Scale-free and stable structures in complex *ad hoc* networks

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Unlike the well-studied models of growing networks, where the dominant dynamics consist of insertions of new nodes and connections and rewiring of existing links, we study *ad hoc* networks, where one also has to contend with rapid and random deletions of existing nodes (and, hence, the associated links). We first show that dynamics based only on the well-known preferential attachments of new nodes do not lead to a sufficiently heavy-tailed degree distribution in *ad hoc* networks. In particular, the magnitude of the power-law exponent increases rapidly (from 3) with the deletion rate, becoming ∞ in the limit of equal insertion and deletion rates. We then introduce a local and universal compensatory rewiring dynamic, and show that even in the limit of equal insertion and deletion rates true scale-free structures emerge, where the degree distributions obey a power law with a tunable exponent, which can be made arbitrarily close to 2. The dynamics reported in this paper can be used to craft protocols for designing highly dynamic peer-to-peer networks and also to account for the power-law exponents observed in existing popular services.

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I. INTRODUCTION

Several random protocols (i.e., stochastic rules for adding/deleting nodes and edges) that lead to the emergence of scale-free networks have been recently proposed. Such scale free networks are characterized by so-called power-law degree distributions, where the probability that a randomly picked node in the network has degree k decreases polynomially with increasing k for large values of k, i.e., $P(k) \sim k^{-\gamma}$, where $\gamma > 0$ is referred to as the exponent of the power-law distribution. The underlying dynamics for almost all of these models follow the principle of preferential attachment for targeting or initiating newly created links of the network. The simplest case is the linear preference dynamic: a node is added to the network at each time step and introduces a constant number of new edges or links, where the destination node of each link is picked randomly with probability proportional to the current degree of the node. The resulting network for this simple model has a power-law degree distribution with an exponent $\gamma = 3$. Other variations of this procedure have also been widely studied [1,4–6].

Many of the interesting and potentially useful properties of random power-law networks appear when the degree exponent $\gamma < 3$. These properties include almost constant diameter and zero percolation threshold [5]. Moreover, almost all cases of power laws observed in real life networks, which these models ultimately might want to account for, have exponents less than 3 [5]. Motivated by both these issues, a few stochastic linking rules resulting in exponents with magnitude less than 3 have been introduced. Examples of such protocols include the doubly preferential attachment scheme for links, where both the initiator and the target nodes of an edge are chosen preferentially, as proposed in [3,6], and the rewiring scheme of existing links to preferential targets as proposed in [2]. Such parametrized protocols, where the degree exponent γ can be varied from as close to 2 as desired to higher values have been termed *universal* protocols. Moreover, if each node in the network makes connectivity decisions (e.g., adding or rewiring links) based only on its own information (e.g., the outcome of a random number generator, or the information that one of its edges has been deleted) then the protocol is said to be *local*.

Most of these random protocols have been motivated by the need to model growing and mostly rigid networks, where nodes and links are gradually added. Examples of such graphs are the citation and collaboration networks. Once a connection is made between two nodes in these graphs it is never deleted, and also nodes never leave the network. A second class of networks that has been studied is where the nodes are stable, but the links could be deleted. For example, on the Worldwide Web one can assume that nodes almost always remain in the network once created; however, existing links can easily be deleted and new links created. In this paper, we address a third class of networks (first introduced in [1]), where the nodes themselves are also unstable and unreliable, and in an extreme case, the nodes (and hence all their connections) might leave the network without prior notice and through independent decisions.

Our motivation for considering such dynamic networks comes, in part, from the recent interest in designing less structured or *ad hoc* distributed systems, with peer-to-peer (P2P) content sharing networks as a prime example. In an instance of Gnutella, for example, a study [7] shows that almost half of all nodes log off within two hours from their log in. Hence, the time scale within which the network assumes its structure is much shorter than the time scale within which it grows. A number of crawls of these networks show that at least in some regimes they follow a power law [7]. However, a stochastic model that can lead to the emergence of such complex networks has not been proposed. Another significant example is the *ad hoc* and mobile communication paradigm where each member can provide a short-time unreliable service and yet a global topological structure with reliable properties is to be ensured at all times.

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We first use the continuous rate equation approach introduced in [1] (see Sec. II) to predict the power-law exponent for stochastic models, where new nodes joining the network make links preferentially, and existing nodes in the network are uniformly deleted at a constant rate. We show that for such models the power-law degree distribution of the resulting network has an exponent $\gamma > 3$, and that it rapidly approaches ∞ as the deletion and insertion rates become equal. Thus a network with even small deletion rates (see Fig. 2 below) will essentially have characteristics that are more similar to an exponential degree distribution. In Sec. III, we introduce a compensatory rewiring procedure to exploit the deletion dynamic of the nodes itself to maintain a scale-free structure. In this protocol, in addition to the new nodes making preferential attachments, existing nodes compensate for lost links by initiating new preferential attachments. We show that the exponent of the power law for the degree distributions of the resulting networks for any deletion rate can be tuned as close to 2 as desired, and hence the proposed protocol is universal. Thus, we provide a local random protocol for generating scale-free networks even in the limit where the deletion and addition rates are equal and the network size is almost constant. Applications of the protocol designed in Sec. III to both analysis and design of complex and P2P networks are discussed in Sec. IV.

II. GROWING NETWORKS IN THE PRESENCE OF PERMANENT NODE DELETION

The scale-free properties of growing networks that incorporate preferential attachment with *permanent deletion of randomly chosen links* was considered by Dorogovtsev and Mendes [1]. They concluded that the scale-free properties of the emerging network depend strongly on the deletion rate of the links and are observed only for low deletion rates. However, the analysis of the effect of *random deletions of nodes at a fixed rate* was incomplete. A comprehensive analysis is presented in this section, and, as noted in the Introduction, the associated results are shown to have far-reaching consequences for *ad hoc* networks.

A. Preferential attachment and random node deletions

We consider the following model. At each time step, a node is inserted into the network and it makes m attachments to m preferentially chosen nodes. That is, for each of the links, a node with degree k is chosen as a target with probability proportional to k. Then with probability c a randomly chosen node is deleted.

We adopt the same approach as introduced in [1] for our analysis. Let each node in the network be labeled by the time it was inserted, i.e., the ith node in the network is the node that was inserted at time step i. Next define $k(i,t)$ as the degree of the ith node at time t, where $t > i$. Let $D(i,t)$ be the probability that the ith node is not deleted (i.e., it is still in the network) until time t, where $t > i$. Assuming the ith node to be in the network at time t, the rate at which its expected degree increases is

$$\frac{\partial k(i,t)}{\partial t} = m\frac{k(i,t)}{S(t)} - c\frac{k(i,t)}{N(t)}, \tag{1}$$

where

$$S(t) = \int_0^t D(i,t)k(i,t)di \tag{2}$$

is the sum of the degrees of all nodes that are present in the network at time t, and $N(t) = (1-c)t$ is the total number of nodes in the network. Note that the first term in Eq. (1) is simply the expected number of links node i receives as a result of the m preferential attachments made by the newly introduced node. The probability that a randomly chosen node is among the neighbors of node i, and hence the probability that node i loses a link, is, of course, $k(i,t)/N(t)$, which accounts for the second term in Eq. (1).

One can solve for the various unknown quantities in the following order: $D(i,t)$, $S(t)$, and then $k(i,t)$ using the approach of [1] (see the Appendix for detailed derivations). The results are quoted below:

$$D(i,t) = \left(\frac{t}{i}\right)^{c/(c-1)} \tag{3}$$

and

$$S(t) = 2m\frac{1-c}{1+c}t = 2m\frac{N(t)}{1+c}. \tag{4}$$

Inserting Eq. (4) back into the rate equation, we get

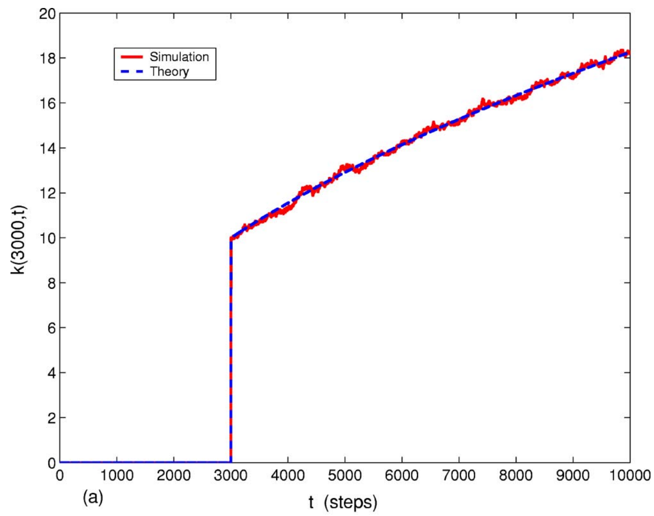
$$\begin{aligned}\frac{\partial k(i,t)}{\partial t} &= m\frac{(1+c)k(i,t)}{2m(1-c)t} - \frac{c}{1-c}\frac{k(i,t)}{t} \\ &= \frac{(1+c-2c)k(i,t)}{2(1-c)t} = \frac{k(i,t)}{2t},\end{aligned} \tag{5}$$

which implies that

$$k(i,t) = m\left(\frac{t}{i}\right)^{\beta}, \tag{6}$$

where $\beta = 1/2$. Equation (6) is quite significant since it states that the degree of a node in the network (when it is not deleted) does not depend on the deletion rate. To verify this, we have made numerous simulations for a wide range of deletion rates. Figure 1 shows the results for two rather extreme cases of 20% and 70% deletion rates, respectively.

Now, to calculate the probability $P(k,t)$ that a randomly chosen node at time t will have degree k we need to calculate the expected number of nodes at time t with degree k and divide it by the total number of nodes, $N(t)$. Let $I_k(t)$ be the set of all nodes i with degree k at time t. Since we are following a continuous-time rate equation approach, the number of nodes in $I_k(t)$ is the number of i's for which $k \leqslant k(i,t) < k+1$, which can be approximated as $|\partial k(i,t)/\partial i|_{i=i_k}^{-1}$, where i_k is the solution to the equation $k(i,t) = k$. Hence, we get



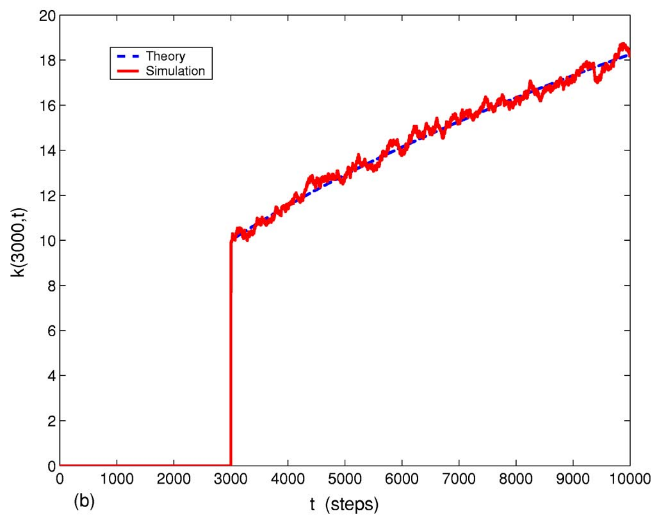


FIG. 1. The evolution of the degree of a node inserted in the network. The power-law growth, and its independence of the deletion rate, are at the heart of the results of Sec. II. (a) is the plot for the case of 20% deletion rate while (b) is for the case of 70% deletion rate. A node is inserted at time step $t=3000$, and its degree is recorded at future time steps (for $m=10$) until it gets deleted. Over 1000 trials, the degrees of this node (for the trials where it was not deleted until time step 10 000) are averaged, and the results are compared to predictions.

$$P(k,t)=\frac{[(\text{No. of nodes with degree})=k]}{(\text{Total number of nodes})}$$
$$=\frac{1}{N(t)}\sum_{i\in I_k(t)} D(i,t)=\frac{1}{N(t)}D(i_k,t)\left|\frac{\partial k(i,t)}{\partial i}\right|^{-1}_{i=i_k}. \tag{7}$$

From Eq. (6), we obtain

$$\frac{t}{i_k}=m^{-1/\beta}k^{1/\beta},$$

and thus

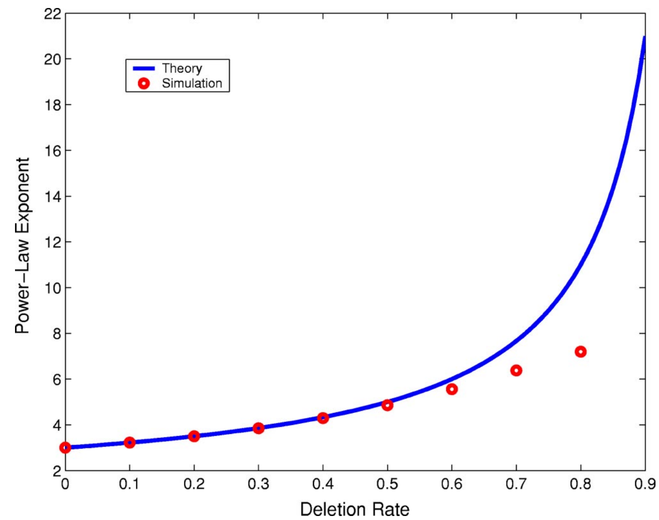


FIG. 2. The power-law exponent for the degree distribution of networks generated with the model discussed in Sec. II. The time steps at which snapshots are taken vary from 20 000 to 100 000 based on the deletion rate, so that at the time of the snapshot almost 20 000 nodes are in the network for all cases. The theory and the simulation results are in perfect agreement for $c\leqslant 0.6$. For larger values of c, however, tracking the rapidly growing exponent is rather hard, and the deviation is due to the finite number of time steps in the simulations. Note that the value of the exponent for $c>50\%$ is too large for the network to display any of the desirable properties usually associated with scale-free networks.

$$\left|\frac{\partial i}{\partial k(i,t)}\right|_{i=i_k}=m^{1/\beta}k^{-1/\beta-1}(-1/\beta)t. \tag{8}$$

Inserting it in Eq. (7), we get

$$P(k,t)=\frac{k^{-c/[(1-c)\beta]}}{(1-c)m^{-1/[\beta(1-c)]}}k^{-1/\beta-1}=\frac{k^{-1-(1/[(1-c)\beta])}}{(1-c)m^{-1/[\beta(1-c)]}}, \tag{9}$$

which is a power-law distribution with the exponent

$$\gamma=1+\frac{1}{(1-c)\beta}. \tag{10}$$

This equation for obtaining the power-law exponent from Eq. (6) for a general β will be used later on too. Note that $(\gamma-1)\beta=1/(1-c)$, which is a violation of the naive scaling relation which suggests $(\gamma-1)\beta=1$ (see [1,3] for a discussion about this general scaling rule). The reason for such violation is the effective renormalization of the number of nodes with a given degree (due to deletion), as also suggested in [1].

For our case of $\beta=1/2$ we get the exponent of

$$\gamma=1+\frac{2}{1-c}. \tag{11}$$

As illustrated in Fig. 2, simulation results provide a verification of Eq. (11).

B. Additional preferentially targeted links will not help

We now show that introducing new preferential attachments, as introduced in [2], will not help control the divergence of the exponent. To see this, let us modify the protocol as follows: At each time step, a new node is added and it makes m preferential attachments; c randomly chosen nodes are deleted; and a randomly chosen node initiates b preferentially targeted links.

Following the same steps as in the preceding section, one can show that $S(t)=2m(1+c)(b+1)t/(1-c)$, and one can verify that the rate equation would simplify to

$$\frac{\partial k(i,t)}{\partial t}=m\frac{(1+c)k(i,t)}{2m(1-c)t}-\frac{c}{1-c}\frac{k(i,t)}{t}=\frac{k(i,t)}{2t},$$

which is identical to Eq. (5), and hence results in the same power-law exponent as in Eq. (11).

C. The expected degree of any particular node

The degree of an existing node is governed by Eq. (6) until it gets deleted, when its degree can be assumed to be 0. Thus, the expected degree of the ith node at time t is given by (see [1,3])

$$\begin{aligned}E(i,t)&=K(i,t)D(i,t)\\&=m\left(\frac{t}{i}\right)^{c/(1-c)+\beta}\\&=m\left(\frac{t}{i}\right)^{[-(\beta+1)c+\beta]/(1-c)}.\end{aligned}\tag{12}$$

Hence, if we define $c_0=\beta/(\beta+1)=1/3$, then, for $c<c_0$, $E(i,t)\rightarrow\infty$, and, for $c>c_0$, $E(i,t)\rightarrow 0$ when $t/i\rightarrow\infty$.

III. THE COMPENSATION PROCESS

We now introduce a local and universal random protocol that will lead to the emergence of true scale-free networks when nodes are deleted at a fixed rate.

A. Deletion-compensation protocol

Consider the following process, where at each time step (1) a new node is inserted and it makes m connections to m preferentially chosen nodes; (2) with probability c, a uniformly chosen node and all its links are deleted; (3) If a node loses a link, then to compensate for the lost link it initiates $n<n_{\mathrm{crit}}(c)$ (n is real) links, the targets of which are chosen preferentially [the upper-bound $n_{\mathrm{crit}}(c)$ is specified later].

This protocol is simple in its description as well as in implementation. It is also truly local, i.e., the decisions for all nodes (whether to be deleted or to initiate a compensatory link) are independent and based on the node's own state.

B. Properties of the emergent network

1. Degree distribution

In order to write the rate equation for $k(i,t)$, the degree of the ith node at time t, we first recognize that the two terms on the right-hand side of the rate equation in Eq. (1) capture the dynamics introduced by the insertion of a new node and the random deletion of an existing node and hence will also be in the present rate equation. In addition, we need to include terms that capture the compensatory dynamics of the protocol outlined above. Let $S(t)$ be the sum of the degrees of all the nodes in the network at time t [as defined in Eq. (2)], and let $\langle k(t)\rangle=S(t)/N(t)$ be the average degree of nodes at time t. Then we note that (i) the probability that the ith node loses a link is $ck(i,t)/N(t)$, and hence the expected number of links it picks up is $nck(i,t)/N(t)$, and (ii) since each of the nodes that loses an edge [there are $c\langle k(t)\rangle$ such expected nodes] makes n new preferential attachments, the number of these new preferential edges picked up by the ith node is $nc\langle k(t)\rangle k(i,t)/S(t)$. Hence, the rate equation for $k(i,t)$ can be stated as

$$\frac{\partial k(i,t)}{\partial t}=m\frac{k(i,t)}{S(t)}-c\frac{k(i,t)}{N(t)}+nc\frac{k(i,t)}{N(t)}+nc\langle k(t)\rangle\frac{k(i,t)}{S(t)}.\tag{13}$$

Note that $D(i,t)$ is still given by Eq. (3). Next, for computing $S(t)$ we provide a direct method as an alternative to the approach taken in [1] and in the Appendix. Let $\mathcal{E}(t)=S(t)/2$ be the total number of edges/links in the network at time t. Then, $\mathcal{E}(t)$ is altered at the tth time step as follows: (i) the new node brings in m edges, (ii) with probability c, $\langle k(t)\rangle$ expected edges are removed due to the random deletion of a node, and (iii) with probability c, $n\langle k(t)\rangle$ expected new edges are added as part of the compensatory wiring aspect of the protocol. Hence, the rate equation for $\mathcal{E}(t)$ is

$$\frac{d\mathcal{E}(t)}{dt}=m-(c-nc)\langle k(t)\rangle=m-(c-nc)\frac{S(t)}{N(t)}.\tag{14}$$

Substituting $S(t)=2\mathcal{E}(t)$ and $N(t)=(1-c)t$, we get

$$S(t)=\frac{2m(1-c)}{1+c-2nc}t\quad\text{and}\quad\langle k(t)\rangle=\frac{2m}{1+c-2nc}=\langle k\rangle_c.\tag{15}$$

Inserting $S(t)$ back into Eq. (13) we get

$$\begin{aligned}\frac{\partial k(i,t)}{\partial t}&=\frac{k(i,t)}{2(1-c)t}(1+c-2nc-2c+4nc)\\&=\frac{k(i,t)}{2(1-c)t}(1-c+2nc).\end{aligned}\tag{16}$$

Hence, $k(i,t)=m(t/i)^{\beta}$, where $\beta=(1-c+2nc)/[2(1-c)]$. Next, applying Eq. (10), we get the power-law exponent to be

$$\gamma=1+\frac{2}{1-c+2nc}.\tag{17}$$

Note that, in this case, there is no singularity when $c\rightarrow 1$. In fact, for $c=1$ and $0<n\leqslant n_{\mathrm{crit}}(1)=1$, we get

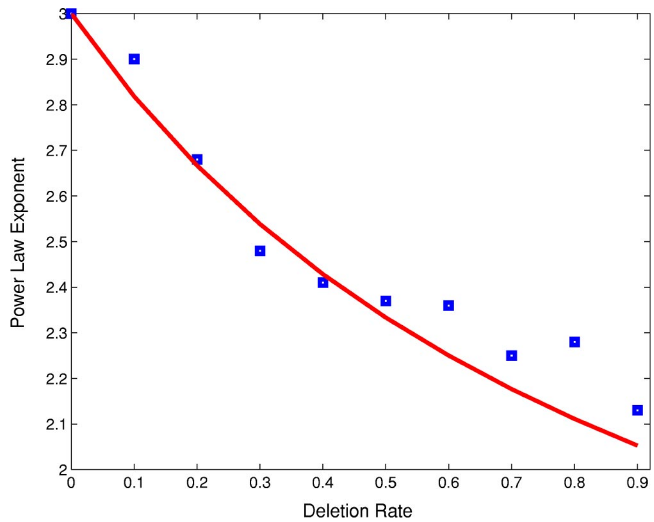


FIG. 3. The power-law exponent for different values of the deletion rate. The parameter n is taken to be 1. For all values of c, the number of nodes at the times the snapshots were taken was kept to be at least 20 000. The simulation results are indicated by □.

$$\gamma = 1 + \frac{1}{n}. \tag{18}$$

The power-law exponents are computed for the range $c = 0-90\,\%$ and $n=1$, and the results are checked against predictions in Fig. 3.

Note that Eq. (15) is valid only when the denominator is positive, which is equivalent to a finite average degree. So $1+c-2nc>0$, which implies that $-c+2nc<1$ and $\gamma>1+2/(1+1)=2$. This also implies that, for any given c, $0<n<n_{\rm crit}=(1+c)/(2c)$. Thus, *for any given deletion rate c, by varying the average number of compensatory edges for each deleted edge n, one can program the power-law exponent γ to be anywhere in* $(2,\infty)$. Of course, the price one pays for getting close to 2 is the associated increase in the average degree, as implied by Eq. (15). This also provides a hint for designing network protocols, that is, too many compensatory links might make the network unstable.

2. The expected degree of a random node at time t

Let us look at the quantity $E(i,t)$ as defined in Eq. (12):

$$\begin{aligned} E(i,t) = D(i,t)K(i,t) &= m\left(\frac{t}{i}\right)^{-c/(1-c)+\beta} \\ &= m\left(\frac{t}{i}\right)^{[-2c+(1-c+2nc)]/[2(1-c)]} \\ &= m\left(\frac{t}{i}\right)^{[1-c(3-2n)]/[2(1-c)]}. \end{aligned} \tag{19}$$

Hence, for any $n<n_{\rm crit}$ the expected degree of a random node would remain finite. A number of interesting observations can be made from the form of the dependence of γ on the parameters c and n. For example, for any $n>0.5$, and irrespective of the value of c, we have $\gamma<3$ and the underlying degree distribution has unbounded variance. Thus, one might want to work in the regime $1>n>0.5$ and $1\geqslant c>1/(3-2n)$. For example, if $n=0.75$ and $c\to 1$, then one can get an exponent of 2.33, and yet have the expected degree of any node be bounded.

IV. CONCLUDING REMARKS

We first point out a conceptual link between our compensatory rewiring scheme discussed in Sec. III, and the doubly preferential attachment scheme, as introduced in [3,6]. In the doubly preferential attachment protocol, some of the links have both the initiator and the target nodes chosen preferentially based on their degrees. For example, consider the following random protocol. At each time step, a new node is inserted that makes m connections to m preferentially chosen nodes. From the nodes in the network, l nodes are chosen with probabilities proportional to their degrees. Each of these selected nodes initiates m new links to m preferentially chosen targets. It can be shown [3,6] that the power-law exponent $\gamma=2+1/(1+2l)$, and hence, by choosing l, one can make the exponent as close to 2 as desired. In this regard, our compensatory rewiring scheme can be considered as a natural means for introducing doubly preferential attachments. By uniformly deleting nodes, a node loses links with probability proportional to its degree. So a node initiating a compensatory preferential attachment intrinsically introduces doubly preferential attachments. The random deletion of nodes is thus being used in our stochastic protocol to lead to the emergence of truly scale-free networks.

One of our main motivations for this work was to design random protocols that will solve the problem of organizing a highly dynamic content sharing network. The first step in this direction would be to design a local and easily implementable protocol that would lead to the emergence of a prespecified network structure under the usage constraints imposed by the users. As mentioned in the Introduction, although the network size usually grows for such networks (more people join such networks), the time scale within which the size changes is much larger than the time scale within which the old members of the network log in and log off. Hence, the desired form of the network structure should emerge almost solely due to the dynamics of the protocol and cannot rely too much on the growth rate itself. As regarding the desired structure of the network, motivated by many advantageous aspects of scale-free networks, one might want to come up with protocols that could make the network self-organize into a scale-free structure with a desired power-law exponent (usually around 2.5).

There has been some concern that searches on such power-law networks might not be scalable; however, our recent results show that, by using bond percolation on the underlying networks, one can make such networks very efficiently searchable. In particular, we show that for networks having a power-law degree distribution with exponent close to 2, a traffic efficient search strategy can be locally implemented. Specifically, we show that $O(\sqrt{N}\log^2(N))$ communications on those networks are sufficient to find each content with probability 1. This is to be compared to

$\Theta(N\log(N))$ communications for currently used broadcast protocols. Also, the search takes only $O(\log(N))$ time steps [9]. Thus, scale-free structures with exponents close to 2 not only are observed in current P2P systems, but also are the desirable structures for realizing a truly distributed and unstructured P2P database system.

The very high rate of log offs in real P2P networks prevents the ordinary preferential attachment scheme from forming a scale-free network with exponent less than 3 (as shown in Sec. II). The local compensation process introduced in Sec. III, however, imposes a scale-free structure with an exponent that can always be kept below 3. All a node has to do is to start a new preferential connection, whenever it loses one. Note that this compensatory procedure is quite natural (and probably essential) for networks in which the members have to be part of the giant connected component to be able to have access to almost all other nodes. In fact, in many clients of the existing P2P networks, this condition is imposed by always keeping a constant number of links to active IP addresses. Our numerical simulations show that graphs resulting from our compensatory protocol are almost totally connected; that is, a randomly chosen node with probability 1 belongs to the giant connected component of the graph even in the limit of $c=1$. Thus, *using our decentralized compensatory rewiring protocol one can launch, tune, and maintain a dynamic and searchable P2P content-sharing system*.

We also believe that our model can, at least intuitively, account for the degree distributions found in some crawls of P2P networks like Gnutella. As an example, in [7], the degree distribution of the nodes in a crawl of the network was found to be a power law with an exponent of 2.3. Although the Gnutella protocol [8] does not impose an explicit standard on how an agent should act when it loses a connection, there are certain software implementations of Gnutella which try always to maintain a minimum number of connections by trying to make new ones when one is lost. Thus, while all clients might not be compensating for lost edges, it is reasonable to assume that at least a certain fraction are. As shown in Sec. III, if we pick $n=0.75$ (i.e., 75% of the lost links are compensated for), and as $c\to 1$, the degree distribution is indeed a power law with exponent 2.33.

To summarize, we have designed truly local and yet universal protocols which, when followed by all nodes, result in robust, totally connected, and scale-free networks with exponents arbitrarily close to 2 even in an *ad hoc*, rapidly changing, and unreliable environment.

APPENDIX

We provide details for the derivations of the quantities $D(i,t)$ (the probability that the ith node is still in the network at time t, $t\geqslant i$) and $S(t)$ (the expected sum of the degrees of all the nodes in the network at time t) as introduced in Sec. II A. First, using the independence of the events corresponding to random deletions of nodes at each time step, it is easy to verify that $D(i,t+1)=D(i,t)[1-c/N(t)]$. Hence, the continuous version of the dynamic of $D(i,t)$ can be stated as follows:

$$\frac{\partial D(i,t)}{\partial t}=-c\frac{D(i,t)}{N(t)}=-\frac{c}{1-c}\frac{D(i,t)}{t}.$$

Since $D(t,t)=1$, we get $D(i,t)=(t/i)^{c/(c-1)}$. Note that $D(i,t)$ is solely determined by the deletion rate c, and hence its expression remains unchanged for the compensatory rewiring protocol introduced in Sec. III.

To find $S(t)$, we first multiply both sides of Eq. (1) by $D(i,t)$ and integrate out i from 0 to t. Then,

$$\int_0^t D(i,t)\frac{\partial k(i,t)}{\partial t}di=m-c\frac{S(t)}{(1-c)t}. \tag{A1}$$

The left-hand side of the above equation can now be simplified as follows:

$$\begin{aligned}\int_0^t \frac{\partial}{\partial t}\{D(i,t)k(i,t)\}di-\int_0^t k(i,t)\frac{\partial}{\partial t}D(i,t)di\\ =\frac{\partial}{\partial t}\left[\int_0^t\{D(i,t)k(i,t)\}di\right]-k(t,t)D(t,t)\\ -\int_0^t k(i,t)\frac{c}{t(c-1)}D(i,t)di.\end{aligned}$$

Substituting the above expression in Eq. (A1), and noting that $k(t,t)=m$, $D(t,t)=1$, and $S(t)=\int_0^t\{D(i,t)k(i,t)\}di$, we get

$$\frac{\partial S(t)}{\partial t}-m-\frac{c}{(c-1)t}S(t)=m-c\frac{S(t)}{(1-c)t}. \tag{A2}$$

The solution to the above equation is

$$S(t)=2m\frac{1-c}{1+c}t=2m\frac{N(t)}{1+c}.$$

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[9] N. Sarshar, P. O. Boykin, and V. Roychowdhury (unpublished).