Solve: $Ax = b$
where A is indefinite and non-symmetric

Define the residual as
$$r = b - Ax$$

We seek to find $\underbrace{\text{a solution}}_{x_m}$ that minimizes:

$$\|r_m\|_2 = \|b - Ax_m\|_2$$

Iteratively, we can start by seeking a correction Δx that minimizes:
$$\|b - A(x_0 + \Delta x)\|_2$$
$$r = b - Ax$$
$$r_0 = b - Ax_0$$

the new residual is:
$$r(x_0 + \Delta x) = b - A(x_0 + \Delta x)$$
$$= (b - Ax_0) - A\Delta x$$
$$= r_0 - A\Delta x$$

if $A = I$ then $\Delta x = r_0$
if $A = sI$ then $r = r_0 - s\alpha r_0 \Rightarrow \alpha = \frac{1}{s}$

where we assumed $\Delta x = \alpha r_0$

In practice one direction r_0 is not enough:

if $A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $x_0 = 0$ $r_0 = b$

for $\Delta x = \alpha r_0 \Rightarrow r = r_0 - \alpha A r_0 = \begin{bmatrix} 1 - 2\alpha \\ 1 - 0.5\alpha \end{bmatrix}$

there isn't a value for alpha
which will produce 0 for both
components of r

Add another direction Ar_0:
$$\Delta x = \alpha r_0 + \beta A r_0$$
$$A\Delta x = \alpha A r_0 + \beta A^2 r_0$$
Solving $A\Delta x = r_0$ gives us α and β

In general, the search space we need is the Krylov subspace:
$$K_m(A, r_0) = \text{span}\{r_0, Ar_0, A^2 r_0, \dots, A^{m-1} r_0\}$$

→ directly working with this basis is a bad idea because the vectors quickly become lin. dependent
→ the solution is to construct an orthonormal basis for the Krylov subspace with the Arnoldi iteration
The Arnoldi process gives:
$AQ_m = Q_{m+1}\tilde{H}_m$ where $\tilde{H}_m$ is upper Hessenberg matrix
and Q_m is an orthonormal basis for the Krylov subspace

$$\|r_m\|_2 = \|b - Ax_m\|_2$$
$$= \|r_0 - A\Delta x\|_2$$
$$= \|r_0 - AQ_m y_m\|_2 \quad \text{where } Q_m y_m = \Delta x = x_m - x_0$$

$\underline{Q_m y_m \in \text{Krylov}}$

$$= \|\, \|r_0\| q_1 - AQ_m y_m\|_2$$
$$= \|\beta q_1 - Q_{m+1}\tilde{H}_m y_m\|_2 \quad \text{where } \beta = \|r_0\|$$
$$= \|Q_{n+1}(\beta e_1 - \tilde{H}_m y_m)\|_2 \quad \text{where } e_1 = (1, 0, 0, \dots, 0)^T \in \mathbb{R}^{n+1}$$
$$= \underbrace{\|\beta e_1 - \hat{H}_m y_m\|_2}$$

by the fact that orthonormal matrices preserve the euclidean norm (they are isometries)

find y_m that minimizes this
and update the solution

$$\boxed{x_m = x_0 + Q_m y_m}$$