Critical Rate Determination in a Dynamical Glaciation Model

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Glaciation Model

Glaciation Model

Governing Equations

Our glaciation model is presented in [1], given as:

$$C\dot{T} = \frac{S(t)}{4}(1 - \widehat{\alpha}(T)) - \frac{S_0}{4}(1 - \alpha_0) - a(T - T_0) + b\log\left[\frac{P}{P_0}\right], \tag{1}$$

$$\dot{P} = V - W(T)\exp\left[k(T - T_0)\right]\left(\frac{P}{P_0}\right)^{\beta}. \tag{2}$$

where:

$$\widehat{\alpha}(T) = \alpha_c - (\alpha_c - \alpha_w)\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{T - T_i}{\gamma}\right]\right), \tag{3}$$

$$W(T) = W_w\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{T - T_i}{\gamma}\right]\right), \tag{4}$$

$$S(t) = S_u - \frac{1}{2}(S_u - S_l)(1 + \tanh\left[t\right]). \tag{5}$$

Glaciation Model

Non-dimensionalization

We follow [2] and first non-dimensionalize time:

$$\tau = \epsilon t \quad \text{where} \quad \epsilon = \frac{V}{P_0}. \tag{6}$$

We then introduce two non-dimensionalized variables for the temperature (fast) and carbon-silicate (slow) equations:

$$x = \frac{T}{T_0} \quad \text{and} \quad y = \log\left[\frac{P}{P_0}\right]. \tag{7}$$

Glaciation Model

Non-autonomous Dimensionless System

Using this non-dimensionalization, we arrive at the following system:

$$\delta \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\lambda(\tau)}{4}\left(1-\alpha(x)\right) - \frac{1}{4}\left(1-\alpha_0\right) - a(x-1) + by \qquad (8)$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \left(1 - w(x)e^{k(x-1)+\beta y}\right)e^{-y}. \qquad (9)$$

where:

$$\alpha(x) = \alpha_c - (\alpha_c - \alpha_w)\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{xT_0 - T_i}{\gamma}\right]\right) \qquad (10)$$

$$w(x) = w_w\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{xT_0 - T_i}{\gamma}\right]\right) \qquad (11)$$

$$\lambda(\tau) = s_u - \frac{1}{2}(s_u - s_l)(1 + \tanh\left[\frac{\tau}{\epsilon}\right]). \qquad (12)$$

Glaciation Model

Dimensionless Parameters

Using this non-dimensionalization, we arrive at the following dimensionless parameters:

Parameter	**Expression**	**Value**
a	AT_0/S_0	~ 0.46
b	B/S_0	$\sim 5.9 \times 10^{-3}$
w_w	W_w/V	~ 1.114
k	KT_0	~ 29
s_u	S_u/S_0	~ 0.941
s_l	S_l/S_0	~ 0.931

Glaciation Model

Rate Constant

We think of $0 < \delta \ll 1$ as quantifying the ratio of the x and y time scales and defined as:

$$\delta = \frac{CVT_0}{P_0 S_0} \sim 3 \times 10^{-4}. \tag{13}$$

Also, it is the rate, which we define as $\widehat{v} = 1/\epsilon$, that achieves some critical value, $\widehat{v}_c$, that is of interest in this work.

Specifically, we note that the largest rate, at $\tau = 0$, is defined for the sigmoid by:

$$\frac{\widehat{v}(s_u - s_l)}{2} = v, \tag{14}$$

from which we can derive as a first estimate:

$$v = \frac{P_0(s_u - s_l)}{2V\Delta t} \sim 7 \times 10^{-4}. \tag{15}$$

Glaciation Model

Dimensionless System

Lastly, we can note that our $\lambda(\tau)$ can be viewed as a solution of a Bernoulli Equation and then define:

$$\delta s = \frac{1}{2}(s_u - s_l)\left(1 + \tanh\left[\widehat{v}\tau\right]\right), \tag{16}$$

which allows us to finally deal with the equations:

$$\delta \frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\lambda(\tau)}{4}\left(1 - \alpha(x)\right) - \frac{1}{4}\left(1 - \alpha_0\right) - a(x-1) + by \tag{17}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = \left(1 - w(x)e^{k(x-1)+\beta y}\right)e^{-y}, \tag{18}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = 4v\left(\frac{\delta s}{(s_u - s_l)} - \frac{(\delta s)^2}{(s_u - s_l)^2}\right). \tag{19}$$

Critical Rate Determination: Theory

Critical Rate Determination: Theory

General Dynamical System

We follow [3] and define general dynamical system:

$$\delta \frac{\mathrm{d}x}{\mathrm{d}\tau} = f(x, y, \lambda(\tau), \delta) \tag{20}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = g(x, y, \lambda(\tau), \delta), \tag{21}$$

where:

- x is the fast variable
- y is the slow variable
- f and g are smooth functions
- δ is a small parameter, $0 < \delta \ll 1$ that defines the ratio of time scales values between x and y
- $\lambda_{min} \leq \lambda(\tau) \leq \lambda_{max}$ is a bounded function that evolves in time on the slow time scale, defined by the time scale: $\tau = \epsilon t$.

Critical Rate Determination: Theory

General Dynamical System

Now, we can think of the singular limit of $\delta = 0$ as defining the one-dimensional critical manifold $S(\lambda)$, on which $\mathrm{d}y \,/\, \mathrm{d}\tau = g(x, y, \lambda, 0)$ is constrained. We can therefore define the critical manifold by:

$$S(\lambda) \;:\; f(x, y, \lambda, 0) = 0. \tag{22}$$

In our work, this will look like the function $y = h(x)$.

Critical Rate Determination: Theory

General Dynamical System

By (a2) of [3], our first task is to find the one steady state which is asymptotically state and varies continuously with λ. This is found by calculating the intersection of the nullclines, $\mathrm{d}y \,/\, \mathrm{d}\tau = 0$ and $\mathrm{d}x \,/\, \mathrm{d}\tau = 0$, and determining which equilibrium point is stable.

Critical Rate Determination: Theory

General Dynamical System

Then, by (a1) of [3], we will determine the single fold $F(\lambda)$ that is tangent to the fast x-direction defined by:

$$\left.\frac{\partial f}{\partial x}\right|_{S} = 0. \tag{23}$$

and:

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{S} \neq 0. \tag{24}$$

This will give us values x_F, the one closest to the stable equilibrium we will choose, as is a condition for this theorem. We can then plug this back into our equation for $S(\lambda)$ to get y_F.

Critical Rate Determination: Theory

General Dynamical System

Lastly, we can determine the critical rate, v_c as:

$$v_c \approx \inf\left\{ v > 0 \ : \ m = \left(g\frac{\partial f}{\partial y} + v\frac{\partial f}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \right) \Big|_{F=(x_F, y_F)} = 0 \right\}. \quad (25)$$

Critical Rate Determination: Theory

General Dynamical System

At this point, we are really dealing with two functions:

$$f(x_F, y_F, \lambda(\tau), 0) \text{ and } g(x_F, y_F, \lambda(\tau), 0), \tag{26}$$

so we can think of this problem as a bifurcation problem in just the (τ, m)-plane: at which point v_c is the function defined in Eq. (26) satisfied, i.e. for the single real double root of m to come into existence.

Critical Rate Determination: Application

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Critical Manifold

We first find the critical manifold using (22) to find:

$$S(\lambda)\ :\ y = -\frac{1}{b}\left[\frac{\lambda(\tau)}{4}\left(1-\alpha(x)\right) - \frac{1-\alpha_0}{4} - a(x-1)\right]. \quad (27)$$

Critical Rate Determination: Application

Equilibria

We next calculate the equilibria by finding the intersection of the nullclines defined by:

$$\text{x-nullcline:} \quad y = -\frac{1}{b}\left[\frac{\lambda(\tau)}{4}(1-\alpha(x)) - \frac{1-\alpha_0}{4} - a(x-1)\right] \tag{28}$$

$$\text{y-nullcline:} \quad y = \frac{1}{\beta}\left[k(1-x) - \log\left[w(x)\right]\right]. \tag{29}$$

Critical Rate Determination: Application

Fold

We next compute the fold following (23):

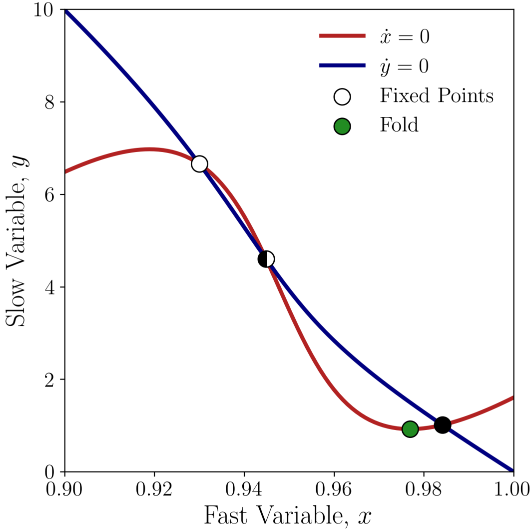
$$F(\lambda)\ :\ x_f = \frac{T_i}{T_0} + \left[\frac{\gamma(\alpha_c - \alpha_w)}{4a\pi T_0}\lambda(\tau) - \left(\frac{\gamma}{T_0}\right)^2\right]^{1/2}, \qquad (30)$$

satisfying (24). Then, we find the corresponding y_F to be:

$$y_F = -\frac{1}{b}\left[\frac{\lambda(\tau)}{4}\left(1 - \alpha(x_F)\right) - \frac{1 - \alpha_0}{4} - a(x_F - 1)\right]. \qquad (31)$$

Critical Rate Determination: Application

Equilibria



Critical Rate Determination: Application

Numerical Estimation of Critical Rate

We now have a minimization problem of the form:

$$v_c = \min_{v>0} \{ v \ : \ m(v) = 0 \} \,. \tag{32}$$

To calculate this, we implementing a custom minimization scheme in `Python` that is given by the following pseudo-code:

Initialize some $v > 0$ sufficiently large

At some `tolEp`, reduce v per `N` many steps, `N` sufficiently large

Per step, calculate $m(v)$:

If $m(v) <$ `tol0`, then break

Otherwise, run until this condition is satisfied.

From this routine, we estimate:

$$\boxed{v_c \sim 8.578 \times 10^{-4},} \tag{33}$$

at which rate we have:

$$\boxed{(x_F, y_F, \lambda_F) = (0.9766, 1.368, 0.9356).} \tag{34}$$

That's within an order of magnitude of the estimated $\sim 7 \times 10^{-4}$... look's like we're getting somewhere.

References

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