# Group exercise 1

Pernille Kjeilen Fauskanger

2022-09-05

## Task 1 – Proof of 2.2 a

We are to prove that

$$\mathrm{E}\big[u(w-X)\big] \leq \mathrm{E}\big[u(w-Y)\big] \iff \mathrm{E}\big[b+a\cdot u(w-X)\big] \leq \mathrm{E}\big[b+a\cdot u(w-Y)\big], \tag{1}$$

for $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Here, $\mathbb{R}^+ = (0,\infty)$. We begin to prove '$\Longrightarrow$':

$$\begin{aligned} &\mathrm{E}\big[u(w-X)\big] \leq \mathrm{E}\big[u(w-Y)\big] \overset{a>0}{\Longrightarrow} a\cdot\mathrm{E}\big[u(w-X)\big] \leq a\cdot\mathrm{E}\big[u(w-Y)\big] \\ &\Longrightarrow \mathrm{E}\big[b\big] + a\cdot\mathrm{E}\big[u(w-X)\big] \leq \mathrm{E}\big[b\big] + a\cdot\mathrm{E}\big[u(w-Y)\big] \\ &\Longrightarrow \mathrm{E}\big[b+a\cdot u(w-X)\big] \leq \mathrm{E}\big[b+a\cdot u(w-Y)\big]\square \end{aligned} \tag{2}$$

We now prove '$\Longleftarrow$':

$$\begin{aligned} &\mathrm{E}\big[b+a\cdot u(w-X)\big] \leq \mathrm{E}\big[b+a\cdot u(w-Y)\big] \overset{\text{lin op}}{\Longrightarrow} \mathrm{E}\big[b\big] + \mathrm{E}\big[a\cdot u(w-X)\big] \leq \mathrm{E}\big[b\big] + \mathrm{E}\big[a\cdot u(w-Y)\big] \\ &\Longrightarrow \mathrm{E}\big[a\cdot u(w-X)\big] \leq \mathrm{E}\big[a\cdot u(w-Y)\big] \Longrightarrow a\cdot\mathrm{E}\big[u(w-X)\big] \leq a\cdot\mathrm{E}\big[u(w-Y)\big] \\ &\overset{a>0}{\Longrightarrow} \mathrm{E}\big[u(w-X)\big] \leq \mathrm{E}\big[u(w-Y)\big]\square \end{aligned} \tag{3}$$

We have now proved that

$$\mathrm{E}\big[u(w-X)\big] \leq \mathrm{E}\big[u(w-Y)\big] \iff \mathrm{E}\big[b+a\cdot u(w-X)\big] \leq \mathrm{E}\big[b+a\cdot u(w-Y)\big]. \tag{4}$$

Let us define

$$v(x) = \frac{u(x)-u(0)}{u'(0)} \tag{5}$$

Assuming that $u(0) = 0$, $u(1) = 1$ and $u'(0) > 0$, we can see that $v$ is a linear transformation of $u$ entailing that $v$ is equivalent to $u$ based on the proof above. Defining $v$ in this way imply that $v(0) = 0$, and $v'(0) = 1$, of course, assuming that $u'(0) \neq 0$, which is the case for $u'(0) > 0$.

## Task 2 – Exercise 1.2.4

Let $u(x) = \sqrt{x},\ x \geq 0 \Rightarrow u(w-Z) = \sqrt{w-Z},\ w \geq Z$. This person will prefer $X$ to $Y$ if the expected utility evaluated at $X$ is larger than at $Y$. So, we need to show that

$$\mathrm{E}\big[u(X)\big] > \mathrm{E}\big[u(Y)\big] \Rightarrow \mathrm{E}\big[\sqrt{X}\big] > \mathrm{E}\big[\sqrt{Y}\big] \Rightarrow \sum P(X=x)\cdot\sqrt{x} > \sum P(X=y)\cdot\sqrt{y} \tag{6}$$

This is straight forward as we know the PMFs for $X$ and $Y$:

$$\begin{aligned} &\sum P(X=x)\cdot\sqrt{x} > \sum P(X=y)\cdot\sqrt{y} \Rightarrow \frac{1}{2}\Big[20+30\Big] > \frac{1}{5}\Big[30+80\Big] \\ &\Rightarrow 25 \geq 22 \Rightarrow \text{prefer X to Y} \end{aligned}$$

He is risk-averse because $u''(x) < 0$. A risk-averse person that now have determined that he want to go for $X$ rather than $Y$, would decline for values of $w$ where facing $X$ is better than facing $\mathrm{E}\big[X\big]$;

$$25 = \mathrm{E}\big[u(X)\big] > u\Big(\mathrm{E}\big[X\big]\Big) = \sqrt{w} \Rightarrow w < 625$$

For example, a linear utility function would results in this person preferring $Y$ over $X$.

### Task 3 – Exercise 1.2.8

Nothing about current wealth are stated in the text, so we assume that $w = 0$ for simplicity. Winning the first game yields a price of 1. Then the utility function evaluated at his current wealth is then $u(1)$ because his new wealth is 1. Assuming that the the chance of winning the game 0.5, we can see that the expected utility playing a second round is $\mathrm{E}\big[u(2)\big]$. The choice of yes, signify that he is a risk-lover:

$$\mathrm{E}\big[u(2)\big] > u(1) \Rightarrow \frac{1}{2}(u(0) + u(2)) > 1 \Rightarrow u(2) > 2 \tag{7}$$

If he plays a third round, his expected utility is $\mathrm{E}\big[u(4)\big]$. Saying yes to this after a huddle, as he does, entails that he is more risk-neutral all of the sudden:

$$\mathrm{E}\big[u(4)\big] = u(2) \Rightarrow \frac{1}{2}(u(0) + u(4)) = u(2) \Rightarrow u(4) = 2u(2) > 4 \tag{8}$$

If he plays a fourth round, his expected utility is $\mathrm{E}\big[u(8)\big]$. Saying no to this, as he does, entails that he now is risk-averse:

$$\mathrm{E}\big[u(8)\big] < u(4) \Rightarrow \frac{1}{2}(u(0) + u(8)) = u(4) \Rightarrow 8 > 2u(4) > u(8) \tag{9}$$

## Task 4 – Exercise 1.3.6

We use the exponential utility function with $\alpha = 0.001$. Recall that the exponential utility function $u$ is defined by a persons wealth $w$, in the following manner:

$$u(w) = -\alpha \exp(-\alpha w) \tag{10}$$

The premium, $P$, may be defined by the equilibrium principle, which is defined by this relationship with the risk, $R$:

$$u(w) = -\alpha \exp(-\alpha w) = \mathrm{E}\big[u(w + P - R)\big] = \mathrm{E}\big[-\alpha \exp(-\alpha(w + P - R))\big]$$

The utility functions evaluated at the wealth $w$ cancels, which simplifies the computation

$$\begin{aligned} 1 = \mathrm{E}\big[\exp(-\alpha(P - R))\big] &\Rightarrow \exp(\alpha P) = \mathrm{E}\big[\exp(\alpha R)\big] \Rightarrow \alpha P = \log \mathrm{E}\big[\exp(\alpha R)\big] \\ &\Rightarrow P = \frac{1}{\alpha}\log(M_R(\alpha)) \end{aligned} \tag{11}$$

We see that the premium $P$ is a function of $\alpha$ and will therefore be denoted by $P(\alpha)$. For the given risks $X$, and $Y$ (that are normally distributed, and therefore the MGFs are well defined) and $\alpha = 0.001$, we have that

$$\log(M_X(0.001)) = \mu_X \cdot 0.001 + \frac{\sigma_X^2\alpha^2}{2} = 400 \cdot 0.001 + \frac{1}{2} \cdot 25000 \cdot 0.001^2 = 0.4125$$
$$\log(M_Y(0.001)) = \mu_Y \cdot 0.001 + \frac{\sigma_Y^2\alpha^2}{2} = 420 \cdot 0.001 + \frac{1}{2} \cdot 20000 \cdot 0.001^2 = 0.43$$

Thus, the premiums for risks $X$ and $Y$ are calculated to be

$$P_X(0.001) = \underline{\underline{412.5}} \tag{12}$$
$$P_Y(0.001) = \underline{\underline{430}} \tag{13}$$

The premium for the risk $Y$ is larger. To see which $\alpha$ satisfying that the premium of $Y$ are higher than that for $X$, we solve the following inequality for $\alpha$:

$$\begin{aligned} &P_Y(\alpha) > P_X(\alpha) \Rightarrow \frac{4.2 \cdot 10^2 \cdot \alpha + 0.5 \cdot 2 \cdot 10^4\alpha^2}{\alpha} > \frac{4 \cdot 10^2 \cdot \alpha + 0.5 \cdot 2.5 \cdot 10^4\alpha^2}{\alpha} \\ &\Rightarrow 4.2 \cdot 10^2 + \alpha \cdot 10^4 > 4.0 \cdot 10^2 + 1.25 \cdot \alpha \cdot 10^4 \\ &\Rightarrow 20 > 2.5 \cdot 10^3\alpha \Rightarrow \underline{\underline{\alpha < 0.008}} \end{aligned} \tag{14}$$

# Task 5 – Exercise 1.3.8

We have that $P(\alpha) \geq 1250$. We can solve the inequality to say something about $\alpha$. As we showed in the previous task, the minimum premium the insurer are willing to agree on, is $P^-$, which was defined based on $\alpha$ and:

$$P^-(\alpha) = \frac{1}{\alpha}\left[\mu_R \cdot \alpha + \frac{1}{2}\sigma_R^2\alpha^2\right] \tag{15}$$

So we are interested in solving

$$1250 \geq \frac{1}{\alpha}\left[\mu_R \cdot \alpha + \frac{1}{2}\sigma_R^2\alpha^2\right] \tag{16}$$

Here $\mu_R = 10^3$ and $\sigma_R^2 = 10^4$. So we have everything we need to obtain $\alpha$ values solving this inequality:

$$\begin{aligned} &1250 \geq \frac{1}{\alpha}\left[10^3 \cdot \alpha + \frac{1}{2}10^4\alpha^2\right] = 10^3 + 5 \cdot 10^3 \cdot \alpha \Rightarrow 50 \geq 10^3 \cdot \alpha \\ &\frac{50}{10^3} \geq \alpha \Rightarrow \alpha \leq \underline{\underline{0.05}} \end{aligned} \tag{17}$$

So what can we say about the risk aversion coefficient when it is restricted to $\alpha \in (0, 0.05]$? See figure @ref(fig:risk-aversion-alpha-T5-1), to see the relationship between wealth $w$ and the exponential utility $u$.

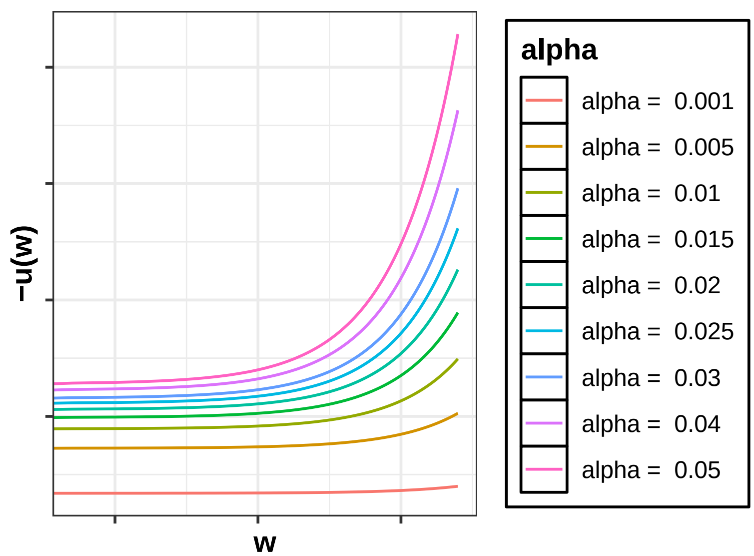


Figure 1: The exponential utility function for the alpha values representing the risk-aversion of the insurer.

What is the dimension of $\alpha$ given that the dimension of $R$ is money? We define $\alpha$ given the distribution of $X$ by

$$\alpha = (P_R(\alpha) - \mu_R) \cdot \frac{2}{\sigma_R^2} \tag{18}$$

Here $\dim(P_R(\alpha)) = \dim(\mu_R) = \dim(\sigma_R) = 1$. The dimension of $\sigma_R^2$ is then 2. Thus the dimension of $\alpha$ is

$$\dim(\alpha) = \text{money} \cdot \frac{1}{\text{money}^2} = \underline{\underline{\text{money}^{-1}}} \tag{19}$$