Chapter 2
Homogeneous Transformation Matrix

Abstract The transformation of frames is a fundamental concept in the modeling and programming of a robot. In this Chapter, we present a notation that allows us to describe the relationship between different frames and objects of a robotic cell. This notation, called homogeneous transformation, has been widely used in computer graphics to compute the projections and perspective transformations of an object on a screen. Currently, this is also being used extensively in robotics. We will show how the points, vectors and transformations between frames can be represented using this approach. We also make an overview of different set of parameters that are used for characterizing the orientation of a body.

2.1 Homogeneous Coordinates and Homogeneous Transformation Matrix

Let $({}^j x_P,\ {}^j y_P,\ {}^j z_P)$ be the Cartesian coordinates of an arbitrary point P with respect to the frame $\mathscr{F}_j$, which is described by the origin O_j and the axes $\boldsymbol{x}_j$, $\boldsymbol{y}_j$, $\boldsymbol{z}_j$ (Fig. 2.1). The homogeneous coordinates of P with respect to frame $\mathscr{F}_j$ are defined by $(w\ {}^j x_P,\ w\ {}^j y_P,\ w\ {}^j z_P,\ w)$, where w is a scaling factor (Newman and Sproull 1979; Roberts 1965). In robotics, w is taken to be equal to 1 (Paul 1981; Pieper 1968). Thus, we represent the homogeneous coordinates of P by the (4×1) column vector:

$$
{}^j\tilde{\mathbf{p}} = \begin{bmatrix} {}^j x_P \\ {}^j y_P \\ {}^j z_P \\ 1 \end{bmatrix}. \tag{2.1}
$$

A direction (free vector) is also represented by four components, but the fourth component is zero, indicating a vector at infinity. If the Cartesian coordinates of a unit vector $\mathbf{u}$ with respect to frame $\mathscr{F}_j$ are $({}^j u_x,\ {}^j u_y,\ {}^j u_z)$, its homogeneous coordinates will be:

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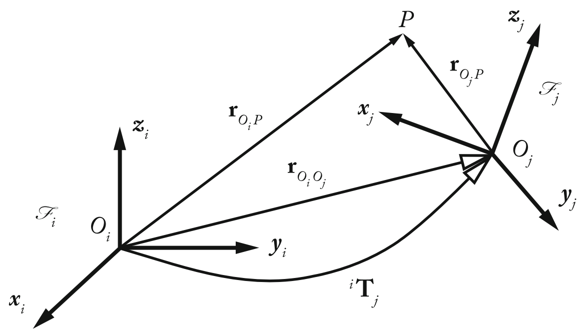

Fig. 2.1 Transformation of a vector

$$
{}^j\tilde{\mathbf{u}} = \begin{bmatrix} {}^ju_x \\ {}^ju_y \\ {}^ju_z \\ 0 \end{bmatrix}. \tag{2.2}
$$

The coordinates of the point P can be defined in another frame $\mathscr{F}_i$ by ${}^i\tilde{\mathbf{p}} = \begin{bmatrix} {}^ix_P & {}^iy_P & {}^iz_P & 1 \end{bmatrix}^T$ and they can be obtained as a function of ${}^j\tilde{\mathbf{p}}$ by (Fig. 2.1):

$$
{}^i\tilde{\mathbf{p}} = {}^jx_P{}^i\tilde{\mathbf{s}}_j + {}^jy_P{}^i\tilde{\mathbf{n}}_j + {}^jz_P{}^i\tilde{\mathbf{a}}_j + {}^i\tilde{\mathbf{r}}_j = {}^i\mathbf{T}_j{}^j\tilde{\mathbf{p}} \tag{2.3}
$$

where ${}^i\mathbf{s}_j$, ${}^i\mathbf{n}_j$ and ${}^i\mathbf{a}_j$ are unit vectors directed along the $\mathbf{x}_j$, $\mathbf{y}_j$ and $\mathbf{z}_j$ axes with corresponding homogeneous coordinates ${}^i\tilde{\mathbf{s}}_j$, ${}^i\tilde{\mathbf{n}}_j$, ${}^i\tilde{\mathbf{a}}_j$, respectively, and are expressed in frame $\mathscr{F}_i$; ${}^i\tilde{\mathbf{r}}_j$ is the homogeneous vector representing the coordinates (parameterized by the 3D vector ${}^i\mathbf{r}_j = {}^i\mathbf{r}_{O_iO_j}$) of the origin O_j of frame $\mathscr{F}_j$ expressed in frame $\mathscr{F}_i$.

In Eq. (2.3), the matrix ${}^i\mathbf{T}_j$ allows us to calculate the coordinates of a vector ${}^j\tilde{\mathbf{p}}$ with respect to frame $\mathscr{F}_i$ in terms of its coordinates in frame $\mathscr{F}_j$. This (4×4) matrix is called the transformation matrix. It permits us to define the transformation, translation and/or rotation, of the frame $\mathscr{F}_i(O_i, \mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ towards the frame $\mathscr{F}_j(O_j, \mathbf{x}_j, \mathbf{y}_j, \mathbf{z}_j)$ (Fig. 2.1) and it is represented by:

$$
{}^i\mathbf{T}_j = \begin{bmatrix} {}^i\tilde{\mathbf{s}}_j & {}^i\tilde{\mathbf{n}}_j & {}^i\tilde{\mathbf{a}}_j & {}^i\tilde{\mathbf{r}}_j \end{bmatrix} = \begin{bmatrix} & {}^i\mathbf{R}_j & & {}^i\mathbf{r}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4}
$$

where ${}^i\mathbf{R}_j$ is the rotation matrix expressing the orientation of the frame $\mathscr{F}_j$ with respect to frame $\mathscr{F}_i$ (see Sects. 2.2.2 and 2.4).

In summary:

- The matrix ${}^i\mathbf{T}_j$ represents the transformation from frame $\mathscr{F}_i$ to frame $\mathscr{F}_j$;
- The matrix ${}^i\mathbf{T}_j$ can be interpreted as representing the frame $\mathscr{F}_j$ (three orthogonal axes and an origin) with respect to frame $\mathscr{F}_i$.

2.2 Elementary Transformation Matrices

2.2.1 Transformation Matrix of a Pure Translation

A general pure translation matrix from frame $\mathscr{F}_i$ to frame $\mathscr{F}_j$ is denoted by $\mathbf{Trans}(a,\ b,\ c)$, where a, b and c denote the translation along the $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ axes respectively, where (Fig. 2.2):

$$ {}^{i}\mathbf{T}_{j} = \mathbf{Trans}(a,\ b,\ c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{Trans}(\boldsymbol{x},\ a)\ \ \mathbf{Trans}(\boldsymbol{y},\ b)\ \ \mathbf{Trans}(\boldsymbol{z},\ c) \tag{2.5} $$

taking any order of the multiplication.

2.2.2 Transformation Matrices of a Rotation About the Principle Axes x, y and z

Let us consider a rotation of angle θ around the axis $\boldsymbol{x}$ and let us denote this transformation as $\mathbf{Rot}(\boldsymbol{x},\ \theta)$. From Fig. 2.3, we deduce that:

$$ {}^{i}\mathbf{T}_{j} = \mathbf{Rot}(\boldsymbol{x},\ \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{c}\theta & -\mathrm{s}\theta & 0 \\ 0 & \mathrm{s}\theta & \mathrm{c}\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & & 0 \\ & \mathbf{rot}(\boldsymbol{x},\ \theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6} $$

where $\mathrm{c}\theta$ and $\mathrm{s}\theta$ represent $\cos\theta$ and $\sin\theta$ respectively, and $\mathbf{rot}(\boldsymbol{x},\ \theta)$ denotes the (3×3) orientation matrix.

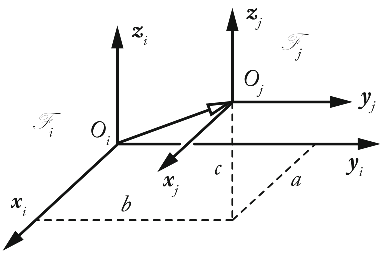

Fig. 2.2 Transformation of pure translation

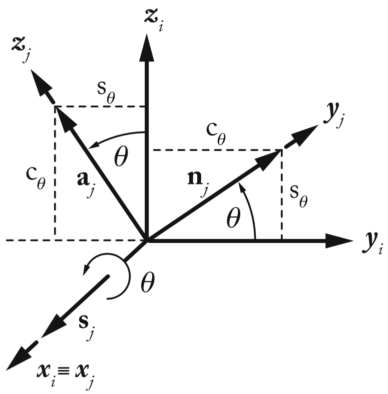

Fig. 2.3 Transformation of a pure rotation about the $\boldsymbol{x}$–axis

Similarly, the rotation of angle θ around the axis $\boldsymbol{y}$ axis is given by:

$$\mathbf{Rot}(\boldsymbol{y},\ \theta) = \begin{bmatrix} c_\theta & 0 & s_\theta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\theta & 0 & c_\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{rot}(\boldsymbol{y},\ \theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

and the rotation of angle θ around the axis $\boldsymbol{z}$ axis is given by:

$$\mathbf{Rot}(z,\ \theta) = \begin{bmatrix} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{rot}(z,\ \theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.8}$$

2.3 Properties of Homogeneous Transformation Matrices

Before going further, we need to define the following properties of the homogeneous transformation matrices.

Property 1 *From (2.4), a transformation matrix can be written as:*

$$\mathbf{T} = \begin{bmatrix} s_x & n_x & a_x & r_x \\ s_y & n_y & a_y & r_y \\ s_z & n_z & a_z & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{s} & \mathbf{n} & \mathbf{a} & \mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{r} \\ 0\ 0\ 0 & 1 \end{bmatrix}. \tag{2.9}$$

The matrix $\mathbf{R} = \left[\mathbf{s}\ \mathbf{n}\ \mathbf{a}\right]$ *represents the rotation whereas the vector* $\mathbf{r}$ *represents the translation. For a transformation of pure translation,* $\mathbf{R} = \mathbf{1}_3$ *(*$\mathbf{1}_3$ *represents the identity matrix of order 3), whereas* $\mathbf{r} = \mathbf{0}$ *for a transformation of pure rotation.*

Property 2 *The matrix* $\mathbf{R}$ *is orthogonal and its determinant is equal to 1. Consequently, its inverse is equal to its transpose:*

$$\mathbf{R}^{-1} = \mathbf{R}^T \tag{2.10}$$

where the superscript "T" indicates the transpose of the matrix.

Property 3 *The inverse of a matrix* ${}^i\mathbf{T}_j$ *is the matrix* ${}^j\mathbf{T}_i$*. Thus, to express the components of a vector* ${}^i\tilde{\mathbf{p}}_1$ *into frame* $\mathscr{F}_j$*, we write:*

$${}^j\tilde{\mathbf{p}}_1 = {}^j\mathbf{T}_i\,{}^i\tilde{\mathbf{p}}_1 \tag{2.11}$$

with:

$${}^j\mathbf{T}_i = {}^i\mathbf{T}_j^{-1}. \tag{2.12}$$

Property 4 *We can easily verify that:*

$$\mathbf{Rot}^{-1}(\boldsymbol{u},\,\theta) = \mathbf{Rot}(\boldsymbol{u},\,-\theta) = \mathbf{Rot}(-\boldsymbol{u},\,\theta) \tag{2.13}$$

$$\mathbf{Trans}^{-1}(\boldsymbol{u},\,d) = \mathbf{Trans}(\boldsymbol{u},\,-d) = \mathbf{Trans}(-\boldsymbol{u},\,d). \tag{2.14}$$

Property 5 *The inverse of a transformation matrix represented by Eq. (2.9) can be obtained as:*

$$\mathbf{T}^{-1} = \begin{bmatrix} & & & -\mathbf{s}^T\mathbf{r} \\ & \mathbf{R}^T & & -\mathbf{n}^T\mathbf{r} \\ & & & -\mathbf{a}^T\mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{R}^T & & -\mathbf{R}^T\mathbf{r} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.15}$$

Property 6 *Composition of two matrices: The multiplication of two transformation matrices gives a transformation matrix:*

$$\begin{aligned} \mathbf{T}_1\mathbf{T}_2 &= \begin{bmatrix} & \mathbf{R}_1 & & \mathbf{r}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} & \mathbf{R}_2 & & \mathbf{r}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} & \mathbf{R}_1\mathbf{R}_2 & & \mathbf{R}_1\mathbf{r}_2 + \mathbf{r}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{2.16}$$

In general, $\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$.

Property 7 *If a frame* $\mathscr{F}_0$ *is subjected to k consecutive transformations (Fig. 2.4) and if each transformation i (*$i = 1, \ldots, k$*) is defined with respect to the current*

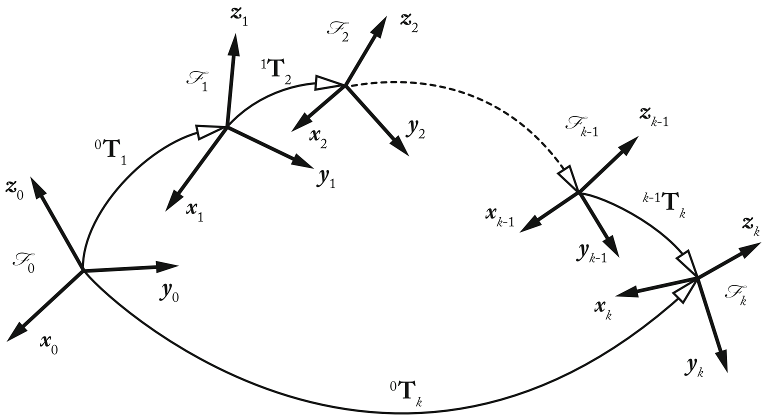

Fig. 2.4 Composition of transformations: multiplication on the right

frame $\mathscr{F}_{i-1}$, then the transformation $^0\mathbf{T}_k$ can be deduced by multiplying all the transformations on the right as

$$^0\mathbf{T}_k = \prod_{i=1}^{k} {}^{i-1}\mathbf{T}_i = {}^0\mathbf{T}_1 \cdot {}^1\mathbf{T}_2 \cdot {}^2\mathbf{T}_3 \cdots {}^{k-1}\mathbf{T}_k. \tag{2.17}$$

Property 8 *Consecutive transformations about the same axis: We note the following properties:*

$$\mathbf{Rot}(\boldsymbol{u},\, \theta_1)\mathbf{Rot}(\boldsymbol{u},\, \theta_2) = \mathbf{Rot}(\boldsymbol{u},\, \theta_1 + \theta_2), \tag{2.18}$$

$$\mathbf{Rot}(\boldsymbol{u},\, \theta)\mathbf{Trans}(\boldsymbol{u},\, d) = \mathbf{Trans}(\boldsymbol{u},\, d)\mathbf{Rot}(\boldsymbol{u},\, \theta). \tag{2.19}$$

2.4 Parameterization of the General Matrices of Rotation

The orientation of a body with respect to any frame can be obtained through the use of the rotation matrix **R**. It can be calculated by using a different set of parameters. The most used representations in parallel robotics are described below.

2.4.1 Rotation About One General Axis u

The pure rotation of angle θ around any axis $\boldsymbol{u}$ parameterized by the unit vector $\mathbf{u} = [u_x \; u_y \; u_z]^T$ can be represented by (Khalil and Dombre 2002):

$$\mathbf{R} = \mathbf{rot}(\boldsymbol{u}, \theta)$$
$$= \begin{bmatrix} u_x^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & u_x u_y(1-\mathrm{c}\theta)-u_z\mathrm{s}\theta & u_x u_z(1-\mathrm{c}\theta)+u_y\mathrm{s}\theta \\ u_x u_y(1-\mathrm{c}\theta)+u_z\mathrm{s}\theta & u_y^2(1-\mathrm{c}\theta)+\mathrm{c}\theta & u_y u_z(1-\mathrm{c}\theta)-u_x\mathrm{s}\theta \\ u_x u_z(1-\mathrm{c}\theta)-u_y\mathrm{s}\theta & u_y u_z(1-\mathrm{c}\theta)+u_x\mathrm{s}\theta & u_z^2(1-\mathrm{c}\theta)+\mathrm{c}\theta \end{bmatrix}. \tag{2.20}$$

Inverse problem. Let **R** be any arbitrary rotational transformation matrix such that:

$$\mathbf{R} = \begin{bmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{bmatrix}. \tag{2.21}$$

We solve the following expression for $\boldsymbol{u}$ and θ:

$$\mathbf{R} = \mathbf{rot}(\boldsymbol{u},\ \theta),\ \text{ with } 0 \le \theta \le \pi. \tag{2.22}$$

Adding the diagonal terms of Eqs. (2.20) and (2.21), we obtain:

$$\mathrm{c}\theta = \frac{1}{2}(s_x + n_y + a_z - 1). \tag{2.23}$$

From the off-diagonal terms, we obtain:

$$\begin{aligned} 2u_x\mathrm{s}\theta &= n_z - a_y \\ 2u_y\mathrm{s}\theta &= a_x - s_z \\ 2u_z\mathrm{s}\theta &= s_y - n_x \end{aligned} \tag{2.24}$$

yielding:

$$\mathrm{s}\theta = \frac{1}{2}\sqrt{(n_z - a_y)^2 + (a_x - s_z)^2 + (s_y - n_x)^2}. \tag{2.25}$$

From Eqs. (2.23) and (2.25), we deduce that:

$$\theta = \mathrm{atan2}(\mathrm{s}\theta,\ \mathrm{c}\theta),\ \text{ with } 0 \le \theta \le \pi \tag{2.26}$$

where “atan2” is the four-quadrant inverse tangent function.

u_x, u_y and u_z are calculated using Eq. (2.24) if $\mathrm{s}\theta \neq 0$. When $\mathrm{s}\theta$ is small, the elements u_x, u_y and u_z cannot be determined with good accuracy by this equation. However, in the case where $\mathrm{c}\theta < 0$, we obtain u_x, u_y and u_z more accurately using the diagonal terms of $\mathbf{rot}(\boldsymbol{u},\ \theta)$ as follows:

$$u_x = \mathrm{sign}(n_z - a_y)\sqrt{\frac{s_x - \mathrm{c}\theta}{1 - \mathrm{c}\theta}}$$
$$u_y = \mathrm{sign}(a_x - s_z)\sqrt{\frac{n_y - \mathrm{c}\theta}{1 - \mathrm{c}\theta}}$$
$$u_z = \mathrm{sign}(s_y - n_x)\sqrt{\frac{a_z - \mathrm{c}\theta}{1 - \mathrm{c}\theta}} \tag{2.27}$$

where "sign(.)" indicates the sign function of the expression between brackets, thus $\mathrm{sign}(e) = +1$ if $e > 0$, $\mathrm{sign}(e) = -1$ if $e < 0$ and $\mathrm{sign}(e) = 0$ if $e = 0$.

2.4.2 Quaternions

The quaternions are also called *Euler* parameters or *Olinde-Rodrigues* parameters. This is another way of parameterizing the rotation of an angle θ $(0 \le \theta \le \pi)$ about an axis $\boldsymbol{u}$. In this representation, the orientation is expressed by four parameters. We define the quaternions as:

$$Q_1 = \mathrm{c}_{\theta/2}$$
$$Q_2 = u_x \mathrm{s}_{\theta/2}$$
$$Q_3 = u_y \mathrm{s}_{\theta/2}$$
$$Q_4 = u_z \mathrm{s}_{\theta/2}. \tag{2.28}$$

From these relations, we obtain:

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = 1. \tag{2.29}$$

The transformation matrix $\mathbf{T}$ is deduced from Eq. (2.20), defining $\mathbf{rot}(\boldsymbol{u}, \theta)$ (Sect. 2.4.1), after rewriting its elements as a function of Q_j. Thus, the orientation matrix is given as:

$$\mathbf{R} = \begin{bmatrix} 2(Q_1^2 + Q_2^2) - 1 & 2(Q_2Q_3 - Q_1Q_4) & 2(Q_2Q_4 + Q_1Q_3) \\ 2(Q_2Q_3 + Q_1Q_4) & 2(Q_1^2 + Q_3^2) - 1 & 2(Q_3Q_4 - Q_1Q_2) \\ 2(Q_2Q_4 - Q_1Q_3) & 2(Q_3Q_4 + Q_1Q_2) & 2(Q_1^2 + Q_4^2) - 1 \end{bmatrix}. \tag{2.30}$$

Inverse problem. Let us find the expression of the quaternions as functions of the direction cosines of the general matrix $\mathbf{R}$ of (2.21). Equating the elements of the diagonals of the right sides of Eqs. (2.21) and (2.30) leads to:

$$Q_1 = \frac{1}{2}\sqrt{s_x + n_y + a_z + 1} \tag{2.31}$$

which is always positive. If we then subtract the second and third diagonal elements from the first diagonal element, we can write after simplifying:

$$4Q_2^2 = s_x - n_y - a_z + 1. \tag{2.32}$$

This expression gives the magnitude of Q_2. For determining the sign, we consider the difference of the (3,2) and (2,3) matrix elements, which leads to:

$$4Q_1 Q_2 = n_z - a_y. \tag{2.33}$$

The parameter Q_1 being always positive, the sign of Q_2 is that of $(n_z - a_y)$, which allows us to write:

$$Q_2 = \frac{1}{2}\mathrm{sign}(n_z - a_y)\sqrt{s_x - n_y - a_z + 1}. \tag{2.34}$$

Similar reasoning for Q_3 and Q_4 gives:

$$Q_3 = \frac{1}{2}\mathrm{sign}(a_x - s_z)\sqrt{-s_x + n_y - a_z + 1} \tag{2.35}$$

$$Q_4 = \frac{1}{2}\mathrm{sign}(s_y - n_x)\sqrt{-s_x - n_y + a_z + 1}. \tag{2.36}$$

Contrary to Euler angles, roll-pitch-yaw angles and *T&T* angles (see next sections), quaternion representation is free of singularity. For more information on the algebra of quaternions, the reader can refer to (de Casteljau 1987).

2.4.3 Euler Angles

The orientation of frame $\mathscr{F}_k$ expressed in frame $\mathscr{F}_i$ can be determined by specifying three angles, ϕ, θ and ψ corresponding to a sequence of three rotations (Fig. 2.5).

Let us consider two intermediate frames $\mathscr{F}_j$ and $\mathscr{F}_{j'}$ defined by $\mathscr{F}_j$ $(O_i, \boldsymbol{x}_j, \boldsymbol{y}_j, \boldsymbol{z}_j)$ and $\mathscr{F}_{j'}$ $(O_{j'}, \boldsymbol{x}_{j'}, \boldsymbol{y}_{j'}, \boldsymbol{z}_{j'})$ and characterized by:

- $\boldsymbol{z}_j \equiv \boldsymbol{z}_i$ and $\boldsymbol{y}_j$ is the intersection between the two planes $(O_i, \boldsymbol{x}_i, \boldsymbol{y}_i)$ and $(O_i, \boldsymbol{x}_k, \boldsymbol{y}_k)$,
- $\boldsymbol{y}_{j'} \equiv \boldsymbol{y}_j$ and $\boldsymbol{z}_{j'} \equiv \boldsymbol{z}_k$.

Taking into account these considerations, the Euler angles are defined as:

- ϕ: *precession* angle between $\boldsymbol{y}_i$ and $\boldsymbol{y}_j$ about $\boldsymbol{z}_i \equiv \boldsymbol{z}_j$, with $0 \le \phi < 2\pi$; that angle characterizes the pure rotation of angle ϕ around $\boldsymbol{z}_i$ (see Sect. 2.2.2) that transforms the frame $\mathscr{F}_i$ into the frame $\mathscr{F}_j$;

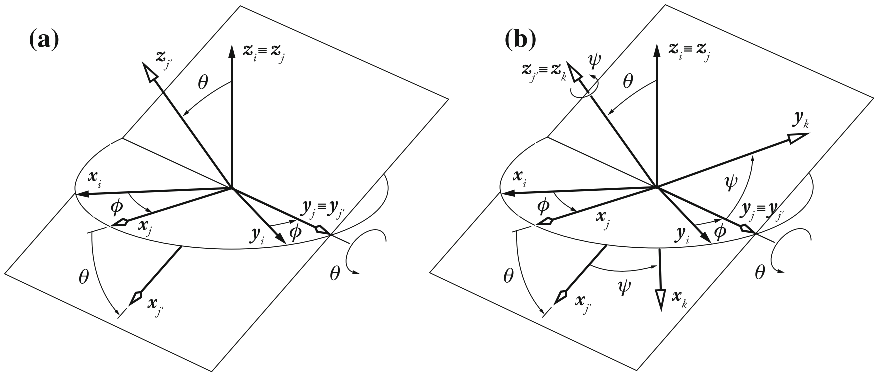

Fig. 2.5 The successive rotations that define the ZYZ Euler angles: **a** precession and nutation, **b** spin

- θ: *nutation* angle between z_i and $z_{j'}$ about $y_j \equiv y_{j'}$, with $0 \leq \theta < \pi$; that angle characterizes the pure rotation of angle θ around y_j (see Sect. 2.2.2) that transforms the frame $\mathscr{F}_j$ into the frame $\mathscr{F}_{j'}$;
- ψ: *spin* angle between $y_{j'} \equiv y_j$ and y_k about $z_{j'} \equiv z_k$, with $0 \leq \psi < 2\pi$; that angle characterizes the pure rotation of angle ψ around $z_{j'}$ (see Sect. 2.2.2) that transforms the frame $\mathscr{F}_{j'}$ into the frame $\mathscr{F}_k$.

The transformation matrix is given by:

$$\begin{aligned}\mathbf{R} &= \mathbf{rot}(z,\ \phi)\mathbf{rot}(y,\ \theta)\mathbf{rot}(z,\ \psi)\\ &= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}.\end{aligned} \tag{2.37}$$

Inverse problem. Let us find the expression of the Euler angles as functions of the direction cosines of the general matrix **R** of (2.21). Premultiplying Eq. (2.37) by $\mathbf{rot}(z, \phi)$, we obtain:

$$\mathbf{rot}(z, \phi)\mathbf{R} = \mathbf{rot}(y,\ \theta)\mathbf{rot}(z,\ \psi) \tag{2.38}$$

which results in

$$\begin{bmatrix} c_\phi s_x + s_\phi s_y & c_\phi n_x + s_\phi n_y & c_\phi a_x + s_\phi a_y \\ -s_\phi s_x + c_\phi s_y & -s_\phi n_x + c_\phi n_y & -s_\phi a_x + c_\phi a_y \\ s_z & n_z & a_z \end{bmatrix} = \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ s_\psi & c_\psi & 0 \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}. \tag{2.39}$$

From the elements on the second raw, third column of (2.45), we obtain:

$$-\, s_\phi a_x + c_\phi a_y = 0 \tag{2.40}$$

thus:

$$\begin{aligned}\phi &= \operatorname{atan2}(a_y, a_x)\\ \phi' &= \operatorname{atan2}(-a_y, -a_x) = \phi + \pi.\end{aligned} \tag{2.41}$$

There is a singularity if a_x and a_y are zero. In that case, $\theta = k\pi$ $(k = 0, 1)$.

In the same way, from the elements on the first and third rows, and third column of (2.39), and then from those of the second row, first and second columns, we deduce that:

$$\begin{aligned}\theta &= \operatorname{atan2}(\mathrm{c}_\phi a_x + \mathrm{s}_\phi a_y, a_z)\\ \psi &= \operatorname{atan2}(-\mathrm{s}_\phi s_x + \mathrm{c}_\phi s_y, -\mathrm{s}_\phi n_x + \mathrm{c}_\phi n_y).\end{aligned} \tag{2.42}$$

The described Euler angles convention is denoted as the ZYZ convention, where ZYZ denotes that we have a first rotation around $\boldsymbol{z}_i$, then a second rotation around $\boldsymbol{y}_j$ and finally a last rotation around $\boldsymbol{z}_{j'}$. There exists in total 12 different sequences of the three rotations, and, hence, there can be 12 Euler conventions: XYZ, XZY, YXZ, YZX, ZXY, ZYX, XYX, XZX, YXY, YZY, ZXZ, and ZYZ, where the convention PQR denotes that we have a first rotation around $\boldsymbol{p}_i$–axis, then a second rotation around $\boldsymbol{q}_j$–axis and finally a last rotation around $\boldsymbol{r}_{j'}$–axis.

2.4.4 Roll-Pitch-Yaw Angles

Following the convention shown in Fig. 2.6, the angles ϕ, θ and ψ indicate roll, pitch and yaw respectively. If we suppose that the direction of motion (by analogy to the

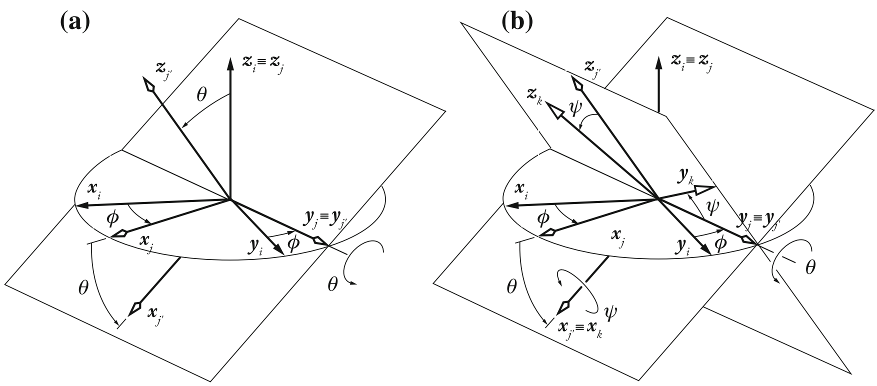

Fig. 2.6 Roll-Pitch-Yaw angles

direction along which a ship is sailing) is along the z_i axis, the transformation matrix can be written as:

$$\begin{aligned}\mathbf{R} &= \mathbf{rot}(\boldsymbol{z},\ \phi)\mathbf{rot}(\boldsymbol{y},\ \theta)\mathbf{rot}(\boldsymbol{x},\ \psi) \\ &= \begin{bmatrix} \mathrm{c}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi \\ -\mathrm{s}_\theta & \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{c}_\theta \mathrm{c}_\psi \end{bmatrix}.\end{aligned} \tag{2.43}$$

This description is analogous to the ZYX Euler angle convention.

Inverse problem. Let us find the expression of the Roll-Pitch-Yaw angles as functions of the direction cosines of the general matrix **R**. We use the same method discussed in the previous section. Premultiplying Eq. (2.43) by $\mathbf{rot}(z, \phi)$, we obtain:

$$\mathbf{rot}(z, \phi)\mathbf{R} = \mathbf{rot}(\boldsymbol{y},\ \theta)\mathbf{rot}(\boldsymbol{x},\ \psi) \tag{2.44}$$

which results in:

$$\begin{bmatrix} \mathrm{c}_\phi s_x + \mathrm{s}_\phi s_y & \mathrm{c}_\phi n_x + \mathrm{s}_\phi n_y & \mathrm{c}_\phi a_x + \mathrm{s}_\phi a_y \\ -\mathrm{s}_\phi s_x + \mathrm{c}_\phi s_y & -\mathrm{s}_\phi n_x + \mathrm{c}_\phi n_y & -\mathrm{s}_\phi a_x + \mathrm{c}_\phi a_y \\ s_z & n_z & a_z \end{bmatrix} = \begin{bmatrix} \mathrm{c}_\theta & \mathrm{s}_\theta \mathrm{s}_\psi & \mathrm{s}_\theta \mathrm{c}_\psi \\ 0 & \mathrm{c}_\psi & -\mathrm{s}_\psi \\ -\mathrm{s}_\theta & \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{c}_\theta \mathrm{c}_\psi \end{bmatrix}. \tag{2.45}$$

From the elements on the second raw, first column of (2.45), we obtain:

$$-\,\mathrm{s}_\phi s_x + \mathrm{c}_\phi s_y = 0 \tag{2.46}$$

thus:

$$\begin{aligned}\phi &= \mathrm{atan2}(s_y, s_x) \\ \phi' &= \mathrm{atan2}(-s_y, -s_x) = \phi + \pi.\end{aligned} \tag{2.47}$$

There is a singularity if s_x and s_y are zero. In that case, $\theta = \pm\pi/2$.

In the same way, from the elements on the first and third rows, and first column of (2.45), and then from those of the second row, second and third columns, we deduce that:

$$\begin{aligned}\theta &= \mathrm{atan2}(-s_z, \mathrm{c}_\phi s_x + \mathrm{s}_\phi s_y) \\ \psi &= \mathrm{atan2}(\mathrm{s}_\phi a_x - \mathrm{c}_\phi a_y, -\mathrm{s}_\phi n_x + \mathrm{c}_\phi n_y).\end{aligned} \tag{2.48}$$

2.4.5 Tilt-and-Torsion Angles

A novel three-angle orientation representation, later called the Tilt-and-Torsion (*T&T*) angles, was proposed in (Bonev and Ryu 1999). These angles were also independently introduced in (Huang et al. 1999), (Crawford et al. 1999) and (Wang 1999). They had been also proposed in (Korein 1984) under the name halfplane-deviation-twist angles. In (Bonev et al. 2002a), the advantages of the *T&T* angles in the study of spatial parallel mechanisms were further demonstrated. It was shown that there is a class of 3–*DOF* mechanisms that have always a zero torsion, that we now call zero-torsion parallel mechanisms. Furthermore, it was demonstrated in (Bonev and Gosselin 2005a) and (Bonev and Gosselin 2006) that the workspace and singularities of symmetric spherical parallel mechanisms are best analyzed using the *T&T* angles.

The *T&T* angles are defined in two stages: a tilt and a torsion. This does not, however, mean that only two angles define the *T&T* angles but simply that the axis of tilt is defined by another angle. In the first stage, illustrated in Fig. 2.7a, the body frame is tilted about a horizontal axis, $\boldsymbol{u}$, at an angle θ, referred to as the *tilt*. The axis $\boldsymbol{u}$ is defined by an angle ϕ, called the *azimuth*, which is the angle between the axes $\boldsymbol{u}$ and $\boldsymbol{y}_i$, $\boldsymbol{u}$ being at the intersection of the planes $(O_i, \boldsymbol{x}_i, \boldsymbol{y}_i)$ and $(O_i, \boldsymbol{x}_k, \boldsymbol{y}_k)$. In the second stage, illustrated in Fig. 2.7, the body frame is rotated about the body $\boldsymbol{z}_k$ axis at an angle σ, called the *torsion*.

For space limitations, we will omit the otherwise quite interesting details of the derivation process [see (Bovev et al. 2002a)], and write directly the resulting transformation matrix of the *T&T* angles, which is

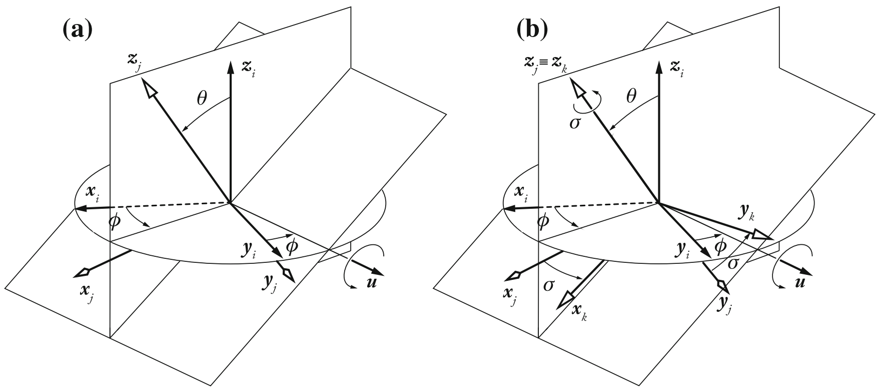

Fig. 2.7 The successive rotations of the *T&T* angles: **a** tilt, **b** torsion

$$
\begin{aligned}
\mathbf{R} &= \mathbf{rot}(\boldsymbol{u},\ \theta)\mathbf{rot}(z,\ \sigma) \\
&= \begin{bmatrix} c_\phi c_\theta c_{\sigma-\phi} - s_\phi s_{\sigma-\phi} & -c_\phi c_\theta s_{\sigma-\phi} - s_\phi c_{\sigma-\phi} & c_\phi s_\theta \\ s_\phi c_\theta c_{\sigma-\phi} + c_\phi s_{\sigma-\phi} & -s_\phi c_\theta s_{\sigma-\phi} + c_\phi c_{\sigma-\phi} & s_\phi s_\theta \\ -s_\theta c_{\sigma-\phi} & s_\theta s_{\sigma-\phi} & c_\theta \end{bmatrix}
\end{aligned} \tag{2.49}
$$

where $\mathbf{rot}(\boldsymbol{u},\ \theta) = \mathbf{rot}(z,\ \phi)\mathbf{rot}(\boldsymbol{y},\ \theta)\mathbf{rot}(z,\ -\phi)$.

From the above, we see that the *T&T* angles $(\phi,\ \theta,\ \sigma)$ are equivalent to the ZYZ Euler angles $(\phi,\ \theta,\ \sigma - \phi)$, i.e., the *spin* angle ψ has been replaced with $\sigma - \phi$.

Inverse problem. From the previous consideration, the inverse problem of the *T&T* angles can be solved as shown in Sect. 2.4.3, from which we find that:

$$
\begin{aligned}
\phi &= \operatorname{atan2}(a_y, a_x) \text{ or } \phi = \operatorname{atan2}(-a_y, -a_x) \\
\theta &= \operatorname{atan2}(c_\phi a_x + s_\phi a_y, a_z) \\
\sigma &= \operatorname{atan2}(-s_\phi s_x + c_\phi s_y, -s_\phi n_x + c_\phi n_y) + \phi.
\end{aligned} \tag{2.50}
$$

There is a singularity if $\theta = 0 + k\pi\ (k = 0, 1)$.

One of the properties of three-angle orientation representation is that a given orientation can be represented by at least two triplets of angles. In our case, the triplets $\{\phi, \theta, \sigma\}$ and $\{\phi \pm \pi, -\theta, \sigma\}$ are equivalent. To avoid this and the representational singularity at $\theta = \pi$ (which is hardly achieved by any parallel mechanism), we set the ranges of the *azimuth*, *tilt*, and *torsion* as, respectively, $\phi \in (-\pi, \pi]$, $\theta \in [0, \pi)$, and $\sigma \in (-\pi, \pi]$. Then, probably the most valuable property of the *T&T* angles is that for the ranges just defined, the angles $(\theta,\ \phi,\ \sigma)$ can be represented in a cylindrical coordinate system (r, ϕ, h) through a one-to-one mapping. In other words, any orientation (except $\theta = \pi$) corresponds to a unique point within a cylinder in the cylindrical coordinate system, and vice versa. The reason is that the *T&T* representational singularity at $\theta = 0$ is of the same nature as the singularity of the cylindrical coordinate system occurring at zero-radius $(r = 0)$.

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