Week 5

March 2025

Problem 1

Let's assume that the contrary is false, that is that every path has length less than d first. We will prove this problem by contradiction.

Let's pick the longest simple path in the graph and let's say that it goes like this: $v_a \to v_b \to v_c$, now let's concentrate on v_c. Since v_c has degree at least d, then it must have at least d neighbours and all of those neighbours are distinct. There are $c + 1 \leq d$ vertices in this path therefore there are at least $d - k \geq 1$ neighbours of v_c that are not a part of the path. We can therefore continue this path by one of those therefore contradicting the original assumption that this was the longest simple path. This completes the proof by contradiction.

Problem 2

a) If G is disconnected, then it has at least two components say C_1 and C_2. By the definition of disconnected graph, there is no edges between any component in C_1 and C_2. Therefore with the definition of the complement graph, for all x in C_1 and for y in C_2 there exists an edge (x, y). This completes the proof.

b)

$V = \{a, b, c, d\}$ $G = \{(a, b), (b, c), (c, d)\}$

The reasoning is that if you take the complement graph, it is connected.

Problem 3

Let tau be t for easier writing.

We can see that $\rho = (1\ 2\ 4)$ and $\sigma = (3\ 4\ 5)$

Let's first define tau to be:

$t(1) = 3$ $t(2) = 4$ $t(4) = 5$

such that it maps to σ:s cycle. Let's define $t(3) = 1$ and $t(5) = 2$ let's define t to be a permutation with the first row as $(1\ 2\ 3\ 4\ 5)$ and second row as $(3\ 4\ 1\ 5\ 2)$. If you go through the computations for 1 through 5 and compute the result, you can see that this makes the left hand side equal to the right hand side, therefore tau is:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$$

Problem 4

a) There are 4 rotations, 0 degrees, 90 degrees, 180 degrees and 270 degrees. There is also 4 reflections. Horizontal, vertical and through both of the diagonals.

b) ρ has order of 4, since there are 4 rotations and π has order 2 since π is just the reflection along the diagonal from bottom left to top right.

Therefore all of the permutations are the rotations ρ^0, ρ^1, ρ^2, ρ^3 and the reflections $\rho^0\pi$, $\rho^1\pi$, $\rho^2\pi$, $\rho^3\pi$ and when you go through each of these, none of these are duplicates. Since we argued in part a) that there are 8 total permutations, therefore these are all of the permutations. Therefore all of the permutations can be written in the form $\rho^n\pi^m$, when $n \in \{0, 1, 2, 3\}$ and $m \in \{0, 1\}$ This completes the proof.