

# Note 35

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## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Problems</b>	<b>2</b>
2.1	Problem 1: Conditional Probability Density Functions . . . . .	2
2.2	Problem 2: Lifetime Model of Electrical Components . . . . .	3
2.3	Problem 3: Independent Exponential Variables . . . . .	3
2.4	Problem 4: Joint Distribution of Normal Variables . . . . .	4
2.5	Problem 5: Bivariate Transformation Method . . . . .	4
2.6	Problem 6: Bivariate Random Variable . . . . .	4
2.7	Problem 7: Joint Density Function Analysis . . . . .	5
2.8	Problem 8: Covariance Definition . . . . .	5
<b>3</b>	<b>Conclusion</b>	<b>5</b>
<b>4</b>	<b>References</b>	<b>5</b>

# 1 Introduction

This document presents a series of probability theory problems, focusing on various aspects of probability density functions, joint distributions, and covariance. Each problem challenges the reader to apply their understanding of random variables and their relationships, providing a solid exercise for those studying advanced probability concepts.

## 2 Problems

### 2.1 Problem 1: Conditional Probability Density Functions

Given the data from Problems 2, Question 2, we explore the conditional relationships between two random variables,  $X$  and  $Y$ .

#### 1. Find the conditional probability density function of $X$ given $Y$ :

- *Definition:* The conditional probability density function  $f_{X|Y}(x|y)$  is defined as:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

- *Steps:*

- Calculate the joint probability density function  $f_{X,Y}(x,y)$ .
- Determine the marginal probability density function  $f_Y(y)$ .
- Substitute into the formula.

#### 2. Find the conditional probability density function of $Y$ given $X$ :

- *Definition:* The conditional probability density function  $f_{Y|X}(y|x)$  is similarly defined as:

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

- *Steps:*

- Compute the joint probability density function  $f_{X,Y}(x,y)$ .
- Determine the marginal probability density function  $f_X(x)$ .
- Substitute the values into the formula.

#### 3. Find $P(X + Y > 3|X = 1)$ :

- *Method:*

- Use the conditional probability formula:

$$P(X + Y > 3|X = 1) = P(Y > 2|X = 1)$$

- Calculate this probability accordingly based on the conditional density of  $Y$ .

## 2.2 Problem 2: Lifetime Model of Electrical Components

In this problem, we examine the lifetime of electrical components modeled by two random variables: quality factor  $Q$  and lifetime  $T$ .

### 1. Model Description:

- Each component has a quality factor  $q$  and a lifetime  $t$ .
- $Q$  follows an exponential distribution:

$$Q \sim \text{Exp}(\lambda), \quad \lambda > 0$$

- The lifetime given quality  $q$  is represented as:

$$T|Q=q) \sim \text{Exp}(q)$$

### 2. Joint Probability Density Function:

- Find the joint distribution  $f_{T,Q}(t,q)$ :

$$f_{T,Q}(t,q) = f_{T|Q}(t|q) \cdot f_Q(q)$$

- Substitute the relevant exponential density functions.

### 3. Marginal Probability Density Function of $T$ :

- To find  $f_T(t)$ , integrate over all possible values of  $Q$ :

$$f_T(t) = \int_0^\infty f_{T,Q}(t,q) dq$$

- This results in a probability density function that depends on  $\lambda$ .

## 2.3 Problem 3: Independent Exponential Variables

Let  $X$  and  $Y$  be independent exponential random variables with the same distribution,  $X \sim \text{Exp}(\lambda)$  and  $Y \sim \text{Exp}(\lambda)$ .

### 1. Determine the Range of $(U,V)$ :

- Define  $U$  and  $V$  as follows:
  - $U = \frac{X}{X+Y}$
  - $V = X + Y$
- *Sketching the Diagram:*  $U$  ranges from 0 to 1, while  $V$  spans from 0 to  $\infty$ .

### 2. Joint Probability Density Function of $(U,V)$ :

- Find  $f_{U,V}(u,v)$  using transformation techniques.

### 3. Prove $U$ has a Uniform Distribution on $(0,1)$ :

- Show that:

$$P(U \leq u) = u \quad \text{for } u \in (0,1)$$

## 2.4 Problem 4: Joint Distribution of Normal Variables

Consider two independent random variables  $X$  and  $Y$  that each follow a standard normal distribution, denoted as  $X \sim N(0, 1)$  and  $Y \sim N(0, 1)$ .

### 1. Define $U$ and $V$ :

- Let:
  - $U = X$
  - $V = \rho X + \sqrt{1 - \rho^2}Y$ , where  $|\rho| < 1$ .

### 2. Find the Joint Probability Density Function of $(U, V)$ :

- Use the properties of normal distributions to derive this function.

### 3. Prove the Covariance:

- Show that:

$$\text{Cov}(U, V) = \rho$$

## 2.5 Problem 5: Bivariate Transformation Method

Given independent random variables  $X$  and  $Y$ , with  $Z = X + Y$ , we apply the bivariate transformation method to demonstrate a fundamental theorem in probability.

### 1. Define $W$ :

- Let  $W = X$ .

### 2. Prove the Distribution:

- Show that:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x) dx$$

- This provides an alternative proof of the convolution theorem.

## 2.6 Problem 6: Bivariate Random Variable

Consider the discrete bivariate random variable  $(X, Y)$  with a given joint probability mass function.

### 1. Determine Independence:

- *Question:* Are  $X$  and  $Y$  independent?
- *Method:* Check if  $P(X, Y) = P(X)P(Y)$ .

### 2. Find Covariance:

- Calculate  $\text{Cov}(X, Y)$ .

### 3. Calculate Correlation:

- Determine the correlation coefficient  $\rho_{X,Y}$ .

## 2.7 Problem 7: Joint Density Function Analysis

Let  $(X, Y)$  have the joint density function defined by:

$$f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } y > 0, x+y < 1 \text{ and } y < x+1 \\ 0 & \text{otherwise} \end{cases}$$

### 1. Find Covariance:

- Calculate  $\text{Cov}(X, Y)$ .

### 2. Independence Check:

- Prove that  $X$  and  $Y$  are not independent.

### 3. Significance Commentary:

- Discuss the implications of their dependency.

### 4. Covariance of Squares:

- Calculate  $\text{Cov}(X^2, Y^2)$  and provide another proof of their dependence.

## 2.8 Problem 8: Covariance Definition

Prove that:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

using the definition of covariance presented in lectures.

## 3 Conclusion

This document has explored various complex problems in probability theory, requiring detailed calculations and a robust understanding of statistical principles. Each problem not only tests theoretical knowledge but also applies practical techniques for solving real-world problems involving random variables and their interactions.

## 4 References

- Introduction to Probability Theory
- Advanced Statistical Methods
- Joint Probability Distribution Literature