

# Highlights of Calculus

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## 1 $\epsilon, \delta$ for Limits

Danger case:

$$\begin{aligned} \infty - \infty \\ 0 \cdot \infty \\ \frac{0}{0} \\ 0^0 \text{ or } 1^\infty \end{aligned}$$

L'Hospital Rule:

$$\frac{f(x)}{g(x)} \rightarrow \frac{\frac{\Delta f}{\Delta x}}{\frac{\Delta g}{\Delta x}} \rightarrow \frac{f'}{g'}$$

For any small  $\epsilon$  chosen, we can find  $\delta > 0$ , so that if  $|f(x) - f(a)| < \epsilon$ , then  $|f(x) - f(a)| < \delta$

## 2 Fundamental Theorem of Calculus

$$\begin{aligned}f'(x) &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \frac{df}{dx}, \text{ When } \Delta x \rightarrow 0\end{aligned}$$

Here,  $\Delta x$  means two point difference in  $x$ ,  $df$  means difference in function value caused by the difference in  $x$ .  $\Delta x \rightarrow 0$  就是代数到微积分的过程。

考虑函数值  $x$  在点  $x_1, x_2, \dots, x_n$  的函数值  $f_1, f_2, \dots, f_n$ , 进而考虑其两者之间的差值  $(f_2 - f_1) + (f_3 - f_2) + \dots + (f_n - f_{n-1}) = f_n - f_1$ 。从这里可以简单的理解为, 你可以将一个函数, 利用其差值累加还原为原函数的值, 这就是积分的过程;

$$\begin{aligned}\sum \Delta y &= y_{\text{last}} - y_{\text{first}} \\f(x) &= \int f'(x) dx = \sum \frac{\Delta y}{\Delta x} \cdot \Delta x, \text{ Where } f'(x) dx = df, \text{ when } \Delta x \rightarrow 0\end{aligned}$$

从这里可以看出, 对于导函数可将其视为用高度函数表示原函数的函数, 其高度与其底部“面积”的乘积表示了其空间大小, 即原函数的差值。

对于微分还有另一种理解为变换的视角, 即从一个函数变换到另一个函数-线性映射, 这个映射操作的符号记做  $\frac{d}{dx}$ , 它将  $y$  进行变换到  $y'$ ,  $y' = \frac{d}{dx} \cdot y$

二阶导数的定义如下:

$$y'' = \frac{d^2 y}{dx^2}$$

对于这里的符号解释如下:

对于  $dx^2$ , 只是对于  $x$  只是进行了两次除法操作即  $\frac{\Delta \Delta f}{\Delta x \cdot \Delta x}$ , 但是对于  $y$  而言则是在第一次的  $df$  之上再次取差值即  $d(df)$ , 也就是求差值这个操作  $d(\text{diffence})$  重复了两次。

$$\begin{aligned}f''(x) &> 0 \rightarrow \text{convex function} \\f''(x) &< 0 \rightarrow \text{concave function}\end{aligned}$$

关于一阶, 以及二阶导数的主要应用在于寻找各个特殊的点。

$f'(x) \rightarrow$  stationary point

$f''(x) \rightarrow$  inflection point

$f'(x) = 0$ , and  $f''(x) > 0 \rightarrow$  Local max

$f'(x) = 0$ , and  $f''(x) < 0 \rightarrow$  Local min

对于函数的最值，则需要比较所有极值点以及边界点确定。

### 3 Derivatives of $e^x, \sin x, \cos x, x^n$

#### 3.1 Exponential Function

Key: Which function's derivatives are equal to the function itself?

$$\frac{df}{dx} = y \rightarrow \text{first differential equation}$$

Construction:

$$y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$
$$\frac{df}{dx} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

这里思想在于当  $x = 0, e^x = 1$ , 那么其导数也为 1; 导数为 1, 原函数为什么其导数才为 1 呢? 如此反复迭代; 显然当  $n \rightarrow \infty$ , 两式才相等。该级数称之为指数级数。

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2 \cdot 1}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

set  $x = 0, e = 1 + 1 + \cdots = 2.71828\ldots \rightarrow$  Euler's Number

用指数级数可证明指数函数下面的性质

$$e^a \cdot e^b = e^{a+b}$$

Euler's Number 也可以通过如下方式计算得到

$$e = \left(1 + \frac{1}{N+1}\right)^N, \text{ When } N \rightarrow \infty$$

对于该式子的展开基于二项式定理 (Binomial Theorem).

$$\frac{dy}{dx} = y$$

$$y = f(x) = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n + \dots = e$$

### 3.2 Trigonometric Function

三角函数起源于勾股定理

$$a^2 + b^2 = c^2$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$$(\sin \theta)^2 + (\cos \theta)^2 = 1$$

三角函数求导关键在于用半径为 1 的圆描述周期运动，以及其中的三角形。

下面给两个重要的极限

$$\sin \theta < \theta \rightarrow \frac{\sin \theta}{\theta} < 1$$

$$\tan \theta > \theta \rightarrow \frac{\sin \theta}{\theta} > \cos \theta$$

$$\frac{\sin \theta}{\theta} = 1, \text{ when } \theta \rightarrow 0$$

前两个式子可由弧度制的弧长和面积证明，该极限可认为是  $\sin \theta$  在  $\theta = 0$  处的导数，由上面两个式子夹逼准则定义。

下面给出另一个重要的极限。

$$\frac{\cos \theta - 1}{\theta} = 0, \text{ when } \theta \rightarrow 0$$

该极限可认为是  $\cos 0$  处的导数。

$$\begin{aligned}\frac{\Delta \sin x}{\Delta x} &= \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\ &= \frac{\sin x (\cos \Delta x - 1)}{\Delta x} + \frac{\sin \Delta x \cos x}{\Delta x} \\ &= \cos x\end{aligned}$$

仿照上例子可得到  $\cos \theta$  的导数；下面不加证明地给出  $\cos x$  的导数

$$\frac{d \cos x}{dx} = -\sin x$$

### 3.3 Product Rule, Quotient Rule, Derivatives to Power Function

$$q(x) = f(x)g(x)$$

考虑边长分别为  $f(x), g(x)$  的长方形，当两边分别改变  $\Delta x$ ，其面积的变化：

$$\Delta \text{area} = f(x)g(x + \Delta x) - g(x) + g(x)(f(x + \Delta x) - f(x)) + \Delta x^2$$

When  $\Delta x \rightarrow 0$ ,

$$\begin{aligned}dq &= f(x)dg + g(x)df \\ \frac{dq}{dx} &= f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}\end{aligned}$$

Quation rule 可由乘法法则推导得到。

$$\frac{f(x)}{g(x)} = \frac{f(x)g' - g(x)f'}{g(x)^2}$$

## 4 Chain Rule, and Derivatives of Inverse Function $\ln x, \sin^{-1} x, \cos^{-1} x$

### 4.1 Chain Rule

$$f'(y(x)) = \frac{df}{dx} = \frac{df}{dy} \frac{dy}{dx}$$

对于偶函数，其导数为奇函数。对于奇函数，其导数为偶函数。

$$y = f(x) \rightarrow x = f^{-1}(y)$$

需要注意的是只有在单调区间内，才有逆函数，且  $f$  与  $f^{-1}$  的函数图像关于原点对称。

### 4.2 Logarithmic Function

指数函数的逆函数为对数函数，其求的是指数的值。

$$x = \ln y$$

其具有如下性质

$$\ln ab = \ln a + \ln b$$

$$\ln y^n = n \ln y$$

### 4.3 Derivatives for $\ln x, \sin^{-1} x, \cos^{-1} x$

set

$$y = e^x$$

$$x = \ln y$$

Then

$$y = e^x \rightarrow e^{\ln y} = y$$

$$e^{\ln y} \cdot \frac{d \ln y}{dy} = 1, \text{ Where } e^{\ln y} = y$$

set

$$y = \sin x$$

$$x = \sin^{-1} y$$

Then

$$\sin \sin^{-1} y = y$$

$$\cos \sin^{-1} y \cdot \frac{d \sin^{-1} y}{dy} = 1, \text{ Where } \cos \sin^{-1} y = \frac{1}{\sqrt{1-y^2}}$$

Note that the  $\sin^{-1} y$  is an angle.

Give the  $\frac{d \cos^{-1} y}{dy}$  without proof.

$$\frac{d \cos^{-1} y}{dy} = -\frac{1}{\sqrt{1-y^2}}$$

Note that:

$$\frac{d \cos^{-1} y}{dy} + \frac{d \sin^{-1} y}{dy} = 0$$

Where  $\theta + \alpha = \frac{\pi}{2}$  is a constant.

Some other derivatives:

$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

$$\frac{d \operatorname{arccot} x}{dx} = -\frac{1}{1+x^2}$$

$$\frac{d a^x}{dx} = a^x \ln a$$

Converion between different base.

$$\log_a |x| = \frac{1}{x \ln a}$$

$$\log_a b = \frac{\ln b}{\ln a} = \frac{\log_n b}{\log_n a}$$

## 5 Growth Rate and Logarithmic Plot

各函数的增长速度如下，其倒数就是减慢的速度。

$$CX \dots \quad x^2, x^3 \dots 2^x, e^x, 10^x \dots \quad x! x^x$$

Linear   Polynomial   Exponential   Factorial

对数尺度能够处理极大或者极小 ( $x \rightarrow 0$ ) 的值, 但是该尺度下是没有 0 的。

对数尺度能够将非线性问题转换为线性问题

$$y = AX^n \rightarrow \log y = \log A + n \log X, \text{logarithmic plot}$$

$$y = B10^{Cx} \rightarrow \log y = \log B + Cx, \text{semi-logarithmic plot}$$

## 6 Linear Approximation/Newton's Method

$$f(x) = f(a) + f'(a)(x - a)$$

$$F(x) = 0 \rightarrow x - a = \frac{F(a)}{F'(a)}$$

The core of Newton's method is iteration.



## 7 Power Series/Euler's Great Formula

幂级数的核心在于用多项式进行函数的近似，用多项式近似的好处在于其  $n$  阶导数只和第  $n$  阶项有关，其它在此之前的多项式都为 0，第  $n$  阶项的系数为  $n!$ 。

考虑指数级数，在 0 处的  $0, 1, 2, \dots, n$  导数值。

$$1, 1, 1, \dots, 1$$

为了匹配这个系数，对于幂函数的  $n$  阶项的导数系数  $n!$  除  $n!$  则可匹配每一阶的系数。

$$e^x = 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n + \dots$$

仿照上例，给出  $\sin x, \cos x$  的幂级数

$$\begin{aligned}\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\end{aligned}$$

对于欧拉公式，可由上面三个级数给出

$$e^{i\theta} = 1 + ix + \frac{1}{2}(ix)^2 + \frac{1}{6}(ix)^3 + \dots$$

整理之后可见，右边即为  $\sin x, \cos x$  的幂级数。

$$e^{i\theta} = \cos x + i \sin x$$

欧拉公式给出了在横轴为实数，纵轴为复数的复平面上，数据之间的关系。

下面给出两个其它重要的幂级数

$$\text{Geometrix series } \frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots, \text{ Where } 0 < |x| < 1$$

$$\text{Integrated from the above equation } -\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots \text{ Where } x < 1$$

## 8 Differential Equations

### 8.1 Differential Equations of Motion

Linear, and Second order equation.

$$m \frac{d^2 y}{dt^2} + 2r \frac{dy}{dt} + ky = 0$$

When  $m = 0$

$$\frac{dy}{dt} = ay \rightarrow y = ce^{at}$$

When  $r = 0$

$$\frac{d^2 y}{dt^2} = \frac{k}{m} y = -\omega^2 y \rightarrow y = C \cos \omega t + D \sin \omega t$$

When  $m = r = 0$

$$\frac{d^2 y}{dt^2} = 0 \rightarrow y = C + Dt$$

General solutaion - Try  $y = e^{\lambda t}$

$$m\lambda^2 + 2r\lambda + K = 0$$

Three Cases:

$$y'' + 6y' + 8y = 0 \rightarrow y(t) = Ce^{-2t} + De^{-4t}$$

$$y'' + 6y' + 10y = 0 \rightarrow y(t) = Ce^{(-3-i)t} + De^{(-3+i)t}$$

$$y'' + 6y' + 9 = 0 \rightarrow y(t) = Ce^{-3t} + Dte^{-3t}$$

## 8.2 Differential Equations of Growth

The growth rate proportional to itself.

$$\begin{aligned}\frac{dy}{dt} &= cy \\ y(0) &\rightarrow \text{Given start} \\ y(t) &= y(0)e^{ct}\end{aligned}$$

Add source term:

$$\begin{aligned}\frac{dy}{dt} &= cy + s \text{ Where } s \text{ is source term} \\ \frac{d(y + \frac{s}{c})}{dt} &= c(y + \frac{s}{c}) \\ y + \frac{s}{c} &= (y(0) + \frac{s}{c})e^{ct}\end{aligned}$$

For Linear eq, the solutions to eq have form below

$$y(t) = y_{\text{particular}}(t) + y_{\text{right side 0}}(t)$$

Specially for  $\frac{dy}{dt} = cy + s$

$$\begin{aligned}y_{\text{particular}} &= -\frac{s}{c} \\ y_{\text{set } s=0} &= Ae^{ct}\end{aligned}$$

Then

$$y = -\frac{s}{c} + Ae^{ct}$$

To find  $A$ , put  $t = 0$ ,  $y(0) = \frac{s}{c} + A$

Non-linear equation for population:

$$\frac{dp}{dt} = cp - sp^2$$

To solve this equation, set  $y = \frac{1}{p}$  to turn this equation to linear equation.

Equation for predators and prey

$$\begin{aligned}\frac{du}{dt} &= -cu + suv \\ \frac{dv}{dt} &= cv - suv\end{aligned}$$

## 9 Six Functions, Six Rules, and Six Theorems

Six Functions

$$\begin{aligned}\frac{1}{n+1}x^{n+1} &\rightarrow x^n && \rightarrow (n-1)x^{n-1} \\ -\cos x &\rightarrow \sin x && \rightarrow \cos x \\ \sin x &\rightarrow \cos x && \rightarrow -\sin x \\ \frac{1}{c}e^{cx} &\rightarrow e^{cx} && \rightarrow ce^{cx} \\ x \ln x - x &\rightarrow \ln x && \rightarrow \frac{1}{x} \text{power -1}\end{aligned}$$

Ramp Function

Six Rules

$$\begin{aligned}af(x) + bg(x) &\rightarrow a\frac{df}{dx} + b\frac{dg}{dx} \\ f(x)g(x) &\rightarrow f(x)\frac{dg}{dx} + \frac{df}{dx}(gx) \\ \frac{f(x)}{g(x)} &\rightarrow \frac{gf' - fg'}{g^2} \\ x = f^{-1}(y) &\rightarrow \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \\ f(g(x)) &\rightarrow \frac{df}{dy} \cdot \frac{dy}{dx}\end{aligned}$$

$$\text{L'Hospital} \frac{f}{g} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} \text{ When } x \rightarrow a, f(a), g(a) \rightarrow 0$$

Six Theorems

- Fundamental Theorem of Calculus
- Mean Values Theorem

- Taylors Series/Theorem
- Bionomial Theorem - Taylor at  $a = 0 \rightarrow$  Pascal triangle

$$f(x) = (1+x)^p = 1 + px + \frac{p(p-1)}{2 \cdot 1}x^2 + \dots$$