U34, Lecture 1 and 2. Utility function

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Eg 1.

b	$EX = b\cdot 0.01 + 0\cdot 0.99$	Premium which individual is willing to pay.
1	0.01	0.01 suffice (small loss).
1000	10	somewhat more than 10
100 0000	10000	a lot more than 10000 (very big losss, and individual want to insure aagainst it with strong attention and can pay large fee for it).

2.2. a. Note: if Z is random variable, a, b is constant $\Rightarrow E(aZ+b) = aE(Z)+b$.

eg. Let $u(x) = ax+b \Rightarrow u(0) = b$ and $u'(x) = a$.

$$\therefore V(x) = \frac{u(x)-u(0)}{u'(x)} = \frac{ax+b-b}{a} = x.$$

Thus $u(x)$ is just linear transform of $V(x)$

Eg 3. From 2.2 a), Let $u(0) = 0$, $u(-1) = -1$.

If facing random loss, then the "expected" utility is:

$$E[u(-Ib)] = P(I=0) \cdot u(0) + P(I=1) \cdot u(-b)$$

$$= \frac{1}{2} \cdot u(0) + \frac{1}{2} \cdot u(-b)$$

If just take a fixed loss, the utility is $u(-\frac{b}{2})$.

b	$u(-\frac{b}{2})$		$\frac{1}{2}u(0) + \frac{1}{2}u(-b)$	
1	$u(-0.5)$	$<$	$\underbrace{\frac{1}{2}u(0) + \frac{1}{2}u(-1)}_{-0.5}$	$\Rightarrow u(-0.5) < -0.5$
2	$u(-1)$	$=$	$\frac{1}{2}u(0) + \frac{1}{2}u(-2)$	$\Rightarrow -1 = \frac{1}{2}u(-2)$ $\Downarrow$ $u(-2) = -2$
4	$u(-2)$	$>$	$\frac{1}{2}u(0) + \frac{1}{2}u(-4)$	$\Rightarrow u(-2) > \frac{1}{2}u(-4)$ $\Downarrow$ $u(-4) < -4$

$\therefore$ For this person, the utility function satisfy:

$u(0) = 0$, $u(-0.5) < -0.5$; $u(-1) = -1$.

$u(-2) = -2$, $u(-4) < -4$.

suppose $u(-0.5) = -0.8$ and $u(-4) = -6 \Rightarrow$ Figure 2.

Approximation for p^+, proof

As $Ex = m$, $E(x-m)^2 = \sigma^2$

Then: $E\{u(w-x)\} \approx E\{u(w-m)\} + E\{u'(w-m)(m-x)\}$

$+ \frac{1}{2} E\{u''(w-m)(m-x)^2\}$

only x is random variable

$= u(w-m) + u'(w-m)\underbrace{E(m-x)}_{m-Ex=0}$.

$+ \frac{1}{2} u''(w-m) \cdot \sigma^2$.

From (1): $E\{u(w-x)\} = u(w-p^+)$.

$\Rightarrow u(w-m) + \frac{1}{2} u''(w-m)\sigma^2 \approx u(w-m) + (m-p^+)u'(w-m)$.

$$\Rightarrow m - p^+ \approx \frac{\frac{1}{2}\sigma^2 u''(w-m)}{u'(w-m)}$$

$$\Rightarrow p^+ \approx m - \frac{\frac{1}{2}\sigma^2 u''(w-m)}{u'(w-m)}$$

Eg 4. With linear utility function, we have:

$$\left.\begin{array}{ll} U(w+p^- - X) = w + p^- - X \\ U(w) \quad\quad\quad = w \end{array}\right\} \xrightarrow{Eq.(2)} E(w+p^- - X) = w$$

$$\therefore \quad w + p^- - EX = w \Rightarrow p^- = EX$$

Eg 5: With exponential utility function, we have:

$$\left.\begin{array}{l} u(w-X) = -\alpha e^{-\alpha(w-X)} \\ u(w-p^+) = -\alpha e^{-\alpha(w-p^+)} \end{array}\right\} \xrightarrow{Eq(1)} E[-\alpha e^{-\alpha(w-X)}] = -\alpha e^{-\alpha(w-p^+)}.$$

$$\therefore \quad -\alpha e^{-\alpha w} \cdot E e^{\alpha X} = -\alpha e^{-\alpha w} \cdot e^{-\alpha p^+}$$

$$\therefore \quad E e^{\alpha X} = e^{\alpha p^+} \Rightarrow \log E e^{\alpha X} = \alpha p^+ \Rightarrow p^+ = \frac{\log E e^{\alpha X}}{\alpha}$$

$E e^{\alpha X} = M_X(\alpha)$ where $M_X(\cdot)$ is moment generating function (mgf) for X.

$$\therefore \quad p^+ = \frac{1}{\alpha} \log M_X(\alpha).$$

Eg 6. By applying result from Eg.5, with $X \sim exp(\lambda)$:

$$p^+ = \frac{1}{\alpha} \log M_X(\alpha) = \frac{1}{\alpha} \log \frac{\lambda}{\lambda-\alpha} \overset{\substack{\lambda=0.01 \\ \alpha=0.005}}{=\!=} \frac{1}{0.005} \log \frac{0.01}{0.01-0.005}$$

$$= 200 \cdot \log 2 \doteq 138.6.$$

by appoximation method from (3):

$$p^+ \simeq m + \frac{1}{2}\sigma^2 \underbrace{r(w-m)}_{\alpha} \overset{\substack{m=0.01^{-1} \\ \sigma^2 = 0.01^{-2}}}{=} 0.01^{-1} + \frac{1}{2} \cdot 0.01^{-2} \cdot 0.005$$

$$= 125$$

Eg 8. $\boxed{11}$ The distribution for X is:

$X=0$: $\quad Pr[X=0] = 0.5$

$X>0$: $\quad f(x) = Pr[X>0] \cdot \lambda \cdot e^{-\lambda x}$

$$\overset{\lambda=0.1}{=\!=} 0.5 \cdot 0.1 \cdot e^{-0.1x}$$

As $f(x \mid X>0) \sim exp(0.1)$

Thus $\frac{f(x, X>0)}{Pr(X>0)} \sim exp(0.1)$

From Riemann - Strieltjes integral:

$$E[X] = 0 \cdot Pr[X=0] + \int_0^{+\infty} x \cdot 0.5 \cdot 0.1\, e^{-0.1x}\, dx$$

$$= 0.5 \int_0^{+\infty} 0.1 x e^{-0.1x}\, dx = 0.05 \int_0^{+\infty} x e^{-0.1x}\, dx$$

As $\int x e^{cx}\, dx = e^{cx}\left(\frac{cx-1}{c^2}\right) + \text{constant}$.

$$\int_0^{+\infty} x e^{-0.1x}\, dx = e^{-0.1x}\left(\frac{-0.1x-1}{0.01}\right)\Big|_0^{+\infty} = 100 e^{-0.1x}(-0.1x-1)\Big|_0^{+\infty}$$

$$= 100\, e^{-0.1 \cdot +\infty}(-0.1 \cdot \infty - 1) - 100 \cdot e^0 (0-1)$$

$= 0 - 100 \cdot (-1) = 100$

$\therefore EX = 0.05 \cdot 100 = 5$

$\boxed{2}$ From Eg 5:

$$p^+ = \frac{1}{\alpha} \log M_X(\alpha)$$

$$M_X(\alpha) = Ee^{\alpha X} = e^{\alpha \cdot 0} \cdot P_r(X=0] + \int_0^{+\infty} e^{\alpha x} \cdot 0.5 \cdot 0.1 \cdot e^{-0.1x} dx$$

$$= 0.5 + 0.05 \int_0^{+\infty} e^{(-0.1+\alpha)x} dx$$

As $\int e^{cx} dx = \frac{1}{c} e^{cx}$ + constant

$$\therefore \int_0^{+\infty} e^{(\alpha-0.1)x} dx = \frac{1}{\alpha-0.1} e^{(\alpha-0.1)x} \Big|_0^{+\infty}$$

$$= \frac{1}{\alpha-0.1} e^{(\alpha-0.1)\cdot+\infty} - \frac{1}{\alpha-0.1} \cdot e^0$$

$$\overset{\alpha=0.01}{=} 0 - \frac{1}{0.01-0.1} = \frac{1}{0.09}$$

$$\therefore M_X(\alpha) \overset{\alpha=0.01}{=} 0.5 + 0.05 \cdot \frac{1}{0.09} = \frac{1}{2} + \frac{5}{9} = \frac{19}{18}$$

$$\therefore p^+ = 100 \cdot \log \frac{19}{18} \approx 5.406722$$