Heat equation

Method: Separation of variables

$u_t = k u_{xx}$

with i.c. $u(x,0) = f(x)$

b.c. $u(0,t) = u(L,t) = 0$ (L is the length of the rod)

We assume solutions of the form:

$u(x,t) = X(x)T(t)$; plugged in the heat equation gives:

$$X(x)T'(t) = kX''(x)T(t) \quad \text{(here we used the chain-rule)}$$

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} \quad \text{(holds for all } x \text{ and } t\text{)}$$

If we vary x, then the LHS cannot change due to the independence of x, hence it must be the constant function. The same holds if we vary t.

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \quad \text{(the minus sign is a convention due to convenience)}$$

So, we have two ODEs:

(1) $X''(x) + \lambda X(x) = 0$

(2) $T'(t) + k\lambda T(t) = 0$

The i.c. and b.c. imply that:

$u(0,t) = X(0)T(t) = 0 \implies X(0) = 0$ for all t

$u(L,t) = X(L)T(t) = 0 \implies X(L) = 0$ for all t

(1) Assume solutions of the form:

$X(x) = e^{rt}$ then

$X''(x) = r^2 e^{rt}$ substituting gives

$r^2 e^{rt} + \lambda e^{rt} = 0$

$(r^2+\lambda)e^{rt} = 0 \implies r^2 + \lambda = 0 \implies r = \pm i\sqrt{\lambda} \quad \lambda > 0$ hence

$X(x) = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$

By Euler $e^{i\theta} = \cos\theta + i\sin\theta \quad e^{-i\theta} = \cos\theta - i\sin\theta$

$X(x) = C_1(\cos\theta + i\sin\theta) + C_2(\cos\theta - i\sin\theta)$

$= (C_1 + C_2)\cos\theta + i(C_1 - C_2)\sin\theta$

$X(x) = C_1\cos\theta + C_2\sin\theta$

$= C_1\cos(\sqrt{\lambda}x) + C_2(\sqrt{\lambda}x)$

This analysis may be unnecessary since the ODE:

$$X'' + \lambda X = 0$$

directly suggests solutions of the form:

$$X(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$$

using $X(0) = 0$ forces $C_1 = 0$ because $\cos(0) = 1$

so we have $X(0) = C_2\sin(\sqrt{\lambda}x)$

$X(L) = C_2\sin(\sqrt{\lambda}L)$

$\sin(\sqrt{\lambda}L) = 0$ for $\sqrt{\lambda} = \frac{n\pi}{L}$ $(n \in \mathbb{Z})$ $\quad C_2 = D$

the eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ with eigenfunctions $X_n(x) = D_n\sin\left(\frac{n\pi}{L}x\right)$

(which are solutions to (1))

(2) $T'(t) = -\lambda k T(t)$

$T_n(t) = Ae^{-\lambda k t}$

The solutions to the Heat equation are:

$u_n(x,t) = T_n(t)X_n(x)$

Any combination of these is also a solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) e^{-k\lambda_n t}$$

$\boxed{A_n = D_n C_n}$

We need to choose A_n so the i.c. are satisfied

$u(x,0) = \sum_n A_n X_n(x) = f(x)$

Since this is combination of eigenfunctions (sines) we use the orthogonality property:

$f(x) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)$

$f(x)\sin\left(\frac{m\pi}{L}x\right) = \sum_n A_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx$

$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_n A_n \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$

if $m \neq n$ then $L\int_0^1 \sin(m\pi y)\sin(n\pi y)\,dy \quad y = \frac{x}{L}$

if $m = n$ then $L\int_0^1 \sin^2(n\pi y)\,dy = \frac{L}{2}$

$= \sum_n A_n\left(\frac{L}{2}\delta_{mn}\right) = \frac{L}{2}A_m \implies A_m = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 x}$$