Distance & Similarity

Boston University CS 506 - Lance Galletti

Data

$$\textbf{n}\ \text{data points}\ \left\{ \underbrace{\begin{pmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1m} \\ \vdots & \ddots & \vdots & & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{im} \\ \vdots & & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nm} \end{pmatrix}}_{\textbf{m}\ \text{features}} \right.$$

Feature Space

From our data we can generate a **feature space** of all possible values for the set of features in our data.

name	age	balance
Jane	25	150
John	30	100

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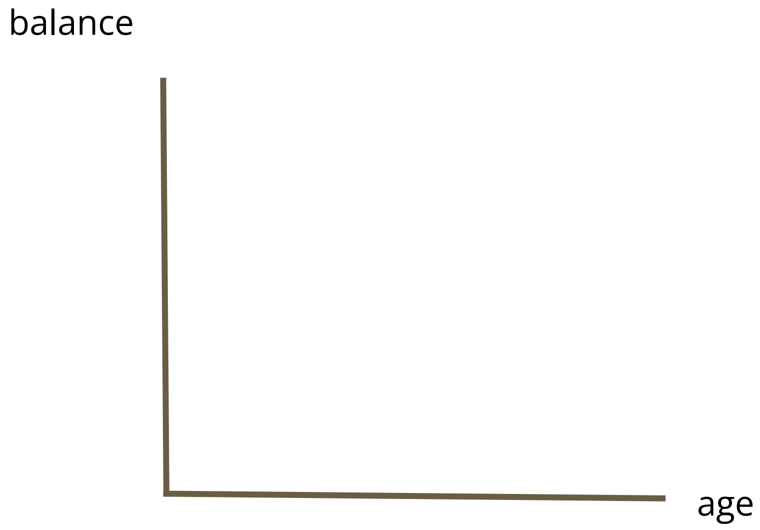

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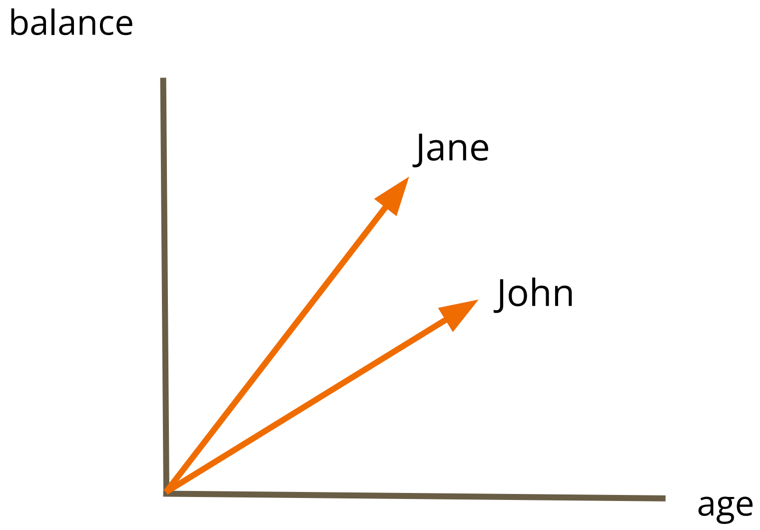


Our feature space is the Euclidean plane

Dissimilarity

In order to uncover interesting structure from our data, we need a way to **compare** data points.

A **dissimilarity function** is a function that takes two objects (data points) and returns a **large value** if these objects are **dissimilar**.

A special type of dissimilarity function is a **distance** function

Distance

d is a distance function if and only if:

- $d(i, j) = 0$ if and only if $i = j$
- $d(i, j) = d(j, i)$
- $d(i, j) \leq d(i, k) + d(k, j)$

We don't **need** a distance function to compare data points, but why would we prefer using a distance function?

$\boxed{\text{it's intuitive.}}$

Minkowski Distance

For **x**, **y** points in **d**-dimensional real space

I.e. $\mathbf{x} = [\mathbf{x}_1, \ldots, \mathbf{x}_d]$ and $\mathbf{y} = [\mathbf{y}_1, \ldots, \mathbf{y}_d]$

p ≥ 1

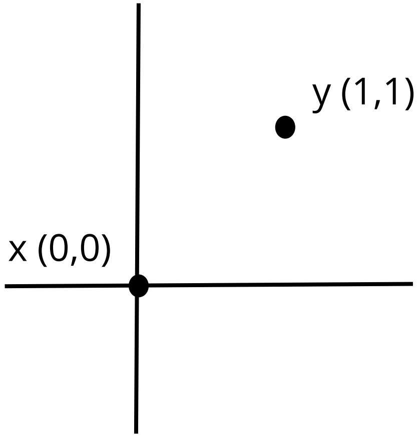
$$L_p(x, y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

When **p** = 2 -> Euclidean Distance

When **p** = 1 -> Manhattan Distance

Example

d = 2



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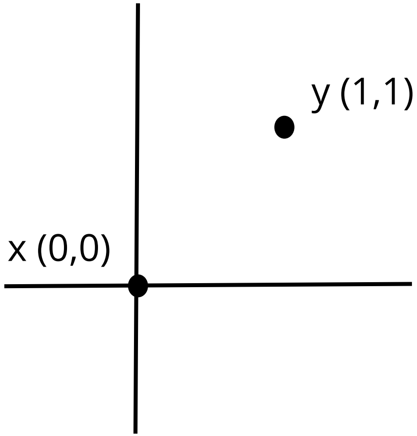

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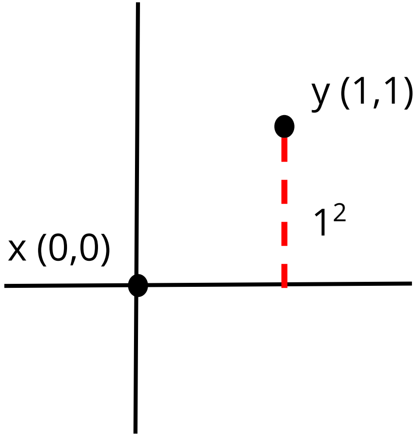

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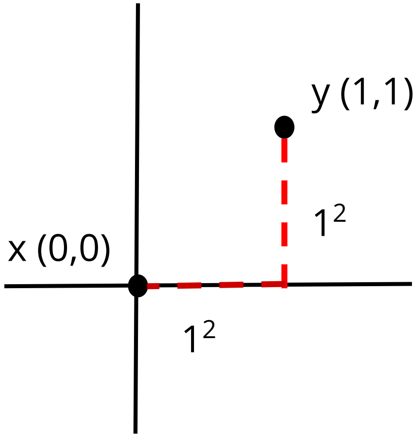

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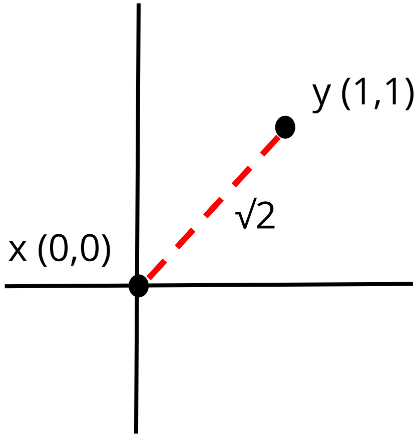

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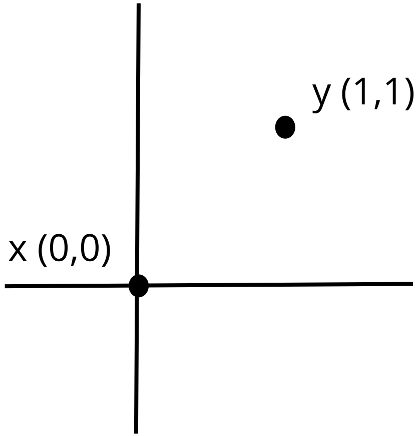

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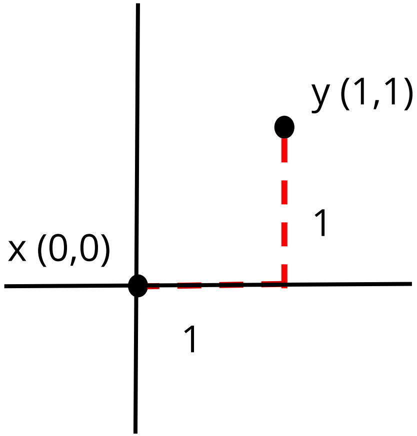

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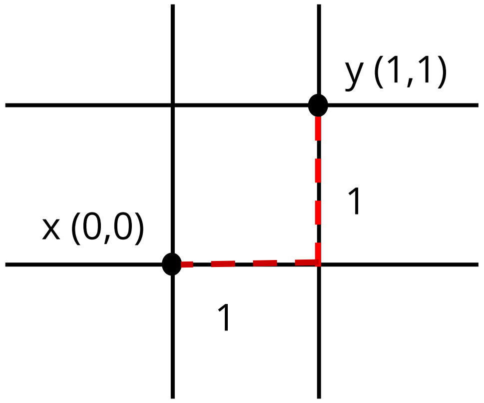

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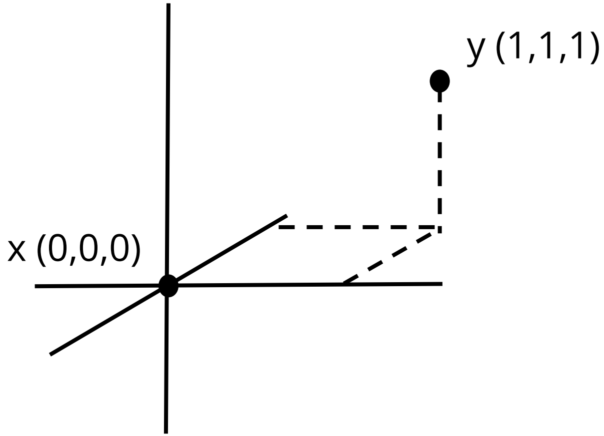

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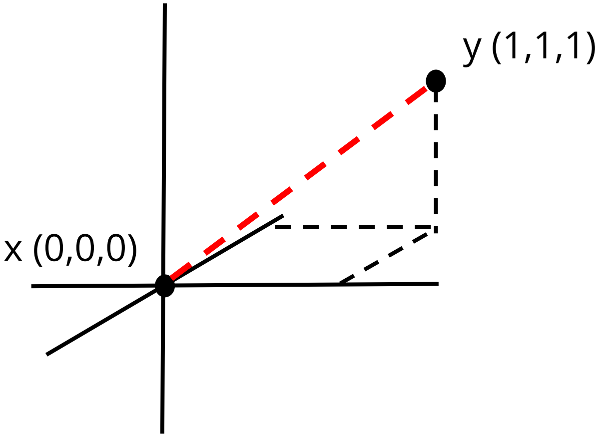

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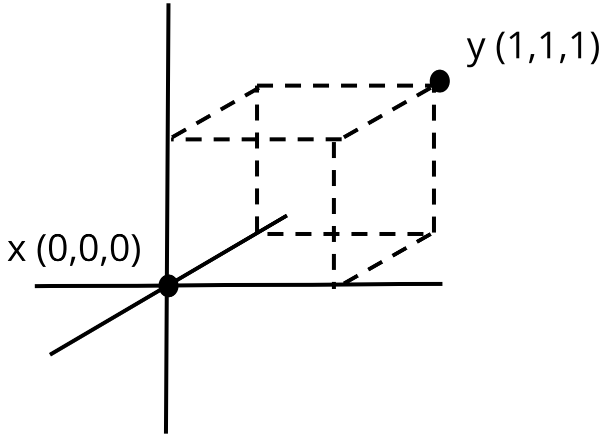

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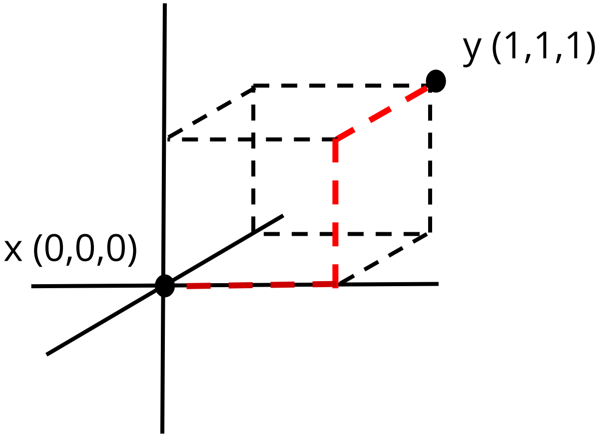

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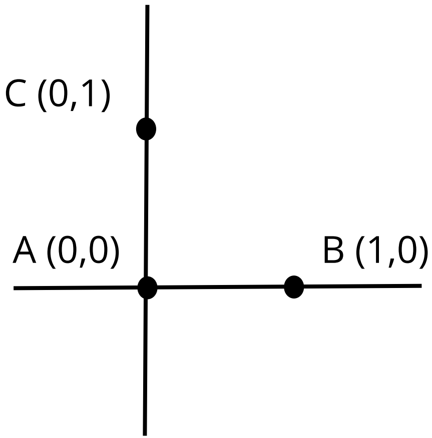

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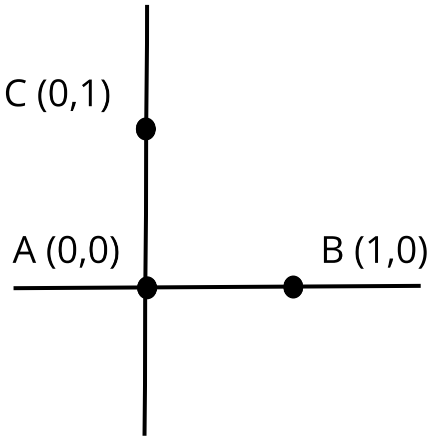
D(B,A) = D(A, C) = 1

D(B, C) = $2^{1/p}$



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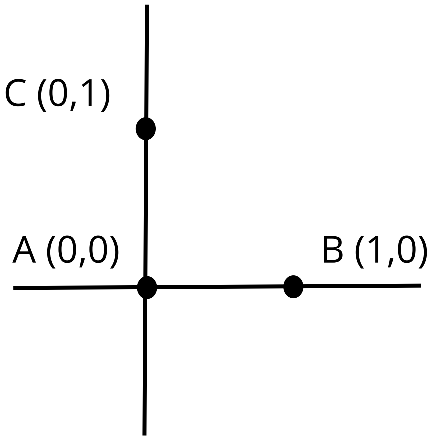

D(B,A) + D(A, C) = 2

D(B, C) = $2^{1/p}$

But... if $p < 1$ then $1/p > 1$

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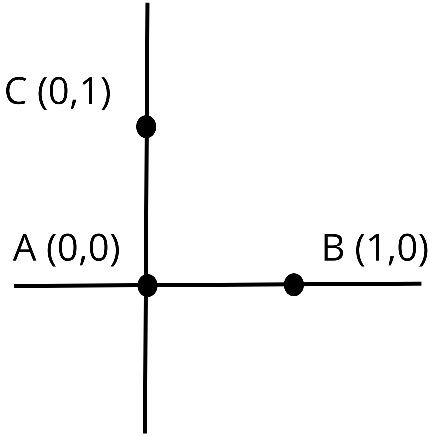

D(B,A) + D(A, C) = 2

D(B, C) = $\mathbf{2^{1/p}}$

So **D(B, C) > D(B, A) + D(A, C)** which violates the triangle inequality

Cosine Similarity

A **similarity** function is a function that takes two objects (data points) and returns a **large value** if these objects are **similar**.

$$\mathbf{s(x, y) = \cos(\theta)}$$

where $\boldsymbol{\theta}$ is the angle between **x** and **y**

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Cosine Similarity

To get a corresponding **dissimilarity** function, we can usually try

$$d(x, y) = 1 / s(x, y)$$

or

$$d(x, y) = k - s(x, y) \text{ for some } k$$

Here, we can use

$$d(x, y) = 1 - s(x, y)$$

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When **direction** matters more than **magnitude**

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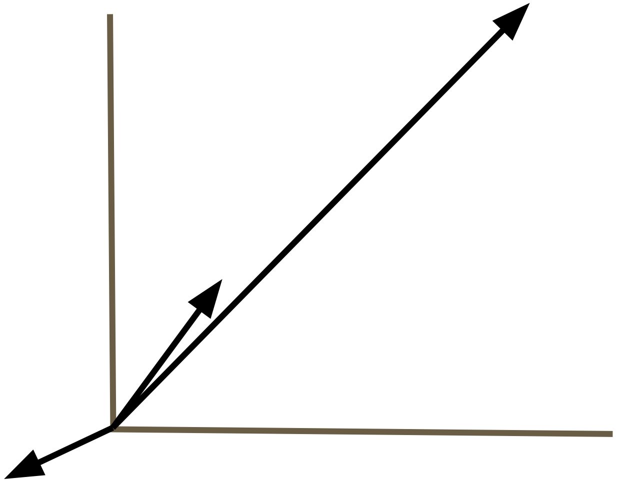

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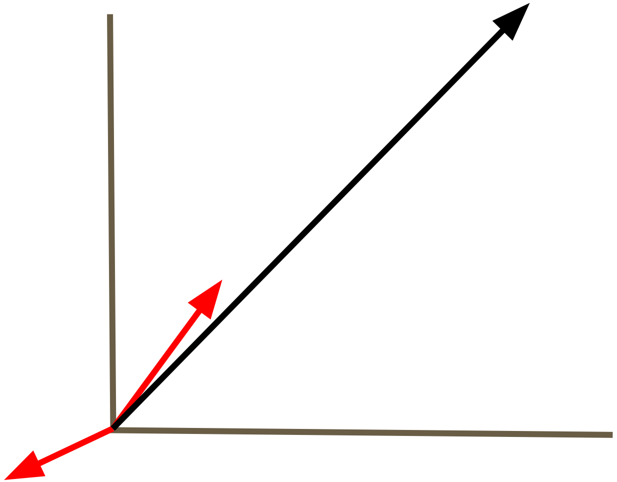

Close under
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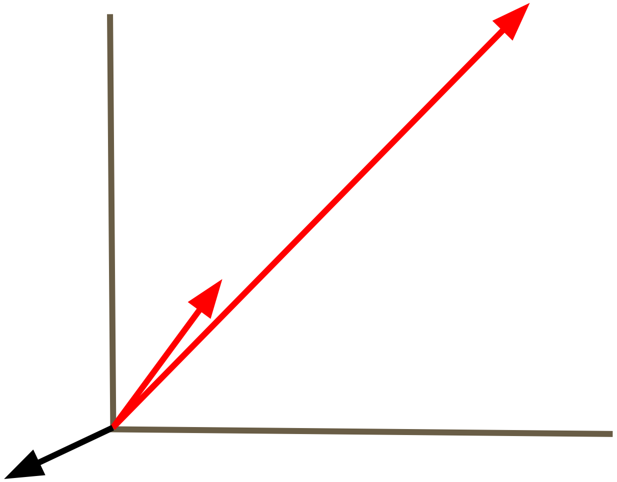

Close under Cosine Similarity

Jaccard Similarity

How similar are the following documents?

	w_1	w_2	...	w_d
x	1	0	...	1
y	1	1	...	0

Jaccard Similarity

One way is to use the Manhattan distance which will return the size of the set difference

	w_1	w_2	...	w_d
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$$L_1(x, y) = \sum_{i=1}^{d} |x_i - y_i|$$

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Will only be 1 when $x_i \neq y_i$

Jaccard Similarity

But how can we distinguish between these two cases?

	w_1	w_2	...	w_{d-1}	w_d
x	1	1	1	0	1
y	1	1	1	1	0

Only differ on the last two words

	w_1	w_2
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Completely different

Jaccard Similarity

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Both have Manhattan distance of 2

Jaccard Similarity

We need to account for the size of the intersection!

Given two documents x and y:

$$JSim(x, y) = \frac{|x \cap y|}{|x \cup y|}$$

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What is the jaccard distance in each?

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A quick note on Norms

- Distance from the origin
  - Minkowski Distance <=> Lp Norm
  - Not all distances can create a norm

- Notion of size
- Has the following properties:
  - $p(x + y) \leq p(x) + p(y)$
  - $p(ax) = |a|p(x)$
  - $p(x) = 0$ iff $x = 0$
  - $p(x) \geq 0$ for all x