

# Probability Distribution

2022-06-10

## 1 离散分布

### 1.1 二项分布 (Binomial Distribution)

多重伯努利实验中, 已知事件  $A$  成功的概率为  $p$ , 且实验次数  $n$  固定, 那么随机变量  $X$  —— 事件  $A$  发生次数  $X$  :

$$P(X = k) = C_n^k p^k (1 - p)^{n-k}, k = 0, 1, \dots, n.$$

记为:

$$X \sim b(n, p) \text{ Where } E(X) = np, D(X) = np(1 - p)$$

```
curve(dbinom(x, 100, 0.3), 0, 80, col = "red")
```

Warning in dbinom(x, 100, 0.3): non-integer x = 0.800000

Warning in dbinom(x, 100, 0.3): non-integer x = 1.600000

Warning in dbinom(x, 100, 0.3): non-integer x = 2.400000

Warning in dbinom(x, 100, 0.3): non-integer x = 3.200000

Warning in dbinom(x, 100, 0.3): non-integer x = 4.800000

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Warning in dbinom(x, 100, 0.3): non-integer x = 10.400000

Warning in dbinom(x, 100, 0.3): non-integer x = 11.200000

Warning in dbinom(x, 100, 0.3): non-integer x = 12.800000

Warning in dbinom(x, 100, 0.3): non-integer x = 13.600000

Warning in dbinom(x, 100, 0.3): non-integer x = 14.400000

Warning in dbinom(x, 100, 0.3): non-integer x = 15.200000

Warning in dbinom(x, 100, 0.3): non-integer x = 16.800000

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Warning in dbinom(x, 100, 0.3): non-integer x = 18.400000

Warning in dbinom(x, 100, 0.3): non-integer x = 19.200000

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Warning in dbinom(x, 100, 0.3): non-integer x = 21.600000

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Warning in dbinom(x, 100, 0.3): non-integer x = 32.800000

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Warning in dbinom(x, 100, 0.3): non-integer x = 79.200000

```
curve(dbinom(x, 100, 0.5), 0, 80, col = "blue", add = TRUE)
```

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Warning in dbinom(x, 100, 0.5): non-integer x = 74.400000

Warning in dbinom(x, 100, 0.5): non-integer x = 75.200000

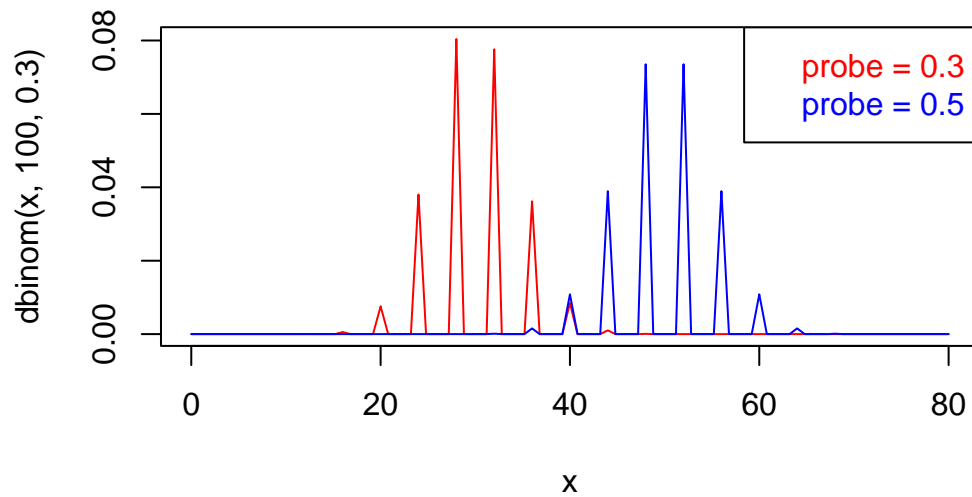
Warning in dbinom(x, 100, 0.5): non-integer x = 76.800000

Warning in dbinom(x, 100, 0.5): non-integer x = 77.600000

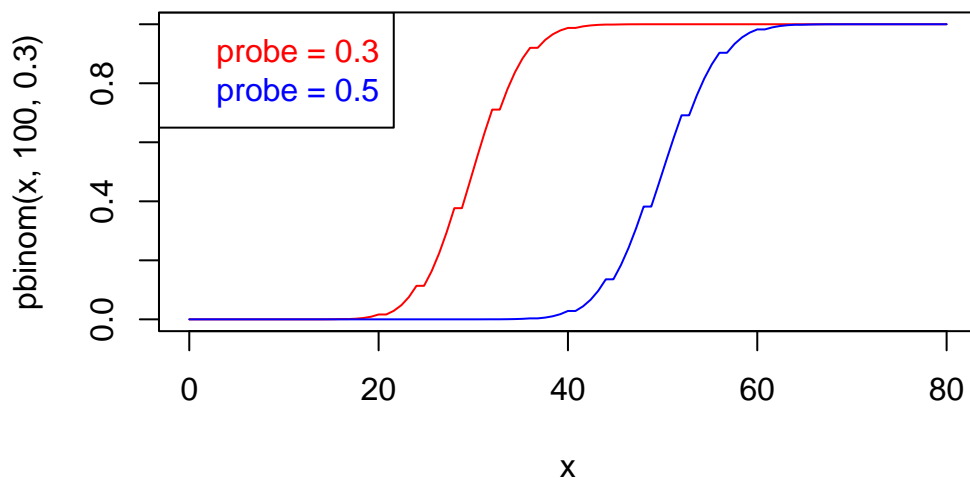
Warning in dbinom(x, 100, 0.5): non-integer x = 78.400000

Warning in dbinom(x, 100, 0.5): non-integer x = 79.200000

```
legend("topright",  
  legend = paste0("probe = ", c(0.3, 0.5)),  
  text.col = c("red", "blue")  
)
```



```
curve(pbinom(x, 100, 0.3), 0, 80, col = "red")  
curve(pbinom(x, 100, 0.5), 0, 80, col = "blue", add = TRUE)  
legend("topleft",  
  legend = paste0("probe = ", c(0.3, 0.5)),  
  text.col = c("red", "blue")  
)
```



**两点分布 (Bernoulli Distribution)** , 即一重伯努利实验, 为二项分布的特殊分布。

## 1.2 负二项分布 (Negative Binomial Distribution)

多重伯努利实验中, 已知事件  $A$  发生的概率为  $p$ , 那么当事件  $A$  第  $r$  次发生, 那么随机变量  $X$  —— 伯努利实验次数:

$$P(X = K) = C_{k-1}^{r-1} p^r (1-p)^{k-r}, k = r, r+1, \dots$$

记作:

$$X \sim Nb(r, p), \text{ Where } E(X) = \frac{r}{p}, D(X) = \frac{r(1-p)}{p^2}$$

**几何分布 (Geometric Distribution)** 为负二项分布的特殊分布, 即当  $r = 1$  时的负二项分布。

记为:

$$X \sim Ge(p)$$

### 1.3 超几何分布

不放回的随机抽样，设有  $N$  件产品，其中  $M$  件不合格品，从中不放回的随机抽取  $n$  件，则其中的不合格的件数服从超几何分布：

$$P(X = k) = \frac{C_M^K C_{N-M}^{n-k}}{C_N^n}$$

记为：  $X \sim h(n, N, M)$

$$E(X) = n \frac{M}{N}$$

$$D(X) = \frac{nM(N-M)(N-n)}{N^2(N-1)}$$

### 1.4 泊松分布 (Poisson Distribution)

涉及到单位时间，面积，体积的计数过程，数量  $X$ ：

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

记为：

$$X \sim P(\lambda)$$

$$E(X) = \lambda$$

$$D(X) = \lambda$$

## 2 连续分布

### 2.1 正态分布

正态分布含有两个参数  $\mu, \sigma$ ，其中  $\mu$  为位置参数，控制曲线在  $x$  轴上的位置； $\sigma$  为尺度参数，用于控制曲线的参数。记为：

$$X \sim N(\mu, \sigma)$$

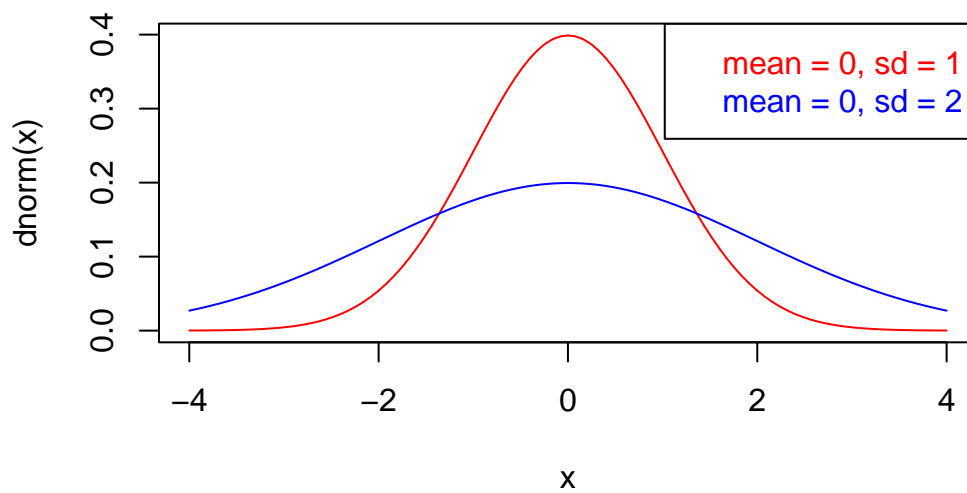
$$E(X) = \mu$$

$$D(X) = \sigma^2$$

概率密度函数：

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

```
curve(dnorm(x), from = -4, 4, col = "red")
curve(dnorm(x, 0, 2), from = -4, 4, add = TRUE, col = "blue")
legend(
  "topright",
  paste0("mean = 0, sd = ", c(1, 2)),
  text.col = c("red", "blue")
)
```



分布函数：

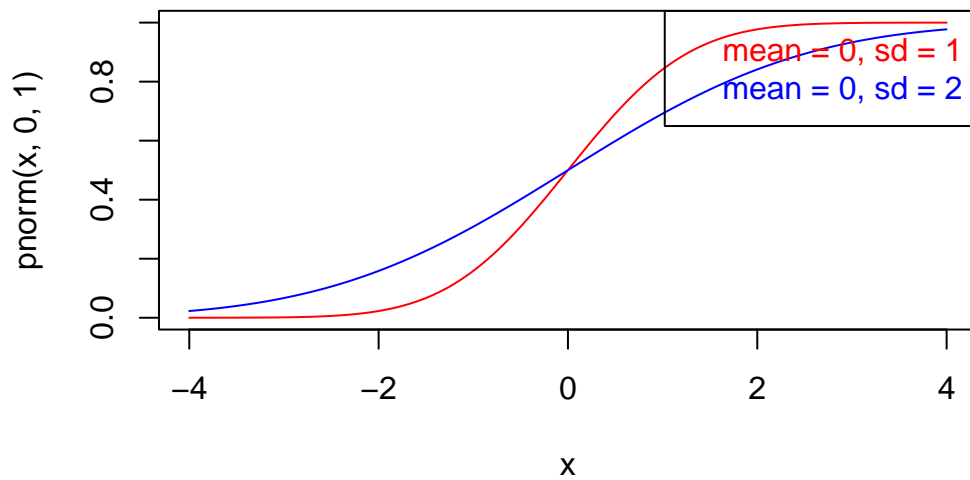
$$F(x) = \int_{-\infty}^x p(t)dt = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

```
curve(pnorm(x, 0, 1), from = -4, 4, col = "red")
curve(pnorm(x, 0, 2), from = -4, 4, add = TRUE, col = "blue")
legend(
```

```

"topright",
paste0("mean = 0, sd = ", c(1, 2)),
text.col = c("red", "blue")
)

```



## 2.2 均匀分布

记为:

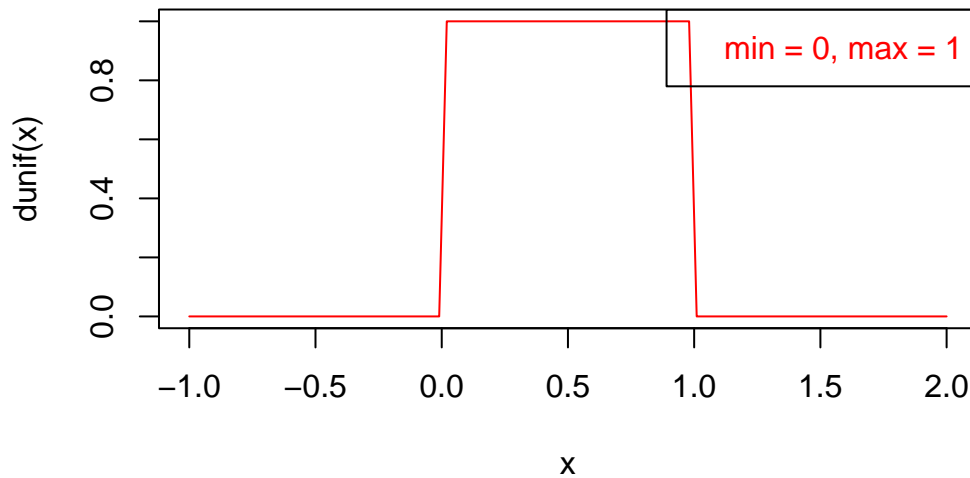
$$\begin{aligned}
 X &\sim U(a, b) \\
 E(X) &= \frac{a + b}{2} \\
 D(X) &= \frac{(b - a)^2}{12}
 \end{aligned}$$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

```

curve(dunif(x), -1, 2, col = "red")
legend("topright",
      legend = "min = 0, max = 1",
      text.col = "red"
)

```



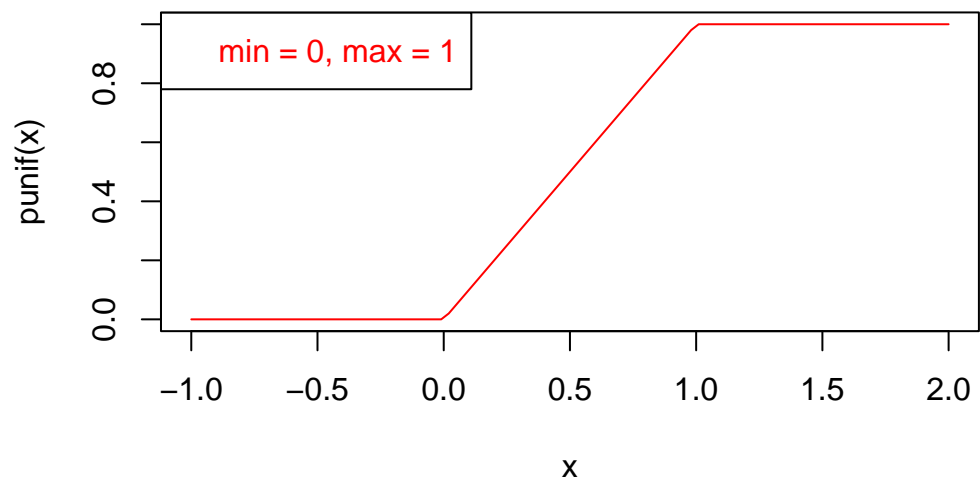
$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ \frac{x-a}{b-a} & \text{for } a \leq x < b, \\ 1 & \text{for } x \geq b. \end{cases}$$

```

curve(punif(x), -1, 2, col = "red")
legend("topleft",
      legend = "min = 0, max = 1",
      text.col = "red"
)

```





## 2.3 指数分布

记为:

$$X \sim \text{Exp}(\lambda)$$

$$E(X) = \frac{1}{\lambda}$$

$$D(x) = \frac{1}{\lambda^2}$$

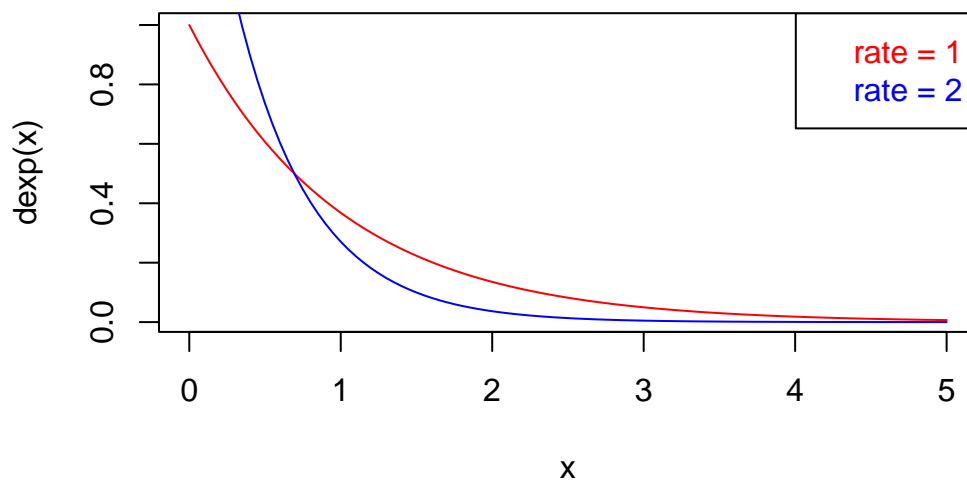
密度函数

$$f(x; \lambda) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0 \text{ and } \lambda > 0.$$

```

curve(dexp(x), 0, 5, col = "red")
curve(dexp(x, rate = 2), 0, 5, col = "blue", add = TRUE)
legend("topright",
      legend = paste0("rate = ", c(1, 2)),
      text.col = c("red", "blue")
)

```

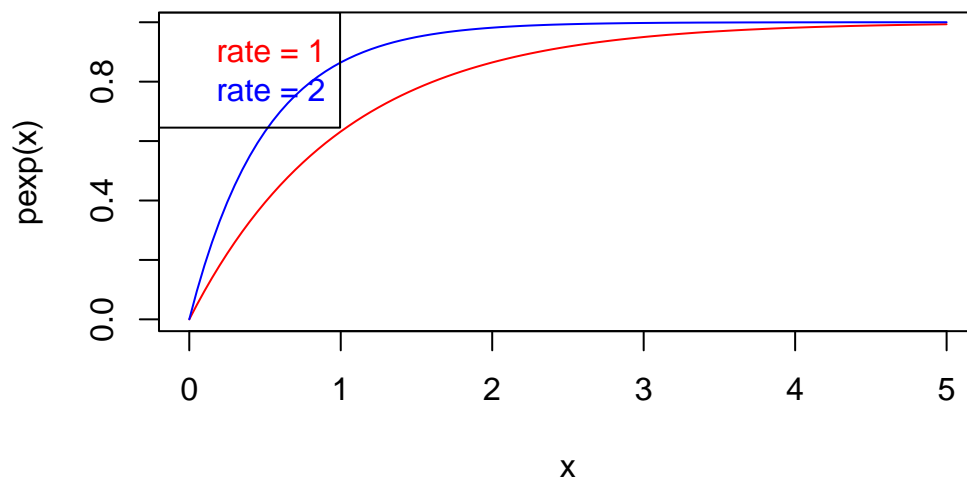


$$F(x; \lambda) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0 \text{ and } \lambda > 0.$$

```

curve(pexp(x), 0, 5, col = "red")
curve(pexp(x, rate = 2), 0, 5, col = "blue", add = TRUE)
legend("topleft",
      legend = paste0("rate = ", c(1, 2)),
      text.col = c("red", "blue")
)

```



## 2.4 $\Gamma$ 分布

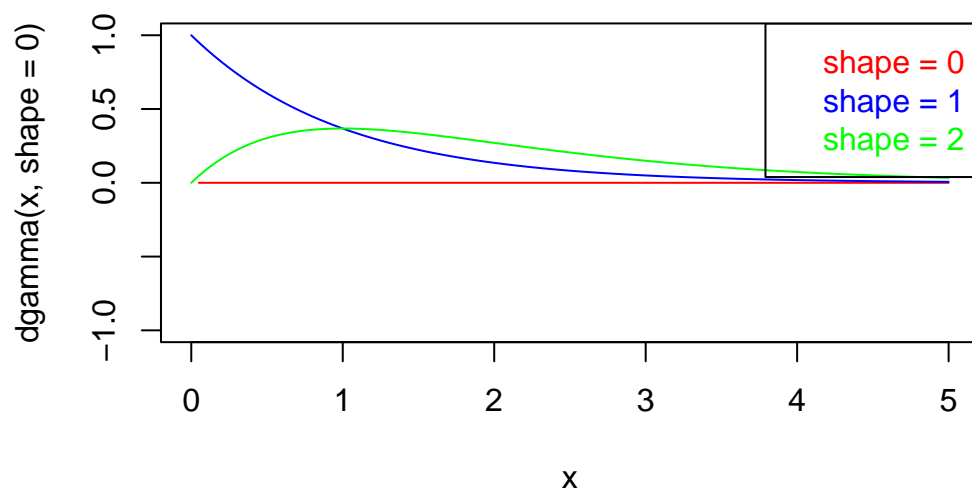
记为:  $X \sim Ga(\alpha, \lambda)$   $E(X) = \frac{\alpha}{\lambda}$ ,  $D(X) = \frac{\alpha}{\lambda^2}$

密度函数

$$f(x; k, \theta) = \frac{x^{k-1} e^{-\frac{x}{\theta}}}{\theta^k \Gamma(k)} \quad \text{for } x > 0 \text{ and } k, \theta > 0.$$

其中,  $k$  是形状参数 (也称为度数),  $\theta$  是尺度参数 (与标准差成比例), 而  $\Gamma(k)$  是伽马函数。

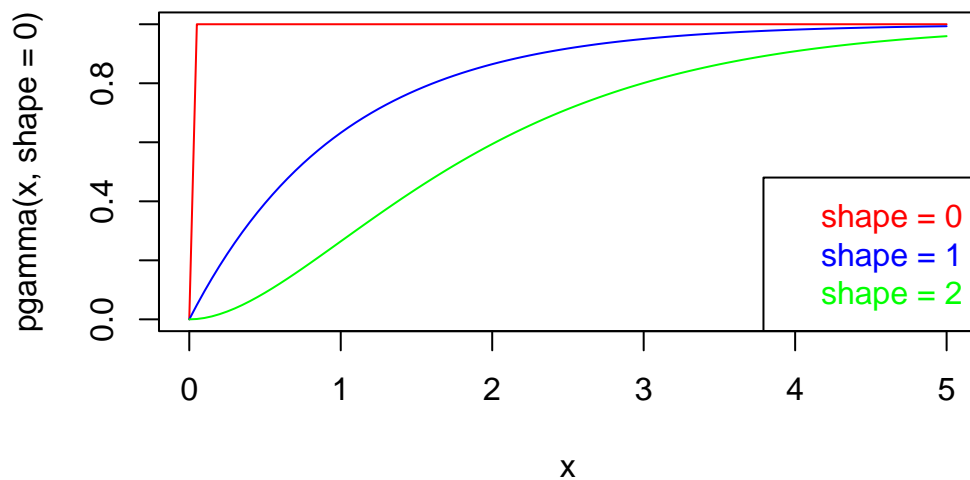
```
curve(dgamma(x, shape = 0), 0, 5, col = "red")
curve(dgamma(x, shape = 1), 0, 5, col = "blue", add = TRUE)
curve(dgamma(x, shape = 2), 0, 5, col = "green", add = TRUE)
legend("topright",
      legend = paste0("shape = ", c(0, 1, 2)),
      text.col = c("red", "blue", "green")
)
```



分布函数

$$F(x; k, \theta) = \int_0^x \frac{t^{k-1} e^{-\frac{t}{\theta}}}{\theta^k \Gamma(k)} dt = \frac{\gamma(k, \frac{x}{\theta})}{\Gamma(k)} \quad \text{for } x > 0 \text{ and } k, \theta > 0.$$

```
curve(pgamma(x, shape = 0), 0, 5, col = "red")
curve(pgamma(x, shape = 1), 0, 5, col = "blue", add = TRUE)
curve(pgamma(x, shape = 2), 0, 5, col = "green", add = TRUE)
legend("bottomright",
      legend = paste0("shape = ", c(0, 1, 2)),
      text.col = c("red", "blue", "green")
)
```



## 2.5 $\beta$ 分布

记为:  $X \sim Be(a, b)$   $E(X) = \frac{a}{a+b}$ ,  $D(x) = \frac{ab}{(a+b)^2(a+b+1)}$

密度函数

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)} \quad \text{for } 0 < x < 1 \text{ and } \alpha, \beta > 0,$$

分布函数

$$F(x; \alpha, \beta) = I_x(\alpha, \beta) = \frac{B_x(\alpha, \beta)}{B(\alpha, \beta)} \quad \text{for } 0 \leq x \leq 1 \text{ and } \alpha, \beta > 0$$

## 3 三大抽样分布

抽样分布指的是从总体中抽取样本，样本统计量的分布。这里首先给出三大抽样分布构造的定义；

卡方分布为特殊的伽玛分布，在概率论中其定义如下：

$$\chi = \gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

- 若  $\{X_i\}_{i=1}^n$  独立同分布于  $N(0, 1)$ ，那么  $\sum X_i^2 \sim \chi(n)$ ，其  $E(\chi^2) = n, \text{Var}(\chi^2) = 2n$ 。
- 若有  $\chi_1(m), \chi_2(n)$ ，那么  $\frac{\frac{\chi_1^2}{m}}{\frac{\chi_2^2}{n}} \sim F(m-1, n-1)$ 。
- 若有  $X \sim N(0, 1)$ ，以及  $\chi$ ，那么  $\frac{X}{\sqrt{\frac{\chi(n)}{n}}} \sim t(n-1)$

关于抽样分布的几个定理

### 定理一

若  $\{x_i\}_{i=1}^n$  是来自正态总体  $\mathcal{N}(\mu, \sigma^2)$  的样本，其样本均值和方差分别为

$$\bar{x} = \frac{1}{n} \sum x_i, s^2 = \frac{1}{n-1} \sum (x - \bar{x})^2$$

则：

1.  $\bar{X}$  与  $s^2$  相互独立。
2.  $\bar{X} \sim \mathcal{N}(\mu, \frac{1}{n}\sigma^2) \rightarrow \frac{\bar{X}-\mu}{\sigma \cdot \sqrt{\frac{1}{n}}} \sim \mathcal{N}(0, 1)$
3.  $(n-1)\frac{s^2}{\sigma^2} \sim \chi(n)$

### 定理二

若  $x, y$  分别是来自正态总体  $X, Y$  的样本，其样本方差分别为  $s_x, s_y$ ，则：

$$\frac{s_x^2/\sigma_x^2}{s_y^2/\sigma_y^2} \sim F(m-1, n-1)$$

### 定理三

设  $\{X_i\}_{i=1}^n$  是来自正态总体  $\mathcal{N}(\mu, \sigma)$  的样本，则：

$$\frac{\bar{x} - \mu}{s \cdot \sqrt{\frac{1}{n}}} \sim t(n-1)$$

test