Lecture 1 & 2. Introduction and Utility functions, mixed distributions and risks (Chap. 1.1-1.3, Chap 2.2)

1. Life vs Non-life

Insurance is protection towards uncertainty (random loss). Uncertainty is essential in insurance.

Life Insurance: it is related to a human life. It's a *long term* investment and requires periodic payments, either monthly or quarterly or annually. The risks that are covered by life insurance include – premature death, income during retirement, illness. The time perspective is long (can larger than 30 years). The contract is binding for the company. This means the *interest* on the capital is important.

Nonlife Insurance: covers things apart from the things covered in life insurance, such as car, house, travel…. The contract is of *short duration*, often one year, whereby premiums are normally paid on a one time basis. Thus unlike life insurance, non-life insurance is usually limited to *annual* policies. The interest and inflation are of smaller importance. The risks that are covered by non-life insurance is property loss (stolen car or burnt house), liability arising from damage caused by an individual to a third party, accidental death or injury.

Central questions: risk models, ruin probability, premium principle, risk measures, reinsurance, reserves…

2. Utility function (Chap 1.1-1.2, H.L. means handwriting on lecture)

2.1 Introduction to utility function

Situation**:** if an individual wants to insure himself against **random** loss X, it is supposed that the *insurer* (*eg*. insurance company) will require a certain **premium**. If both parties (the individual and insurer) entry an insurance policy and the individual paid the premium, we can call the individual as the *insured* (insured can be a group of people, a company). Then if the loss happens, the insured will get X subsidy from the insurer. If the loss does not happen, the insured get no subsidy.

Thus, a non-life insurance contract is an agreement between insurer and insured. Based on this agreement, the insurer will compensate the insured for the **random, unpredictable losses** during a time period (usually one year for the non-life insurance), against certain fee (premium). By the insurance contract, the unpredictable economic risk is transferred from the insured to the insurer.

A natural **proposal** for the amount of premium is **net premium** $\mathrm{E}[X]$. Both insured and insurer are decision makers. Now two questions:

I. Can insured and insurer come agree with a premium = $\mathrm{E}[X]$ in an insurance policy?

II. Will a decision maker who facing *random* loss X pay premium to insure against the loss, or will he or she face the random loss self ?

To answer the question I, we need to know that, in real life, many decision makers' decisions are **NOT** based on the actual wealth *x*:

Eg.1, Let an individual face a random loss *X* which equals *b* with probability 0.01, and equals zero otherwise. The individual can insure himself against this loss, and is willing to pay a premium *P* for this insurance policy. We have three situations: *b* = 10, 1000, 1000 000.

(H.L.)

*The premium for a risk is not *homogeneous*: not proportional to the risk

Eg.2, For a price *P*, you may enter the following game: a fair coin is tossed until head appears. If this happens on trial *n*, the gain is 2^n. The probability that the head appears until the *n*th trial is $(\frac{1}{2})^n$. Then by entering this game, the expected gain equals $\sum_{n=1}^{\infty} 2^n (\frac{1}{2})^n - P = \infty$. However, empirical studies show that people are prepared to enter this game only for very moderate values of *P*

Expected utility hypothesis: generally, the decisions for a decision maker are **not** based on his actual wealth *w* but rather on *u*(*w*) for some *utility function* *u*(·), with *u*(*w*) representing the 'utility' attached to capital *w*. So if his capital is *w*, he prefers losing *X* than losing *Y* when

$$\mathrm{E}[u(w-X)] \geq \mathrm{E}[u(w-Y)]$$

(*Complex decisions are thus reduced to comparisons of real numbers.*)

2.2 Properties of Utility functions

Different decision makers may have different utility functions, however, the utility function generally has the following properties:

a. Any **linear transformation** $au(\cdot)+b$ (a and b are constant) with $a > 0$ is equivalent to $u(\cdot)$ as they lead to the same decisions:

$$\mathrm{E}[u(w-X)] \le \mathrm{E}[u(w-Y)] \Longleftrightarrow \mathrm{E}[au(w-X)+b] \le \mathrm{E}[au(w-Y)+b]$$

*From each class of equivalent utility functions, we can select one standard function by requiring that $u(0) = 0$ and $u(1) = 1$ (or $u(-1) = -1$).

* Also $v(x) = \dfrac{u(x)-u(0)}{u'(x)}$ and $u(x)$ are equivalent.

(H.L.)

b. $u(\cdot)$ **is non decreasing function, thus the marginal utility** $u'(w) \ge 0$: 'The richer, the happier'.

c. If the decision maker has a concave $u(\cdot)$ function, $u''(w) \le 0$, then he or she is has **non increasing** marginal utility $u'(w)$ ('the first \$ counts more than the next') and **risk-averse** . (However, if $u''(w) = 0$, we should view the decision maker is risk neutral.)

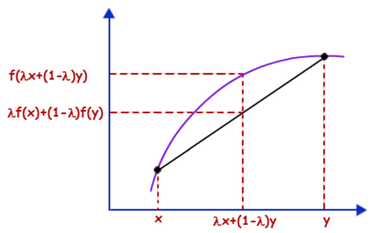

Figure 1 Concave function

Theorem *Jensen's inequality:* if $u(\cdot)$ is concave, then $\mathrm{E}[u(X)] \le u(\mathrm{E}[X])$ for all random variables X (Brief proof available in "Extra material" at Mitt uib).

Jensen implies that $\mathrm{E}[u(w-X)] \le u(\mathrm{E}[w-X]) = u(w-\mathrm{E}[X])$ .Thus decision maker with concave utility function would rather pay a deterministic fee $\mathrm{E}[X]$ to against the loss than facing the random loss X itself, and is therefore *risk-averse*. (Answer to question II.)

On the contrary, if $\mathrm{E}[u(X)] > u(\mathrm{E}[X])$, $u(\cdot)$ is convex and the corresponding decision function is *risk- loving.*

***Eg. 3*, (**Risk averse versus risk loving) Assume a person owns capital w = 0; his utility function is $u(\cdot)$. Random variable I ~ Bernoulli(0.5), thus Pr[I=0] = Pr[I=1] = 0.5. Let b

be constant. He is given the choice by choosing random loss Ib or fixed loss $b/2$. Suppose that:

- losing Ib is preferred if $b = 1$ *(riskiness)*
- there is indifference if $b = 2$
- losing $b/2$ is preferred if $b = 4$ *(security)*

What does this tell us about this person's utility function $u(\cdot)$?

(H.L.)

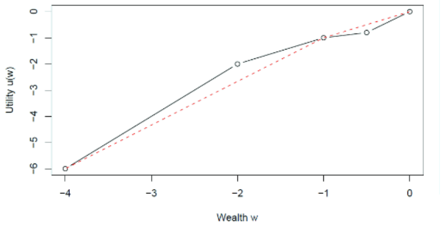

Figure 2 A example of utility function

**So u(w) for this person is neither convex everywhere, nor concave everywhere)*

**Generally, $u(\cdot)$ should be increase, but not must be concave. Concave $u(\cdot)$ corresponds to a risk averse decision maker*

2.3 Applying utility function to decide premium in an insurance policy

(1) For the insured with utility function $u(\cdot)$ and own capital w: the insured is willing to pay premium P (P is deterministic) if $E[u(w-X)] \le u(w-P)$ for avoiding random loss X. For the insured: the lower P, the better.

***Maximal premium** P^+ that the insured with $u(\cdot)$ and own capital w is willing to pay:

Largest P for which $E[u(w-X)] \le u(w-P)$, hence that P^+ for which:

$$E[u(w-X)] = u(w-P^+) \quad (1)$$

(2) For the insurer with utility function $U(\cdot)$ and own capital W, the insurer will accept a premium P and entry an insure policy only if $E[U(W+P-X)] \ge U(W)$. For the insurer: the higher premium, the better.

Thus Minimal premium P^- that the insurer needs: Least P for which $E[U(W+P-X)] \ge U(W)$, hence that P^- for which:

$$E[U(W+P^- - X)] = U(W) \quad (2)$$

*If $P^- < P^+$, both the insurer and the insured gain utility from an insurance policy with premium $P \in (P^-, P^+)$. And it is possible to entry an insurance policy between insurer and the insured (Answer to question I). The premiums P^+ and P^- are called **zero-utility premiums**. They keep utility at the same level before and after insurance. We can solve P^+ and P^- by resolving equations (1) and (2).

For P^+, we have the following approximations.

Definition For utility function $u(\cdot)$, its *risk aversion coefficient* is $r(w) = -u''(w)/u'(w)$.

Approximation for P^+: Let m and σ^2 denote the mean and variance of random loss X. When $u(\cdot)$ and own capital w for the insured is known, we have the following *approximation for* P^+:

$$P^+ \approx m - \frac{1}{2}\sigma^2 u''(w-m)/u'(w-m) = m + \frac{1}{2}\sigma^2 r(w-m) \qquad (3)$$

(*Taylor expansion*: A one-dimensional Taylor series of a real function $f(x)$ is an expansion of a real function $f(x)$ about a point a and is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots) \quad (4)$$

Proof: first two terms of Taylor expansion for $u(w-P^+)$ at point $w-m$ leads to the following approximation:

$$u(w-P^+) \approx u(w-m) + u'(w-m)(m-P^+) \qquad (5)$$

[components in (4) correspond to those in (5) as $x = w - P^+$, $a = w - m$, $x - a = m - P^+$]

First three terms of Taylor expansion for $u(w-X)$ at point $w-m$ leads to the following approximation:

$$u(w-X) \approx u(w-m) + u'(w-m)(m-X) + \frac{1}{2}u''(w-m)(m-X)^2 \qquad (6)$$

[components in (4) correspond to those in (6) as $x = w - X$, $a = w - m$, $x - a = m - X$]

Further Proof: (H.L.)

*$r(w)$ does not change when $u(\cdot)$ is replaced by $au(\cdot)+b$, where a and b are constant

*$r(w)$ reflects the degree of risk aversion: the more risk averse one has, the larger the premium one is willing to pay.

3. Classes of utility functions (Chap 1.3)

The following utility functions (including their linear transforms) are "non-negative" and "non increasing" marginal utility $u'(w)$ and have properties that make them suitable for practical use:

linear utility:	$u(w) = w$	
quadratic utility:	$u(w) = -(\alpha - w)^2$	$(w \leq \alpha)$
logarithmic utility:	$u(w) = \log(\alpha + w)$	$(w > -\alpha)$
exponential utility:	$u(w) = -\alpha e^{-\alpha w}$	$(\alpha > 0)$
power utility:	$u(w) = w^c$	$(w > 0,\ 0 < c \leq 1)$

(R plot of different utility function)

a. "non-negative" marginal utility: $u'(w) \geq 0$, the utility function self is not decreasing

b. "non increasing" marginal utility: $u''(w) \leq 0$. If $u''(w) = 0$ (eg, linear utility), the decision maker is risk neutral, if $u''(w) < 0$, the utility function is concave decision maker is risk averse

c. For exponential utility function, $u'(w) = \alpha^2 e^{-\alpha w}$, $u''(w) = -\alpha^3 e^{-\alpha w}$, then the *risk aversion coefficient* $r(w) = -u''(w)/u'(w) = \alpha$ and does not depend on *w*.

Eg. 4, Find P^- for the insurer with $U(\cdot)$ being linear utility function and own capital *W, by setting* $E[U(W + P^- - X)] = U(W)$

(H.L.)

*The risk aversion coefficient *r*(*w*) for the linear utility function is 0. Thus linear utility leads to the *principle of equivalence* or *risk neutrality*

Eg. 5, Find P^+ for the insured with $u(\cdot)$ being *exponential* utility function and own capital *w* by solving $E[u(w - X)] = u(w - P^+)$

(H.L.)

*As the exponential utility function has a constant risk aversion $r(w) \equiv \alpha$, the approximate premium in (I) above also does not depend on *w*.

*If $\mathrm{Var}[X] > 0$, then $P^+(\alpha) = \frac{1}{\alpha}\log(m_X(\alpha))$ and it *increases* in α. (Proof can refer to Theorem 1.3.2 at page 7.)

*For the insurer with *exponential* utility function, the P^- has the same expression as $P^- = \frac{1}{\alpha}\log(m_X(\alpha))$, with α representing the risk aversion of the insurer.

Eg. 6, Anta at random loss $X \sim \exp(0.010)$. Find P^+ for the insured with $u(\cdot)$ being *exponential* utility function and $\alpha = 0.005$, by applying result from ***Eg. 5*** and by approximation method from (3)

($X \sim \exp(\lambda)$: $f(x) = \lambda e^{-\lambda x}$, E[$X$]= λ^{-1}, Var[X] = λ^{-2}, $m_X(\alpha)$ =E[$e^{\alpha X}$]= $\frac{\lambda}{\lambda - \alpha}$)

(H.L.)

4. Distribution of individual claim X_i, *i* = 1, . . . , *n*. (Chap 2.2)

Let X be claim severity (insurance risk or loss, or claim amount) for an insured individual. The distribution for individual claim X is in general neither *continuous*, nor *discrete*, X is *mixed continuous/discrete distributions*

- there are some *'special'* discrete values of claims, *e.g.* $X = 0$ when there is no claim
- there is a *continuum* of non-special values of a claim follow certain continuous distribution, when claim occurs with certain probability

Eg.7 Let the risk X have the following distributional properties:

- the probability of no claim, hence $X = 0$, equals 0.5;
- if there is a claim, it is exponentially distributed as exp(0.1)

Let $F(x) = \Pr[X \leq x]$ denote the cumulative distribution function (*cdf*) for X.

(1) X pure *discrete*: $F(x)$ is a non-decreasing and right- continuous step function. The associated probability mass function (*pmf*) $f(x)$ (the literature book call this function as probability density function (*pdf*), same as the continuous case)) represents the height of the step at discrete value *x:* $f(x) = F(x) - F(x-0) = \Pr[X = x]$ for all $x \in (-\infty, \infty)$. $F(x-0)$ is shorthand for $\lim_{\varepsilon \downarrow 0} F(x - \varepsilon)$

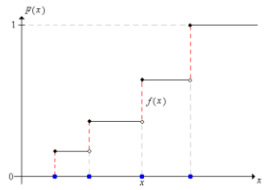

Figure 1 *cdf* and *pdf* for pure discrete random variable

(2) X pure *continuous*: density $f(x) = F'(x)$

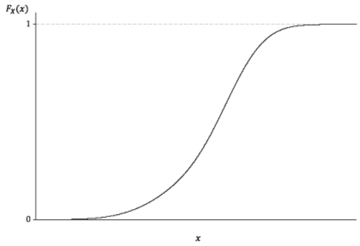

Figure 2 *cdf* for pure continuous random variable

(3) X is *mixed distributed* (like many insurance risks), we use **Riemann-Stieltjes differential** $dF(x)$ to represent the probability distribution of X:

$$dF(x) = F(x) - F(x-dx) = \begin{cases} F(x) - F(x-0) & \text{(when } X \text{ is discrete)} \\ f(x)dx & \text{(when } X \text{ is continous)} \end{cases}$$

And for any constants a and b: $\Pr(a < X \leq b) = \int_a^b dF(x)$

Riemann-Stieltjes integral of E[X] for mix distributed X:

$$\begin{aligned} \mathrm{E}[X] &= \int_x x dF(x) = \sum_{x \text{ discrete}} x dF(x) + \int_{x \text{ continous}} x dF(x) \\ &= \sum_{x \text{ discrete}} x \underbrace{[F(x) - F(x-0)]}_{f(x) = \Pr[X=x]} + \int_{x \text{ continous}} x f(x) dx \end{aligned}$$

Let $g(\cdot)$ be a function and *Riemann-Stieltjes* notation for $E[g(X)]$

$$\mathrm{E}[g(X)] = \int_x g(x) dF(x) = \sum_{x \text{ discrete}} g(x) \underbrace{[F(x) - F(x-0)]}_{f(x) = \Pr[X=x]} + \int_{x \text{ continous}} g(x) f(x) dx$$

Eg.8 Follow ***Eg.7***,

1) Find the expected amount of claim E[X] in ***Eg 7***.

2) Find $P^{+}(\alpha)$, the exponential premium the insured will pay for risk X, with risk aversion coefficient $\alpha = 0.01$

(H.L.)

- **Compulsory reading** after Lecture 1 and 2: Theorem 1.3.2, Example 1.3.3 and 1.3.4. Example 2.2.5
- **Group Exercise 1 (**Thursday 1.Sep. 08:15-10:00, RFB, Hjørnet)**: Literature book - Tasks 1.2.4, 1.2.8, 1.3.6, 1.3.8.**