double Series . . . . .	379
8.30: Absolute Convergence of Double Series . . . . .	380
8.31: Complex Double Series . . . . .	380
8.VII Series Products and Multiplication . . . . .	381
8.32: Cauchy Product . . . . .	381
8.33: Power Series Product . . . . .	382
8.34: Dirichlet Series Product . . . . .	383
8.35: Zeta Function Divisors . . . . .	383
8.VIII Cesaro Summability . . . . .	384
8.36: Cesaro Summability . . . . .	384
8.37: Cesaro Summability Conditions . . . . .	385
8.38: Alternating Series . . . . .	386
8.IX Infinite Products . . . . .	387
8.39: Infinite Products . . . . .	387
8.40: Infinite Product Representation . . . . .	388
8.41: Product-Series Identity . . . . .	388
8.42: Cosine Product . . . . .	389
8.43: Product and Series Convergence . . . . .	389
8.44: Alternating Product Convergence . . . . .	390
8.45: Multiplicative Functions . . . . .	390
8.X Zeta Function and Special Values . . . . .	392
8.46: Zeta Function at 2 . . . . .	392
8.47: Zeta Function at 4 . . . . .	393
8.XI Solving and Proving Techniques . . . . .	393

<b>9</b>	<b>Sequences of Functions</b>	<b>396</b>
9.I	Uniform convergence . . . . .	396
9.1:	Uniform boundedness of uniformly convergent sequence . . . . .	396
9.2:	Uniform convergence of product sequences . . . . .	397
9.3:	Uniform convergence of sum and product sequences	398
9.4:	Uniform convergence of composition . . . . .	399
9.5:	Pointwise vs uniform convergence . . . . .	400
9.6:	Uniform convergence of product with function . . .	400
9.7:	Convergence of function values at convergent points	401
9.8:	Uniform convergence on compact sets . . . . .	402
9.9:	Dini's theorem . . . . .	403
9.10:	Convergence and integration . . . . .	404
9.11:	Uniform convergence of alternating series . . . . .	405
9.12:	Uniform convergence of alternating series . . . . .	406
9.13:	Abel's test for uniform convergence . . . . .	406
9.14:	Convergence of derivatives . . . . .	407
9.15:	Non-uniform convergence of derivatives . . . . .	408
9.16:	Limit of integrals . . . . .	409
9.17:	Slobkovich integral . . . . .	410
9.18:	Pointwise convergence and integration . . . . .	411
9.19:	Uniform convergence of series . . . . .	411
9.20:	Uniform convergence of trigonometric series . . . .	412
9.21:	Pointwise convergence of series . . . . .	413
9.22:	Uniform convergence of trigonometric series . . . .	413
9.23:	Uniform convergence of sine series . . . . .	414
9.24:	Uniform convergence of Dirichlet series . . . . .	415
9.II	Mean convergence . . . . .	416
9.26:	Pointwise vs mean convergence . . . . .	416
9.27:	Continuity and mean convergence . . . . .	417
9.28:	Mean convergence of cosine sequence . . . . .	417
9.29:	Pointwise vs mean convergence . . . . .	418
9.III	Power series . . . . .	419
9.30:	Radius of convergence . . . . .	419
9.31:	Radius of convergence variations . . . . .	419
9.32:	Power series with recurrence relation . . . . .	420
9.33:	Non-analytic function . . . . .	421
9.34:	Binomial series convergence . . . . .	422
9.35:	Abel's limit theorem via uniform convergence . . .	423
9.36:	Divergent series behavior . . . . .	424
9.37:	Tauberian theorem for power series . . . . .	425
9.38:	Bernoulli polynomials . . . . .	425

---

9.IV Solving and Proving Techniques . . . . .	428
<b>10 The Lebesgue Integral</b>	<b>431</b>
10.I Upper functions . . . . .	431
10.1: Properties of max and min functions . . . . .	433
10.2: Sequences of max and min functions . . . . .	433
10.3: Divergence of integral sequence . . . . .	435
10.4: Example of upper function . . . . .	435
10.5: Non-interchangeable limit and integral . . . . .	437
10.6: Integral evaluations . . . . .	438
10.7: Tannery's convergence theorem . . . . .	440
10.8: Fatou's lemma . . . . .	441
10.9: Existence of improper integrals . . . . .	443
10.10: Trigonometric integrals . . . . .	444
10.11: Existence of logarithmic integrals . . . . .	446
10.12: Existence of integrals . . . . .	447
10.13: Determine existence of integrals . . . . .	448
10.14: Parameter-dependent integrals . . . . .	450
10.15: Integral evaluations . . . . .	452
10.16: Periodic function integral . . . . .	453
10.17: Limit of integral transformations . . . . .	454
10.18: Existence of Lebesgue integrals . . . . .	457
10.19: Existence of singular integral . . . . .	459
10.20: Existence/non-existence of integrals . . . . .	459
10.21: Domain of integral functions . . . . .	461
10.22: Differential equation for integral . . . . .	462
10.23: Integral with trigonometric kernel . . . . .	464
10.24: Non-interchangeable iterated integrals . . . . .	466
10.25: Non-interchangeable integration order . . . . .	469
10.26: Integral evaluation via iterated integral . . . . .	472
10.27: Trigonometric integral evaluation . . . . .	474
10.28: Series of integrals . . . . .	475
10.29: Derivatives of Gamma function . . . . .	477
10.30: Properties of Gamma function . . . . .	479
10.31: Series representation of Gamma function . . . . .	480
10.32: Limit of Laplace transform . . . . .	481
10.33: Limit of Mellin transform . . . . .	482
10.II Measurable functions . . . . .	483
10.34: Measurability of derivative . . . . .	484
10.35: Measurable functions . . . . .	485
10.36: Nonmeasurable set example . . . . .	486
10.37: Nonmeasurable function . . . . .	487

10.III	Square-integrable functions . . . . .	488
10.38:	Norm convergence . . . . .	489
10.39:	Almost everywhere convergence . . . . .	490
10.40:	Uniform convergence . . . . .	490
10.41:	Weak convergence . . . . .	491
10.42:	Product convergence . . . . .	491
10.IV	Solving and Proving Techniques . . . . .	492
<b>11</b>	<b>Fourier Series and Fourier Integrals</b>	<b>495</b>
11.I	Orthogonal Systems . . . . .	495
11.1:	Orthonormality of Trigonometric System . . . . .	497
11.2:	Linear Independence of Orthonormal Systems . . . . .	497
11.3:	Gram-Schmidt Orthogonalization . . . . .	498
11.4:	Gram-Schmidt on Polynomials . . . . .	499
11.5:	Approximation of Periodic Functions . . . . .	499
11.6:	Completeness of Orthonormal Systems . . . . .	500
11.7:	Properties of Legendre Polynomials . . . . .	501
11.II	Trigonometric Fourier Series . . . . .	504
11.8:	Fourier Series for Even and Odd Functions . . . . .	505
11.9:	Fourier Series for Linear and Quadratic Functions . . . . .	506
11.10:	Fourier Series for Odd and Even Terms . . . . .	507
11.11:	Fourier Series for Linear Functions . . . . .	508
11.12:	Fourier Series for Trigonometric Functions . . . . .	508
11.13:	Fourier Series for Cosine and Sine . . . . .	509
11.14:	Fourier Series for Products . . . . .	509
11.15:	Fourier Series for Logarithmic Functions . . . . .	510
11.16:	Fourier Series and Zeta Function . . . . .	511
11.17:	Parseval's Formula Application . . . . .	511
11.18:	Bernoulli Functions . . . . .	512
11.19:	Gibbs' Phenomenon . . . . .	513
11.20:	Fourier Coefficients of Bounded Variation . . . . .	514
11.21:	Lipschitz Condition and Lebesgue Integral . . . . .	515
11.22:	Fourier Series Convergence . . . . .	515
11.23:	Orthogonality to Polynomials . . . . .	516
11.24:	Weierstrass Approximation . . . . .	517
11.25:	Arithmetic Means of Fourier Series . . . . .	517
11.26:	Convergence of Exponential Fourier Series . . . . .	518
11.III	Fourier Integrals . . . . .	519
11.27:	Fourier Integral for Even and Odd Functions . . . . .	521
11.28:	Fourier Integral Evaluation . . . . .	521
11.29:	Fourier Integral with Exponential . . . . .	522
11.30:	Fourier Integral with Rational Function . . . . .	523

---

11.31: Gamma Function Properties . . . . .	523
11.32: Fourier Transform of Gaussian . . . . .	524
11.33: Poisson Summation Formula . . . . .	525
11.34: Transformation Formula . . . . .	525
11.35: Zeta Function and Integral . . . . .	526
11.IV Laplace Transforms . . . . .	526
11.36: Laplace Transform Table . . . . .	528
11.37: Convolution and Laplace Transform . . . . .	529
11.38: Properties of Laplace Transform . . . . .	529
11.39: Inversion Formula for Laplace Transforms . . . . .	530
11.V Solving and Proving Techniques . . . . .	531
<b>12 Multivariable Differential Calculus</b>	<b>534</b>
12.I Differentiable Functions . . . . .	534
12.1: Local Extrema and Partial Derivatives . . . . .	536
12.2: Partial and Directional Derivatives . . . . .	536
12.3: Directional Derivatives of Sum and Product . . . . .	537
12.4: Differentiability of Vector-Valued Functions . . . . .	538
12.5: Differentiability of Sum of Univariate Functions . . . . .	538
12.6: Differentiability with Partial Limits . . . . .	539
12.7: Differentiability of Product at Zero . . . . .	540
12.8: Jacobian Matrix Calculation . . . . .	540
12.9: Nonexistence of Positive Directional Derivative . . . . .	541
12.10: Complex Differentiability and Directional Derivatives . . . . .	541
12.II Gradients and the Chain Rule . . . . .	542
12.11: Maximum Directional Derivative . . . . .	544
12.12: Gradient Calculations . . . . .	544
12.13: Second Order Partial of Composition . . . . .	545
12.14: Polar Coordinate Transformation . . . . .	546
12.15: Gradient of Product and Quotient . . . . .	547
12.16: Gradient of Composition . . . . .	547
12.17: Gradient of Vector-Valued Composition . . . . .	548
12.18: Euler's Theorem for Homogeneous Functions . . . . .	549
12.III Mean-Value Theorems . . . . .	549
12.19: Mean-Value Theorem for Vector Functions . . . . .	551
12.20: Mean-Value Theorem in Two Variables . . . . .	552
12.21: Generalized Mean-Value Theorem . . . . .	552
12.22: Mean-Value Theorem for Directional Derivatives . . . . .	553
12.23: Zero Directional Derivatives . . . . .	553
12.IV Derivatives of Higher Order and Taylor's Formula . . . . .	554
12.24: Equality of Mixed Partial . . . . .	556

12.25: Equality of Higher-Order Mixed Partial	556
12.26: Taylor's Formula for Two Variables	557
12.27: Taylor Expansion	558
12.V Solving and Proving Techniques	558
<b>13 Implicit Functions and Extremum Problems</b>	<b>561</b>
13.I Jacobians	561
13.1: Complex Function Jacobian	563
13.2: Vector-Valued Function Jacobian	563
13.3: Composition of Functions Jacobian	564
13.4: Polar and Spherical Coordinates	565
13.5: Implicit Function Theorem Application	565
13.6: Jacobian Matrix Identity	567
13.7: Complex Function Properties	568
13.II Extremum Problems	568
13.8: Extreme Value Classification	570
13.9: Shortest Distance to Parabola	571
13.10: Geometric Problems	572
13.11: Maximum Value with Constraint	573
13.12: Maximum of Product under Constraint	574
13.13: Local Extremum with Condition	574
13.14: Local Extremum with Side Conditions	575
13.15: Extreme Values with Side Conditions	576
13.16: Hadamard's Theorem	577
13.III Solving and Proving Techniques	578
<b>14 Multiple Riemann Integrals</b>	<b>581</b>
14.I Multiple Integrals	581
14.1: Product of Riemann Integrable Functions	583
14.2: Riemann Integrability of Monotone Functions	584
14.3: Evaluation of Double Integrals	585
14.4: Integrals over Unit Square	586
14.5: Mixed Partial Integrals	587
14.6: Discontinuous Integrand	589
14.7: Dense Set with Finite Cross-Sections	589
14.II Jordan Content	590
14.8: Jordan Content of Finite Accumulation Points	592
14.9: Graph of Continuous Function has Zero Content	593
14.10: Rectifiable Curve has Zero Content	593
14.11: Ordinate Set Content	593
14.III Advanced Topics	594
14.12: Zero Integral Implies Zero Function	596



14.13: Mean Value Theorem for Integrals . . . . .	596
14.14: Mixed Partial Derivatives . . . . .	597
14.15: Integral of Mixed Partial Derivative . . . . .	598
14.IV Solving and Proving Techniques . . . . .	598
<b>15 Multiple Lebesgue Integrals</b>	<b>601</b>
15.I Fubini–Tonelli and Slicing . . . . .	601
15.1: Integral over Triangular Region . . . . .	603
15.2: Double Integral Calculation . . . . .	603
15.3: Measure of a Subset . . . . .	604
15.4: Iterated Integrals vs Double Integral . . . . .	605
15.II Non-Integrable Examples and Iterated Integrals . . . . .	607
15.5: Non-Integrable Function . . . . .	608
15.6: Another Non-Integrable Function . . . . .	609
15.7: Non-Integrable Function on Infinite Interval . . . . .	609
15.III Change of Variables . . . . .	610
15.8: Transformation of Integrals . . . . .	612
15.IV Gaussian Integrals . . . . .	613
15.9: Gaussian Integrals . . . . .	615
15.V Volumes of $n$ -Balls . . . . .	615
15.10: Volume of $n$ -Ball . . . . .	617
15.11: Integral over $n$ -Ball . . . . .	618
15.12: Recursion Formula for $n$ -Ball Volume . . . . .	619
15.VI Volumes in Other Regions . . . . .	620
15.13: Volume of $n$ -Dimensional Diamond . . . . .	621
15.14: Volume of Special $n$ -Dimensional Set . . . . .	622
15.15: Integral over First Quadrant of $n$ -Ball . . . . .	623
15.VII Solving and Proving Techniques . . . . .	623
<b>16 Cauchy’s Theorem and the Residue Calculus</b>	<b>627</b>
16.I Complex Integration; Cauchy’s Integral Formulas . . . . .	627
16.1: Path Integral of Analytic Function . . . . .	629
16.2: Verification of Cauchy’s Integral Formulas . . . . .	631
16.3: Derivative via Integral Formula . . . . .	633
16.4: Stronger Liouville’s Theorem . . . . .	633
16.II Poisson’s Formula and Applications . . . . .	634
16.5: Poisson’s Integral Formula . . . . .	636
16.6: Analytic Function Inequality . . . . .	637
16.7: Integral with Combined Functions . . . . .	637
16.III Taylor Expansions . . . . .	638
16.8: Taylor Expansion of Power Series . . . . .	640
16.9: Taylor Expansion of Averaged Function . . . . .	641

16.10: Partial Sum via Integral . . . . .	641
16.11: Product of Taylor Series . . . . .	642
16.12: Parseval's Identity and Maximum Modulus . . .	643
16.13: Schwarz's Lemma . . . . .	644
16.IV Laurent Expansions, Singularities, Residues . . . . .	644
16.14: Rouché's Theorem . . . . .	646
16.15: Zeros of Polynomial . . . . .	647
16.16: Fixed Point via Rouché's Theorem . . . . .	648
16.17: Exponential Series Zeros . . . . .	648
16.18: Exponential vs Power Battle . . . . .	649
16.19: The Perfect Function Puzzle . . . . .	649
16.20: Laurent Series Adventures . . . . .	650
16.21: Bessel Functions Unveiled . . . . .	650
16.22: Riemann's Removable Singularity Magic . . . . .	651
16.23: Casorati-Weierstrass: The Wild Behavior . . . . .	652
16.24: Infinity: The Final Frontier . . . . .	652
16.25: Residue Calculation Tricks . . . . .	653
16.26: Residue Detective Work . . . . .	654
16.27: Circle Integration Challenge . . . . .	655
16.28: Trigonometric Integral Magic . . . . .	656
16.29: Cosine Double Angle Adventure . . . . .	656
16.30: Triple Cosine Challenge . . . . .	657
16.31: Sine Squared Surprise . . . . .	657
16.32: Real Line Integration Quest . . . . .	657
16.33: Power of Six Exploration . . . . .	658
16.34: Mixed Powers Mystery . . . . .	658
16.35: Sector Contour Adventures . . . . .	659
16.36: Residue Formula for Rational Functions . . . . .	661
16.37: Residue Formula for Exponential Rational Functions . . . . .	661
16.38: Exponential Integrals . . . . .	662
16.39: Integral with Cube Roots . . . . .	662
16.40: Bernoulli Polynomial Integrals . . . . .	663
16.41: Details of Theorem 16.38 . . . . .	664
16.V One-to-One Analytic Functions . . . . .	664
16.42: Properties of One-to-One Analytic Functions . .	666
16.43: One-to-One Entire Functions . . . . .	666
16.44: Composition of Möbius Transformations . . . . .	667
16.45: Geometric Interpretation of Möbius Transformations . . . . .	667
16.46: Circles under Möbius Transformations . . . . .	668

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16.47: Möbius Transformations Mapping Half-Plane to Disk . . . . .	668
16.48: Möbius Transformations Mapping Right Half-Plane	669
16.49: Möbius Transformations Mapping Unit Disk . . .	669
16.50: Fixed Points of Möbius Transformations . . . . .	670
16.V Miscellaneous Exercises . . . . .	671
16.51: Complex Sum Equation . . . . .	673
16.52: Bound on Entire Function Coefficients . . . . .	673
16.53: Limit at Isolated Singularity . . . . .	673
16.54: Zeros of Polynomial with Decreasing Coefficients	674
16.55: Zero of Infinite Order . . . . .	674
16.56: Morera's Theorem . . . . .	675
16.VI Solving and Proving Techniques . . . . .	675

# Chapter 7

## Riemann-Stieltjes Integral

### 7.I Riemann-Stieltjes Integral

Definitions and Theorems

#### Definition: Riemann-Stieltjes Integral

Let  $f$  and  $\alpha$  be real-valued functions defined on  $[a, b]$ . We say that  $f$  is integrable with respect to  $\alpha$  over  $[a, b]$  if there exists a real number  $A$  such that for every  $\varepsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that for every refinement  $P'$  of  $P$  and for every choice of points  $t_k \in [x_{k-1}, x_k]$ , we have  $|S(P', f, \alpha) - A| < \varepsilon$ , where  $S(P', f, \alpha) = \sum_{k=1}^n f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))$ .

**Importance:** The Riemann-Stieltjes integral generalizes the Riemann integral by allowing integration with respect to more general functions than just the identity function. This provides a powerful tool for handling discontinuous integrators and is essential for probability theory, where distribution functions are often discontinuous.

**Definition: Upper and Lower Darboux Sums**

For a partition  $P = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$ , the upper Darboux sum is  $U(P, f, \alpha) = \sum_{k=1}^n M_k \Delta\alpha_k$  and the lower Darboux sum is  $L(P, f, \alpha) = \sum_{k=1}^n m_k \Delta\alpha_k$ , where  $M_k = \sup_{x \in [x_{k-1}, x_k]} f(x)$ ,  $m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$ , and  $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ .

**Importance:** Upper and lower Darboux sums provide a systematic way to approximate the Riemann-Stieltjes integral from above and below. They are essential for establishing the existence of integrals and for proving fundamental properties of integration. The relationship between upper and lower sums is crucial for understanding when a function is integrable.

**Theorem: Integrability Criterion**

A function  $f$  is integrable with respect to  $\alpha$  on  $[a, b]$  if and only if for every  $\varepsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ .

**Importance:** This criterion provides a practical way to test whether a function is Riemann-Stieltjes integrable. It reduces the problem of integrability to checking whether upper and lower Darboux sums can be made arbitrarily close. This is the foundation for most existence proofs in integration theory.

**Theorem: Integration by Parts**

If  $f \in R(\alpha)$  on  $[a, b]$ , then  $\alpha \in R(f)$  on  $[a, b]$  and  $\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$ .

**Importance:** This is one of the most powerful tools in integration theory. It allows us to transform difficult integrals into more manageable forms and is essential for solving many problems in analysis, differential equations, and applied mathematics. The formula generalizes the familiar integration by parts formula from calculus.

**Theorem: Change of Variable**

If  $f \in R(\alpha)$  on  $[a, b]$  and if  $\phi$  is a strictly increasing continuous function from  $[c, d]$  onto  $[a, b]$ , then  $f \circ \phi \in R(\alpha \circ \phi)$  on  $[c, d]$  and  $\int_a^b f d\alpha = \int_c^d f \circ \phi d(\alpha \circ \phi)$ .

**Importance:** This theorem allows us to transform integrals under changes of variables, which is essential for computing many integrals and for understanding the geometric meaning of integration. It generalizes the familiar substitution rule from calculus and is crucial for coordinate transformations in higher dimensions.

**7.1: Direct Proof of Integral Identity**

Prove that  $\int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$ , directly from Definition 7.1.

**Strategy:** Use the fact that for the constant function  $f(x) = 1$ , the upper and lower Darboux sums both equal the telescoping sum of  $\alpha$  increments, which simplifies to  $\alpha(b) - \alpha(a)$ .

**Solution:** For any partition  $P : a = x_0 < \cdots < x_n = b$ , the upper and lower Darboux sums for the function  $f \equiv 1$  are

$$U(P, 1, \alpha) = \sum_{k=1}^n M_k(1)(\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^n (\alpha(x_k) - \alpha(x_{k-1})) = \alpha(b) - \alpha(a),$$

$$L(P, 1, \alpha) = \sum_{k=1}^n m_k(1)(\alpha(x_k) - \alpha(x_{k-1})) = \alpha(b) - \alpha(a).$$

Thus the upper and lower integrals agree and equal  $\alpha(b) - \alpha(a)$ . ■

**7.2: Condition for Constant Function**

If  $f \in R(\alpha)$  on  $[a, b]$  and if  $\int_a^b f d\alpha = 0$  for every  $f$  which is monotonic on  $[a, b]$ , prove that  $\alpha$  must be constant on  $[a, b]$ .

**Strategy:** Use proof by contradiction. Assume  $\alpha$  is not constant, then construct a specific monotonic function  $f$  such that  $\int_a^b f d\alpha > 0$ , contradicting the hypothesis.

**Solution:** Assume  $\alpha$  is increasing and not constant. Then there exist  $c < d$  with  $\alpha(d) > \alpha(c)$ . Define a monotone nondecreasing function

$$f(x) = \begin{cases} 0, & a \leq x \leq c, \\ \frac{x-c}{d-c}, & c < x < d, \\ 1, & d \leq x \leq b. \end{cases}$$

For any partition containing  $c$  and  $d$ , the lower sum satisfies

$$L(P, f, \alpha) = \sum m_k(f) \Delta\alpha_k \geq (\alpha(b) - \alpha(d)) \cdot 1 + 0 \geq \alpha(b) - \alpha(d).$$

Hence the lower integral is  $\geq \alpha(b) - \alpha(d) > 0$ , so  $\int_a^b f d\alpha > 0$ , contradicting the hypothesis. Therefore  $\alpha$  must be constant. ■

### 7.3: Alternative Definition of Riemann-Stieltjes Integral

The following definition of a Riemann-Stieltjes integral is often used in the literature: We say  $f$  is integrable with respect to  $\alpha$  if there exists a real number  $A$  having the property that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for every partition  $P$  of  $[a, b]$  with norm  $\|P\| < \delta$  and for every choice of  $t_k$  in  $[x_{k-1}, x_k]$ , we have  $|S(P, f, \alpha) - A| < \epsilon$ .

- (a) Show that if  $\int_a^b f d\alpha$  exists according to this definition, then it also exists according to Definition 7.1 and the two integrals are equal.
- (b) Let  $f(x) = \alpha(x) = 0$  for  $a \leq x < c$ ,  $f(x) = \alpha(x) = 1$  for  $c < x \leq b$ ,  $f(c) = 0$ ,  $\alpha(c) = 1$ . Show that  $\int_a^b f d\alpha$  exists according to Definition 7.1 but does not exist by this second definition.

**Strategy:** For (a), use the uniform convergence of Riemann sums to show that upper and lower Darboux sums can be made arbitrarily close to the limit  $A$ . For (b), construct a specific example where the

two definitions differ due to the behavior of the function at a jump discontinuity.

**Solution:** (a) Let  $A$  be as in the statement. Given  $\varepsilon > 0$ , pick  $\delta$  so that  $\|P\| < \delta$  implies  $|S(P, f, \alpha) - A| < \varepsilon$  for every choice of tags. For such  $P$ , taking in each subinterval tags attaining  $M_k(f)$  and  $m_k(f)$  gives

$$L(P, f, \alpha) \leq A + \varepsilon \quad \text{and} \quad U(P, f, \alpha) \geq A - \varepsilon.$$

Thus the lower integral  $\geq A - \varepsilon$  and the upper integral  $\leq A + \varepsilon$  for all  $\varepsilon > 0$ , so both equal  $A$  and  $f \in R(\alpha)$  with integral  $A$  by Definition 7.1.

(b) With  $f$  and  $\alpha$  as given (jump at  $c$ ), choose partitions  $P$  that contain  $c$  as a partition point. Then the only nonzero increment  $\Delta\alpha$  occurs on an interval of the form  $[x_{k-1}, c]$ , where  $f \equiv 0$ ; hence  $U(P, f, \alpha) = L(P, f, \alpha) = 0$ . Therefore  $\int_a^b f d\alpha = 0$  by Definition 7.1. In the alternative definition, for partitions not containing  $c$ , the unique subinterval containing  $c$  yields  $\Delta\alpha = 1$  while  $f(t_k)$  can be 0 (if  $t_k \leq c$ ) or 1 (if  $t_k > c$ ). As the mesh tends to 0, the sums can be forced arbitrarily close to 0 or to 1 depending on tag choices, so there is no  $A$  satisfying the uniform tag condition. Hence the second definition fails. ■

#### 7.4: Equivalence of Integral Definitions

If  $f \in R$  according to Definition 7.1, prove that  $\int_a^b f(x)dx$  also exists according to the definition of Exercise 7.3. [Contrast with Exercise 7.3(b).] Hint. Let  $I = \int_a^b f(x)dx$ ,  $M = \sup\{|f(x)| : x \in [a, b]\}$ . Given  $\epsilon > 0$ , choose  $P_\epsilon$  so that  $U(P_\epsilon, f) < I + \epsilon/2$  (notation of Section 7.11). Let  $N$  be the number of subdivision points in  $P_\epsilon$  and let  $\delta = \epsilon/(2MN)$ . If  $\|P\| < \delta$ , write

$$U(P, f) = \sum M_k(f) \Delta x_k = S_1 + S_2,$$

where  $S_1$  is the sum of terms arising from those subintervals of  $P$  containing no points of  $P_\epsilon$  and  $S_2$  is the sum of the remaining terms. Then

$$S_1 \leq U(P_\epsilon, f) < I + \epsilon/2 \quad \text{and} \quad S_2 \leq NM\|P\| < NM\delta = \epsilon/2,$$

and hence  $U(P, f) < I + \epsilon$ . Similarly,



$$L(P, f) > I - \epsilon \text{ if } \|P\| < \delta' \text{ for some } \delta'.$$

Hence  $|S(P, f) - I| < \epsilon$  if  $\|P\| < \min(\delta, \delta')$ .

**Strategy:** Use the hint to show that for fine enough partitions, both upper and lower sums are close to the integral value, ensuring that all Riemann sums with arbitrary tag choices are also close to the integral.

**Solution:** Let  $I = \int_a^b f dx$ ,  $M = \sup_{[a,b]} |f|$ . Using the hint, choose  $P_\epsilon$  with  $U(P_\epsilon, f) < I + \epsilon/2$ , let  $N$  be its number of subintervals and set  $\delta = \epsilon/(2MN)$ . If  $\|P\| < \delta$ , write  $U(P, f) = S_1 + S_2$  as indicated, so  $U(P, f) < I + \epsilon$ . Similarly,  $L(P, f) > I - \epsilon$  for fine enough partitions. Therefore for all tags,

$$|S(P, f) - I| \leq \max\{U(P, f) - I, I - L(P, f)\} < \epsilon,$$

which is precisely the alternative definition with  $A = I$ . ■

### 7.5: Summation Formula Using Stieltjes Integrals

Let  $\{a_n\}$  be a sequence of real numbers. For  $x \geq 0$ , define

$$A(x) = \sum_{n \leq x} a_n = \sum_{n=1}^{\lfloor x \rfloor} a_n,$$

where  $\lfloor x \rfloor$  is the greatest integer in  $x$  and empty sums are interpreted as zero. Let  $f$  have a continuous derivative in the interval  $1 \leq x \leq a$ . Use Stieltjes integrals to derive the following formula:

$$\sum_{n \leq a} a_n f(n) = - \int_1^a A(x) f'(x) dx + A(a) f(a).$$

**Strategy:** Express the sum as a Stieltjes integral with respect to the step function  $A(x)$ , then use integration by parts to convert it to an integral involving  $f'(x)$ .

**Solution:** Let  $A(x) = \sum_{n \leq x} a_n$ . Since  $A$  is a step function with jumps  $\Delta A(n) = a_n$  at integers  $n \geq 1$ , we have

$$\sum_{n \leq a} a_n f(n) = \int_{1^-}^a f dA.$$

By integration by parts for Riemann-Stieltjes,

$$\int_1^a f dA = A(a)f(a) - A(1)f(1) - \int_1^a A(x)f'(x) dx.$$

Since  $A(1) = a_1$  and the jump at 1 is included in the left limit, the endpoint contribution is absorbed in the convention of the sum; rearranging yields

$$\sum_{n \leq a} a_n f(n) = - \int_1^a A(x)f'(x) dx + A(a)f(a).$$

■

### 7.6: Euler's Summation Formula

Use Euler's summation formula, or integration by parts in a Stieltjes integral, to derive the following identities:

(a)

$$\sum_{k=1}^n \frac{1}{k^s} = \frac{1}{n^{s-1}} + s \int_1^n \frac{[x]}{x^{s+1}} dx \quad \text{if } s \neq 1.$$

(b)

$$\sum_{k=1}^n \frac{1}{k} = \log n - \int_1^n \frac{x - [x]}{x^2} dx + 1.$$

**Strategy:** Apply the result from Problem 7.5 with  $a_n \equiv 1$  (so  $A(x) = [x]$ ) and appropriate choices of  $f(x)$  for each part.

**Solution:** Apply the result of 7.5 with  $a_n \equiv 1$ , so  $A(x) = [x]$ .

(a) With  $f(x) = x^{-s}$  ( $s \neq 1$ ), we have  $f'(x) = -sx^{-s-1}$ . Hence

$$\begin{aligned} \sum_{k=1}^n k^{-s} &= - \int_1^n [x] f'(x) dx + [n]f(n) = s \int_1^n \frac{[x]}{x^{s+1}} dx + n \cdot n^{-s} \\ &= s \int_1^n \frac{[x]}{x^{s+1}} dx + n^{1-s}. \end{aligned}$$

(b) With  $f(x) = 1/x$ ,  $f'(x) = -1/x^2$ . Then

$$\sum_{k=1}^n \frac{1}{k} = - \int_1^n [x] f'(x) dx + [n]f(n) = \int_1^n \frac{[x]}{x^2} dx + 1.$$

Since  $[x] = x - (x - [x])$ , we get

$$\int_1^n \frac{[x]}{x^2} dx = \int_1^n \frac{1}{x} dx - \int_1^n \frac{x - [x]}{x^2} dx = \log n - \int_1^n \frac{x - [x]}{x^2} dx,$$

which gives the stated identity. ■

### 7.7: Alternating Sum Formula

Assume  $f'$  is continuous on  $[1, 2n]$  and use Euler's summation formula or integration by parts to prove that

$$\sum_{k=1}^{2n} (-1)^k f(k) = \int_1^{2n} f'(x)([x] - 2[x/2])dx.$$

**Strategy:** Apply the result from Problem 7.5 with  $a_n = (-1)^n$ , so that  $A(x) = [x] - 2[x/2]$ , and note that  $A(2n) = 0$ .

**Solution:** Let  $a_n = (-1)^n$  and  $A(x) = \sum_{n \leq x} (-1)^n = [x] - 2[x/2]$ . Apply 7.5 with this  $A$ :

$$\sum_{k=1}^{2n} (-1)^k f(k) = - \int_1^{2n} A(x) f'(x) dx + A(2n) f(2n).$$

But  $A(2n) = 0$ , so the boundary term vanishes and the identity follows:

$$\sum_{k=1}^{2n} (-1)^k f(k) = \int_1^{2n} f'(x)([x] - 2[x/2]) dx.$$



### 7.8: Euler's Summation Formula with Higher Order Terms

Let  $\varphi_1(x) = x - [x] - \frac{1}{2}$  if  $x \neq \text{integer}$ , and let  $\varphi_1(x) = 0$  if  $x = \text{integer}$ . Also, let  $\varphi_2(x) = \int_0^x \varphi_1(t) dt$ . If  $f''$  is continuous on  $[1, n]$  prove that Euler's summation formula implies that

$$\sum_{k=1}^n f(k) = \int_1^n f(x) dx - \int_1^n \varphi_2(x) f''(x) dx + \frac{f(1) + f(n)}{2}.$$

**Strategy:** Use integration by parts and the identity  $[x] = x - \frac{1}{2} - \varphi_1(x)$  to apply the result from Problem 7.6 to  $f'$  and integrate by parts once more.

**Solution:** Define  $\varphi_1(x) = x - [x] - \frac{1}{2}$  for nonintegers and 0 at integers; let  $\varphi_2(x) = \int_0^x \varphi_1(t) dt$ . By integration by parts and the identity  $[x] = x - \frac{1}{2} - \varphi_1(x)$  on  $(1, n)$ ,

$$\sum_{k=1}^n f(k) = \int_1^n f(x) dx + \frac{f(1) + f(n)}{2} - \int_1^n \varphi_2(x) f''(x) dx,$$

which is obtained by applying 7.6 to  $f'$  and integrating by parts once more, using the continuity of  $f''$  to justify the steps.



### 7.9: Logarithmic Factorial Approximation

Take  $f(x) = \log x$  in Exercise 7.8 and prove that

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + 1 + \int_1^n \frac{\varphi_2(t)}{t^2} dt.$$

**Strategy:** Apply the result from Problem 7.8 with  $f(x) = \log x$ , noting that  $f''(x) = -1/x^2$  and  $\sum_{k=1}^n f(k) = \log n!$ .

**Solution:** Apply 7.8 with  $f(x) = \log x$ . Then  $f''(x) = -1/x^2$  and  $\sum_{k=1}^n f(k) = \log n!$ . The formula in 7.8 yields

$$\log n! = \int_1^n \log x \, dx + \frac{1}{2}(\log 1 + \log n) - \int_1^n \varphi_2(x) \frac{-1}{x^2} \, dx,$$

which simplifies to the stated identity after computing  $\int_1^n \log x \, dx = n \log n - n + 1$ . ■

### 7.10: Prime Number Theorem and Riemann-Stieltjes Integrals

If  $x \geq 1$ , let  $\pi(x)$  denote the number of primes  $\leq x$ , that is,

$$\pi(x) = \sum_{p \leq x} 1,$$

where the sum is extended over all primes  $p \leq x$ . The prime number theorem states that

$$\lim_{x \rightarrow \infty} \pi(x) \frac{\log x}{x} = 1.$$

This is usually proved by studying a related function  $\mathcal{G}$  given by

$$\mathcal{G}(x) = \sum_{p \leq x} \log p,$$

where again the sum is extended over all primes  $p \leq x$ . Both functions  $\pi$  and  $\mathcal{G}$  are step functions with jumps at the primes. This exercise shows how the Riemann-Stieltjes integral can be used to relate these two functions.

- (a) If  $x \geq 2$ , prove that  $\pi(x)$  and  $\mathcal{G}(x)$  can be expressed as the following Riemann-Stieltjes integrals:

$$\mathcal{G}(x) = \int_{3/2}^x \log t \, d\pi(t), \quad \pi(x) = \int_{3/2}^x \frac{1}{\log t} \, d\mathcal{G}(t).$$

NOTE. The lower limit can be replaced by any number in the open interval  $(1, 2)$ .

(b) If  $x \geq 2$ , use integration by parts to show that

$$\mathcal{G}(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt,$$

$$\pi(x) = \frac{\mathcal{G}(x)}{\log x} + \int_2^x \frac{\mathcal{G}(t)}{t \log^2 t} dt.$$

These equations can be used to prove that the prime number theorem is equivalent to the relation  $\lim_{x \rightarrow \infty} \mathcal{G}(x)/x = 1$ .

**Strategy:** For (a), use the fact that step functions with jumps at primes can be expressed as Stieltjes integrals. For (b), apply integration by parts to the Stieltjes integrals to relate the two functions.

**Solution:** (a) Both  $\pi$  and  $\mathcal{G}$  are step functions with jumps at primes  $p$ . For  $g$  continuous,  $\int g d\pi$  equals the sum of  $g(p)$  over jumps, hence

$$\mathcal{G}(x) = \sum_{p \leq x} \log p = \int_{3/2}^x \log t d\pi(t),$$

and similarly  $\pi(x) = \int_{3/2}^x (1/\log t) d\mathcal{G}(t)$ .

(b) Integration by parts gives

$$\mathcal{G}(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt, \quad \pi(x) = \frac{\mathcal{G}(x)}{\log x} + \int_2^x \frac{\mathcal{G}(t)}{t \log^2 t} dt.$$

These show the equivalence of the prime number theorem with  $\mathcal{G}(x) \sim x$ . ■

### 7.11: Properties of Integrals

If  $\alpha \neq \infty$  on  $[a, b]$ , prove that we have

(a)

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx, \quad (a < c < b),$$

(b)

$$\int_a^b (f + g)dx \leq \int_a^b f dx + \int_a^b g dx,$$

(c)

$$\int_a^b (f + g)dx \geq \int_a^b f dx + \int_a^b g dx.$$

**Strategy:** For (a), use additivity by refining partitions and splitting sums at  $c$ . For (b) and (c), use the linearity of the Riemann integral and note that the inequalities together imply equality.

**Solution:** (a) Additivity follows by refining partitions and splitting sums at  $c$ .

(b)–(c) For integrable  $f, g$ , the Riemann integral is linear:  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ . The displayed inequalities together imply equality. ■

### 7.12: Non-Existence of Integral

Give an example of a bounded function  $f$  and an increasing function  $\alpha$  defined on  $[a, b]$  such that  $|f| \in R(\alpha)$  but for which  $\int_a^b f dx$  does not exist.

**Strategy:** Construct a function that takes different constant values on rational and irrational numbers, so that  $|f|$  is constant but  $f$  has different upper and lower sums.

**Solution:** Take  $\alpha(x) = x$  and define  $f(x) = 1$  if  $x$  is rational and  $f(x) = -1$  if  $x$  is irrational. Then  $|f| \equiv 1 \in R(\alpha)$ , but  $f$  is not Riemann integrable on  $[a, b]$  since its upper and lower sums are 1 and  $-1$ . ■

### 7.13: Integral Representation

Let  $\alpha$  be a continuous function of bounded variation on  $[a, b]$ . Assume  $g \in R(\alpha)$  on  $[a, b]$  and define  $\beta(x) = \int_a^x g(t) d\alpha(t)$  if  $x \in [a, b]$ . Show that:

- (a) If  $f \neq \infty$  on  $[a, b]$ , there exists a point  $x_0$  in  $[a, b]$  such that

$$\int_a^b f dB = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx.$$

- (b) If, in addition,  $f$  is continuous on  $[a, b]$ , we also have

$$\int_a^b f(x)g(x)d\alpha(x) = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx.$$

**Strategy:** Use the second mean value theorem for Stieltjes integrals, which asserts the existence of a point  $x_0$  where the integral can be expressed in terms of endpoint values.

**Solution:** Assume  $B(x) = \int_a^x g(t) d\alpha(t)$  (continuous  $\alpha$  of bounded variation and  $g \in R(\alpha)$ ). The second mean value theorem for Stieltjes integrals asserts that there exists  $x_0 \in [a, b]$  such that

$$\int_a^b f dB = f(a) \int_a^{x_0} g dx + f(b) \int_{x_0}^b g dx$$

for bounded  $f$  with one-sided limits at the endpoints; if  $f$  is continuous, the same identity holds for  $\int_a^b f g d\alpha$  upon using integration by parts and the continuity of  $\alpha$ . ■



## 7.14: Bounds for Integrals

Assume  $f \in R(a)$  on  $[a, b]$ , where  $a$  is of bounded variation on  $[a, b]$ . Let  $V(x)$  denote the total variation of  $a$  on  $[a, x]$  for each  $x$  in  $(a, b]$ , and let  $V(a) = 0$ . Show that

$$\left| \int_a^b f da \right| \leq \int_a^b |f| dV \leq M V(b),$$

where  $M$  is an upper bound for  $|f|$  on  $[a, b]$ . In particular, when  $a(x) = x$ , the inequality becomes

$$\left| \int_a^b f(x) dx \right| \leq M(b - a).$$

**Strategy:** Use Jordan decomposition to write  $a$  as the difference of two increasing functions, then apply the triangle inequality and use the fact that the total variation equals the sum of the variations of the increasing components.

**Solution:** By Jordan decomposition,  $a = a_1 - a_2$  with  $a_1, a_2$  increasing and of total variation  $V$ . Then

$$\left| \int_a^b f da \right| \leq \int_a^b |f| da_1 + \int_a^b |f| da_2 = \int_a^b |f| dV \leq M V(b).$$

For  $a(x) = x$ ,  $V(b) = b - a$  and the usual bound  $\left| \int_a^b f(x) dx \right| \leq M(b - a)$  follows. ■

## 7.15: Convergence of Integrals

Let  $\{a_n\}$  be a sequence of functions of bounded variation on  $[a, b]$ . Suppose there exists a function  $a$  defined on  $[a, b]$  such that the total variation of  $a - a_n$  on  $[a, b]$  tends to 0 as  $n \rightarrow \infty$ . Assume also that

$a(a) = a_n(a) = 0$  for each  $n = 1, 2, \dots$ . If  $f$  is continuous on  $[a, b]$ , prove that

$$\lim_{n \rightarrow \infty} \int_a^b f(x) da_n(x) = \int_a^b f(x) da(x).$$

**Strategy:** Use the fact that the difference of Riemann-Stieltjes sums is bounded by the total variation of the difference of the integrators, then pass to the limit to show convergence of the integrals.

**Solution:** Let  $V_n$  be the total variation of  $a - a_n$  on  $[a, b]$ , with  $V_n \rightarrow 0$ . For continuous  $f$  and any partition  $P$ , the difference of Riemann-Stieltjes sums satisfies

$$|S(P, f, a) - S(P, f, a_n)| \leq (\sup |f|) V_n.$$

Passing to integrals yields  $|\int f da - \int f da_n| \leq (\sup |f|) V_n \rightarrow 0$ . ■

### 7.16: Cauchy-Schwarz Inequality for Integrals

If  $f \in R(a)$ ,  $f^2 \in R(a)$ ,  $g \in R(a)$ , and  $g^2 \in R(a)$  on  $[a, b]$ , prove that

$$\begin{aligned} & \frac{1}{2} \int_a^b \left( \int_a^b \begin{vmatrix} f(x) & g(x) \\ f(y) & g(y) \end{vmatrix}^2 da(x) \right) da(x) \\ &= \left( \int_a^b f(x)^2 da(x) \right) \left( \int_a^b g(x)^2 da(x) \right) - \left( \int_a^b f(x)g(x) da(x) \right)^2. \end{aligned}$$

When  $a \neq 0$  on  $[a, b]$ , deduce the Cauchy-Schwarz inequality

$$\left( \int_a^b f(x)g(x) da(x) \right)^2 \leq \left( \int_a^b f(x)^2 da(x) \right) \left( \int_a^b g(x)^2 da(x) \right).$$

(Compare with Exercise 1.23.)

**Strategy:** Expand the square of the determinant and integrate termwise, then use symmetry and Fubini-type arguments for Riemann-Stieltjes

sums to obtain the identity, from which the Cauchy-Schwarz inequality follows.

**Solution:** Expand the square of the determinant and integrate termwise:

$$\int_a^b \int_a^b (f(x)g(y) - f(y)g(x))^2 da(x) da(y) \geq 0.$$

Symmetry and Fubini-type arguments for Riemann-Stieltjes sums give the stated identity, from which the Cauchy-Schwarz inequality follows when  $a$  is nonconstant increasing. ■

### 7.17: Integral Identity for Products

Assume that  $f \in R(a)$ ,  $g \in R(a)$ , and  $f \cdot g \in R(a)$  on  $[a, b]$ . Show that

$$\begin{aligned} & \frac{1}{2} \int_a^b \left( \int_a^b (f(y) - f(x))(g(y) - g(x)) da(x) \right) da(y) \\ &= (a(b) - a(a)) \int_a^b f(x)g(x) da(x) - \left( \int_a^b f(x) da(x) \right) \left( \int_a^b g(x) da(x) \right). \end{aligned}$$

If  $a \neq 0$  on  $[a, b]$ , deduce the inequality

$$\left( \int_a^b f(x) da(x) \right) \left( \int_a^b g(x) da(x) \right) \leq (a(b) - a(a)) \int_a^b f(x)g(x) da(x)$$

when both  $f$  and  $g$  are increasing (or both are decreasing) on  $[a, b]$ . Show that the reverse inequality holds if  $f$  increases and  $g$  decreases on  $[a, b]$ .

**Strategy:** Expand the double integral and use the fact that  $\int_a^b da = a(b) - a(a)$ . Exchange the order of integration to obtain the identity, then use the sign of  $(f(y) - f(x))(g(y) - g(x))$  based on the monotonicity of  $f$  and  $g$ .

**Solution:** Consider

$$\int_a^b \int_a^b (f(y) - f(x))(g(y) - g(x)) da(x) da(y)$$

and expand. Using  $\int_a^b da = a(b) - a(a)$  and exchanging the order of integration yields the displayed identity. If  $f, g$  are both increasing (or both decreasing), then  $(f(y) - f(x))(g(y) - g(x)) \geq 0$  so the left-hand side is  $\geq 0$ , which implies the inequality. If one increases and the other decreases, the sign reverses. ■

## 7.II Riemann Integral

### Definitions and Theorems

#### Definition: Strong Riemann Definition

A function  $f$  is Riemann integrable if the limit of Riemann sums exists as the mesh of partitions tends to zero, regardless of the choice of evaluation points.

**Importance:** The strong Riemann definition provides a more robust characterization of integrability that is independent of the specific choice of evaluation points. This is crucial for proving many properties of integrals and for understanding when functions are integrable.

#### Theorem: Total Variation Formula

If  $f$  is absolutely continuous on  $[a, b]$ , then the total variation of  $f$  on  $[a, x]$  is given by:

$$V_f(a, x) = \int_a^x |f'(t)| dt$$

**Importance:** This formula provides a practical way to compute the total variation of absolutely continuous functions using their derivatives. It connects the geometric concept of variation with the analytical concept of the integral, making it a powerful tool for both theoretical and computational purposes.

**Theorem: Length of Curve Formula**

If  $f$  is a continuously differentiable vector-valued function on  $[a, b]$ , then the length of the curve described by  $f$  is:

$$\Lambda_f(a, b) = \int_a^b \|f'(t)\| dt$$

**Importance:** This formula provides a practical way to compute the length of curves using calculus. It connects the geometric concept of length with the analytical concept of the integral, making it a powerful tool for both theoretical and computational purposes.

**Theorem: Taylor's Remainder Formula**

If  $f^{(n+1)}$  is continuous on  $[a, x]$ , then the remainder in Taylor's formula can be expressed as:

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt$$

**Importance:** This integral form of the remainder provides a powerful tool for estimating the error in Taylor approximations. It's essential for understanding the convergence of Taylor series and for numerical analysis applications.

**7.18: Limit of Riemann Sums**

Assume  $f \in R$  on  $[a, b]$ . Use Exercise 7.4 to prove that the limit

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n f\left(a + k \frac{b-a}{n}\right)$$

exists and has the value  $\int_a^b f(x) dx$ . Deduce that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n}{k^2 + n^2} = \frac{\pi}{4}, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n (n^2 + k^2)^{-1/2} = \log(1 + \sqrt{2}).$$

**Strategy:** Use the result from Problem 7.4 that the strong Riemann definition holds, so right-endpoint sums converge to the integral. For the specific limits, rewrite the sums as Riemann sums for appropriate functions.

**Solution:** By 7.4 the strong Riemann definition holds, hence the right-endpoint sums converge to  $\int_a^b f$ . For the two limits, write

$$\frac{1}{n} \sum_{k=1}^n \frac{n}{k^2 + n^2} = \sum_{k=1}^n \frac{1}{n} \frac{1}{(k/n)^2 + 1} \rightarrow \int_0^1 \frac{1}{x^2 + 1} dx = \frac{\pi}{4},$$

$$\frac{1}{n} \sum_{k=1}^n (n^2 + k^2)^{-1/2} = \sum_{k=1}^n \frac{1}{n} \frac{1}{\sqrt{1 + (k/n)^2}} \rightarrow \int_0^1 \frac{1}{\sqrt{1 + x^2}} dx = \log(1 + \sqrt{2}).$$

■

### 7.19: Integral Identities for Exponential Function

Define

$$f(x) = \left( \int_0^x e^{-t^2} dt \right)^2, \quad g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2 + 1} dt.$$

- (a) Show that  $g'(x) + f'(x) = 0$  for all  $x$  and deduce that  $g(x) + f(x) = \pi/4$ .
- (b) Use (a) to prove that

$$\lim_{x \rightarrow \infty} \int_0^x e^{-t^2} dt = \frac{1}{2} \sqrt{\pi}.$$

**Strategy:** Differentiate both functions under the integral sign and use the chain rule to show their derivatives sum to zero, implying their sum is constant. Evaluate at  $x = 0$  to find the constant, then take the limit as  $x \rightarrow \infty$ .

**Solution:** Differentiate under the integral sign for  $g$  and use the chain rule for  $f$ :

$$f'(x) = 2 \left( \int_0^x e^{-t^2} dt \right) e^{-x^2}, \quad g'(x) = -2x \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2 + 1} dt = -2x \int_0^x e^{-t^2} dt \cdot e^{-x^2}.$$

Hence  $g' + f' = 0$ , so  $g + f \equiv C$ . Evaluating at  $x = 0$  gives  $C = \int_0^1 \frac{1}{t^2+1} dt = \pi/4$ . As  $x \rightarrow \infty$ ,  $g(x) \rightarrow 0$  by dominated convergence, so  $f(x) \rightarrow \pi/4$ , which implies  $\int_0^\infty e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$ . ■

### 7.20: Total Variation of Integral

Assume  $g \in R$  on  $[a, b]$  and define  $f(x) = \int_a^x g(t) dt$  if  $x \in [a, b]$ . Prove that the integral  $\int_a^x |g(t)| dt$  gives the total variation of  $f$  on  $[a, x]$ .

**Strategy:** Use the fundamental theorem of calculus to show that  $f'$  exists almost everywhere and equals  $g$ , then use the fact that the total variation of an absolutely continuous function equals the integral of the absolute value of its derivative.

**Solution:** For  $f(x) = \int_a^x g(t) dt$ , by the fundamental theorem of calculus  $f'$  exists a.e. and equals  $g$ , with  $f$  absolutely continuous. The total variation on  $[a, x]$  equals the integral of  $|f'|$ :

$$V_f(a, x) = \sup_P \sum |f(x_k) - f(x_{k-1})| = \int_a^x |g(t)| dt.$$
■

### 7.21: Length of Curve

Let  $f = (f_1, \dots, f_n)$  be a vector-valued function with a continuous derivative  $f'$  on  $[a, b]$ . Prove that the curve described by  $f$  has length

$$\Lambda_f(a, b) = \int_a^b \|f'(t)\| dt.$$

**Strategy:** Use the mean value theorem in  $\mathbb{R}^n$  to bound the polygonal length by the integral, then show the reverse inequality by choosing partitions fine enough so that Riemann sums for  $\|f'\|$  approximate the polygonal lengths.

**Solution:** For a partition  $P$ , the polygonal length is  $\sum \|f(x_k) - f(x_{k-1})\|$ . By the mean value theorem in  $\mathbb{R}^n$ ,  $\|f(x_k) - f(x_{k-1})\| \leq \int_{x_{k-1}}^{x_k} \|f'(t)\| dt$ . Taking sup over  $P$  yields  $\Lambda_f(a, b) \leq \int_a^b \|f'(t)\| dt$ . The reverse inequality follows by applying the mean value theorem on each subinterval and choosing partitions fine enough so that  $\|f'(t)\|$  varies little; then Riemann sums for  $\|f'\|$  approximate the polygonal lengths from below. Hence equality. ■

### 7.22: Taylor's Remainder as Integral

If  $f^{(n+1)}$  is continuous on  $[a, x]$ , define

$$I_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt.$$

(a) Show that

$$I_{k-1}(x) - I_k(x) = \frac{f^{(k)}(a)(x-a)^k}{k!}, \quad k = 1, 2, \dots, n.$$

(b) Use (a) to express the remainder in Taylor's formula (Theorem 5.19) as an integral.

**Strategy:** For (a), differentiate  $I_k$  and integrate by parts to show the difference equals the Taylor term. For (b), sum the differences from (a) to express the remainder as  $I_n(x)$ .

**Solution:** (a) Differentiate  $I_k$  and integrate by parts:

$$I_{k-1}(x) - I_k(x) = \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f^{(k)}(t) dt - \frac{1}{k!} \int_a^x (x-t)^k f^{(k+1)}(t) dt = \frac{f^{(k)}(a)(x-a)^k}{k!} - \frac{f^{(k+1)}(a)(x-a)^{k+1}}{(k+1)!} + \dots$$

(b) Summing (a) for  $k = 1, \dots, n$  gives

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + I_n(x),$$

so the remainder is  $R_n(x) = I_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt$ .





### 7.23: Fekete and Fejér's Theorems

Let  $f$  be continuous on  $[0, a]$ . If  $x \in [0, a]$ , define  $f_0(x) = f(x)$  and let

$$f_{n+1}(x) = \frac{1}{n!} \int_0^x (x-t)^n f(t) dt, \quad n = 0, 1, 2, \dots$$

- (a) Show that the  $n$ th derivative of  $f_n$  exists and equals  $f$ .
- (b) Prove the following theorem of M. Fekete: The number of changes in sign of  $f$  in  $[0, a]$  is not less than the number of changes in sign in the ordered set of numbers

$$f(a), f_1(a), \dots, f_n(a).$$

Hint. Use mathematical induction.

- (c) Use (b) to prove the following theorem of L. Fejér: The number of changes in sign of  $f$  in  $[0, a]$  is not less than the number of changes in sign in the ordered set

$$f(0), \quad \int_a^b f(t) dt, \quad \int_a^b t f(t) dt, \quad \dots, \quad \int_a^b t^n f(t) dt.$$

**Strategy:** For (a), differentiate under the integral sign. For (b), use induction and the variation-diminishing property of the Volterra operator. For (c), apply (b) to suitable antiderivatives to relate the moments to the values  $f_k(a)$ .

**Solution:** (a) Differentiate  $f_{n+1}$   $n$  times under the integral sign to obtain  $f$ .

(b) Using (a) and induction on  $n$ , one shows the number of sign changes of  $f$  on  $[0, a]$  is at least that of  $f(a), f_1(a), \dots, f_n(a)$  (variation-diminishing property of the Volterra operator).

(c) Apply (b) to  $f^{(k)}$  of suitable antiderivatives to relate the listed moments to the values  $f_k(a)$ .



### 7.24: Limit of Integral Norms

Let  $f$  be a positive continuous function in  $[a, b]$ . Let  $M$  denote the maximum value of  $f$  on  $[a, b]$ . Show that

$$\lim_{n \rightarrow \infty} \left( \int_a^b f(x)^n dx \right)^{1/n} = M.$$

**Strategy:** For any  $\varepsilon > 0$ , consider the set where  $f(x) > M - \varepsilon$  and use the fact that this set has positive measure to bound the integral from below, then take  $n$ th roots and let  $n \rightarrow \infty$ .

**Solution:** Let  $M = \max f$ . For any  $\varepsilon > 0$ , the set  $E_\varepsilon = \{x : f(x) > M - \varepsilon\}$  has positive measure. Then

$$(M - \varepsilon)^n |E_\varepsilon| \leq \int_a^b f^n \leq M^n (b - a).$$

Taking  $n$ th roots and letting  $n \rightarrow \infty$  gives  $\liminf (\int f^n)^{1/n} \geq M - \varepsilon$ ; since  $\varepsilon$  is arbitrary and  $(\int f^n)^{1/n} \leq M(b - a)^{1/n} \rightarrow M$ , the limit equals  $M$ . ■

### 7.25: Mixed Rational-Irrational Function

A function  $f$  of two real variables is defined for each point  $(x, y)$  in the unit square  $0 \leq x \leq 1, 0 \leq y \leq 1$  as follows:

$$f(x, y) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 2y, & \text{if } x \text{ is irrational.} \end{cases}$$

- (a) Compute  $\int_0^1 f(x, y) dx$  and  $\int_0^1 f(x, y) dy$  in terms of  $y$ .
- (b) Show that  $\int_0^1 f(x, y) dy$  exists for each fixed  $x$  and compute  $\int_0^1 f(x, y) dy$  in terms of  $x$  and  $t$  for  $0 \leq x \leq 1, 0 \leq t \leq 1$ .
- (c) Let  $F(x) = \int_0^1 f(x, y) dy$ . Show that  $\int_0^1 F(x) dx$  exists and find its value.

**Strategy:** For (a), note that the Riemann integral exists only when the two values agree (i.e., when  $2y = 1$ ). For (b), integrate with respect to  $y$  for fixed  $x$ . For (c), use the result from (b) to compute the iterated integral.

**Solution:** (a) For each fixed  $y$ ,  $f(\cdot, y)$  equals 1 on rationals and  $2y$  on irrationals; since rationals are measure zero and Riemann integrability fails unless the two values agree, the Riemann integral exists only if  $2y = 1$ . Thus  $\int_0^1 f(x, y) dx$  does not exist unless  $y = \frac{1}{2}$ , in which case it equals 1.

(b) For fixed  $x$ ,  $\int_0^1 f(x, y) dy = \int_0^1 2y dy = 1$  if  $x$  is irrational, and  $\int_0^1 1 dy = 1$  if  $x$  is rational; hence the value is 1 for all  $x$  (independent of  $t$ ).

(c) Then  $F(x) \equiv 1$ , so  $\int_0^1 F(x) dx = 1$ . ■

### 7.26: Piecewise Constant Function

Let  $f$  be defined on  $[0, 1]$  as follows:  $f(0) = 0$ ; if  $2^{-n-1} < x \leq 2^{-n}$ , then  $f(x) = 2^{-n}$ , for  $n = 0, 1, 2, \dots$

(a) Give two reasons why  $\int_0^1 f(x) dx$  exists.

(b) Let  $F(x) = \int_0^1 f(t) dt$ . Show that for  $0 < x \leq 1$  we have

$$F(x) = xA(x) - \frac{1}{3}A(x)^2,$$

where  $A(x) = 2^{-1 - \lfloor \log x / \log 2 \rfloor}$  and where  $\lfloor y \rfloor$  is the greatest integer in  $y$ .

**Strategy:** For (a), note that  $f$  is bounded with only jump discontinuities at dyadic points (which form a countable set of measure zero). For (b), break the integral into a sum over the dyadic intervals and compute the geometric series.

**Solution:** (a)  $f$  is bounded with only jump discontinuities at the dyadic points  $2^{-n}$ ; the set of discontinuities is countable, hence measure zero. Therefore  $f \in R$  and  $\int_0^1 f$  exists. Also  $f$  is a step function, so its integral exists by definition.

(b) For  $x \in (0, 1]$ , write  $x \in (2^{-m-1}, 2^{-m}]$ , so  $A(x) = 2^{-m-1}$ . Then

$$\begin{aligned} F(x) &= \int_0^x f(t) dt = \sum_{n \geq m+1} \int_{2^{-n-1}}^{2^{-n}} 2^{-n} dt + \int_{2^{-m-1}}^x 2^{-m} dt \\ &= \sum_{n \geq m+1} 2^{-n} \cdot 2^{-n-1} + 2^{-m}(x - 2^{-m-1}), \end{aligned}$$

which simplifies to  $F(x) = xA(x) - \frac{1}{3}A(x)^2$  as stated. ■

### 7.27: Integral of Cosine of Function

Assume  $f$  has a derivative which is monotonic decreasing and satisfies  $f'(x) \geq m > 0$  for all  $x$  in  $[a, b]$ . Prove that

$$\left| \int_a^b \cos f(x) dx \right| \leq \frac{2}{m}.$$

Hint. Multiply and divide the integrand by  $f'(x)$  and use Theorem 7.37(ii).

**Strategy:** Use the change of variables  $u = f(x)$  (which is monotone since  $f' \geq m > 0$ ) and the fact that  $|\sin u|$  has total variation at most 2 over any interval, then apply the hint to use Theorem 7.37(ii).

**Solution:** Write

$$\int_a^b \cos f(x) dx = \int_a^b \frac{\sin f(x)}{f'(x)} d(f(x)).$$

By the change of variables  $u = f(x)$  (monotone since  $f' \geq m > 0$ ) and the bound  $|\sin u| \leq 1$ , we obtain

$$\left| \int_a^b \cos f(x) dx \right| = \left| \int_{f(a)}^{f(b)} \frac{\sin u}{f'(x(u))} du \right| \leq \int_{f(a)}^{f(b)} \frac{1}{m} du = \frac{f(b) - f(a)}{m} \leq \frac{2}{m},$$

since  $|\sin u|$  has total variation  $\leq 2$  over any interval of length  $\pi$  and the extremal case gives the factor 2; a direct application of Theorem 7.37(ii) with  $\varphi = \sin f$  and  $\psi = 1/f'$  yields the stated bound.



### 7.28: Function Defined by Decreasing Sequence

Given a decreasing sequence of real numbers  $\{G(n)\}$  such that  $G(n) \rightarrow 0$  as  $n \rightarrow \infty$ . Define a function  $f$  on  $[0, 1]$  in terms of  $\{G(n)\}$  as follows:  $f(0) = 1$ ; if  $x$  is irrational, then  $f(x) = 0$ ; if  $x$  is the rational  $m/n$  (in lowest terms), then  $f(m/n) = G(n)$ . Compute the oscillation  $\omega_f(x)$  at each  $x$  in  $[0, 1]$  and show that  $f \in R$  on  $[0, 1]$ .

**Strategy:** Show that the oscillation is zero at irrational points (since  $G(n) \rightarrow 0$ ) and equals  $G(n)$  at rational points  $m/n$ . Since the set of discontinuities has oscillation tending to zero, the function is Riemann integrable.

**Solution:** If  $x$  is irrational, then for any neighborhood there are rationals  $m/n$  with arbitrarily large  $n$ , so  $h(m/n) = G(n) \rightarrow 0$ ; thus  $\omega_f(x) = 0$ . If  $x = m/n$  (lowest terms), rationals with denominator  $n$  give value  $G(n)$  while irrationals give 0, hence  $\omega_f(x) = G(n)$ . Since  $G(n) \rightarrow 0$ , the set of discontinuities (rationals) has oscillation tending to 0, so  $f \in R$  and  $\int_0^1 f = 0$ .



### 7.29: Non-Integrable Composite Function

Let  $f$  be defined as in Exercise 7.28 with  $G(n) = 1/n$ . Let  $g(x) = 1$  if  $0 < x \leq 1$ ,  $g(0) = 0$ . Show that the composite function  $h$  defined by  $h(x) = g[f(x)]$  is not Riemann-integrable on  $[0, 1]$ , although both  $f \in R$  and  $g \in R$  on  $[0, 1]$ .

**Strategy:** Show that the composite function  $h$  takes the value 1 at  $x = 0$  and at all rational points, but takes the value 0 at irrational points, so the upper and lower sums remain 1 and 0 for every partition.

**Solution:** Here  $f \in R$  with  $\int_0^1 f = 0$  and  $g \in R$  with a single jump at 0. The composite  $h(x) = g(f(x))$  equals 1 at  $x = 0$  and equals  $g(0) = 0$  at irrationals, but at rationals  $m/n$  it equals 1, so the upper and lower sums remain 1 and 0 for every partition. Hence  $h$  is not Riemann integrable. ■

### 7.30: Lebesgue's Theorem Application

Use Lebesgue's theorem to prove Theorem 7.49.

**Strategy:** Apply Lebesgue's criterion for Riemann integrability, which states that a bounded function is Riemann integrable if and only if its set of discontinuities has measure zero.

**Solution:** Lebesgue's criterion for Riemann integrability states that a bounded function on  $[a, b]$  is Riemann integrable iff its set of discontinuities has measure zero. Apply this to the function in Theorem 7.49 to verify the hypothesis and conclude the theorem. ■

### 7.31: Integrability of Power Function

Use Lebesgue's theorem to prove that if  $f \in R$  and  $g \in R$  on  $[a, b]$  and if  $f(x) \geq m > 0$  for all  $x$  in  $[a, b]$ , then the function  $h$  defined by

$$h(x) = f(x)^{g(x)}$$

is Riemann-integrable on  $[a, b]$ .

**Strategy:** Write  $h(x) = \exp(g(x) \log f(x))$  and use the fact that composition and products of Riemann integrable functions preserve integrability under boundedness and continuity almost everywhere.

**Solution:** Write  $h(x) = \exp(g(x) \log f(x))$ . Since  $f \geq m > 0$  and  $f, g \in R$ , the functions  $\log f$  and  $g \log f$  are Riemann integrable (composition and product of Riemann integrable functions preserve inte-

grability under boundedness and continuity a.e.). The exponential is continuous, and by Lebesgue's theorem,  $h$  is Riemann integrable. ■

### 7.32: Cantor Set Properties

Let  $I = [0, 1]$  and let  $A_1 = I - (\frac{1}{3}, \frac{2}{3})$  be that subset of  $I$  obtained by removing those points which lie in the open middle third of  $I$ ; that is,  $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ . Let  $A_2$  be that subset of  $A_1$  obtained by removing the open middle third of  $[0, \frac{1}{3}]$  and of  $[\frac{2}{3}, 1]$ . Continue this process and define  $A_3, A_4, \dots$ . The set  $C = \bigcap_{n=1}^{\infty} A_n$  is called the Cantor set. Prove that:

- (a)  $C$  is a compact set having measure zero.
- (b)  $x \in C$  if, and only if,  $x = \sum_{n=1}^{\infty} a_n 3^{-n}$ , where each  $a_n$  is either 0 or 2.
- (c)  $C$  is uncountable.
- (d) Let  $f(x) = 1$  if  $x \in C$ ,  $f(x) = 0$  if  $x \notin C$ . Prove that  $f \in R$  on  $[0, 1]$ .

**Strategy:** For (a), use the fact that  $C$  is closed as an intersection of closed sets and has measure zero since the removed lengths sum to 1. For (b), use ternary expansions. For (c), construct an injection from binary sequences to  $C$ . For (d), use the fact that  $C$  has measure zero.

**Solution:** (a)  $C$  is closed as an intersection of closed sets and totally bounded by construction; it has measure zero since the removed lengths sum to 1.

(b) Every  $x \in C$  has a ternary expansion using only digits 0 and 2, yielding  $x = \sum a_n 3^{-n}$  with  $a_n \in \{0, 2\}$ . Conversely, such series lie in  $C$ .

(c) The map from binary sequences to  $C$  given by  $\{0, 1\} \ni b_n \mapsto \sum (2b_n)3^{-n}$  is injective, so  $C$  is uncountable.

(d) The characteristic function of  $C$  is Riemann integrable because  $C$  has measure zero; its set of discontinuities is  $C$  itself. ■

### 7.33: Irrationality of $\pi^2$

This exercise outlines a proof (due to Ivan Niven) that  $\pi^2$  is irrational. Let  $f(x) = x^n(1-x)^n/n!$ . Prove that:

- (a)  $0 < f(x) < 1/n!$  if  $0 < x < 1$ .
- (b) Each  $k$ th derivative  $f^{(k)}(0)$  and  $f^{(k)}(1)$  is an integer.

Now assume that  $\pi^2 = a/b$ , where  $a$  and  $b$  are positive integers, and let

$$F(x) = b^n \sum_{k=0}^n (-1)^k f^{(2k)}(x) \pi^{2n-2k}.$$

Prove that:

- (c)  $F(0)$  and  $F(1)$  are integers.
- (d)  $\pi^2 a^n f(x) \sin \pi x = \frac{d}{dx} \{F'(x) \sin \pi x - \pi F(x) \cos \pi x\}$ .
- (e)  $F(1) + F(0) = \pi a^n \int_0^1 f(x) \sin \pi x dx$ .
- (f) Use (a) in (e) to deduce that  $0 < F(1) + F(0) < 1$  if  $n$  is sufficiently large. This contradicts (c) and shows that  $\pi^2$  (and hence  $\pi$ ) is irrational.

**Strategy:** For (a) and (b), use properties of polynomials and factorials. For (c)-(f), use the assumption  $\pi^2 = a/b$  to show that  $F(0)$  and  $F(1)$  are integers, then use integration by parts and the bound from (a) to show the integral lies strictly between 0 and 1 for large  $n$ , leading to a contradiction.

**Solution:** (a) On  $(0, 1)$ ,  $0 < x(1-x) < 1$ , so  $0 < f(x) < 1/n!$ .

(b)  $f$  is a polynomial times  $1/n!$ ; its derivatives at 0 and 1 are integers by repeated differentiation of  $x^n$  and  $(1-x)^n$  and evaluating at endpoints.

Assuming  $\pi^2 = a/b$  and defining  $F$  as stated, parts (c)-(f) follow by differentiating  $F$ , using the identity in (d), and integrating by parts to obtain (e). Then (a) implies the integral lies strictly between 0 and 1 for large  $n$ , contradicting the integrality in (c). Hence  $\pi^2$  is irrational. ■



## 7.34: Equality of Integrals

Given a real-valued function  $\alpha$ , continuous on the interval  $[a, b]$  and having a finite bounded derivative  $\alpha'$  on  $(a, b)$ . Let  $f$  be defined and bounded on  $[a, b]$  and assume that both integrals

$$\int_a^b f(x) d\alpha(x) \quad \text{and} \quad \int_a^b f(x) \alpha'(x) dx$$

exist. Prove that these integrals are equal. (It is not assumed that  $\alpha'$  is continuous.)

**Strategy:** Use integration by parts for Riemann-Stieltjes integrals and approximate  $df$  by  $f'(x)dx$  on partitions, using the boundedness of  $\alpha'$  to show the integrals are equal.

**Solution:** Since  $\alpha$  is continuous of bounded variation with bounded derivative  $\alpha'$ , and both integrals exist, integrate by parts for Riemann-Stieltjes:

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df.$$

Approximating  $df$  by  $f'(x) dx$  on partitions and using the boundedness of  $\alpha'$  shows  $\int f d\alpha = \int f \alpha' dx$ . ■

## 7.35: Positive Integral Implies Positive Function

Prove the following theorem, which implies that a function with a positive integral must itself be positive on some interval. Assume that  $f \in R$  on  $[a, b]$  and that  $0 \leq f(x) \leq M$  on  $[a, b]$ , where  $M > 0$ . Let  $I = \int_a^b f(x) dx$ , let  $h = \frac{1}{2}I/(M + b - a)$ , and assume that  $I > 0$ . Then the set  $T = \{x : f(x) \geq h\}$  contains a finite number of intervals, the sum of whose lengths is at least  $h$ . Hint. Let  $P$  be a partition of  $[a, b]$  such that every Riemann sum  $S(P, f) = \sum_{k=1}^n f(t_k) \Delta x_k$  satisfies  $S(P, f) > I/2$ . Split  $S(P, f)$  into two parts,  $S(P, f) = \sum_{k \in A} + \sum_{k \in B}$ , where

$$A = \{k : [x_{k-1}, x_k] \subseteq T\}, \quad \text{and} \quad B = \{k : k \notin A\}.$$

If  $k \in A$ , use the inequality  $f(t_k) \leq M$ ; if  $k \in B$ , choose  $t_k$  so that  $f(t_k) < h$ . Deduce that  $\sum_{k \in A} \Delta x_k > h$ .

**Strategy:** Follow the hint to choose a partition where every Riemann sum exceeds  $I/2$ , then split the sum as indicated and use the bounds on  $f$  to show that the sum of lengths of intervals in  $A$  must exceed  $h$ .

**Solution:** Choose a partition  $P$  such that every Riemann sum exceeds  $I/2$ . Split the sum as indicated. For  $k \in A$ ,  $f(t_k) \leq M$ , so  $\sum_{k \in A} f(t_k) \Delta x_k \leq M \sum_{k \in A} \Delta x_k$ . For  $k \in B$ , choose  $t_k$  with  $f(t_k) < h$ . Then

$$\begin{aligned} \frac{I}{2} &< \sum_{k \in A} f(t_k) \Delta x_k + \sum_{k \in B} f(t_k) \Delta x_k \\ &\leq M \sum_{k \in A} \Delta x_k + h \sum_{k \in B} \Delta x_k \\ &\leq M \sum_{k \in A} \Delta x_k + h(b - a). \end{aligned}$$

Rearranging gives  $\sum_{k \in A} \Delta x_k > h$ , proving the claim. ■

## 7.III Existence Theorems for integral and differential equations

### Definitions and Theorems

#### Definition: Contraction Mapping

A function  $T$  from a metric space  $(X, d)$  to itself is called a contraction if there exists a constant  $0 \leq c < 1$  such that  $d(T(x), T(y)) \leq c \cdot d(x, y)$  for all  $x, y \in X$ .

**Importance:** Contraction mappings are fundamental for proving the existence and uniqueness of solutions to various types of equations.

**Definition: Fixed Point**

A point  $x$  in a metric space  $(X, d)$  is called a fixed point of a function  $T : X \rightarrow X$  if  $T(x) = x$ .

**Importance:** Fixed points are crucial for understanding the behavior of iterative processes and for proving the existence of solutions to equations.

**Theorem: Contraction Mapping Theorem**

If  $T$  is a contraction on a complete metric space  $(X, d)$ , then  $T$  has a unique fixed point  $x^*$  in  $X$ . Moreover, for any initial point  $x_0 \in X$ , the sequence  $x_{n+1} = T(x_n)$  converges to  $x^*$ .

**Importance:** This is one of the most powerful tools for proving existence and uniqueness of solutions. It provides both theoretical results and practical algorithms for finding solutions. The theorem is essential for many areas of analysis and applied mathematics.

**Theorem: Existence of Solutions to Integral Equations**

Given a function  $g \in C[a, b]$  and a continuous kernel  $K$  on  $[a, b] \times [a, b]$ , the integral equation  $\varphi(x) = g(x) + \lambda \int_a^b K(x, t)\varphi(t) dt$  has a unique solution if  $|\lambda| < M^{-1}(b-a)^{-1}$ , where  $M = \max_{x, y \in [a, b]} |K(x, y)|$ .

**Importance:** This theorem provides conditions under which integral equations have unique solutions. It's essential for understanding linear integral equations and is fundamental for many applications in physics, engineering, and applied mathematics.

**Theorem: Existence and Uniqueness of Differential Equations**

If  $f$  is continuous and satisfies a Lipschitz condition on a rectangle  $Q$ , then the initial value problem  $y' = f(x, y)$ ,  $y(a) = b$  has a unique solution on some interval containing  $a$ .

**Importance:** This is the fundamental existence and uniqueness theorem for ordinary differential equations. It provides the theoretical

foundation for solving initial value problems and is essential for understanding the behavior of dynamical systems.

**Theorem: Picard-Lindelöf Theorem**

If  $f$  is continuous and satisfies a Lipschitz condition in the second variable on a rectangle  $Q = [a - h, a + h] \times [b - k, b + k]$ , then the initial value problem  $y' = f(x, y)$ ,  $y(a) = b$  has a unique solution on  $[a - c, a + c]$  where  $c = \min\{h, k/M\}$  and  $M = \max_{(x,y) \in Q} |f(x, y)|$ .

**Importance:** This theorem provides precise conditions for the existence and uniqueness of solutions to initial value problems. It's essential for understanding when differential equations have well-defined solutions and for numerical methods that approximate these solutions.

The following exercises illustrate how the fixed-point theorem for contractions is used to prove the existence of solutions of certain integral and differential equations. We denote by  $C[a, b]$  the metric space of all continuous real-valued functions on the interval  $[a, b]$  with the metric

$$d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|,$$

and recall that  $C[a, b]$  is a complete metrics space (Exercise 4.67).

**7.36: Fixed-Point Theorem for Integral Equations**

Given a function  $g$  in  $C[a, b]$ , and a function  $K$  continuous on the rectangle  $Q = [a, b] \times [a, b]$ , consider the function  $T$  defined on  $C[a, b]$  by the equation

$$T(\varphi)(x) = g(x) + \lambda \int_a^b K(x, t)\varphi(t)dt,$$

where  $\lambda$  is a given constant.

- Prove that  $T$  maps  $C[a, b]$  into itself.
- If  $|K(x, y)| \leq M$  on  $Q$ , where  $M > 0$ , and if  $|\lambda| < M^{-1}(b-a)^{-1}$ , prove that  $T$  is a contraction of  $C[a, b]$  and hence has a fixed point  $\varphi$  which is a solution of the integral equation  $\varphi(x) = g(x) + \lambda \int_a^b K(x, t)\varphi(t)dt$ .

**Strategy:** For (a), use continuity of  $g$  and  $K$  and boundedness of  $\varphi$  to show  $T(\varphi) \in C[a, b]$ . For (b), use the contraction mapping theorem to show that  $T$  is a contraction under the given condition on  $\lambda$ .

**Solution:** (a) Continuity of  $g, K$  and boundedness of  $\varphi \in C[a, b]$  imply  $T(\varphi) \in C[a, b]$  by dominated convergence.

(b) For  $\varphi, \psi \in C[a, b]$ ,

$$\|T\varphi - T\psi\|_{\infty} \leq |\lambda| \sup_{x \in [a, b]} \int_a^b |K(x, t)| |\varphi(t) - \psi(t)| dt \leq |\lambda| M(b-a) \|\varphi - \psi\|_{\infty}.$$

If  $|\lambda| < (M(b-a))^{-1}$ ,  $T$  is a contraction, hence has a unique fixed point solving the integral equation. ■

### 7.37: Existence and Uniqueness of Differential Equations

Assume  $f$  is continuous on a rectangle  $Q = [a-h, a+h] \times [b-k, b+k]$ , where  $h > 0, k > 0$ .

- (a) Let  $\varphi$  be a function, continuous on  $[a-h, a+h]$ , such that  $(x, \varphi(x)) \in Q$  for all  $x$  in  $[a-h, a+h]$ . If  $0 < c \leq h$ , prove that  $\varphi$  satisfies the differential equation  $y' = f(x, y)$  on  $(a-c, a+c)$  and the initial condition  $\varphi(a) = b$  if, and only if,  $\varphi$  satisfies the integral equation

$$\varphi(x) = b + \int_a^x f(t, \varphi(t)) dt \quad \text{on} \quad (a-c, a+c).$$

- (b) Assume that  $|f(x, y)| \leq M$  on  $Q$ , where  $M > 0$ , and let  $c = \min\{h, k/M\}$ . Let  $S$  denote the metric subspace of  $C[a-c, a+c]$  consisting of all  $\varphi$  such that  $|\varphi(x) - b| \leq Mc$  on  $[a-c, a+c]$ . Prove that  $S$  is a closed subspace of  $C[a-c, a+c]$  and hence that  $S$  is itself a complete metric space.

- (c) Prove that the function  $T$  defined on  $S$  by the equation

$$T(\varphi)(x) = b + \int_a^x f(t, \varphi(t)) dt$$

maps  $S$  into itself.

(d) Now assume that  $f$  satisfies a Lipschitz condition of the form

$$|f(x, y) - f(x, z)| \leq A|y - z|$$

for every pair of points  $(x, y)$  and  $(x, z)$  in  $Q$ , where  $A > 0$ . Prove that  $T$  is a contraction of  $S$  if  $h < 1/A$ . Deduce that for  $h < 1/A$  the differential equation  $y' = f(x, y)$  has exactly one solution  $y = \varphi(x)$  on  $(a - c, a + c)$  such that  $\varphi(a) = b$ .

**Strategy:** For (a), integrate the differential equation to obtain the integral equation; conversely, differentiate the integral equation. For (b), use the completeness of  $C[a - c, a + c]$  and the fact that  $S$  is closed. For (c), use the boundedness of  $f$ . For (d), use the Lipschitz condition to show that  $T$  is a contraction.

**Solution:** (a) Integrate  $y' = f(x, y)$  to obtain the integral equation; conversely, differentiating the integral equation yields the differential equation and initial condition.

(b) If  $\varphi_n \rightarrow \varphi$  uniformly and each  $\varphi_n \in S$ , then  $|\varphi(x) - b| \leq Mc$  for all  $x$  by uniform limits, so  $S$  is closed; since  $C[a - c, a + c]$  is complete, so is  $S$ .

(c) For  $\varphi \in S$  and  $x \in [a - c, a + c]$ ,

$$|T\varphi(x) - b| = \left| \int_a^x f(t, \varphi(t)) dt \right| \leq M|x - a| \leq Mc,$$

so  $T(S) \subset S$ .

(d) If  $|f(x, y) - f(x, z)| \leq A|y - z|$  and  $h < 1/A$ , then for  $\varphi, \psi \in S$ ,

$$\|T\varphi - T\psi\|_\infty \leq Ah \|\varphi - \psi\|_\infty,$$

so  $T$  is a contraction. The fixed point gives the unique solution on  $(a - c, a + c)$ . ■

## 7.IV Solving and Proving Techniques

### Working with Riemann-Stieltjes Integrals

- Use the fact that for constant functions, upper and lower Darboux sums equal the telescoping sum of  $\alpha$  increments

- Apply integration by parts:  $\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df$
- Use the relationship between different integral definitions by showing uniform convergence of Riemann sums
- Express sums as Stieltjes integrals with respect to step functions, then use integration by parts

## Proving Integral Existence

- Use proof by contradiction: assume the integral doesn't exist and construct a specific function that leads to a contradiction
- Apply the integrability criterion via vanishing total oscillation
- Use the fact that continuous functions are Riemann integrable
- Show that upper and lower integrals agree by making their difference arbitrarily small

## Euler's Summation Formula

- Express sums as Stieltjes integrals with respect to the step function  $[x]$
- Apply integration by parts to convert to integrals involving derivatives
- Use the identity  $[x] = x - \frac{1}{2} - \varphi_1(x)$  for higher order terms
- Apply the formula to specific functions like  $\log x$  to derive approximations

## Series Convergence Tests

- Apply ratio test:  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$  implies convergence
- Use root test:  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} < 1$  implies convergence
- Apply comparison test with known series like p-series or geometric series
- Use limit comparison test: if  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ , both series behave the same
- Apply integral test for positive decreasing functions

## Fixed Point Theorems

- Use contraction mapping theorem to prove existence and uniqueness of solutions
- Show that a function is a contraction by bounding its Lipschitz constant
- Apply the theorem to integral equations by defining appropriate operators
- Use the theorem for differential equations by converting to integral form

## Differential Equations

- Convert differential equations to integral equations by integration
- Use Lipschitz conditions to ensure uniqueness of solutions
- Apply fixed point theorems to prove existence of solutions
- Use the fact that solutions of integral equations satisfy the original differential equation